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STATISTICS 610-601: HW # 2

SEPTEMBER 21, 2017

DISTRIBUTION THEORY

Prof. Irina Gaynanova

Homework 2 - Due Thu, Sep 21st, in class

Problems 1-3: numbers 1.36, 1.39 and 1.44 from the textbook.

Problem 4. Suppose there were seven road accidents in one week. What is the probability that they all happened on different days?

Problem 5. Consider

$$\mathcal{S} = \{1,2,3,4\},\ A = \{1,2\},\ B = \{2,3\};$$
$$P = \text{discrete uniform distribution on } \mathcal{S}.$$

Find an event $C \subset \mathcal{S}$ such that A, B and C are pairwise independent, but not mutually independent. Justify your answer.

Problem 6. Consider the probability mass function of the binomial distribution

$$f(j) = b_j(n,p) = \binom{n}{j} p^j (1-p)^{n-j}, \quad j = 0,1,\ldots,n,\ n \in \mathbb{N},\ 0 < p < 1.$$

Show that the binomial distribution can sometimes be approximated by the Poisson distribution. Formally, for $p = p_n \in (0,1)$, $\lim_{n\to\infty} p_n = 0$, $\lim_{n\to\infty} np_n = \lambda > 0$ holds

$p = f(n)$ $\quad$ $\propto \frac{1}{n}$

$$\lim_{n\to\infty} b_j(n,p_n) = e^{-\lambda}\frac{\lambda^j}{j!} \quad \text{for every } j \in \{0,1,\ldots\}.$$

Hint: $\lim_{n\to\infty}\left(1 + \frac{\lambda+\varepsilon_n}{n}\right)^n = e^\lambda$ for $\lambda \in \mathbb{R}$, $\lim_{n\to\infty}\varepsilon_n = 0$.

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Please label each page of your homework clearly with your name IN BLOCK CAPITALS and your UIN. If you use more than one sheet of paper, please staple the sheets together.

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Solution 1:

$$P(\text{"hit atleast twice"}) = 1 - P(\text{"no hit"}) - P(\text{"hit once"})$$

$$= 1 - \left(\frac{4}{5}\right)^{10} - {}^{10}C_1 \times \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^{9}$$

$$= 1 - \left(\frac{4}{5}\right)^{9}\left(\frac{4}{5} + \frac{10}{5}\right)$$

$$= 1 - \left(\frac{4}{5}\right)^{9}\left(\frac{14}{5}\right)$$

(Calculated in R) $= 0.6242$

$$P(\text{"hit atleast twice"} \mid \text{"hit atleast once"})$$

$$= 1 - P(\text{"no hit"} \mid \text{hit atleast once}) - P(\text{hit once} \mid \text{hit atleast once})$$

$$\left[\because P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B) \quad (\text{for disjoint } A_1, A_2)\right]$$

$$= 1 - 0 - P(\text{hit once} \mid \text{hit atleast once})$$

$$= 1 - \frac{P(\text{hit once} \cap \text{hit atleast once})}{P(\text{hit atleast once})}$$

$$= 1 - \frac{P(\text{hit once})}{P(\text{hit atleast once})}$$

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$$= 1 - \frac{{}^{10}C_1 \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^9}{1 - P(\text{no hit})}$$

$$= 1 - \frac{10 \times 4^9}{1 - \left(\frac{4}{5}\right)^{10}} \times \frac{1}{5^{10}} = 1 - \frac{10 \times 4^9}{5^{10} - 4^{10}}$$

(calc. in R) $= 0.6993$

As expected probability should increase.

Solution 2:

(a) $P(A) > 0$ & $P(B) > 0$

Given A and B are mutually exclusive (disjoint) $\Rightarrow P(A \cap B) = 0$ — ①

But product of two non-zero real numbers cannot be zero $\Rightarrow P(A)P(B) \neq 0$ — ②

From ① and ②

$$P(A \cap B) \neq P(A) \cdot P(B)$$

$\therefore$ A and B are not independent

(b) Given A and B are independent,

$\Rightarrow P(A \cap B) = P(A) \cdot P(B)$ — ①

Since product of two non zero real numbers cannot be zero, $P(A) \cdot P(B) \neq 0$ — ②

From ① and ② $P(A \cap B) \neq 0$

$\Rightarrow$ A and B are not mutually exclusive.

Solution 3

$P_k = P(\text{"exactly } k \text{ questions correct"}) = {}^{20}C_k \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{20-k}$

$P(\text{atleast 10 questions correct}) = \sum_{k=10}^{20} P_k$

(calculated using R) $= 0.0139$.

Solution 5

$P(A) = \frac{1}{2}$ $P(B) = \frac{1}{2}$

$A \cap B = \{2\} \Rightarrow P(A \cap B) = \frac{1}{4}$

For A, B, C to be pairwise independent

$P(A \cap C) = P(A) \cdot P(C) = \frac{P(C)}{2}$

& $P(C \cap B) = P(C) \cdot P(B) = \frac{P(C)}{2}$

Assume $P(A \cap C)$ has k_1 elements and $P(C \cap B)$ has k_2 elements then,

$P(A \cap C) = \frac{K_1}{4} = \frac{P(C)}{2} \Rightarrow P(C) = \frac{k_1}{2}$

$P(B \cap C) = \frac{K_2}{4} = \frac{P(C)}{2} \Rightarrow P(C) = \frac{K_2}{2}$

$0 < P(C) < 1$ We have $k_1 = k_2 = K$

$P(C)$ cannot be 0 since $P(A \cap B \cap C) = 0$
$= P(A)\, P(B)\, P(C)$

$P(C)$ cannot be 1 since in that case
$P(A \cap B \cap C) = P(A \cap B)$
$= P(A)\, P(B)$
$= P(A)\, P(B)\, P(C)$

K can only take value 1

$P(A \cap C)$ has 1 element
$P(B \cap C)$ has 1 element } (1)

Also $P(C) = \frac{k}{2} = \frac{1}{2} = \frac{2}{4}$

$\Rightarrow$ C has 2 elements —— (2)

From (1) & (2) $C = \{2,4\}$ or $C = \{1,3\}$

For $C = \{2,4\}$

$P(A \cap B \cap C) = \frac{1}{4}$ & $P(A)\, P(B)\, P(C) = \frac{1}{8}$

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For $C = \{1, 3\}$

$P(A \cap B \cap C) = 0$ & $P(A)P(B)P(C) = \frac{1}{8}$

$\therefore$ $C = \{1, 3\}$ and $C = \{2, 4\}$ are both valid choices.

From above proof, these are also the only two choices.

Solution 4

Total no. of ways

Each accident can happen on any of the 7 days

$$\underset{Acc1}{7} \quad \underset{Acc2}{7} \quad \cdots \quad \underset{Acc7}{7} = 7^7 \text{ ways}$$

(accidents on same day - order not considered)

Favourable no. of cases.

Acc 1 can happen on any one of the 7 days, acc. 2 can happen on any one of the remaining 6 day and so on

$\Rightarrow$ There are $7!$ ways of all happening on different days.

$\therefore$ $P(\text{all accident on diff days}) = \frac{7!}{7^7}$

(calc in R) $= 0.0061$

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Solution 6.

Let $P_n = \frac{\lambda}{n} \Rightarrow \lim_{n\to\infty} P_n = 0$ & $\lim_{n\to\infty} n P_n = \lambda$

$$f(j) = \binom{n}{j} p^j (1-p)^{n-j} = b_j(n,p)$$

$$\Rightarrow \lim_{n\to\infty} b_j(n, P_n) = \lim_{n\to\infty} \frac{n!}{j!(n-j)!} \left(\frac{\lambda}{n}\right)^j \left(1-\frac{\lambda}{n}\right)^{n-j}$$

$$= \frac{\lambda^j}{j!} \lim_{n\to\infty} \frac{n!}{(n-j)!} \times \frac{1}{n^j} \left(1-\frac{\lambda}{n}\right)^{n-j}$$

$$= \frac{\lambda^j}{j!} \lim_{n\to\infty} \underbrace{\frac{n(n-1)\cdots(n-j+1)}{n\cdot n\cdots n}}_{\substack{\text{approaches } 1 \\ \text{as } n\to\infty}} \times \overbrace{\left(1-\frac{\lambda}{n}\right)^n}^{\substack{\text{given app to } e^{-\lambda} \\ \text{with } \epsilon=0}} \times \underbrace{\left(1-\frac{\lambda}{n}\right)^{-j}}_{\substack{\text{approaches } 1 \\ \text{as } n\to\infty}}$$

$$= \frac{\lambda^j}{j!} \times e^{-\lambda} \times 1 \times 1$$

$$= \frac{\lambda^j}{j!} e^{-\lambda}$$

Homework 2 Calculations

R Notebook for calculations in homework 2 for STAT 610: Distribution Theory, Fall 2017.

Calculation for Problem 1.

```
# Part 1
1-(14/5)*(4/5)^9

## [1] 0.6241904

# Part 2
1-(10*4^9)/(5^10-4^10)

## [1] 0.6992744
```

Calculation for Problem 3.

```
# Define function comb() to calculate number of combinations n choose x
comb = function(n, x) {
  return(factorial(n) / (factorial(x) * factorial(n-x)))
}
k = seq(from = 10, to = 20, by = 1)
sum(comb(20,k)*(1/4)^k*(3/4)^(20-k))

## [1] 0.01386442
```

Calculation for Problem 4.

```
factorial(7)/(7^7)

## [1] 0.006119899
```