89/100

RAJAN KAPOOR

STATISTICS 610-601: HW # 8

NOVEMBER 9, 2017

DISTRIBUTION THEORY

Prof. Irina Gaynanova

Homework 8 (due on Thursday November 9, 2017)

Problem 1: Consider independent random variables X and Y with joint pdf $f_{X,Y}$. Show that the independence of X and Y implies $E(X|Y) = EX$, and, by an example in a sample space of three points, that the reverse implication is false.

Problem 2: Independent random variables X and Y (with finite second moments) are always uncorrelated but the reverse implication is not necessarily true. Illustrate this statement with the following example.

		Y = 0	Y = 1	
X	0	1/6	1/4	10/24
	1	1/6	0	1/6
	2	1/6	1/4	10/24
		3/6 = 1/2	1/2	

Problem 3: Find E(V) and Var(V) if

(a) $W \sim$ exponential(β) (with $\beta > 0$) and $V|W \sim$ normal(W, 1),

(b) $W \sim$ gamma(α, 1) (with $\alpha > 0$) and $V|W \sim$ uniform($-W$, W),

(c) $W \sim$ beta(α, β) (with $\alpha > 0, \beta > 0$) and $V|W \sim$ Bernoulli(W).

Problems 4 to 5: numbers 4.30 and 4.47 from the textbook.

UIN: 225001987 RAJAN KAPOOR r.kapoor@tamu.edu

Solution 3

(a) $W \sim \text{expo}(\beta)$ $V|W \sim \text{normal}(W, 1)$

$$E(V) = E(E(V|W))$$
$$= E(W)$$
$$= \beta$$

$$Var(V) = E(Var(V|W)) + Var(E(V|W))$$
$$= E(1) + Var(W)$$
$$= 1 + \beta^2$$

(b) $W \sim \text{gamma}(\alpha, 1)$ $V|W \sim \text{uniform}(-W, W)$

$$E(V) = E(E(V|W))$$
$$= E(0)$$
$$= 0$$

$$Var(V) = E(Var(V|W)) + Var(E(V|W))$$
$$= E\left(\frac{4W^2}{12}\right) + Var(0)$$
$$= \frac{1}{3} E(W^2)$$
$$= \frac{1}{3}(Var(W) + E^2(W))$$
$$= \frac{1}{3}(\alpha + \alpha^2)$$

UIN: 225001987 RAJAN KAPOOR r.kapoor@tamu.edu

Solution2

$$Cov(X,Y) = E(XY) - \mu_X \mu_Y$$

$$E(XY) = \sum_{\forall (x,y)} xy\, f_{x,x}(x,y)$$

$$= 0 + 1\times 0\times \frac{1}{4} + 1\times 1\times 0 + 1\times 2\times \frac{1}{4}$$

$$= \frac{1}{2}$$

$$\mu_X = 0\times\left(\frac{10}{24}\right) + 1\times\left(\frac{1}{6}\right) + 2\times\left(\frac{10}{24}\right)$$

$$= \frac{1}{6} + \frac{20}{24} = \frac{1}{6} + \frac{5}{6} = 1$$

$$\mu_Y = 0 + 1\times\frac{1}{2} = \frac{1}{2}$$

$$Cov(X,Y) = \frac{1}{2} - 1\times\frac{1}{2} = 0$$

$$\Rightarrow Corr(X,Y) = 0$$

But X and Y are not independent

$$f_{XY}(0,0) = \frac{1}{6} \neq f_X(0)\, f_Y(0)$$

$$= \frac{10}{24}\times\frac{1}{2} = \frac{5}{24}$$

		Y = 0	Y = 1	
X	0	1/6	1/4	10/24
	1	1/6	0	1/6
	2	1/6	1/4	10/24
		1/2	1/2	

UIN: 225001987 RAJAN KAPOOR r.kapoor@tamu.edu

(c) $W \sim beta(\alpha, \beta)$ $V|W \sim Bernoulli(W)$

$$E(V) = E(E(V|W))$$
$$= E(P(W=1 | W \sim beta(\alpha, \beta)))$$
$$= E(0) = 0$$

$$Var(V) = Var(E(V|W)) + E(Var(V|W))$$
$$= Var(E(P(W=1))) + E(P(W=1) \times (1 - P(W=1)))$$
$$= Var(0) + E(0)$$
$$= 0$$

Solution 4

(a) $X \sim unif(0,1)$ $Y \sim N(x, x^2)$

$$EY = E(E(Y|X))$$
$$= E(x)$$
$$= 1/2$$

$$Var Y = E(Var(Y|X)) + Var(E(Y|X))$$
$$= E(X^2) + Var(X)$$
$$= 2Var(X) + E^2(X)$$
$$= 2\left(\frac{1}{12}\right) + \frac{1}{4}$$
$$= \frac{1}{6} + \frac{1}{4} = \frac{10}{24} = \frac{5}{12}$$

UIN: 225001987 RAJAN KAPOOR r.kapoor@tamu.edu

$$\text{Cov}(X,Y) = E(XY) - \mu_x \mu_Y$$

$$E(XY) = E(E(XY|X))$$

$$= E(X\, E(Y|X))$$

$$= E(X \cdot X)$$

$$= E(X^2) = Var(X) + E^2(X)$$

$$= \frac{1}{12} + \frac{1}{4} = \frac{4}{12} = 1/3$$

$$\text{Cov}(X,Y) = E(XY) - \mu_x \mu_Y$$

$$= \frac{1}{3} - \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{1}{3} - \frac{1}{4} = 1/12$$

(b) $\frac{Y}{X} | X \sim \mathcal{N}(1,1)$

which is independent of X

?

Solution 1

Steps!