(later version). For the discussion here, the pressure has three components, one in the area direction and two perpendicular to the area. The pressure component in the area direction is called pressure (great way to confuse, isn't it?). The other two components are referred as the shear stresses. The units used for the pressure components is $[N/m^2]$.

The density is a property which requires that liquid to be continuous. The density can be changed and it is a function of time and space (location) but must have a continues property. It doesn't mean that a sharp and abrupt change in the density cannot occur. It referred to the fact that density is independent of the sampling size. Figure 1.2 shows the density as a function of the sample size. After certain sample size, the density remains constant. Thus, the density is defined as

Fig. 1.2 – Density as a function of the size of the sample.

$$\rho = \lim_{\Delta V \longrightarrow \varepsilon} \frac{\Delta m}{\Delta V} \qquad (1.1)$$

It must be noted that ε is chosen so that the continuous assumption is not broken, that is, it did not reach/reduced to the size where the atoms or molecular statistical calculations are significant (see Figure 1.2 for point where the green lines converge to constant density). When this assumption is broken, then, the principles of statistical mechanics must be utilized.
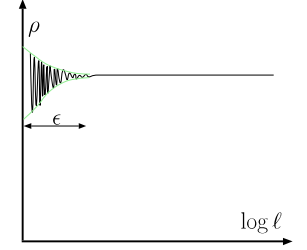
1.4 Shear Stress

The shear stress is part of the pressure tensor. However, here, and many parts of the book, it will be treated as a separate issue. In solid mechanics, the shear stress is considered as the ratio of the force acting on area in the direction of the forces perpendicular to area (Note what the direction of area?). Different from solid, fluid cannot pull directly but through a solid surface. Consider liquid that undergoes a shear stress between a short distance of two plates as shown in Fig. 1.3.
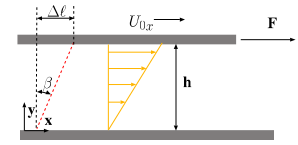

Fig. 1.3 – Schematics to describe the shear stress in fluid mechanics.

The upper plate velocity generally will be

$$U = f(A, F, h) \qquad (1.2)$$

Where A is the area, the F denotes the force, h is the distance between the plates. In this discussion, the aim is to develop differential equation, thus the small distance analysis is applicable. From solid mechanics study, it was shown that when the force per area increases, the velocity of the plate increases also. Experiments show that the increase of height will increase the velocity up to a certain range. Moving the plate with a zero lubricant ($h \sim 0$) results in a large force or conversely a large amount of lubricant

results in smaller force. For cases where the dependency is linear, the following can be written

$$U \propto \frac{h\,F}{A} \tag{1.3}$$

Equations (1.3) can be rearranged to be

$$\frac{U}{h} \propto \frac{F}{A} \tag{1.4}$$

Shear stress was defined as

$$\tau_{xy} = \frac{F}{A} \tag{1.5}$$

The index x represent the "direction of the shear stress while the y represent the direction of the area(perpendicular to the area). From equations (1.4) and (1.5) it follows that ratio of the velocity to height is proportional to shear stress. Hence, applying the coefficient to obtain a new equality as

$$\tau_{xy} = \mu \frac{U}{h} \tag{1.6}$$

Where μ is called the absolute viscosity or dynamic viscosity which will be discussed later in this chapter in a great length.

In steady state, the distance the upper plate moves after small amount of time, δt is

$$d\ell = U\,\delta t \tag{1.7}$$

From Figure 1.4 it can be noticed that for a small angle, $\delta\beta \cong \sin\beta$, the regular approximation provides

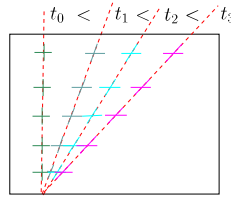$$d\ell = U\,\delta t = \overbrace{h\,\delta\beta}^{geometry} \tag{1.8}$$

From equation (1.8) it follows that

Fig. 1.4 – The deformation of fluid due to shear stress as progression of time.

$$U = h\,\frac{\delta\beta}{\delta t} \tag{1.9}$$

Combining equation (1.9) with equation (1.6) yields

$$\tau_{xy} = \mu \frac{\delta\beta}{\delta t} \tag{1.10}$$

If the velocity profile is linear between the plate (it will be shown later that it is consistent with derivations of velocity), then it can be written for small a angel that

$$\frac{\delta \beta}{\delta t} = \frac{dU}{dy} \tag{1.11}$$

Materials which obey equation (1.10) referred to as Newtonian fluid. For this kind of substance

$$\tau_{xy} = \mu \frac{dU}{dy} \tag{1.12}$$

Newtonian fluids are fluids which the ratio is constant. Many fluids fall into this category such as air, water etc. This approximation is appropriate for many other fluids but only within some ranges.

Equation (1.9) can be interpreted as momentum in the x direction transferred into the y direction. Thus, the viscosity is the resistance to the flow (flux) or the movement. The property of viscosity, which is exhibited by all fluids, is due to the existence of cohesion and interaction between fluid molecules. These cohesion and interactions hamper the flux in y–direction. Some referred to shear stress as viscous flux of x–momentum in the y–direction. The units of shear stress are the same as flux per time as following

$$\frac{F}{A} \left[\frac{kg\,m}{sec^2}\,\frac{1}{m^2} \right] = \frac{\dot{m}\,U}{A} \left[\frac{kg}{sec}\,\frac{m}{sec}\,\frac{1}{m^2} \right]$$

Thus, the notation of τ_{xy} is easier to understand and visualize. In fact, this interpretation is more suitable to explain the molecular mechanism of the viscosity. The units of absolute viscosity are $[N\,sec/m^2]$.

Example 1.1: Shear Between Plane   **Level: Simple**

A space of 1 [cm] width between two large plane surfaces is filled with glycerin. Calculate the force that is required to drag a very thin plate of 1 $[m^2]$ at a speed of 0.5 m/sec. It can be assumed that the plates remains in equidistant from each other and steady state is achieved instantly.

Solution

Assuming Newtonian flow, the following can be written (see equation (1.6))

$$P_{avg} = \frac{\rho\,g\,h}{2}$$

$$F = \frac{A\,\mu\,U}{h} \sim \frac{1 \times 1.069 \times 0.5}{0.01} = 53.45[N]$$

Example 1.2: Concentric Cylinders

Level: Simple

Castor oil at $25°C$ fills the space between two concentric cylinders of 0.2[m] and 0.1[m] diameters with height of 0.1 [m]. Calculate the torque required to rotate the inner cylinder at 12 rpm, when the outer cylinder remains stationary. Assume steady state conditions.

Solution

The velocity is

$$U = r\,\dot\theta = 2\,\pi\,r_i\,\mathsf{rps} = 2 \times \pi \times 0.1 \times \overbrace{12/60}^{\mathrm{rps}} = 0.4\,\pi\,r_i$$

Where rps is revolution per second.

The same way as in Example 1.1, the moment can be calculated as the force times the distance as

$$M = F\,\ell = \frac{\overbrace{\ell}^{r_i}\ \overbrace{A}^{2\,\pi\,r_i\,h}\ \mu U}{r_o - r_i}$$

In this case $r_o - r_i = h$ thus,

$$M = \frac{2\,\pi^2\ \overbrace{0.1^3}^{r_i}\ \not h\ \overbrace{0.986}^{\mu}\ 0.4}{\not h} \sim .0078[N\,m]$$
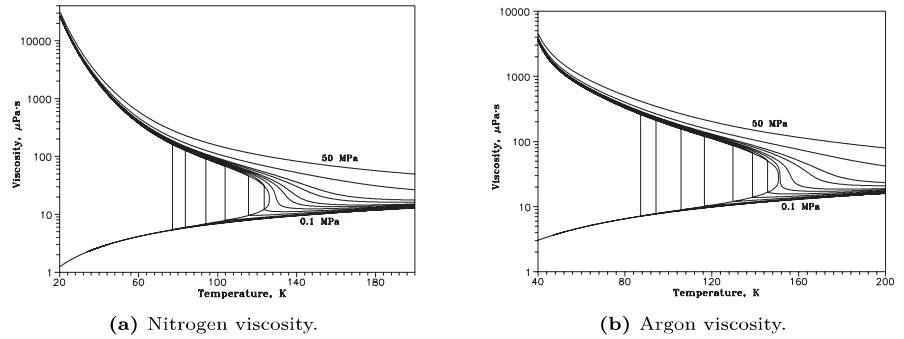
(a) Nitrogen viscosity.

(b) Argon viscosity.

Fig. 1.6 – Nitrogen (left) and Argon (right) viscosity as a function of the temperature and pressure after Lemmon and Jacobsen.

1.5   Viscosity

1.5.1   General Discussion

Viscosity varies widely with temperature. However, temperature variation has an opposite effect on the viscosities of liquids and gases. The difference is due to their fundamentally different mechanism creating viscosity characteristics. In gases, molecules are sparse and cohesion is negligible, while in the liquids, the molecules are more compact and cohesion is more dominate. Thus, in gases, the exchange of momentum between layers brought Viscosity varies widely with temperature as a result of molecular movement normal to the general direction of flow, and it resists the flow. This molecular activity is known to increase with temperature, thus, the viscosity of gases will increase with temperature. This reasoning

Fig. 1.5 – The different of power fluids families. Notice that Bingham fluid has large portion that it is like solid.

is a result of the considerations of the kinetic theory. This theory indicates that gas viscosities vary directly with the square root of temperature. In liquids, the momentum exchange due to molecular movement is small compared to the cohesive forces between the molecules. Thus, the viscosity is primarily dependent on the magnitude of these cohesive forces. Since these forces decrease rapidly with increases of temperature, liquid viscosities decrease as temperature increases.

Fig. 1.6a demonstrates that viscosity increases slightly with pressure, but this variation is negligible for most engineering problems. Well above the critical point, both phases are only a function of the temperature. On the liquid side below the critical point, the pressure has minor effect on the viscosity. It must be stress that the
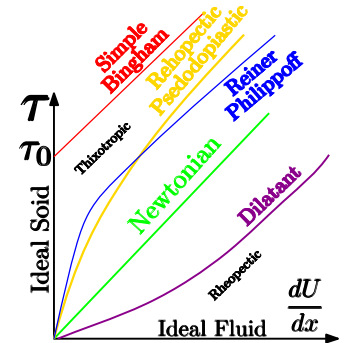
viscosity in the dome is meaningless. There is no such a thing of viscosity at 30% liquid. It simply depends on the structure of the flow as will be discussed in the chapter on multi phase flow. The lines in the above diagrams are only to show constant pressure lines. Oils have the greatest increase of viscosity with pressure which is a good thing for many engineering purposes.

1.5.2 Non–Newtonian Fluids

In equation (1.5), the relationship between the velocity and the shear stress was assumed to be linear. Not all the materials obey this relationship. There is a large class of materials which shows a non–linear relationship with velocity for any shear stress. This class of materials can be approximated by a single polynomial term that is $a = bx^n$. From the physical point of view, the coefficient depends on the velocity gradient. This relationship is referred to as power relationship and it can be written as
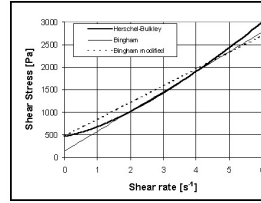

Fig. 1.7 – The shear stress as a function of the shear rate.

$$\tau = K \overbrace{\left(\frac{dU}{dx}\right)^{n-1}}^{viscosity} \left(\frac{dU}{dx}\right) \tag{1.13}$$

The new coefficients (n, K) in equation (1.13) are constant. When $n = 1$ equation represent Newtonian fluid and K becomes the familiar μ. The viscosity coefficient is always positive. When n, is above one, the liquid is dilettante.When n is below one, the fluid is pseudoplastic. The liquids which satisfy equation (1.13) are referred to as purely viscous fluids. Many fluids satisfy the above equation. Fluids that show increase in the viscosity (with increase of the shear) referred to as thixotropic and those that show decrease are called rheopectic fluids (see Figure 1.5).

Materials which behave up to a certain shear stress as a solid and above it as a liquid are referred as Bingham liquids. In the simple case, the "liquid side" is like Newtonian fluid for large shear stress. The general relationship for simple Bingham flow is

$$\tau_{xy} = -\mu \pm \tau_0 \qquad if \ |\tau_{yx}| > \tau_0 \tag{1.14}$$

$$\frac{dU_x}{dy} = 0 \qquad if \ |\tau_{yx}| < \tau_0 \tag{1.15}$$

There are materials that simple Bingham model does not provide adequate explanation and a more sophisticate model is required. The Newtonian part of the model has to be replaced by power liquid. For example, according to Ferraris et al (Ferraris, De Larrard,
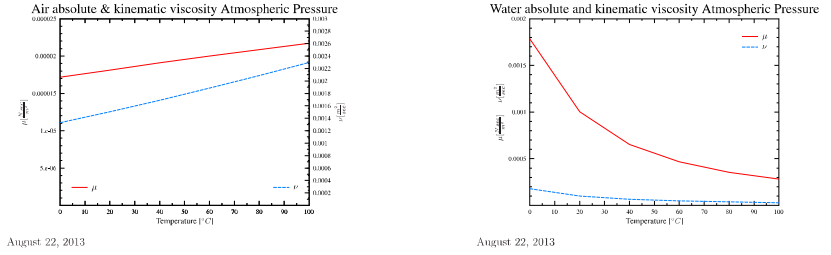
(a) Air viscosity as a function of the temperature.

(b) Water viscosity as a function temperature.

Fig. 1.9 – The effect of the temperature on the absolute the kinematic viscosity of water and air

and Martys 2001) concrete behaves as shown in Figure 1.7. However, for most practical purposes, this kind of figures isn't used in regular engineering practice.

Thixotropic and Rheopectic fluids are two common family of non-Newtonian fluids that additionally are have hysteresis which the shape is time depend. Thixotropic Fluid a fluid wit hysteresis loop is known as thixotropic fluid; the applicable viscosity of a thixotropic fluid reduced with the time for a constant shear stress. For example, the water suspension with bentonitic clay is used in petroleum industry as drilling fluid. Clearly, for long use it advantage to have the viscosity reduced. A dilatant fluid having with increased viscosity for constant shear stress with time. Examples of this category include printer inks and gypsum pastes.
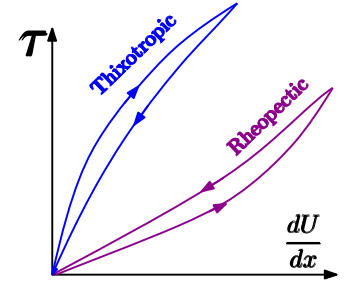
Fig. 1.8 – Thixotropic and Rheopectic fluids with the hysteresis and the time depended.

1.5.3   Kinematic Viscosity

The kinematic viscosity is another way to look at the viscosity. The reason for this new definition is that some experimental data are given in this form. These results also explained better using the new definition. The kinematic viscosity embraces both the viscosity and density properties of a fluid. The above equation shows that the dimensions of ν to be square meter per second, $[m^2/sec]$, which are acceleration units (a combination of kinematic terms). This fact explains the name "kinematic" viscosity. The kinematic viscosity is defined as

$$\nu = \frac{\mu}{\rho} \qquad\qquad (1.16)$$

The gas density decreases with the temperature. However, The increase of the absolute viscosity with the temperature is enough to overcome the increase of density and thus, the kinematic viscosity also increase with the temperature for many materials.

1.5.4   Estimation of The Viscosity

The absolute viscosity of many fluids relatively doesn't change with the pressure but very sensitive to temperature. For isothermal flow, the viscosity can be considered constant in many cases. The variations of air and water as a function of the temperature at atmospheric pressure are plotted in Figures Fig. 1.9.

Some common materials (pure and mixture) have expressions that provide an estimate. For many gases, Sutherland's equation is used and according to the literature, provides reasonable results[3] for the range of $-40°C$ to $1600°C$.

$$\mu = \mu_0 \frac{0.555 \, T_{i0} + Suth}{0.555 \, T_{in} + Suth} \left(\frac{T}{T_0}\right)^{\frac{3}{2}} \tag{1.17}$$

Where

μ	viscosity at input temperature, T
μ_0	reference viscosity at reference temperature, T_{i0}
T_{in}	input temperature in degrees Kelvin
T_{i0}	reference temperature in degrees Kelvin
$Suth$	Suth is Sutherland's constant and it is presented in the Table 1.1. .

Example 1.3: Viscosity Estimation with Sutherland   **Level: Simple**

Calculate the viscosity of air at 800K based on Sutherland's equation. Use the data provide in Table 1.1.

Solution

Applying the constants from Suthelnd's table provides

$$\mu = 0.00001827 \times \frac{0.555 \times 524.07 + 120}{0.555 \times 800 + 120} \times \left(\frac{800}{524.07}\right)^{\frac{3}{2}} \sim 2.51\,10^{-5} \left[\frac{N\,sec}{m^2}\right]$$

The viscosity increases almost by 40%. The observed viscosity is about $\sim 3.710^{-5} \left[\frac{N\,sec}{m^2}\right]$.

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[3]This author is ambivalent about this statement.