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STATISTICS 610-601: HW # 7

NOVEMBER 2, 2017

DISTRIBUTION THEORY

Prof. Irina Gaynanova

Homework 6 (due on Thursday November 2, 2017)

Problem 1: Consider r.v.'s X and Y with joint pdf

$$f_{X,Y}(x,y) = \frac{1}{8}1_{(0,y)}(x)1_{(0,4)}(y).$$

Find $P(X \geq 1 | Y \leq 2)$.

Problems 2-3: numbers 4.16 (a) and (c), and 4.17 from the textbook.

Problem 4: Let the random variable X represent the number of successes in n independent Bernoulli trials with success probability p. Let Y be the number of successes in the first m trials, where $m < n$. Find the conditional pmf of Y given $X = x$ and the conditional mean. *Hint:* consider Y and $X - Y$.

${}^{n-m}C_t\, p^t (1-p)^{n-m-t}$ * $P(X=x, Y=y)$ $P(Y=y)$ $P(X-Y=x-y)$

Problem 5: Find the pdf of the ratio $U = X/Y$ for the random vector (X, Y) with joint pdf

$$f_{X,Y}(x,y) = e^{-(x+y)}1_{(0,\infty)\times(0,\infty)}(x,y).$$

$P(Y = y | X = x) =$

Problem 6: Consider the random vector (X, Y) with distribution given by the below table.

		X = 1	X = 2	X = 3	
Y	1	1/12	2/12	1/12	4/12
	2	1/6	0	1/6	4/12
	3	0	1/3	0	4/12
		3/12	6/12	3/12	1

$P(X=1) = \frac{3}{12}$

$P(X=3) = \frac{3}{12}$

$P(X=2) = 1 - \frac{6}{12} = \frac{6}{12}$

(a) Complete the table by filling in the missing value and by specifying the marginal distributions of X and Y.

$P(X=1, Y=1) = \frac{1}{12} \neq \frac{3}{12} \cdot \frac{4}{12}$

$P(X=2, Y=1) + \frac{1}{3} = \frac{6}{12}$

$\hookrightarrow \frac{6}{12} - \frac{4}{12} = \frac{2}{12} = \frac{1}{6}$

(b) Show that X and Y are dependent.

(c) Determine the distribution of random variables $\tilde{X}$ and $\tilde{Y}$ that have the same marginal distributions as X and Y but are independent.

$P(X=2, Y=2) = 0 \neq \frac{4}{12} \cdot \frac{6}{12}$

1/12	2/12	1/12	4/12
1/12	2/12	1/12	4/12
1/12	2/12	1/12	4/12
3/12	6/12	3/12	

$18 \to 24$

$34 \to 44$

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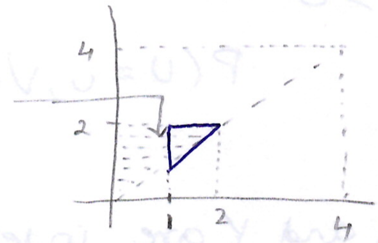
Solution 1

$$f_{X,Y}(x,y) = \frac{1}{8} 1_{(0,y)}(x)\, 1_{(0,4)}(y)$$



$y \geq x$

$0 \leq y \leq 2$

$P(X \geq 1 \mid Y \leq 2)$

$$= \frac{P(X \geq 1, Y \leq 2)}{P(Y \leq 2)}$$

$$P(Y \leq 2) = \int_0^2 \int_0^y f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_0^2 \int_0^y \frac{1}{8}\,dx\,dy = \int_0^2 \frac{1}{8} y\,dy = \frac{y^2}{16}\Big|_0^2 = \frac{4}{16} = \frac{1}{4}$$

$$P(X \geq 1, Y \leq 2) = \int_{0}^{2} \int_1^y f_{x,y}(x,y)\,dx\,dy$$

1

$X \geq 1,\ Y \leq 2$

$\Downarrow$

$1 \leq X \leq Y \leq 2$

$$= \int_0^2 \frac{1}{8}(y-1)\,dy = \frac{1}{8}\left(\frac{y^2}{2} - y\right) = \frac{1}{8}\left(\frac{4}{2} - 2\right) = 0$$

$$\Rightarrow P(X \geq 1 \mid Y \leq 2) = 0$$

Solution 2

(a) $P(X = x \mid p) = p(1-p)^{x-1} 1_{\{1,2,3\ldots\}}(x)$

$P(Y = y \mid p) = p(1-p)^{y-1} 1_{\{1,2,3\ldots\}}(y)$

$U = \min(X,Y) \qquad V = X - Y$

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$P(U=u, V=v) = P(\min(X,Y)=u, X-Y=v)$

$v > 0$

$P(U=u, V=v) = P(Y=u, X=v+y)$

$= P(Y=u, X=v+u)$

X and Y are indep R.V.

$\Rightarrow P(U=u, V=v) = P(Y=u)\,P(X=v+u)$

$= p(1-p)^{u-1}\,p(1-p)^{v+u-1}\,\mathbb{1}_{\{1,2,3\ldots\}\times\{1,2,3\ldots\}}(u,v)$

$= p^2(1-p)^{2u+v-2}\,\mathbb{1}_{\{1,2,3\ldots\}\times\{1,2,3\ldots\}}(u,v)$ — ①

$v < 0$

$P(U=u, V=v) = P(X=u, X-Y=-|v|)$

$= P(X=u, Y=X+|v|)$

$= P(X=u, Y=u+|v|)$

$= p(1-p)^{u-1}\,p(1-p)^{u+|v|-1}\,\mathbb{1}_{\{1,2,3\ldots\}\times\{1,2,3\ldots\}}(u,-v)$

$= p^2(1-p)^{2u+|v|-2}\,\mathbb{1}_{\{1,2,3\ldots\}\times\{-1,-2,-3,\ldots\}}(u,v)$

$v = 0$

$P(U=u, V=v) = P(Y=u, X=u)$

$= p^2(1-p)^{2u-2}\,\mathbb{1}_{\{1,2,3\ldots\}\times\{0\}}(u,v)$ — ③

From ①, ②, ③

$$f_{U,V}(u,v) = \underbrace{p^2(1-p)^{2u-2}\,\mathbb{1}_{\{1,2,3\ldots\}}(u)}_{f(u)} \cdot \underbrace{(1-p)^{|v|}\,\mathbb{1}_{\{\ldots,-2,-1,0,1,2\ldots\}}(v)}_{f(v)}$$

(c) $P(X=x, Y+X=t)$

$= P(X=x, Y=t-x)$

$= P(X=x)\,P(Y=t-x)$ [X and Y are indep]

$= p(1-p)^{x-1}\, p(1-p)^{t-x-1}\, 1_{\{1,2,3..\}\times\{1,2,3..\}}(x, t-x)$

$= p^2(1-p)^{t-2}\, 1_{\{1,2,3...\}\times\{2,3,...\}}(x,t)$

Solution 3

$X \sim expo(1)$

$f(X=x) = e^{-x}\, 1_{[0,\infty)}(x)$

$Y = i+1$ iff $i \le X < i+1$ $i = 0,1,2,...$

(a) $f(Y=y) = f(y-1 \le X < y) = \int_{y-1}^{y} e^{-x}\, 1_{[0,\infty)}(x)\,dx$

$= \left[-e^{-x}\, 1_{[0,\infty)}(x)\right]_{y-1}^{y}$

$= e^{-(y-1)}\, 1_{\{0,1,2..\}}(y-1) - e^{-y}\, 1_{\{1,2,3..\}}(y)$

$= \left[e^{-(y-1)} - e^{-y}\right] 1_{\{1,2,3..\}}(y)$

$= e^{-(y-1)}\left[1 - e^{-1}\right] 1_{\{1,2,3..\}}(y)$

$= (1-e^{-1})\left(1-(1-e^{-1})\right)^{y-1}\, 1_{\{1,2,3..\}}(y)$

Similar to geometric distribution $P(1-p)^{x-1}\, 1_{\{1,2,..\}}(x)$

with $p = 1-e^{-1}$

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(b) $P(X-4 \le t \mid Y \ge 5) = P(X \le 4+t \mid Y \ge 5)$

$= P(X \le 4+t \mid X \ge 4)$

For expo dis., $P(X > s \mid X > t) = P(X > s-t)$ (memoryless)

$\Rightarrow P(X-4 \le t \mid Y \ge 5) = P(X \le 4+t-4)$

$= P(X \le t)$

$= 1-e^{-t} \mathbb{1}_{[0,\infty)}(t)$

Solution 6

		X: 1	2	3	$P(Y=y)$
Y	1	1/12	p	1/12	$p+\frac{2}{12}=\frac{4}{12}=\frac{1}{3}$
	2	1/6	0	1/6	$\frac{4}{12}=\frac{1}{3}$
	3	0	1/3	0	$\frac{4}{12}=\frac{1}{3}$
$P(X=x)$		$\frac{3}{12}=\frac{1}{4}$	$\frac{1}{3}+p = \frac{2}{12}+\frac{4}{12} = \frac{6}{12}=\frac{1}{2}$	$\frac{3}{12}=\frac{1}{4}$	

$\left(\frac{1}{3}+p\right)+\frac{3}{12}+\frac{3}{12}=1$

$\Rightarrow \frac{4}{12}+p = 1-\frac{6}{12}=\frac{6}{12}$

$\Rightarrow p = \frac{2}{12}=\frac{1}{6}$

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$$P(X=x) = \begin{cases} 1/4 & x=1 \\ 1/2 & x=2 \\ 1/4 & x=3 \\ 0 & \text{else} \end{cases}$$

$$P(Y=y) = \begin{cases} 1/3 & y=1 \\ 1/3 & y=2 \\ 1/3 & y=3 \\ 0 & \text{else} \end{cases}$$

(b) $P(X=2, Y=2) = 0 \neq \frac{4}{12} \cdot \frac{6}{12} = P(X=2)\, P(Y=2)$

(c)

$\tilde{Y}$ \ $\tilde{X}$	1	2	3	$P(Y=y)$
1	$1/12$	$2/12$	$1/12$	$4/12$
2	$1/12$	$2/12$	$1/12$	$4/12$
3	$1/12$	$2/12$	$1/12$	$4/12$
$P(X=x)$	$3/12$	$6/12$	$3/12$	

$$P(X=x_i, Y=y_i) = P(X=x_i)\, P(Y=y_i)$$

$x_i = 1, 2, 3$

$y_i = 1, 2, 3.$

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Solution 5

$$U = \frac{X}{Y} \qquad V = Y$$

$$\Rightarrow X = UV \qquad Y = V$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} V & u \\ 0 & 1 \end{vmatrix} = V.$$

$$f_{U,V}(u,v) = f_{X,Y}(X=uv, Y=v)\,|v|$$

$$= e^{-(uv+v)} 1_{(0,\infty)\times(0,\infty)}(uv,v)\, v$$

$$= e^{-(uv+v)} 1_{(0,\infty)\times(0,\infty)}(u,v)\, v$$

$$f_U(u) = \int_0^\infty e^{-(uv+v)} 1_{(0,\infty)}(u)\, v$$

$$= \left(v \left[\frac{e^{-(u+1)v}}{-(u+1)}\right]_0^\infty - \left[\int_0^\infty \frac{e^{-(u+1)v}}{-(u+1)}\, dv\right]\right) 1_{(0,\infty)}(u)$$

$$= -\left[\frac{e^{-(u+1)v}}{(u+1)^2}\right]_0^\infty 1_{(0,\infty)}(u) = \frac{1}{(u+1)^2} 1_{(0,\infty)}(u)$$

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Solution 4

$$P(Y=y \mid X=x) = \frac{P(Y=y, X=x)}{P(X=x)}$$

$$= \frac{P(Y=y, X-Y=x-y)}{P(X=x)}$$

$T = X - Y$

$= \left(\text{No. of successes in } n \text{ trials}\right) - \left(\text{No. of succ in first } m \text{ trials}\right)$.

$=$ No. of successes in last $n-m$ trials

Clearly T and Y are independent

$m < n$

$$P(Y=y \mid X=x) = \frac{P(Y=y)\,P(T=x-y)}{P(X=x)}$$

$$= \frac{{}^{m}C_y\, p^y (1-p)^{m-y}\, 1_{\{0,1,2..,m\}}(y) \times 1_{\{0,1,2..\}}(x-y)\; {}^{n-m}C_{x-y}\, p^{x-y} (1-p)^{(n-m)-(x-y)}}{{}^{n}C_x\, p^x (1-p)^{n-x}\, 1_{\{0,1,2..\}}(x)}$$

$$= \frac{{}^{m}C_y \; {}^{n-m}C_{x-y}}{{}^{n}C_x}\, 1_{\{0,1,2,..\}}(y)\, 1_{\{0,1,2..\}}(x)$$

Hyper geometric distn

mean $= \dfrac{xm}{n}$