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STATISTICS 610-601: HW # 6

99/100

OCTOBER 19, 2017

DISTRIBUTION THEORY

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Homework 6 (due on Thursday October 19, 2017)

Problem 1: The goal of this problem is to show that if X has a $N(0,1)$ distribution, then $P(X > x) \leq e^{-x^2/2}$ for every $x > 0$. We shall employ the "Cramer" trick or "exponential Markov" trick to show this.

Work through the following steps:

(i) Show that, for any $\lambda > 0$, $P(X \geq x) = P(e^{\lambda X} \geq e^{\lambda x}) \leq e^{-\lambda x} E(e^{\lambda X})$.

(ii) Show that $E(e^{\lambda X}) = e^{\lambda^2/2}$.

(iii) Simplify and minimize over all positive values of λ. [Note: Since the inequality is valid for any $\lambda > 0$, it remains valid if you substitute the minimizing value of λ. You should make sure that the minimizer is positive.]

Problems 2-4: numbers 3.28 (c)-(e), 3.38 and 3.39 from the textbook.

Problem 5: Consider $X \sim expon(\beta)$, that is

$$f_X(x) = \frac{1}{\beta}\exp(-x/\beta), \ x > 0.$$

(i) Show that the exponential distribution is memoryless, that is

$$P(X > x + t | X > t) = P(X > x), \ x, t > 0.$$

(ii) Let $Y = X^{1/\gamma}$ for $\gamma > 0$, then Y has the Weibull(γ, β) distribution. Derive its pdf, mean and variance.

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Solution 5

$$f_X(x) = \frac{1}{\beta}\exp\left(-\frac{x}{\beta}\right) \qquad x > 0$$

(i) $P(X > x+t \mid X > t) = \dfrac{P(X > x+t, X > t)}{P(X > t)}$

$t > 0 \Rightarrow x+t > t$
$x > 0$
$\Rightarrow P(X > x+t, X > t) = P(X > x+t)$

$$\Rightarrow P(X > x+t \mid X > t) = \frac{P(X > x+t)}{P(X > t)}$$

$$= \frac{\int_{x+t}^{\infty} \frac{1}{\beta}\exp\left(-\frac{x}{\beta}\right)dx}{\int_{t}^{\infty} \frac{1}{\beta}\exp\left(-\frac{x}{\beta}\right)dx}$$

$$= \frac{\left[-\frac{1}{\beta}\exp\left(-\frac{x}{\beta}\right)\beta\right]_{x+t}^{\infty}}{\left[-\frac{1}{\beta}\exp\left(-\frac{x}{\beta}\right)\beta\right]_{t}^{\infty}}$$

$$= \frac{\exp\left(-\frac{(x+t)}{\beta}\right)}{\exp\left(-\frac{t}{\beta}\right)}$$

$$= \exp\left(-\frac{x}{\beta}\right)$$

$$= \int_{x}^{\infty} \frac{1}{\beta}\exp\left(-\frac{t}{\beta t}\right)dt$$

$$= P(X > x)$$

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(ii) $Y = X^{1/\gamma}$ for $\gamma > 0$

$y = x^{1/\gamma}$ is an increasing f^n of x for $x > 0$, $\gamma > 0$

$$\Rightarrow f_y(y) = f_x(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)$$

$$= \frac{1}{\beta} \exp\left(-\frac{y^\gamma}{\beta}\right) \gamma y^{\gamma-1}$$

$$= \frac{\gamma}{\beta} y^{\gamma-1} \exp\left(-\frac{y^\gamma}{\beta}\right) \qquad \gamma > 0,\ \beta > 0,\ 0 < y < \infty$$

Mean

$$\mu = \int_{-\infty}^{\infty} f_y(y)\,dy = \int_0^\infty y \frac{\gamma}{\beta} y^{\gamma-1} \exp\left(-\frac{y^\gamma}{\beta}\right) dy$$

$\frac{y^\gamma}{\beta} = t$ $\quad y = (t\beta)^{1/\gamma}$

$$= \int_0^\infty (t\beta)^{1/\gamma} \exp(-t)\,dt$$

$\Rightarrow \frac{\gamma y^{\gamma-1}}{\beta} dy = dt$

$t = 0$ to ∞

$$= \beta^{1/\gamma} \int_0^\infty t^{(1+\frac{1}{\gamma})-1} \exp(-t)\,dt$$

$$= \beta^{1/\gamma}\, \Gamma\left(1 + \frac{1}{\gamma}\right)$$

Variance

$$\operatorname{Var} Y = E(Y) - [E(Y)]^2$$

$$E(Y^2) = \int_0^\infty y^2 \frac{\gamma}{\beta} y^{\gamma-1} \exp\left(-\frac{y^\gamma}{\beta}\right) dy$$

Using similar substitution as in mean calculation,

$$E(Y^2) = \Gamma\left(1+\frac{2}{\gamma}\right) \cdot \beta^{2/\gamma}$$

$$\Rightarrow \operatorname{Var} Y = \beta^{2/\gamma}\,\Gamma\left(1+\frac{2}{\gamma}\right) - \beta^{2/\gamma}\,\Gamma^2\left(1+\frac{1}{\gamma}\right)$$

$$= \beta^{2/\gamma}\left[\Gamma\left(1+\frac{2}{\gamma}\right) - \Gamma^2\left(1+\frac{1}{\gamma}\right)\right]$$

Solution 3

$$P(X > x_\alpha) = \int_{x_\alpha}^{\infty} f_X(x)\,dx$$

$$= \int_{x_\alpha}^{\infty} \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) dx$$

$$\frac{x-\mu}{\sigma} = z \quad \Rightarrow \frac{dx}{\sigma} = dz \qquad z : \frac{x_\alpha - \mu}{\sigma} \rightarrow \infty$$

$$P(X > x_\alpha) = \int_{\frac{x_\alpha-\mu}{\sigma}}^{\infty} f(z)\,dz$$

$$= \int_{z_\alpha}^{\infty} f(z)\,dz = P(Z > z_\alpha) = \alpha$$

Solution 4

(a) $$P(X \geq \mu) = \int_{\mu}^{\infty} \frac{1}{\sigma\pi\left[1+\left(\frac{x-\mu}{\sigma}\right)^2\right]}\,dx$$

$$\frac{x-\mu}{\sigma} = z \qquad \frac{dx}{\sigma} = dz \qquad z : 0 \rightarrow \infty$$

$$P(X \geq \mu) = \int_0^{\infty} \frac{1}{\pi}\frac{1}{1+z^2}\,dz = \frac{1}{\pi}\left[\tan^{-1}\infty - \tan^{-1}0\right]$$

$$= \frac{1}{\pi} \times \pi/2 = 1/2$$

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Similarly

$$P(X \le \mu) = \int_{-\infty}^{0} \frac{1}{\pi} \frac{1}{1+z^2} dz = \frac{1}{\pi}\left[-\left(-\frac{\pi}{2}\right)\right] = \frac{1}{2}$$

(b) let $Z = \frac{X-\mu}{\sigma}$ $\Rightarrow f_Z(z) = f_X(\sigma z+\mu)\cdot\sigma$

$$= \frac{1}{\pi}\frac{1}{1+z^2} \quad z \in \mathbb{R}$$

$X = \mu \Rightarrow z = 0$

$$P(X \ge \mu+\sigma) = P\left(\frac{X-\mu}{\sigma} \ge 1\right)$$

$$= P(Z \ge 1)$$

$$= \int_1^\infty f_Z(z)\,dz$$

$$= \int_1^\infty \frac{1}{\pi}\frac{1}{1+z^2} dz$$

$$= \frac{1}{\pi}\left[\tan^{-1}(\infty) - \tan^{-1}(1)\right]$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{1}{4}$$

$$P(X \le \mu-\sigma) = P\left(\frac{X-\mu}{\sigma} \le -1\right)$$

$$= P(Z \le -1)$$

$$= \int_{-\infty}^{-1} \frac{1}{\pi}\frac{1}{1+z^2} dz$$

$$= \frac{1}{\pi}\left(-\frac{\pi}{4} - \left(-\frac{\pi}{2}\right)\right) = \frac{1}{4}$$

Solution 1

$X \sim N(0,1)$

(i) $P(X \geq x) = P(e^{\lambda X} \geq e^{\lambda x})$

$\because$ for $\lambda > 0$, $x > 0$

$e^{\lambda x}$ is an increasing f^n

taking exp both sides doesn't change the inequality

$$P(e^{\lambda X} \geq e^{\lambda x}) = E[\mathbb{1}_{(|e^{\lambda X}| \geq e^{\lambda x})}]$$

$$\leq E\left[\frac{|e^{\lambda X}|^m}{(e^{\lambda x})^m}\right]$$

$m = 1$

$$P(e^{\lambda X} \geq e^{\lambda x}) \leq e^{-\lambda x} E[e^{\lambda X}]$$

(ii) $E(e^{\lambda X})$

$$MGF_X(t) = e^{t^2/2}$$

$t = \lambda$

$$E(e^{\lambda X}) = MGF_X(t=\lambda) = e^{\lambda^2/2}$$

(iii) $P(X \geq x) \leq e^{\lambda^2/2} \cdot e^{-\lambda x}$

$$\leq e^{(\lambda - x)^2/2} \, e^{-x^2/2}$$

Since this is true for all $\lambda > 0$, and since $x > 0$. Taking the limiting case $\lambda \to x$

$$P(X \geq x) \leq \lim_{0<\lambda\to x} e^{(\lambda-x)^2/2} \cdot e^{-x^2/2}$$

$$= e^{-x^2/2}$$

Solution 2

(d) Poisson family

$$f(x|\lambda) = e^{-\lambda} \frac{\lambda^x}{x!} \qquad \lambda \geq 0, \quad x = 0,1,2,\ldots$$

$$= e^{-\lambda} \frac{e^{\log \lambda^x}}{x!}$$

$$= e^{-\lambda} \cdot \frac{1}{x!} \exp(x \log \lambda)$$

$$h(x) = \frac{1}{x!} \mathbb{1}_{\{0,1..\}} \quad c(\lambda) = e^{-\lambda} \quad t(x) = x$$

$$w(\lambda) = \log \lambda$$

(e) Neg. binomial with r known $(0<p<1)$

$$f_X(x|p) = \binom{r+x-1}{x} p^r (1-p)^x \mathbb{1}_{\{0,1,..\}}$$

$$= \binom{r+x-1}{x} e^{\log p^r} \cdot e^{\log(1-p)^x} \mathbb{1}_{\{0,1..\}}$$

$$= \binom{r+x-1}{x} e^{r \log p} \cdot \exp\left(x \log(1-p)\right) \mathbb{1}_{\{0,1..\}}$$

$$h(x) = \binom{r+x-1}{x} \mathbb{1}_{\{0,1,..\}} \qquad c(p) = r \log p$$

$$w(p) = \log(1-p)$$

$$t(x) = x$$

(c) beta family with either α or β or both unknown

— α unknown

$$f(x|\alpha) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \cdot \frac{1}{\Gamma(\beta)} \cdot (1-x)^{\beta-1} \cdot x^{\alpha-1} \mathbb{1}_{(0,1)}$$

$$= \frac{1}{\Gamma(\beta)} \cdot \frac{(1-x)^{\beta}}{x(1-x)} \cdot \exp\left[\log\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \cdot x^{\alpha}\right)\right] \mathbb{1}_{(0,1)}$$

$$= \frac{\mathbb{1}_{(0,1)}}{\Gamma(\beta)} \frac{(1-x)^{\beta}}{x(1-x)} \cdot \exp\left[\log\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\right) + \alpha \log x\right]$$

$$= \frac{\mathbb{1}_{(0,1)}}{\Gamma(\beta)} \frac{(1-x)^{\beta-1}}{x} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \cdot \exp(\alpha \log x)$$

$$h(x) = \frac{\mathbb{1}_{(0,1)} (1-x)^{\beta-1}}{\Gamma(\beta)\, x} \qquad c(\alpha) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}$$

$$w(\alpha) = \alpha \qquad t(x) = \log x$$

— β unknown

replacing x by $1-x$

& α by β

$$h(x) = \frac{\mathbb{1}_{(0,1)}\, x^{\alpha-1}}{\Gamma(\alpha)\,(1-x)} \qquad c(\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}$$

$$w(\beta) = \beta \qquad t(x) = \log(1-x)$$

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– α, β both unknown

$$f(x \mid \alpha, \beta) = \frac{\mathbb{1}_{(0,1)}}{x(1-x)} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \exp\left[\log x^{\alpha} + \log(1-x)^{\beta}\right]$$

$$= \frac{\mathbb{1}_{(0,1)}}{x(1-x)} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \exp\left[\alpha \log x + \beta \log(1-x)\right]$$

$$h(x) = \frac{\mathbb{1}_{(0,1)}}{x(1-x)} \qquad c(\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

w_1, w_2 are real valued f^n of possibly vector-valued parameters

$$\Rightarrow w_1(\alpha,\beta) = \alpha$$
$$w_2(\alpha,\beta) = \beta$$

$$t_1(x) = \log x$$
$$t_2(x) = \log(1-x)$$