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STATISTICS 610-601: HW # 10

NOVEMBER 30, 2017

DISTRIBUTION THEORY

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Homework 10 (due on Thursday November 30, 2017)

Problem 1:

(a) Let $X_n \sim U(-1/n, 1/n)$. Show $X_n \xrightarrow{P} 0$.

(b) Let $X_n \sim Exp(1/n)$ (exponential with parameter $1/n$). Show $X_n \xrightarrow{d} 0$.

Problem 2:

Consider $X_1, \ldots, X_n$ i.i.d. random variables with density function

$$f(x|\theta) = e^{-(x-\theta)}, \quad x \geq \theta,$$

and $f(x|\theta) = 0$ otherwise. Show that the estimator $\frac{1}{n}\sum_{i=1}^{n}(X_i - 1)$ is strongly consistent for θ, that is

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - 1) \to \theta \quad \text{almost surely.}$$

Problem 3:

Consider $U_1, \ldots, U_{300}$ iid from $U(-1, 1)$. Use the central limit theorem to find an approximate distribution for $\frac{1}{10}\sum_{i=1}^{300} U_i$.

Problem 4:

Let $X_1, \ldots, X_n$ be iid random variables with $E(X_1) = 0$ and $var(X_1) = 1$. Let

$$T = \sum_{i=1}^{n}\sum_{j=1}^{n}(X_i - X_j)^2.$$

Find $E(T)$.

Problem 5:

Number 5.40 from the book.

Solution 1

(a) $X_n \sim U\left(-\frac{1}{n}, \frac{1}{n}\right)$

To Prove: $\lim_{n\to\infty} P(|X_n - 0| < \epsilon) = 1 \quad \forall \epsilon > 0$

Proof

$$P(|X_n| < \epsilon) = P(X_n < \epsilon) - P(X_n < -\epsilon)$$
$$= P(X_n < \epsilon) - 1 + P(X_n < \epsilon)$$
$$= 2P(X_n < \epsilon) - 1$$

$X_n \sim U(a, b)$

$$Z_n = \frac{X_n - a}{b - a} \sim U(0,1)$$

$$P(|X_n| < \epsilon) = 2P\left(\frac{X_n + \frac{1}{n}}{2/n} < \frac{\epsilon + 1/n}{2/n}\right) - 1$$
$$= 2P\left(\frac{nX_n + 1}{2} < \frac{n\epsilon + 1}{2}\right) - 1$$
$$= 2P\left(Z_n < \frac{n\epsilon + 1}{2}\right) - 1$$

$$\lim_{n\to\infty} P(|X_n| < \epsilon) = \lim_{n\to\infty} 2P\left(Z_n < \frac{n\epsilon+1}{2}\right) - 1$$

$Z_n \sim U(0,1)$

$$= 2(1) - 1$$
$$= 1.$$

(b) $X_n \sim \text{Expo}\left(\frac{1}{n}\right)$

To prove $\lim_{n\to\infty} F_{X_n}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$

Proof

$$X \sim \frac{1}{\beta} \exp(-x/\beta) \Rightarrow \frac{X}{\beta} \sim \exp(-x)$$

$$P(X_n \leq x) = 0 \quad \text{if } x \leq 0 \qquad \beta = \frac{1}{n}$$

for $x > 0$ $\quad P(X_n \leq x) = P(nX_n \leq nx)$

$$= P(Z_n \leq nx) \qquad Z_n \sim \text{Expo}(1)$$

$$F_{X_n}(x) = 1 - \exp(-nx)$$

$$\lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} 1 - \exp(-nx)$$

$$= 1 - 0$$

$$= 1 \qquad \text{if } x \geq 0$$

$$\Rightarrow X_n \xrightarrow{d} 0$$

Solution 2

To Prove: $\frac{1}{n}\sum_{i=1}^{n}(X_i - 1) \xrightarrow{a.s.} \theta$ as $n \to \infty$

Proof

By SLLN,

$$\bar{X}_n \xrightarrow{a.s.} E(X_1) \text{ as } n \to \infty$$

if $X \sim \frac{1}{\beta}\exp\left(-\frac{x}{\beta}\right) \Rightarrow EX = \beta$, $Var X = \beta^2$

$f(x) = e^{-(x-\theta)} = f_{Exp}(x-\theta)$ (shifted in location exponential)

$$\Rightarrow EX = EZ + \theta \qquad Z \sim Expo(1)$$

$$= 1 + \theta$$

and $Var X = Var Z = 1$

Now, $\bar{X}_n \xrightarrow{a.s.} E(X_1)$ as $n \to \infty$

$$\Rightarrow P\left(\lim_{n\to\infty} |\bar{X}_n - 1 - \theta| < \epsilon\right) = 1 \quad \forall \epsilon > 0$$

$$\Rightarrow P\left(\lim_{n\to\infty} \left|\frac{1}{n}\sum_{i=1}^{n} X_i - 1 - \theta\right| < \epsilon\right) = 1$$

$$\Rightarrow P\left(\lim_{n\to\infty} \left|\frac{1}{n}\sum_{i=1}^{n}(X_i - 1) - \theta\right| < \epsilon\right) = 1 \qquad \left(\because 1 = \frac{1}{n}\cdot n = \frac{1}{n}\sum_{i=1}^{n} 1\right)$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}(X_i - 1) \xrightarrow{a.s.} \theta \text{ as } n \to \infty$$

$\Rightarrow$ The estimator is strongly consistent

Solution 3

$U_1 \text{ --- } U_{300} \sim U(-1,1)$

$EU_i = 0 \qquad Var\ U_i = \frac{(1+1)^2}{12} = \frac{4}{12} = 1/3$

$\bar{U} = \frac{1}{300}\sum_{i=1}^{300} u_i$

CLT : $\left[\frac{\bar{U} - 0}{\frac{1}{\sqrt{3}}}\right]\sqrt{300} \sim N(0,1)$

$\Rightarrow \bar{U} \times 30 \sim N(0,1)$

$\Rightarrow \frac{1}{10}\sum_{i=1}^{300} u_i \sim N(0,1)$

Solution 4

$\sum_{i=1}^{n} E(X_i - \bar{X})^2 = (n-1)S^2$

$E(X_i - \mu)^2 = Var(X_i) = 1$

$E(T) = E\left(\sum_{j=1}^{n}\sum_{i=1}^{n}(X_i - X_j)^2\right)$

$= \sum_{j=1}^{n}\sum_{i=1}^{n} E(X_i - \mu + \mu - X_j)^2$

$= \sum_{j=1}^{n}\left(\sum_{i=1}^{n} E(X_i - \mu)^2\right) + \sum_{i=1}^{n}\left(\sum_{j=1}^{n} E(X_j - \mu)^2\right)$

$-2\sum_{j=1}^{n}\sum_{i=1}^{n} \underbrace{E(X_i - \mu, X_j - \mu)}_{\cancel{Cov(X_i, X_j)} \to 0}$

$$\Rightarrow E(T) = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} Var(X_i)\right) + \sum_{i=1}^{n}\sum_{j=1}^{n} Var(X_j)$$

$$= 2n^2$$

Solution 5

(a) To Prove : $P(X \le t-\epsilon) \le P(X_n \le t) + P(|X_n - X| \ge \epsilon)$

Proof : By Law of total Prob.

$$P(X \le t-\epsilon) = P(X \le t-\epsilon, X_n \le t) + P(X \le t-\epsilon, X_n > t)$$

$$\because P(A \cap B) = P(A|B)P(B) \le P(B)$$

$$\Rightarrow P(X \le t-\epsilon) \le P(X_n \le t) + P(X \le t-\epsilon, X_n > t)$$

$$= P(X_n \le t) + P(X_n - X \ge X_n - t + \epsilon, X_n - t > 0)$$

$$\le P(X_n \le t) + P(X_n - X \ge \epsilon)$$

$$\le P(X_n \le t) + P(X_n - X \ge \epsilon) + P(X_n - X \le -\epsilon)$$

$$= P(X_n \le t) + P(|X_n - X| \ge \epsilon)$$

$$\Rightarrow P(X \le t-\epsilon) \le P(X_n \le t) + P(|X_n - X| \ge \epsilon)$$

(b) Interchanging X_n & X and replacing $t-\epsilon$ by t' and following exactly same steps as in (a),

$$P(X_n \leq t') \leq P(X \leq t'+\epsilon) + P(|X_n - X| \geq \epsilon)$$

(c) $P(X \leq t-\epsilon) - P(|X_n - X| \geq \epsilon) \leq P(X_n \leq t) \leq P(X \leq t+\epsilon) + P(|X_n - X| \geq \epsilon)$

in the limit $\epsilon \to 0$ and $\lim n \to \infty$

$$F_X(t) \leq \lim_{n\to\infty} P(X_n \leq t) \leq F_X(t)$$

thus $X_n \xrightarrow{d} X$. $\left(\because \lim_{n\to\infty} P(|X_n - X| \geq \epsilon = 0)\right)$