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STATISTICS 610-601: HW # 5

97/100

OCTOBER 12, 2017

DISTRIBUTION THEORY

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Homework 5 (due on Thursday October 12, 2017)

Problem 1: Consider a r.v. X and the transformation $Y = e^X$ with $E(Y) = 1$. Can we determine the sign of $E(X)$? If no: why not? If yes: find it.

Problem 2: numbers 2.30 (a) - (c) from the textbook.

Problems 3 to 5: numbers 2.34, 3.6 and 3.18 from the textbook.

Problem 6: Suppose you toss a coin about which you know that, on average, in one third of all cases it shows "Tails" and in two thirds of all cases "Heads". Consider the following situation: each time you observe "Heads" you roll a fair die and note the number of dots x. If the coin shows "Tails" you do *not* roll the die but, instead, note $x = 0$ for the observed number of dots. Find the moment generating function of X = "number of dots".

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Solution 1

(a) $f(x) = \frac{1}{c}$ $\quad 0 < x < c$

$$MGF_X(t) = \int_0^c e^{tx} \frac{1}{c} dx$$

$$= \left.\frac{e^{tx}}{t}\right|_0^c \frac{1}{c} = \left(\frac{e^{tc}}{t} - \frac{1}{t}\right)\frac{1}{c}$$

$$= \frac{1}{ct}(e^{tc} - 1)$$

(b) $f(x) = \frac{2x}{c^2}$ $\quad 0 < x < c$

$$MGF_X(t) = \int_0^c e^{tx} \frac{2x}{c^2} dx$$

$$= \left.\frac{2x}{c^2} \cdot \frac{e^{tx}}{t}\right|_0^c - \int_0^c \frac{2}{c^2} \cdot \frac{e^{tx}}{t} dx$$

$$= \frac{2c}{c^2} \frac{e^{ct}}{t} - \frac{2}{tc^2}\left.\left(\frac{e^{tx}}{t}\right)\right|_0^c$$

$$= \frac{2e^{ct}}{ct} - \frac{2e^{ct}}{t^2c^2} + \frac{2}{t^2c^2}$$

$$= \frac{2}{c^2t^2}(ct\, e^{ct} - e^{ct} + 1)$$

(c) $f(x) = \frac{1}{2\beta} e^{-|x-\alpha|/\beta}$ $\quad -\infty < x < \infty$, $\quad -\infty < \alpha < \infty$

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$$= \begin{cases} \frac{1}{2\beta} e^{-(x-\alpha)/\beta} & x-\alpha > 0 \text{ or } x > \alpha \\ \frac{1}{2\beta} e^{(x-\alpha)/\beta} & x \le \alpha \end{cases}$$

$$E(e^{tx}) = \int_{-\infty}^{\alpha} \frac{e^{tx}}{2\beta} e^{(x-\alpha)/\beta}\, dx + \int_{\alpha}^{\infty} \frac{e^{tx}}{2\beta} e^{-(x-\alpha)/\beta}\, dx$$

$$= \int_{-\infty}^{\alpha} \frac{\exp\left[\left(t+\frac{1}{\beta}\right)x\right]}{2\beta} \cdot \exp\left(-\frac{\alpha}{\beta}\right) dx +$$

$$\int_{\alpha}^{\infty} \frac{\exp\left[\left(t-\frac{1}{\beta}\right)x\right]}{2\beta} \exp\left(\frac{\alpha}{\beta}\right) dx$$

$$= \exp\left(t+\frac{1}{\beta}\right)x \Big|_{-\infty}^{\alpha} \frac{e^{-\alpha/\beta}}{2\beta\left(t+\frac{1}{\beta}\right)} + \exp\left(t-\frac{1}{\beta}\right)x \Big|_{\alpha}^{\infty} \frac{e^{\alpha/\beta}}{2\beta\left(t-\frac{1}{\beta}\right)}$$

$$= e^{t\alpha} \cdot e^{\alpha/\beta} \cdot \frac{e^{-\alpha/\beta}}{2(\beta t+1)} - e^{t\alpha} \cdot \frac{e^{-\alpha/\beta} \cdot e^{\alpha/\beta}}{2(\beta t-1)}$$

for $-\frac{1}{\beta} < t < \frac{1}{\beta}$

$$= \frac{e^{\alpha t}}{2} \left[\frac{1}{\beta t+1} - \frac{1}{\beta t-1}\right]$$

$$= \frac{e^{\alpha t}}{2} \frac{-2}{\beta^2 t^2 - 1} = \frac{e^{\alpha t}}{1-\beta^2 t^2}$$

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$$\Rightarrow MGF_X(t) = \begin{cases} \dfrac{e^{\alpha t}}{1-\beta^2 t^2} & -\frac{1}{\beta} < t < \frac{1}{\beta} \\ \infty & \text{else} \end{cases}$$

Solution 1

$$E(Y) = E(e^X) = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} e^x f_X(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} \left(1 + x + \sum_{k=2}^{\infty} \frac{x^k}{k!}\right) f_X(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} f_X(x)\,dx + \int_{-\infty}^{\infty} x f_X(x)\,dx + \int_{-\infty}^{\infty} \left(\sum_{k=2}^{\infty} \frac{x^k}{k!}\right) f_X(x)\,dx = 1$$

$$\Rightarrow 1 + E(X) = 1 - \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

$$\Rightarrow E(X) = -\int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

$$f_X(x) \geq 0 \quad \forall x$$

$$g(x) = x^2\left(\frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots\right)$$

$$= e^x - (1+x)$$

for $g(x) \geq 0$

$\Rightarrow e^x \geq 1+x$ which is true for all x

$\Rightarrow g(x) \geq 0 \quad \forall x$

$\Rightarrow \int_{-\infty}^{\infty} g(x) f_X(x)\,dx \geq 0 \quad \forall x$

$\Rightarrow E(X) \leq 0$

Yes. $E(X)$ is negative.

Solution 3

$r = 1$ $\quad E(X) = 0$ (mean of std normal)

$E(Y) = \frac{\sqrt{3}}{6} - \frac{\sqrt{3}}{6} + 0 = 0$

$r = 2$ $\quad E(X^2) = Var(X) - E(X) = 1 - 0 = 1$

$E(Y^2) = 3 \times \frac{1}{6} + 3 \times \frac{1}{6} + 0 = \frac{1}{2} + \frac{1}{2} = 1$

$r = 3$ $\quad E(X^3) = 0$ (odd moment of std normal)

$E(Y^3) = \frac{3\sqrt{3}}{6} - \frac{3\sqrt{3}}{6} + 0 = 0$

$r = 5$

$E(X^5) = 0$

$E(Y^5) = \frac{9\sqrt{3}}{6} - \frac{9\sqrt{3}}{6} + 0 = 0$

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$r=4$

$$MGF_X = e^{t^2/2}$$

$$E(X^4) = \frac{d^4}{dt^4} e^{t^2/2} \Big|_{t=0}$$

$$= \frac{d^3}{dt^3}(te^{t^2/2}) \Big|_{t=0}$$

$$= \frac{d^2}{dt^2}\left(e^{t^2/2} + t(te^{t^2/2})\right) \Big|_{t=0}$$

$$= \frac{d}{dt}\left(te^{t^2/2} + 2te^{t^2/2} + t^2(te^{t^2/2})\right) \Big|_{t=0}$$

$$= \frac{d}{dt}\left(3te^{t^2/2} + t^3e^{t^2/2}\right) \Big|_{t=0}$$

$$= 3e^{t^2/2} + 3t(te^{t^2/2}) + \frac{d}{dt}(t^3e^{t^2/2}) \Big|_{t=0}$$

$$= 3 + 0 + 0$$

$$= 3$$

$$E(Y^4) = \frac{9}{6} + \frac{9}{6} + 0 = \frac{18}{6} = 3$$

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Solution 6

X	0	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{3}$	$\frac{1}{6}\times\frac{2}{3}$	$\frac{1}{6}\times\frac{1}{2}$	$\frac{1}{6}\times\frac{1}{2}$	$\frac{1}{6}\times\frac{1}{2}$	$\frac{1}{6}\times\frac{1}{2}$	$\frac{1}{6}\times\frac{1}{2}$

$$MGF_X(t) = \sum_{x=0}^{6} e^{tx} P(X=x)$$

$P(\text{Head occurs}) = \frac{2}{3}$

$P(\text{Tail occurs}) = \frac{1}{3}$

$$= \frac{e^0}{2} + \left(\frac{e^t}{12} + \frac{e^{2t}}{12} + \cdots\right)$$

$$= \frac{1}{2} + \frac{1}{12}\left(e^t + e^t \cdot e^t + (e^t)^3 + \cdots\right)$$

$$= \frac{1}{2} + \frac{1}{12}\,\frac{(e^t)^6 - 1}{e^t - 1}$$

$$= \frac{1}{2} + \frac{1}{12}\,\frac{e^{6t} - 1}{e^t - 1}$$

Solution 5

MGF of Gamma$(r, 1)$

$$M_X(t) = \left(\frac{1}{1-t}\right)^r$$

MGF of pY, $Y \sim$ Negative Binomial (r, p)

$$MGF_{pY}(t) = M_Y(pt)$$

$$= \left(\frac{p}{1-(1-p)e^{pt}}\right)^r$$

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Now,

$$\lim_{p\to 0} \frac{p}{1-(1-p)e^{pt}} = \lim_{p\to 0} \frac{1}{-(1-p)e^{pt}\cdot t-(-1)e^{pt}}$$

$$= \lim_{p\to 0} \frac{1}{t(p-1)e^{pt}+e^{pt}}$$

$$= \lim_{p\to 0} \frac{1}{-te^{pt}+e^{pt}+tpe^{pt}}$$

$$= \frac{1}{1-t}$$

$$\Rightarrow \lim_{p\to 0} MGF_{pY}(t) = \left(\frac{1}{1-t}\right)^{r} = MGF_X(t)$$

Solution 4

(a) Binomial

success = "the insect survives"

$p = 0.01 \qquad n = 2000$

(b) $$P(X<100) = \sum_{k=0}^{199} \binom{2000}{k} p^k (1-p)^{2000-k}$$

$$= \sum_{k=0}^{99} \binom{2000}{k} (0.01)^k (0.99)^{2000-k}$$

(c) Approximate by Poisson distribution with $\lambda = np$

$$\lambda = 2000 \times 0.01$$
$$= 20$$

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$$P(Y < 100) = \sum_{x=0}^{99} \frac{e^{-20}(20)^x}{x!}$$

$$= e^{-20} \sum_{x=0}^{99} \frac{(20)^x}{x!}$$