Fixed Income Tracking Portfolio Optimization - Executive Summary

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Statement of Problem

Suppose that a company faces a stream of liabilities and wants to guarantee that it will be able to pay using its cash reserves today. As a result, a tracking portfolio will be created to match the liabilities of each year while eliminating interest rate risk. This portfolio will be optimized for the cost of the portfolio today; thus, year 0 portfolio value will be minimized.

Discussion

In this scenario, there are three forward bonds and ten regular bonds that can be used to cover the firm's liabilities over eight years.

Also, we are only buying the future bonds and not shorting any newly issued or future bonds. It is also assumed that the coupon payments are yearly and are made at the end of the year.

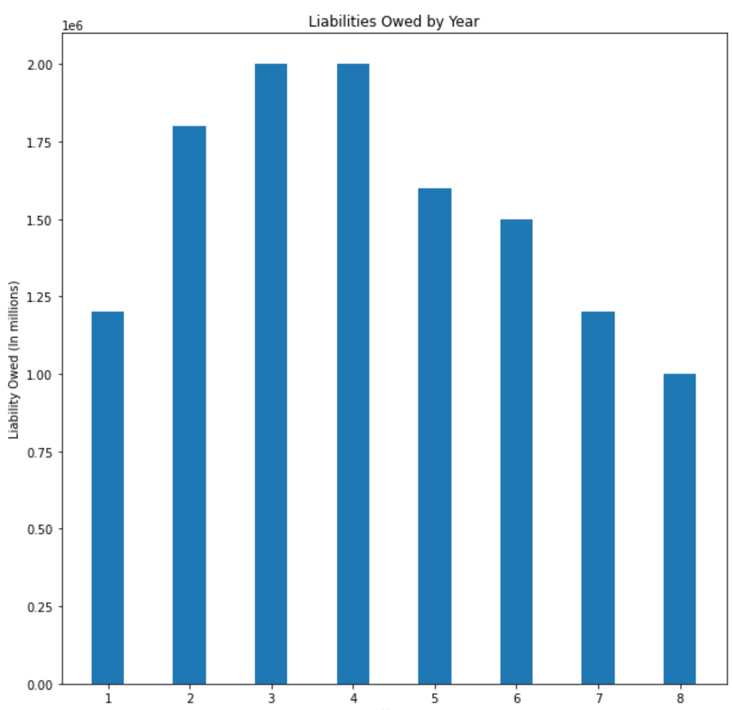

Fig 1) Liabilities owed by year

As you can see, the liabilities are heavily weighted in year 2 - 4. Therefore, these year's constraints will be most impactful on the total portfolio value.

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	107.0	-100.0	3.5	4.0	2.5	4.0	0.0	9.0	6.0	8.0	9.0	7.0	0.0
1	0.0	103.0	102.5	105.0	2.5	4.0	-98.0	9.0	6.0	8.0	9.0	7.0	0.0
2	0.0	0.0	0.0	0.0	100.5	4.0	2.0	9.0	6.0	8.0	9.0	7.0	-91.0
3	0.0	0.0	0.0	0.0	0.0	102.0	100.0	9.0	6.0	8.0	9.0	7.0	3.0
4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	113.0	106.0	8.0	9.0	7.0	3.0
5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	109.0	9.0	7.0	3.0
6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	111.0	7.0	3.0
7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	101.0	94.0

Fig 2) Decision Matrix formulated using earning and spending from bonds

In adherence to the requirements, we need to invest money today by ways of simple bonds to satisfy the liabilities and make any associated buying/transaction of the forward bonds. These constraints will be incorporated into our decision matrix. Additionally, constraints revolve around the assumption that our earnings from interest and principal payments each year will be equal to the sum of spendings and liabilities for that year. Finally, with these constraints, our objective will be to minimize year 0 investment.

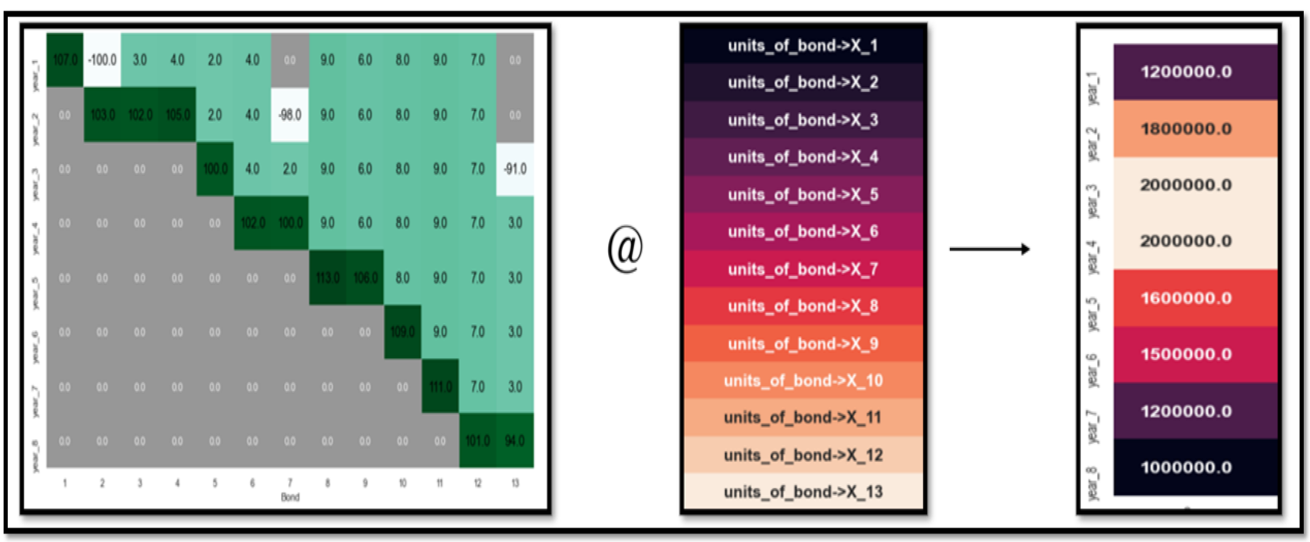

Fig 3) Pictorial representation of our approach

In this illustration of our approach, the decision matrix is based on the returns we earn from each unit bond in the coming 8 years. The multiplicative association of the number of units of each bond, which also serves as our decision variables, with returns, should be equal to the liability we will be facing every year.

Solving for Optimal Portfolio

```
#Model Set UP
model = gp.Model()

modelX = model.addMVar(len(bonds), lb = 0.0 )
modelCon = model.addMConstrs(A, modelX, sense, b)
model.setMObjective(None,obj,0,sense=gp.GRB.MINIMIZE)

model.Params.OutputFlag = 0

model.optimize()

print('\nThe value of the portfolio today is:', round(model.objVal,2))
```

```
The value of the portfolio today is: 9473468.04
```

	Quantity
Bond_1	18905.515879
Bond_2	12920.641375
Bond_3	0.000000
Bond_4	0.000000
Bond_5	15620.674098
Bond_6	16019.160682
Bond_7	0.000000
Bond_8	11864.946053
Bond_9	0.000000
Bond_10	12284.543604
Bond_11	10186.424048
Bond_12	9900.990099
Bond_13	0.000000

Fig 4) Process flow of the optimization along with the obtained results

Using the optimization package Gurobi, we found the optimal portfolio value at $9,473,468.04. This investment consists of the quantities of bonds shown in the right-hand graph. It can be seen that the heavily weighted liabilities are financed by the year 1 simple bond and the forward bond. The firm's liabilities in the later years are financed with respect to the bonds seen in the graph above.

```
[0.9532710280373832,
 0.9255058524634788,
 0.9283886348134114,
 0.850699391556303,
 0.6290196091872993,
 0.6119717786379542,
 0.5217143544841221,
 0.5550099550073301]
```

	Lower Bound	Upper Bound	Actual Liabilities
Year1	-822890.199052	inf	1200000
Year2	469173.938412	inf	1800000
Year3	430122.253167	4.326098e+07	2000000
Year4	366045.610440	2.884069e+07	2000000
Year5	259261.095958	1.535509e+07	1600000
Year6	160984.747123	1.771844e+07	1500000
Year7	69306.930693	1.704376e+07	1200000
Year8	0.000000	1.731429e+07	1000000

Fig 5) Sensitivity analysis of the constants of constraint, i.e the liability value for each year

In the left-hand graph, the shadow prices show that an increase of $1 of liabilities, in their respective years in the data set as well as the order shown here, represent the increase of the price in the year 0 portfolio. The descending nature of the values in the list, shown in this section, roughly shows the concept of the time value of money. For example, if the liabilities due in year 8 were to increase by $1, the investment required to fulfill that extra dollar would only be $0.56 as compared to an increase in year 1 at $0.95. Therefore, the later year liabilities are our least sensitive factors and the nearest liabilities are our most sensitive to variations in the estimates.

	Lower Bound	Upper Bound	Actual Price
Bond_1	98.072248	102.008889	102
Bond_2	-3.814567	0.008634	100
Bond_3	98.200798	inf	99
Bond_4	100.991199	inf	101
Bond_5	72.552007	inf	98
Bond_6	-377.102221	101.848314	98
Bond_7	-3.772857	inf	98
Bond_8	-820.001482	116.128026	104
Bond_9	88.623268	inf	100
Bond_10	-724.877431	inf	101
Bond_11	-670.839431	inf	102
Bond_12	-inf	120.294614	94
Bond_13	-24.472215	inf	91

Fig 6) Sensitivity analysis of coefficients of the equation that needs to be optimized, here actual price of bonds

In this graph, the range of objective values (Year 0 Bond Prices) shows how much each bond's price can vary in year 0 and keep the same minimized portfolio value. Therefore, our shadow prices are still good estimates of the change in portfolio value in year 0. The ranges are extremely large which allows many different scenarios of liability amounts to be analyzed as a function of year 0 portfolio value. The lower bounds that are negative and the upper bounds that are infinity are obviously not realistic or realistically insightful, so they will be ignored. It can see that the optimal portfolio value is most sensitive to Bond 1's price. This is not surprising as it is a one-year bond and our liabilities are most sensitive to changes in year 1.

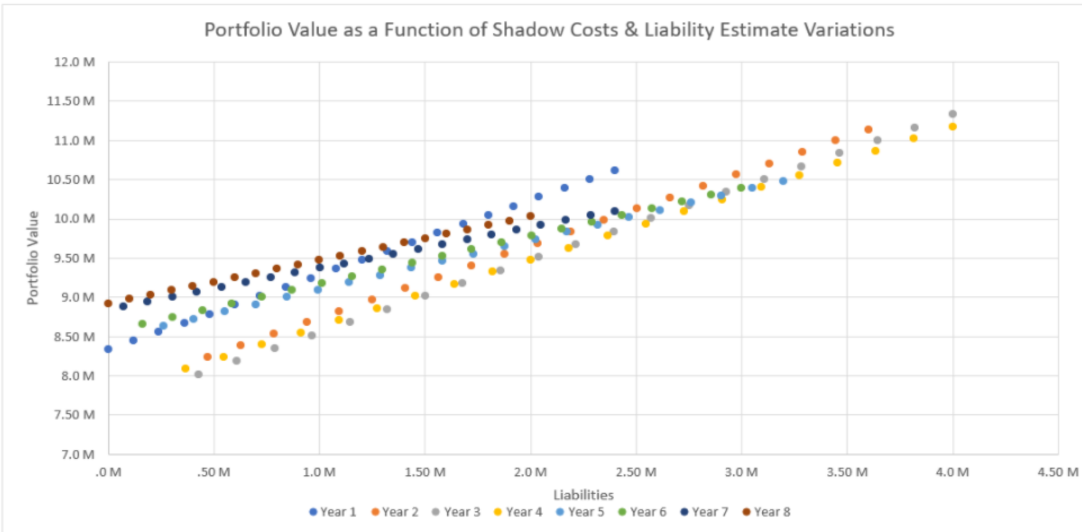

Fig 7) Portfolio value as a function of shadow cost and liability estimate variation

This plot shows the portfolio as a function of the shadow prices, shown above, and change in an individual liability value, keeping all other liability estimates constant. In this analysis, liabilities were varied from their lower bound, seen in the table above, to twice the estimated amount. It can be seen that the slope of the Year 1 portfolio value is the steepest, which confirms our earlier analysis. Additionally, the portfolio values can range from $8 million to almost $11.5 million. Therefore, if the firm does not have confidence in its liability estimates, it is reasonable to advise the firm to adjust its fixed-income investments to reflect the change in liability estimates as seen above.

Finding Optimal Portfolio with Treasury Bonds

```
bond_mat = np.zeros((len(bonds_from_wsj),len(liabilities)+1))

for i in range(len(bonds_from_wsj)):
  bond_mat[i,  int(bonds_from_wsj.iloc[i]['StartTime'])] =  -bonds_from_wsj.iloc[i]['Price']
  bond_mat[i,  int(bonds_from_wsj.iloc[i]['Maturity'])] =  bonds_from_wsj.iloc[i]['Price']+ bonds_from_wsj.iloc[i]['Coupon']

  for j in range( int(bonds_from_wsj.iloc[i]['StartTime']+1) , int(bonds_from_wsj.iloc[i]['Maturity']) ):
    bond_mat[i,j] = bonds_from_wsj.iloc[i]['Coupon']


A = bond_mat.T
bond_df = pd.DataFrame(data = A, columns= ['Bond_%i'%(i) for i in range(1,len(bonds_from_wsj)+1)])
bond_df
```

Bond_6	Bond_7	Bond_8	Bond_9	Bond_10	...	Bond_225	Bond_226	Bond_227	Bond_228	Bond_229	Bond_230	Bond_231	Bond_232	Bond_233	Bond_23
-100.282	-100.140	-100.22	-100.234	-101.010	...	-98.246	-99.184	-112.104	-127.006	-109.002	-127.18	-107.102	-101.266	-135.242	-102.23
102.782	101.265	101.97	102.109	103.385	...	1.125	1.250	3.125	5.250	2.625	5.25	2.375	1.625	6.125	1.75
0.000	0.000	0.00	0.000	0.000	...	1.125	1.250	3.125	5.250	2.625	5.25	2.375	1.625	6.125	1.75
0.000	0.000	0.00	0.000	0.000	...	1.125	1.250	3.125	5.250	2.625	5.25	2.375	1.625	6.125	1.75
0.000	0.000	0.00	0.000	0.000	...	1.125	1.250	3.125	5.250	2.625	5.25	2.375	1.625	6.125	1.75
0.000	0.000	0.00	0.000	0.000	...	1.125	1.250	3.125	5.250	2.625	5.25	2.375	1.625	6.125	1.75
0.000	0.000	0.00	0.000	0.000	...	1.125	1.250	3.125	5.250	2.625	5.25	2.375	1.625	6.125	1.75
0.000	0.000	0.00	0.000	0.000	...	99.371	100.434	115.229	132.256	2.625	5.25	2.375	1.625	6.125	1.75
0.000	0.000	0.00	0.000	0.000	...	0.000	0.000	0.000	0.000	111.627	132.43	109.477	102.891	141.367	103.98

Fig 8) Process flow to optimize portfolio dealing with the Treasury Bonds

Regarding the second tracking portfolio, 234 bonds were sanctioned from current treasury bonds to satisfy our liabilities. The same objective still remains in this problem, but current treasury bonds are used to optimize our objective. As seen above, the bonds vary in their price and coupon payment but there are no forward bonds available to us in this scenario.

```
# set constraint values, equal signs to tell gurobi to match liabilities, and set objective
b = np.array(liabilities['Liability'])
sense = np.array(['=']*len(liabilities))

objective = np.array(bond_df.iloc[0])

wsjMod = gp.Model() # initialize empty model before optimizing
wsjx = wsjMod.addMVar(len(objective)) # tell the model how many variables there are
wsjModCon = wsjMod.addMConstrs(bond_df.iloc[1:len(liabilities)+1], wsjx, sense, b) # add the constraints to the model
wsjMod.setMObjective(None,objective,0,sense=gp.GRB.MAXIMIZE) # add the objective to the model

wsjMod.Params.OutputFlag = 0 #
wsjMod.optimize()

# Total Value of Bonds to be purchased.
print('Total Value of Bonds Purchased ($) = ', -wsjMod.objVal)

# Total Bond Purchase
pd.DataFrame(wsjx.x, index = ['Bond_%i'%(i) for i in range(1,len(bonds_from_wsj)+1)], columns = ['Quantity'] )
```

```
Total Value of Bonds Purchased ($) =  9926378.278402824
```

	Quantity
Bond_1	0.000000
Bond_2	0.000000
Bond_3	0.000000
Bond_4	0.000000
Bond_5	0.000000
...	...
Bond_230	0.000000
Bond_231	0.000000
Bond_232	0.000000
Bond_233	7073.786669
Bond_234	0.000000

234 rows × 1 columns

Fig 9) Results obtained for the Treasury bonds

The graph above shows the optimal portfolio using treasuries priced on October 5th, 2021. This portfolio was calculated within the same constraints as the first portfolio. The optimal portfolio year 0 value calculated with treasuries costs more than the bonds in the first portfolio at $9,926,368.28. If the firm had the option, the liabilities should be funded by the bonds in the first scenario.

Recommendation

The interrelation of the interest rate earned from the bond and the life of the bond serves as the primary factor determining the number of the respective bonds we buy. We should try investing in bonds that pay us interest during our burden years when liability is more. In addition to this, if we have bonds with the same coupon rate, here bond 4 and bond 6, we should prefer investing in the bond that has a lower price and larger life. Moreover, the start time of the forward bonds plays an important role in the decision of whether to include them in our portfolio or not. We should primarily avoid spending on the forward bonds in the years when our estimated liability is high.