Printed by Kapoor, Rajan on Nov 16, 2017 2:41:47 AM

98/100

RAJAN KAPOOR

STATISTICS 610-601: HW # 9

NOVEMBER 16, 2017

DISTRIBUTION THEORY

Prof. Irina Gaynanova

Homework 9 (due on Thursday November 16, 2017)

Problem 1: a random variable T that has a t distribution with $n > 2$ degrees of freedom can be represented as $T = \frac{Z}{\sqrt{Y/n}}$, where $Z \sim$ normal$(0, 1)$ and $Y \sim$ chi-square(n) are independent random variables. Use this to obtain E(T) and var(T).

Problem 2: number 4.44 from the textbook

Problem 3: number 4.58 (a), (b), (c) from the textbook

Problem 4-5: numbers 5.21, 5.22 from the textbook

UIN: 225001987 RAJAN KAPOOR r.kapoor@tamu.edu

Solution 1

(1) $T = \dfrac{Z}{\sqrt{Y/n}}$ $\quad Z \sim \mathcal{N}(0,1)$

$Y \sim \chi^2(n) \quad n > 2$

$$\Rightarrow T|Y \sim \mathcal{N}\left(0, \frac{n}{y}\right)$$

$$E(T) = E_Y(E(T|Y))$$
$$= E(0) = 0$$

$$Var(T) = Var(E(T|Y)) + E(Var(T|Y))$$
$$= Var(0) + E_Y\left(\frac{n}{Y}\right) = n\, E_Y\left(\frac{1}{Y}\right)$$

$$E_Y\left(\frac{1}{Y}\right) = \int_0^\infty \frac{1}{y} \frac{1}{\Gamma(\frac{n}{2})\, 2^{n/2}} y^{\frac{n}{2}-1} e^{-y/2}\, dy$$

$$= \frac{1}{2} \frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n}{2})} \int_0^\infty \underbrace{\frac{1}{\Gamma(\frac{n}{2}-1)\, 2^{n/2-1}} y^{(\frac{n}{2}-1)-1} e^{-y/2}}_{\text{pdf of chi-sq}(n-2)}\, dy$$

$$= \frac{1}{2} \frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n}{2})} \times 1$$

$$= \frac{1}{2} \frac{\Gamma(\frac{n}{2}-1)}{(\frac{n}{2}-1)\Gamma(\frac{n}{2}-1)} \qquad (\Gamma(\alpha+1) = \alpha\Gamma(\alpha))$$

$$E_Y\left(\frac{1}{Y}\right) = \frac{1}{2}\,\frac{1}{\frac{n-2}{2}} = \frac{1}{n-2}$$

$$\Rightarrow Var(T) = \frac{n}{n-2} \qquad n>2$$

Solution 2

(Proof by Induction)

Clearly base case is true by Theorem 4.5.6.

$$Var(X_1+X_2) = Var X_1 + Var X_2 + 2Cov(X_1,X_2) \quad \text{—(1)}$$

Assume it is true for $n=k$ i.e.

$$Var\left(\sum_{i=1}^{k} X_i\right) = \sum_{i=1}^{k} Var X_i + 2\sum_{1\le i<j\le k} Cov(X_i,X_j) \quad \text{—(2)}$$

To Prove: It is true for $n=k+1$ i.e.

$$Var\left(\sum_{i=1}^{k+1} X_i\right) = \sum_{i=1}^{k+1} Var X_i + 2\sum_{1\le i<j\le k+1} Cov(X_i,X_j)$$

Proof -

$$Var\left(\sum_{i=1}^{k+1} X_i\right) = Var\left(X_{k+1} + \sum_{i=1}^{k} X_i\right)$$

Using base case,

$$= Var(X_{k+1}) + Var\left(\sum_{i=1}^{k} X_i\right) + 2\,Cov\left(X_{k+1}, \sum_{i=1}^{k} X_i\right)$$

UIN: 225001987 RAJAN KAPOOR r.kapoor@tamu.edu

$$= Var(X_{k+1}) + Var\left(\sum_{i=1}^{k} X_i\right) + 2\sum_{i=1}^{k} Cov(X_{k+1}, X_i)$$

Using ②

$$= \left(Var(X_{k+1}) + \sum_{i=1}^{k} Var(X_i)\right) + 2\sum_{1\le i<j\le k} Cov(X_i, X_j) +$$

$$2\sum_{i=1}^{k} Cov(X_i, X_{k+1})$$

$$\left[\begin{array}{l} \because \; Cov(X+Y, Z) = Cov(X,Z) + Cov(Y,Z) \\ \text{and} \;\; Cov(X,Y) = Cov(Y,X) \end{array}\right]$$

$$= \sum_{i=1}^{k+1} Var\, X_i + 2\sum_{1\le i<j\le k+1} Cov(X_i, X_j)$$

Solution 3

(a) $Cov(X, Y) = Cov(X, E(Y|X))$

$$Cov(X, E(Y|X)) = E(X \cdot E(Y|X)) - E(X)\,E(E(Y|X))$$

$$= E(E(XY|X)) - E(X)\,E(Y)$$

$$= E(XY) - E(X)E(Y)$$

$$= Cov(X, Y)$$

(b) $Cov(X, Y - E(Y|X))$

$= cov(X, Y) - Cov(X, E(Y|X))$

$= cov(X, Y) - cov(X, Y)$ (Using result from (a))

$= 0$

(c) $Var(Y - E(Y|X)) = E(Var(Y|X))$

(?)

UIN: 225001987 RAJAN KAPOOR r.kapoor@tamu.edu

Solution 4

$$P(\max(X_1, X_2) > m)$$

$$= 1 - P(\max(X_1, X_2) < m)$$

$$= 1 - P(X_1 < m)\,P(X_2 < m)$$

$$= 1 - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$= 1 - \frac{1}{4} = \frac{3}{4}$$

for $n > 2$

$$P(\max(X_1 \ldots X_n) > m) = 1 - P(X_1 < m)\,P(X_2 < m) \cdots P(X_n < m)$$

$$= 1 - \frac{1}{2^n}$$

Solution 5

$Z = \min(X, Y)$

$$F_{Z^2}(z) = P(Z^2 \leq z)$$

$$= P(|Z| \leq \sqrt{z})$$

$$= P(|X| \leq \sqrt{z})\,P(X \leq Y) + P(|Y| \leq \sqrt{z})\,P(Y \geq X)$$

$P(X \leq Y) = P(X \geq Y) = 1/2$ $\quad P(|X| \leq \sqrt{z}) = P(|Y| \leq \sqrt{z})$

(identically distributed)

$$\Rightarrow F_{Z^2}(z) = \frac{1}{2} \times 2 \times P(|X| \le \sqrt{z})$$

$$= P(-\sqrt{z} \le X \le \sqrt{z})$$

$$= P(X \le \sqrt{z}) - P(X \le -\sqrt{z})$$

$$= F_X(\sqrt{z}) - F_X(-\sqrt{z})$$

$$f_{Z^2}(z) = f_X(\sqrt{z}) \frac{1}{2\sqrt{z}} + f_X(\sqrt{z}) \frac{1}{2\sqrt{z}}$$

$$= \frac{f_X(\sqrt{z})}{\sqrt{z}} = \frac{1}{\sqrt{2\pi}} e^{-z/2} \cdot z^{-1/2}$$

which is pdf of $\chi^2(1)$ i.e. chi-square r.v. with 1 df.