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STATISTICS 610-601: HW # 4

OCTOBER 5, 2017

DISTRIBUTION THEORY

Prof. Irina Gaynanova

Homework 4 (due on Thursday October 5, 2017)

Problems 1 - 5: numbers 2.8 (a), 2.11 (b), 2.14 (a), 2.16 and 2.17 from the textbook.

Problem 6: Let X be a random variable with range $\{0, 1, 2, \ldots\}$. The discrete version of the formula from problem 2.14 in the textbook, i.e. 2.14 (b), can be written

$$EX = \sum_{k=1}^{\infty} P(X \geq k).$$

(The proof is analogous to that of 2.14 (a).) Use this formula to solve the following problem.

A fair dice is thrown n times. The sample space is $S = \{1, 2, \ldots, 6\}^n$, the outcomes are of the form $s = (s_1, \ldots, s_n) \in S$. Let Y_n denote the largest of the results thrown, i.e. Y_n is a r.v. with $Y_n(s_1, \ldots, s_n) = \max_{1 \leq k \leq n} s_k$.

a) Find EY_n and show

$$\lim_{n \to \infty} E(Y_n) = 6.$$

b) Show

$$\lim_{n \to \infty} Var(Y_n) = 0.$$

Problem 7: Let X be a discrete random variable that takes on values $0, 1, 2$ with probability $1/2$, $3/8$, $1/8$, respectively.

a) Find $E(X)$.

b) Find the pmf of $Y = X^2$ and use it to find $E(Y)$.

c) Use the definition of $E\{g(X)\}$, where $g(X)$ is a function of X, to find $E(X^2)$ and compare to your answer in part (b).

d) Find $Var(X)$.

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Solution 1

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-x} & \text{if } x \geq 0 \end{cases}$$

1) $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1 - \lim_{x \to \infty} e^{-x} = 1$

2) $\frac{d}{dx} F_X(x) = -e^{-x}(-1) = e^{-x}$ for $x > 0$

$\Rightarrow \frac{d}{dx} F_X(x) > 0 \quad \forall x > 0$

$\therefore F_X(x)$ is non decreasing function of x

3) We only need to check for continuity at $x = 0$.

$\lim_{0 < x \to 0} F_X(x) = \lim_{0 > x \to 0} 1 - e^{-x} = 1 - 1 = 0 = F_X(0)$

$F_X(x)$ is right continuous everywhere

From 1), 2) and 3) F_X is a cdf.

For $x > 0$,

$F_X^{-1}(y) = x \iff F_X(x) = y$ (since F_X is strictly increasing in $x > 0$)

$\Rightarrow 1 - e^{-x} = y$

$\Rightarrow 1 - y = e^{-x}$

$\Rightarrow \log(1 - y) = -x$

$\Rightarrow \log\left(\frac{1}{1-y}\right) = x = F_X^{-1}(y)$

For $x \le 0$,

$F_X(x)$ is constant

$$F_X^{-1}(y) = \inf\{x : F_X(x) \ge y\}$$

$$x > 0 \Rightarrow \log \frac{1}{1-y} > 0 \Rightarrow \frac{1}{1-y} > e^0$$

$$\Rightarrow 1-y < 1$$

$$\Rightarrow y > 0$$

$$\Rightarrow F_X^{-1}(y) = \inf\{x : F_X(x) \ge 0\}$$

$$= -\infty$$

$$F_X^{-1}(y) = \begin{cases} -\infty & y \le 0 \\ \log\left(\frac{1}{1-y}\right) & y > 0 \end{cases}$$

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Solution 2

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$Y = |X|$$

$$F_Y(y) = P(Y \le y) = P(|X| \le y)$$

$$= P(-y \le X \le y)$$

Split this into two disjoint probabilities

$$= P(-y \le X < 0) + P(0 \le X \le y)$$

$$= [P(X \ge -y) - P(X \ge 0)] + [P(X \le y) - P(X < 0)]$$

equality doesn't matter
(X is cont. RV
$\Rightarrow$ P(fixed value)=0)

$$= 1 - P(X \le -y) + P(X \le y) - \underbrace{P(X \ge 0) - P(X < 0)}$$

$$= 1 - P(X \le -y) + P(X \le y) - 1$$

$$= P(X \le y) - P(X \le -y)$$

$$= F_X(y) - F_X(-y)$$

The question asks for pdf of Y

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

(Normal pdf is absolutely cont $\forall x \in \mathbb{R}$)

$$= \frac{d}{dy} F_X(y) - \frac{d}{dy} F_X(-y) \cdot (-1)$$

$$= f_X(y) + f_X(y)$$

$$= 2 f_X(y) = \sqrt{\frac{2}{\pi}} e^{-y^2/2} \qquad 0 < y < \infty$$

E(Y) ?

V(Y) ?

Solution 3

$$EX = \int_{-\infty}^{\infty} x f_X(x)\,dx \qquad \text{(By def)}$$

$$= \int_{0}^{\infty} x f_X(x)\,dx \qquad (f_X(x)=0 \text{ for } x<0)$$

$$= \left[x \int_0^{\infty} f_X(x)\,dx\right]_0^{\infty} - \int_0^{\infty} dx \int_0^{x} f_X(x)\,dx$$

$$= \underline{\left[x \cdot 1\right]_0^{\infty}} - \int_0^{\infty} dx\, F_X(x) \qquad \int_0^{\infty} f_X(x)\,dx = 1 \text{ (by def of pdf)}$$

The value of these integral will be infinite.

$$= \underline{\int_0^{\infty} dx} - \int_0^{\infty} F_X(x)\,dx$$

$$= \int_0^{\infty} [1 - F_X(x)]\,dx$$

Solution 4

Mean duration

$$= E(T) = \int_0^{\infty} (1 - F_T(t))\,dt$$

Time $t > 0$ always $\Rightarrow f_T(t) = 0$ for $t \leq 0$

$$= \int_0^{\infty} (1 - P(T \leq t))\,dt$$

$$= \int_0^{\infty} P(T > t)\,dt$$

$$= \int_0^{\infty} a e^{-\lambda t} + (1-a)\int_0^{\infty} e^{-\mu t}\,dt$$

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$$= \frac{a}{-\lambda}\left[e^{-\lambda t}\right]_0^{\infty} + \frac{1-a}{-\mu}\left[e^{-\mu t}\right]_0^{\infty}$$

$$= \frac{a}{\lambda} + \frac{1-a}{\mu}.$$

Solution 5

(a) $f(x) = 3x^2 \quad 0 < x < 1$

$$\int_{-\infty}^{m} 3x^2 dx \, \mathbb{1}_{(0,1)} = \frac{1}{2} \qquad \int_{m}^{\infty} 3x^2 dx \, \mathbb{1}_{(0,1)} = \frac{1}{2}$$

$$\Rightarrow \int_0^m 3x^2 dx = \frac{1}{2} \qquad \Rightarrow x^3\Big|_m^1 = \frac{1}{2}$$

$$\Rightarrow 1 - m^3 = \frac{1}{2}$$

$$\Rightarrow m^3 = \frac{1}{2} \quad \Rightarrow m = \left(\frac{1}{2}\right)^{1/3} \qquad \Rightarrow m = \left(\frac{1}{2}\right)^{1/3}$$

$$\boxed{m = \left(\frac{1}{2}\right)^{1/3}}$$

(b) $f(x) = \frac{1}{\pi(1+x^2)} \quad -\infty < x < \infty$

$$\int_{-\infty}^{m} \frac{1}{\pi(1+x^2)} dx = \frac{1}{2} \qquad \int_{m}^{\infty} \frac{1}{\pi(1+x^2)} dx = \frac{1}{2}$$

$x = \tan\theta \Rightarrow dx = \sec^2\theta \, d\theta$

$1 + x^2 = \sec^2\theta$

$\theta = \tan^{-1} x$.

$$\Rightarrow \int_{-\pi/2}^{\tan^{-1} m} \frac{1}{\pi} d\theta = \frac{1}{2} \qquad \Rightarrow \int_{\tan^{-1} m}^{\pi/2} \frac{1}{\pi} d\theta = \frac{1}{2}$$

$\Rightarrow \frac{1}{\pi}\left(\tan^{-1} m - \frac{\pi}{2}\right) = \frac{1}{2}$

$\Rightarrow \tan^{-1} m - \frac{\pi}{2} = \frac{\pi}{2}$

$\Rightarrow \tan^{-1} m = \pi$

$\Rightarrow m = \tan \pi = 0$

$\frac{1}{\pi}\left(\frac{\pi}{2} - \tan^{-1} m\right) = \frac{1}{2}$

$\Rightarrow -\tan^{-1} m = 0$

$\Rightarrow m = \tan 0 = 0.$

$\boxed{m = 0}$

Solution 6

A Note to the grader -

Problem 14(b) in the text mentions the formula as

$$EX = \sum_{k=0}^{\infty} (1 - F_X(k))$$

which can be written as

$$EX = \sum_{k=0}^{\infty} (1 - P(X \leq k)) \quad \text{— } ①$$

$$= \sum_{k=0}^{\infty} P(X > k)$$

The homework problem has an equality sign which is confusing because this is a discrete random variable.

For the purpose of this solution, I am assuming this is a typo. Please discuss in class if this is not the case.

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(a) Find EY_n and show $\lim_{n\to\infty} E(Y_n) = 6$.

From (1), we need $P(X \le k)$

$$P(X\le k) = P(\max_{1\le i\le n} S_i \in \{1,2,\dots,k\})$$

$= P$ (in all n throws a number $\le k$ shows up)

$$= \left(\frac{k}{6}\right)^n$$

$$E(Y_n) = \sum_{k=0}^{5} \left(1 - P(X\le k)\right)$$

$\because \forall k \ge 6 \quad P(X\le k) = 1$

$$\Rightarrow 1 - P(X\le k) = 0$$

$$\Rightarrow E(Y_n) = 6 - \sum_{k=1}^{5}\left(\frac{k}{6}\right)^n \qquad \because P(X\le 0) = 0$$

$$\lim_{n\to\infty} E(Y_n) = 6 - \lim_{n\to\infty}\sum_{k=1}^{5}\left(\frac{k}{6}\right)^n$$

$$= 6 - \sum_{k=1}^{5}\lim_{n\to\infty}\left(\frac{k}{6}\right)^n \qquad \frac{k}{6} < 1 \ \forall\ k\in\{1,2,..5\}$$

$$= 6 - 0$$

$$= 6$$

(b) Show $\lim_{n\to\infty} Var(Y_n) = 0$

$$Var(X) = \sum_{k=1}^{6}\left(f_X(k)\cdot k^2\right) - [E(X)]^2$$

N denotes size of sample space of x_i

$k \in \{1,2,3,4,5,6\}$

$\because f_X(0) = 0 = f_X(k) \ \forall k > 6$. Also $f_X(k) = 0 \ \forall$ $k \in (0,6) - \{1,2,3,4,5\}$

$$\Rightarrow f_X(1) = F_X(1) = \left(\frac{1}{6}\right)^n$$

$$f_X(2) = F_X(2) - F_X(1) = \left(\frac{2}{6}\right)^n - \left(\frac{1}{6}\right)^n$$

$$f_X(3) = F_X(3) - F_X(2) = \left(\frac{3}{6}\right)^n - \left(\frac{2}{6}\right)^n$$

$$f_X(k) = \left(\frac{k}{6}\right)^n - \left(\frac{k-1}{6}\right)^n$$

$\because f_X(k)$ is height of the jump in $F_X(x)$ plot at $x = k$.

$$f_X(6) = 1 - \left(\frac{5}{6}\right)^n$$

$$Var(X) = \sum_{k=1}^{6}\left[\left(\frac{k}{6}\right)^n - \left(\frac{k-1}{6}\right)^n\right]k^2 - \left(6 - \sum_{k=1}^{5}\left(\frac{k}{6}\right)^n\right)^2$$

$$= \sum_{k=2}^{5} k^2\left[\left(\frac{k}{6}\right)^n - \left(\frac{k-1}{6}\right)^n\right] + \underset{\substack{\uparrow \\ k=1}}{\left(\frac{1}{6}\right)^n} + \underbrace{36\left[1 - \left(\frac{5}{6}\right)^n\right]}_{k=6} -$$

$$\left(36 + \left[\sum_{k=1}^{5}\left(\frac{k}{6}\right)^n\right]^2 - 12\sum_{k=1}^{5}\left(\frac{k}{6}\right)^n\right)$$

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$$\lim_{n \to \infty} Var(X) = \sum_{K=2}^{5} K^2 \cdot 0 + 0 + 36[1-0] - \left(36 + 0 - 12 \cdot \sum_{K=1}^{5} 0 \cdot\right)$$

$$= 36 - 36$$

$$= 0$$

Solution 7 -

(a) $E(X) = \sum_{i=1}^{3} x_i f_X(x_i)$

$= 0 \times \frac{1}{2} + 1 \times \frac{3}{8} + 2 \times \frac{1}{8} = \frac{3+2}{8} = \frac{5}{8}$

(b) $X = 0, 1, 2$ $\quad Y = X^2$

$Y = 0, 1, 4$

$$f_Y(y) = \begin{cases} 1/2 & y = 0 \\ 3/8 & y = 1 \\ 1/8 & y = 4 \\ 0 & \text{otherwise} \end{cases}$$

$E(Y) = 0 \times \frac{1}{2} + 1 \times \frac{3}{8} + \frac{4}{8}$

$= \frac{7}{8}$

(c) $E(X^2) = \sum x^2 f_X(x) = 0 \times \frac{1}{2} + 1 \times \frac{3}{8} + 4 \times \frac{1}{8}$

$= \frac{7}{8}$

(d) $Var(X) = E(X^2) - [E(X)]^2$

$= \frac{7}{8} - \frac{25}{64} = \frac{56-25}{64}$

$= \frac{31}{64}$