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STATISTICS 610-601: HW # 3

SEPTEMBER 28, 2017

DISTRIBUTION THEORY

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Homework 3 (due on Thursday September 28, 2017)

Problem 1.

a) Consider $f(x) = cx^{-3}1_{(1,\infty)}(x)$ with $c \in \mathbb{R}$. Determine c such that f is a probability density function and find the corresponding cumulative distribution function F.

b) Consider $f(x) = c\sin(x)1_{(0,\pi)}(x)$ with $c \in \mathbb{R}$. Determine c such that f is a probability density function and find the corresponding cumulative distribution function F.

Problem 2. Consider a distribution P given by a pdf $f(\cdot)$ with $f(x) = 0$ for all $x \leq 0$. Assume F is a continuous antiderivative of f with $F'(x) = f(x)$ for all $x > 0$ and $F(x) = 0$ for all $x \leq 0$. Then, by definition,

$$P((a,b]) = \int_a^b f(x)\,dx = F(b) - F(a) \quad \text{for } 0 \leq a \leq b < \infty.$$

Further assume that there is a constant $\lambda > 0$ with

$$(*) \qquad \lim_{0<h\to 0} \frac{P((t,t+h] \mid (t,\infty))}{h} = \frac{1}{\lambda} \quad \text{for all } t > 0.$$

a) Show that $f(x) = \frac{1}{\lambda}e^{-x/\lambda}$ for all $x > 0$ (exponential distribution).

b) If P is the distribution of the lifetime of an individual or object, then $P((t,\infty))$ is the probability that the individual survives at least t time units. Given this, how would you interpret equation $(*)$? Is it reasonable to assume that humans' lifetimes are exponentially distributed?

Problem 3. Consider a random variable X with a Poisson(1) distribution and the transformation $Y = (X+1)^{-1}$. Specify the probability mass function $f_Y(y)$ of Y. Check your answer by verifying that $f_Y(y)$ satisfies the criteria for a probability mass function.

Problems 4 - 6: numbers 1.53, 1.55, 2.1 from the textbook.

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Solution 1

(a) $f(x) = cx^{-3}\,\mathbb{1}_{(1,\infty)}(x) \qquad c \in \mathbb{R}$

(i) $f(x) \geq 0 \ \forall x \Rightarrow c \geq 0$

(ii) $\int_{-\infty}^{\infty} f(t)\,dt = 1$

$$\Rightarrow \int_{1}^{\infty} c\,t^{-3}\,dt = 1$$

$$\Rightarrow c\left[\frac{t^{-2}}{-2}\right]_1^{\infty} = 1$$

$$\Rightarrow \frac{c}{-2}\left[0 - 1\right] = 1 \quad \Rightarrow \quad c = 2$$

CDF

$$f(x) = 2x^{-3}\,\mathbb{1}_{(1,\infty)}(x)$$

$x > 1$

$$F_X(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$= \int_{1}^{x} 2t^{-3}\,dt$$

$$= \frac{2}{-2}\left[t^{-2}\right]_1^{x}$$

$$= -\left(\frac{1}{x^2} - 1\right) = 1 - \frac{1}{x^2}$$

$$F_X(x) = \begin{cases} 0 & x \leq 1 \\ 1 - \frac{1}{x^2} & x > 1 \end{cases}$$

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(b) $f(x) = c\sin(x)\mathbb{1}_{(0,\pi)}(x) \qquad c \in \mathbb{R}$

(i) $f(x) \geq 0 \quad \forall x \Rightarrow c \geq 0 \qquad [\sin(x) > 0 \text{ in } (0,\pi)]$

(ii) $\int_{-\infty}^{\infty} f(t)\,dt = 1$

$\Rightarrow \int_0^{\pi} c\sin(t)\,dt = 1$

$\Rightarrow -c\,[\cos(t)]_0^{\pi} = 1$

$\Rightarrow -c(-2) = 1 \Rightarrow c = \frac{1}{2}$

CDF

$f(x) = \frac{1}{2}\sin(x)\mathbb{1}_{(0,\pi)}(x)$

$x \geq \pi$

$F_X(x) = 1$

$x \leq 0 \qquad F_X(x) = 0$

$x \in (0,\pi) \qquad F_X(x) = \int_{-\infty}^{x} f(t)\,dt$

$= \int_0^{x} \frac{1}{2}\sin(t)\,dt$

$= -\frac{1}{2}\,[\cos(t)]_0^{x}$

$= \frac{1 - \cos(x)}{2}$

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Solution 2

(a) $$\lim_{0<h\to 0} \frac{P((t,t+h] \mid (t,\infty))}{h} = \frac{1}{\lambda} \quad \forall t>0$$

$$\Rightarrow \lim_{0<h\to 0} \frac{P((t,t+h] \cap (t,\infty))}{P((t,\infty))\cdot h} = \frac{1}{\lambda}$$

$$\Rightarrow \lim_{0<h\to 0} \frac{P((t,t+h])}{h} = \frac{P((t,\infty))}{\lambda}$$

$$\Rightarrow \lim_{0<h\to 0} \frac{\int_t^{t+h} f(x)\,dx}{h} = \frac{P((t,\infty))}{\lambda}$$

$$\Rightarrow \lim_{0<h\to 0} \frac{f(t)\cdot h}{h} = \frac{P((t,\infty))}{\lambda}$$

(As $h\to 0$, area under the curve $f(t)$ from t to $t+h$ becomes equal to area of rectangular strip with height $f(t)$ and width h).

$$\Rightarrow f(t) = \frac{\int_t^{\infty} f(x)\,dx}{\lambda} = \frac{F(\infty) - F(t)}{\lambda}$$

$$= \frac{1-F(t)}{\lambda}$$

Taking derivative both sides

$$f'(t) = -\frac{f(t)}{\lambda}$$

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$$\Rightarrow \frac{f'(t)}{f(t)} = \frac{-1}{\lambda}$$

$$\Rightarrow \int_0^t \frac{f'(t)}{f(t)} = \frac{-1}{\lambda} t + c \qquad [f(t) = 0 \ \forall \ t \leq 0]$$

$$\Rightarrow \log f(t) = -\frac{1}{\lambda} t + c \qquad (\log \text{ is base } e)$$

$$\Rightarrow f(t) = e^{-t/\lambda} \cdot e^c$$

Now, since $f(t)$ is a pdf,

$$\int_{-\infty}^{\infty} f(t)dt = 1$$

$$\Rightarrow \int_0^{\infty} e^{-t/\lambda} e^c dt = 1$$

$$\Rightarrow \left[-\lambda e^{-t/\lambda}\right]_0^{\infty} e^c = 1$$

$$\Rightarrow \lambda e^c = 1 \Rightarrow e^c = \frac{1}{\lambda}$$

$$\therefore f(t) = \frac{1}{\lambda} e^{-t/\lambda} \qquad \forall \ t > 0$$

(b) Given that the object/individual survives atleast t time units, it is the probability that it fails/dies at time unit t.

It is not reasonable to assume so because the probability of death is slightly higher during birth, then it reduces and starts increasing again after 65 or so years of age.

Solution 3

The Poisson distribution is given by the pmf

$$f(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!} \mathbb{1}_{\{0,1,2,\ldots\}}(x)$$

$\lambda = 1$

$$f(x) = \frac{e^{-1}}{x!} \mathbb{1}_{\{0,1,2,\ldots\}}(x)$$

The transformation

$Y = (X+1)^{-1}$ is one to one

$\therefore$ the set $\{x : g(x) = y\}$ is a singleton.

The pdf of y

$$f_Y(y) = \sum_{x \in g^{-1}(y)} f_X(x) = f_X\left(\frac{1}{y} - 1\right)$$

$$\Rightarrow f_Y(y) = \frac{e^{-1}}{\left(\frac{1}{y}-1\right)!} \qquad y = \frac{1}{1+x}, \quad x = 0, 1, 2, \ldots$$

$$\Rightarrow f_Y(y) = \frac{e^{-1}}{\left(\frac{1}{y}-1\right)!} \qquad y = 1, \frac{1}{2}, \frac{1}{3}, \ldots$$

$$= \frac{e^{-1}}{\left(\frac{1}{y}-1\right)!} \mathbb{1}_{\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}}(y)$$

Verify

(i) $f_Y(y) \geq 0 \quad \forall\, y$

(ii) $$\sum_{y \in \{1, \frac{1}{2}, \ldots\}} \frac{e^{-1}}{\left(\frac{1}{y}-1\right)!} = e^{-1} \sum_{x \in \{0,1,2,\ldots\}} \frac{1}{x!} \quad \text{—(1)}$$

To evaluate the summation, we use the hint provided in problem 6 of HW 2,

$$\lim_{n \to \infty} \left(1 + \frac{\lambda + \epsilon_n}{n}\right)^n = e^{\lambda} \quad \text{for } \lambda \in \mathbb{R}$$

$$\lim_{n \to \infty} \epsilon_n = 0$$

For $\lambda = 0$, $\epsilon_n = 0$ we have

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e \quad \text{—(2)}$$

Using binomial expansion, the k^{th} term on left side is given by

$$ {}^{n}C_{k} \frac{1}{n^k} = \frac{n(n-1) \cdots (n-k+1)}{n \cdot n \cdots n} \times \frac{1}{k!} $$

which converges to $\frac{1}{k!}$ in the $\lim_{n\to\infty}$.

∴ left side of the equation ② can be expressed as

$$ \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} \sum_{k=0}^{n} {}^{n}C_{k} \frac{1}{n^k} $$

$$ = \sum_{k=0}^{\infty} \frac{1}{k!} $$

$$ \Rightarrow \sum_{k=0}^{\infty} \frac{1}{k!} = e \qquad (\text{using } ②) \quad — ③ $$

Using ③ in ①

$$ e^{-1} \sum_{x\in\{0,1,2,..\}} \frac{1}{x!} = e^{-1} \cdot e = 1. $$

∴ It satisfies the criteria of prof.

You can assume this $\sum_{n=0}^{\infty} \frac{1}{n!} = e$. No need to provide that proof.

Solution 4

$$F_Y(y) = P(Y \le y) = 1 - \frac{1}{y^2} \qquad 1 \le y < \infty$$

(a) Verify that $F_Y(y)$ is a cdf.

(i) $\lim_{y \to \infty} F_Y(y) = \lim_{y \to \infty} 1 - \frac{1}{y^2} = 1 - 0 = 1$

$\lim_{y \to -\infty} F_Y(y) = 0 \qquad (\because F_Y = 0 \;\forall\; y < 1)$

(ii) For $y_2 > y_1$, $y_1, y_2 \ge 1$

$$F_Y(y_2) - F_Y(y_1) = -\frac{1}{y_2^2} + \frac{1}{y_1^2} = \frac{y_2^2 - y_1^2}{y_1^2 y_2^2} > 0$$

$\therefore$ $F_Y(y)$ is non decreasing f^n of y.

(iii) $\lim_{y_0 < y \to y_0} F_Y(y) = \lim_{y_0 < y \to y_0} 1 - \frac{1}{y^2}$

$y_0 \ge 1$

$$= \lim_{0 < h \to 0} 1 - \frac{1}{(y_0 + h)^2} = 1 - \frac{1}{y_0^2} = F_Y(y_0)$$

$\therefore$ $F_Y(y)$ is right continuous.

$\Rightarrow$ $F_Y(y)$ is a cdf.

(b) Find $f_Y(y)$

$y>1$ $\quad f_Y(y) = \frac{d}{dy} F_Y(y) = -\frac{d}{dy}(y^{-2}) = -(-2y^{-3})$

$= \frac{2}{y^3}$

$y \leq 1$ $\quad f_Y(y) = 0$

(c) $\quad Z = 10(Y-1)$

Clearly $Z = g(y)$ is one to one increasing function and random variable Y is continuous (absolutely) with $F_Y'(y) = f_Y(y)$ for $y>1$.

Using the change of variable formula for cdf

$$F_Z(z) = F_Y(g^{-1}(y))$$

$$= F_Y\left(\frac{z}{10}+1\right)$$

$$= 1 - \frac{1}{\left(\frac{z}{10}+1\right)^2}$$

$z = g(y),\ y \geq 1$

$\Rightarrow \frac{z}{10}+1 \geq 1$

$\Rightarrow z \geq 0$

$$= 1 - \frac{1}{\left(\frac{z}{10}+1\right)^2}, \quad z \geq 0$$

and $\quad F_Z(z) = 0 \quad , z<0$

Solution 5

$T \leq 0 \Rightarrow v < 5$

$P(V \leq v) = 0 \qquad v < 5$

$0 < T < 3 \Rightarrow v = 5$

$P(V \leq 5 | T \geq 3) P(T \geq 3) \to 0$

$$P(V \leq v) = P(V \leq 5 | 0 < T < 3) P(0 < T < 3) + 0$$

$$= \left[\cancel{P(V < 5 | 0 < T < 3)}^{0} + \cancel{P(V = 5 | 0 < T < 3)}^{1} \right] P(0 < T < 3)$$

$$= P(0 < T < 3).$$

$$= \int_0^3 \frac{1}{1.5} e^{-t/1.5} dt$$

$$= -\frac{1.5}{1.5} \left[e^{-2} - e^{0} \right]$$

$$= 1 - e^{-2}$$

$T \geq 3 \Rightarrow v \geq 6$

$$P(V \leq v) = \cancel{P(V \leq v | 0 < T < 3)}^{1} P(0 < T < 3) + P(V \leq v | T \geq 3) P(T \geq 3)$$

$$= P(0 < T < 3) + P(T \leq \frac{v}{2} | T \geq 3) P(T \geq 3)$$

$$= P(0 < T < 3) + \frac{P(3 \leq T \leq \frac{v}{2})}{P(T \geq 3)} \cdot P(T \geq 3)$$

$$= P(0 < T < 3) + P(3 \leq T \leq \frac{v}{2})$$

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$$= P\left(0 < T \le \frac{v}{2}\right) \quad , \frac{v}{2} \geq 3$$

$$= \int_0^{v/2} \frac{1}{1.5} e^{-t/1.5}\, dt \quad , v \geq 6$$

$$= -\frac{1.5}{1.5}\left[e^{-v/3} - e^{0}\right] \quad , v \geq 6$$

$$= 1 - e^{-v/3} \quad , v \geq 6$$

$5 < v < 6$

$$P(V \le v) = P(V \le 5) \quad \because \quad P(V = v) = 0 \ \forall \ 5 < v < 6$$

$$F_V(v) = \begin{cases} 0 & v < 5 \\ 1 - e^{-2} & 5 \le v < 6 \\ 1 - e^{-v/3} & v \geq 6 \end{cases}$$

Solution 6

(a) $Y = X^3$, $f_X(x) = 42x^5(1-x)$ $\quad 0 < x < 1$

One to one transformation, continuous random variable X

$g^{-1}(y) = y^{1/3}$ $\quad 0 < y < 1$

Using change of variable formula for pdf,

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{d}{dy} g^{-1}(y) = \frac{1}{3} y^{-2/3}$$

$$f_Y(y) = \begin{cases} 42\, y^{5/3}(1-y^{1/3}) \frac{1}{3} y^{-2/3} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 14y(1-y^{1/3}) & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Verify

$$\int_{-\infty}^{\infty} f_Y(y)\,dy = \int_0^1 (14y - 14y^{4/3})\,dy = \left[\frac{14}{2} y^2 - \frac{14}{7} y^{7/3} \times 3\right]_0^1$$

$$= 7 - 6 = 1$$

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(b) $Y = 4X + 3$ $\qquad f_X(x) = 7e^{-7x}$

One to one transformation, continuous r.v. X

$$g^{-1}(y) = \frac{y-3}{4} \Rightarrow \frac{d}{dy} g^{-1}(y) = \frac{1}{4} \qquad 3 < y < \infty$$

Using change of variable formula for pdf

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| & 3 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{7}{4} \exp\left(-\frac{7}{4}(y-3)\right) & 3 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Verify

$$\int_3^\infty \frac{7}{4} \exp\left(-\frac{7}{4}(y-3)\right) = \frac{\frac{7}{4}}{-\frac{7}{4}} \left[\exp\left(-\frac{7}{4}(y-3)\right)\right]_3^\infty$$

$$= -1 \, [0 - 1]$$

$$= 1$$

(c) $Y = X^2$ $\quad f_X(x) = 30x^2(1-x)^2 \quad 0<x<1$

For given range $0<x<1$, the transformation is one to one. Also X is continuous r.v.

$g^{-1}(y) = \sqrt{y}$ $\quad \frac{d}{dy} g^{-1}(y) = \frac{1}{2} y^{-1/2} \quad 0<y<1$

Using change of variable formula for pdf.

$$f_Y(y) = \begin{cases} 30y(1-\sqrt{y})^2 \cdot \frac{1}{2} y^{-1/2} & 0<y<1 \\ 0 & \text{Otherwise} \end{cases}$$

$$= \begin{cases} 15(1-\sqrt{y})^2\sqrt{y} & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

Verify

$$\int_0^1 15(y^{+1/2} + yy^{+1/2} - 2y^{1/2}y^{+1/2})\,dy$$

$$= 15\int_0^1 (y^{1/2} + y^{3/2} - 2y)\,dy = 15\left[\frac{y^{3/2}}{3/2} + \frac{y^{5/2}}{5/2} - y^2\right]_0^1$$

$$= 15\left(\frac{2}{3} + \frac{2}{5} - 1\right) = 10 + 6 - 15 = 1$$

Solution 6

MATLAB Live script for Solution 6 for verification using MATLAB Symbolic Toolbox.

```
% 1. isInc is 1 or -1 depending on whether the
% transformation is an increasing or
% decreasing function.
% 2. l1y, l2y are lower and upper limits for
% non-zero pdf of Y.
% 3. check indicates whether pdf of Y
% integrates to 1.
x = sym('x');
X = sym('X');
% Part (a)
disp("Solution 6(a)")
```

Solution 6(a)

```
fX = 42*x^5*(1-x)
```

fX = $-42\,x^5\,(x-1)$

```
Y = X^3
```

Y = X^3

```
isInc = 1;
l1y = 0; l2y = 1;
[fY,check] = find_pdf(fX,Y,X,isInc,l1y,l2y)
```

fY = $-14\,y\,\left(y^{1/3}-1\right)$

check = 1

```
% Part (b)
disp("Solution 6(b)")
```

Solution 6(b)

```
fX = 7*exp(-7*x)
```

fX = $7\,e^{-7\,x}$

```
Y = 4*X+3
```

Y = $4\,X+3$

```
isInc = 1;
l1y = 3; l2y = inf;
[fY,check] = find_pdf(fX,Y,X,isInc,l1y,l2y)
```

fY =

$$\frac{7\, e^{\frac{21}{4}-\frac{7\,y}{4}}}{4}$$

check = 1

```
% Part (c)
disp("Solution 6(c)")
```

Solution 6(c)

```
fX = 30*x^2*(1-x)^2
```

fX = $30\,x^2\,(x-1)^2$

```
Y = X^2
```

Y = X^2

```
isInc = 1;
l1y = 0; l2y = 1;
[fY,check] = find_pdf(fX,Y,X,isInc,l1y,l2y)
```

fY = $15\,\sqrt{y}\,(\sqrt{y}-1)^2$

check = 1

Following the function definition for finding pdf of Y:

```
function [fY,check] = find_pdf(fX,Y,X,isInc,l1y,l2y)
    g = finverse(Y);
    y = sym('y');
    g = subs(g,X,y);
    fY = compose(fX,g)*diff(g)*isInc;
    check = int(fY,y,l1y,l2y);
end
```

Good Work!! Keep it up!!