Laboratorium 4

Importowanie bibliotek:

In [ ]:
```
import numpy as np
import networkx as nx
from collections import deque
from math import inf
from sklearn.utils.extmath import randomized_svd
```

Algorytm permutacji minimum degree

Algorytm minimum degree jest używany w analizie numerycznej do permutacji wierszy i kolumn symetrycznej rzadkiej macierzy przed zastosowaniem rozkładu Cholesky'ego. Celem jest zmniejszenie liczby niezerowych elementów w czynniku Cholesky'ego, co prowadzi do zmniejszenia wymagań pamięciowych i umożliwia zastosowanie czynnika Cholesky'ego z mniejszą liczbą operacji arytmetycznych.

Algorytmy minimum degree są często stosowane w metodzie elementów skończonych, gdzie przestawienie węzłów może być przeprowadzone tylko na podstawie topologii siatki, a nie współczynników w równaniu różniczkowym cząstkowym. Daje to oszczędności efektywności, gdy ta sama siatka jest używana dla różnych wartości współczynników.

Pseudokod:

Dla symetrycznej macierzy M o rozmiarze n x n:

- Dla danego grafu G_0 = (V, E), gdzie V to zbiór wierzchołków, a E to zbiór krawędzi, jeżeli w i-tym wierszy i j-tej kolumnie jest wartość niezerowa to w tamtym miejscu znajduje się krawędź, $|V|$ = n
- Zdefiniuj R = [] jako pustą listę permutacji.
- Dla i = 1 do n (gdzie n to liczba wierzchołków w V):
  - Znajdź wierzchołek v w G_{i-1} o najmniejszym stopniu.
  - Dodaj v do R.
  - G_i = G_{i-1} (bez wierzchołka v)
- Zwróć R.

In [ ]:
```
def minimum_degree_permutation(matrix):
    """
    Returns the minimum degree permutation of the given matrix.
    """
    n,m = matrix.shape
    adjacencies_matrix = {i:set() for i in range(n)}

    for i in range(n):
```

```
        for j in range(m):
            if matrix[i,j] != 0 and i != j:
                adjacencies_matrix[i].add(j)

    permutation = []

    for i in range(n):
        min_degree = m+1

        for v, adj in adjacencies_matrix.items():
            if len(adj) < min_degree:
                min_degree = len(adj)
                min_degree_node = v

        for v in adjacencies_matrix:
            adjacencies_matrix[v] = adjacencies_matrix[v].difference([min

        for u in adjacencies_matrix[min_degree_node]:
            adjacencies_matrix[u] = (adjacencies_matrix[u].union(adjacenc

        adjacencies_matrix.pop(min_degree_node)
        permutation.append(min_degree_node)

    return permutation

def permutate(matrix, permutation):
    new_matrix = matrix.copy()
    for i in range(len(permutation)):
        if i == permutation[i]:
            continue
        new_matrix[i,:] = matrix[permutation[i],:].copy()

    matrix = new_matrix.copy()
    for i in range(len(permutation)):
        if i == permutation[i]:
            continue
        new_matrix[:,i] = matrix[:,permutation[i]].copy()

    return new_matrix
```

Algorytm permutacji Cuthill Mckee

Algorytm Cuthill-McKee (CM) w numerycznej algebrze liniowej służy do permutacji rzadkiej macierzy o symetrycznym wzorcu rzadkości do formy macierzy pasmowej o małej szerokości pasma. Odwrócony algorytm Cuthill-McKee (RCM) to ten sam algorytm, ale z odwróconymi wynikowymi numerami indeksów. W praktyce RCM zazwyczaj prowadzi do mniejszego 'fill-in' niż porządek CM po zastosowaniu eliminacji Gaussa.

Pseudokod:

Dla symetrycznej macierzy M o rozmiarze n x n:

- Dla danego grafu G_0 = (V, E), gdzie V to zbiór wierzchołków, a E to zbiór krawędzi, jeżeli w i-tym wierszy i j-tej kolumnie jest wartość niezerowa to w tamtym miejscu znajduje się krawędź, $|V|$ = n

- Zdefiniuj R = [v], gdzie v to wierzchołek o najmniejszym stopniu z G
- Dla i = 1,2 ... dopóki $|R| < n$:
  - A_i = Adj(R_i) \ R
  - sort(A_i) - rosnąco po minimalnym przodku (odwiedzonym sąsiadem z najmniejszym indeksem w R oraz po stopniu jezeli minimalni przodkowie są tacy sami).
  - Dodaj A_i do zbioru R
- Zwróć R

In [ ]:
```
def cuthill_mckee(matrix):
    """
    Returns the cutill mckee permutation of the given matrix.
    """
    n,m = matrix.shape
    adjacencies_matrix = {i:set() for i in range(n)}

    for i in range(n):
        for j in range(m):
            if matrix[i,j] != 0 and i != j:
                adjacencies_matrix[i].add(j)

    v_min = min(adjacencies_matrix, key=lambda x: len(adjacencies_matrix[

    visited = set()
    ordering = []
    queue = deque()

    # Start with the minimum degree node
    queue.append(v_min)

    while queue:
        v = queue.popleft()
        if v not in visited:
            visited.add(v)
            ordering.append(v)
            neighbors = list(adjacencies_matrix[v])
            neighbors.sort(key=lambda x: len(adjacencies_matrix[x]))
            for u in neighbors:
                if u not in visited:
                    queue.append(u)
    return ordering

def reversed_cuthill_mckee(matrix):
    return cuthill_mckee(matrix)[::-1]
```

Funkcja do permutowania macierzy wybranym algorytmem

In [ ]:
```
def permutate_with_algorithm(matrix,algorithm):
    """
    Returns the permutation of the given matrix using the given algorithm
    """
    permutation = algorithm(matrix)
```

```
    return permutate(matrix,permutation)
```

Algorytm kompresji

In [ ]:
```
class MatrixTree:
    def __init__(self, matrix, x1, x2, y1, y2):
        self.matrix = matrix
        self.x1 = x1
        self.x2 = x2
        self.y1 = y1
        self.y2 = y2
        self.leaf = False
        self.children = None


    def compress(self, r, eps):
        U, Sigma, V = randomized_svd(self.matrix[self.x1:self.x2, self.y1
        if self.x1 + r == self.x2 or np.all(Sigma[r] <= eps):
            self.leaf = True
            if not self.matrix[self.x1:self.x2, self.y1: self.y2].any():
                self.rank = 0
            else:
                self.rank = len(Sigma)
                self.u = U
                self.s = Sigma
                self.v = V
        else:
            self.children = []
            self.children.append(MatrixTree(self.matrix, self.x1, (self.x
            self.children.append(MatrixTree(self.matrix, self.x1, (self.x
            self.children.append(MatrixTree(self.matrix, (self.x1 + self.
            self.children.append(MatrixTree(self.matrix, (self.x1 + self.

            for child in self.children:
                child.compress(r, eps)

    def decompress(self, output_matrix):
        if self.leaf:
            if self.rank != 0:
                sigma = np.zeros((self.rank,self.rank))
                np.fill_diagonal(sigma, self.s)
                output_matrix[self.x1:self.x2, self.y1: self.y2] = self.u
            else:
                output_matrix[self.x1:self.x2, self.y1: self.y2] = self.m
        else:
            for child in self.children:
                child.decompress(output_matrix)

    def compute_compression(self):
        if self.leaf:
            x = self.x2 - self.x1
            y = self.y2 - self.y1
            return (2 * self.rank * len(self.u) + self.rank, x * y) if se
        v,s = 0,0
        for child in self.children:
            v1,s1 = child.compute_compression()
            v += v1
            s += s1
```

```
        return v,s

    def compute_ratio(self):
        v,s = self.compute_compression()
        return s/v
```

In [ ]:
```
import matplotlib.pyplot as plt

def create_tree_image(root, title=''):
    image = np.ones(root.matrix.shape)*255
    Q = deque()
    Q.append(root)
    while Q:
        v = Q.pop()
        if v.leaf:
            image[v.x1:v.x2, v.y1:v.y1+v.rank] = np.zeros((v.x2 - v.x1, v
            image[v.x1:v.x1+v.rank, v.y1:v.y2] = np.zeros((v.rank , v.y2
            image[v.x1, v.y1:v.y2] = np.zeros((1,v.y2 - v.y1))
            image[v.x2-1, v.y1:v.y2] = np.zeros((1,v.y2 - v.y1))
            image[v.x1:v.x2,v.y1] = np.zeros(v.x2-v.x1)
            image[v.x1:v.x2,v.y2-1] = np.zeros(v.x2-v.x1)
        else:
            for child in v.children:
                Q.append(child)
    return image
```

Funkcja generowania macierzy

In [ ]:
```
def generate_3d_grid(k):
    size = 2 ** (3 * k)
    side_length = 2 ** k
    matrix = np.zeros((size, size))

    for i in range(size):
        # Calculate the 3D coordinates of the node
        x = i % side_length
        y = (i // side_length) % side_length
        z = i // (side_length ** 2)

        # Add edges to the neighboring nodes
        if x > 0: matrix[i, i - 1] = np.random.rand()
        if x < side_length - 1: matrix[i, i + 1] = np.random.rand()
        if y > 0: matrix[i, i - side_length] = np.random.rand()
        if y < side_length - 1: matrix[i, i + side_length] = np.random.ra
        if z > 0: matrix[i, i - side_length ** 2] = np.random.rand()
        if z < side_length - 1: matrix[i, i + side_length ** 2] = np.rand

    return matrix
```

Przygotowanie macierzy do rysowania

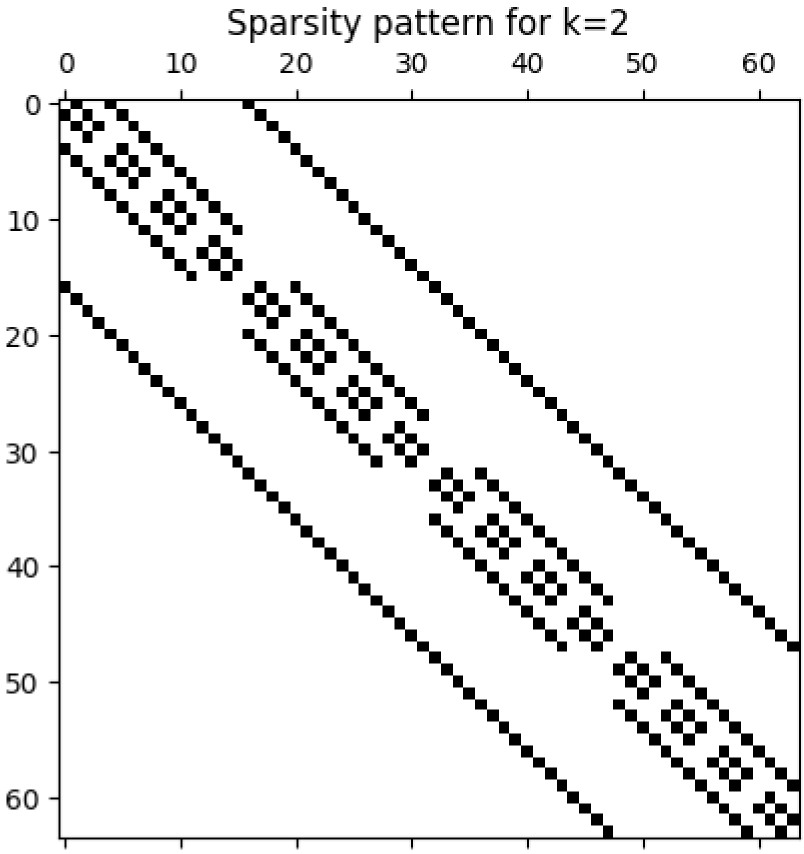
In [ ]:
```
# Generate the matrices for k=2,3,4
matrixes = [generate_3d_grid(k) for k in range(2, 5)]
matrixes_compressed = []

for matrix in matrixes:
```

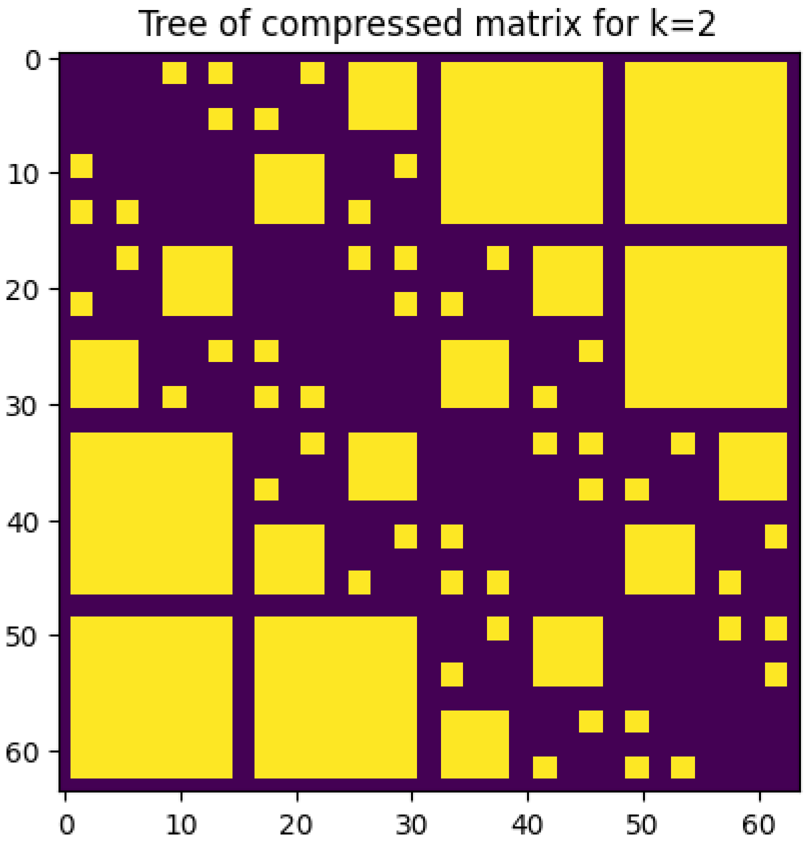
```
    _, eps, _ = randomized_svd(matrix, n_components=matrix.shape[0], rand
    root = MatrixTree(matrix, 0, matrix.shape[0], 0, matrix.shape[1])
    root.compress(1, eps)
    matrixes_compressed.append(root)
```

In [ ]:
```
matrixes_permutated = [(permutate_with_algorithm(matrix, cuthill_mckee),p
matrixes_permutated_compressed = []

for cuthill_mckee, cuthill_mckee_reversed, mdp in matrixes_permutated:

    _, eps, _ = randomized_svd(cuthill_mckee, n_components=cuthill_mckee.
    root1 = MatrixTree(cuthill_mckee, 0, cuthill_mckee.shape[0], 0, cuthi
    root1.compress(1, eps)

    _, eps, _ = randomized_svd(cuthill_mckee_reversed, n_components=cuthi
    root2 = MatrixTree(cuthill_mckee_reversed, 0, cuthill_mckee_reversed.
    root2.compress(1, eps)

    _, eps, _ = randomized_svd(mdp, n_components=mdp.shape[0], random_sta
    root3 = MatrixTree(mdp, 0, mdp.shape[0], 0, mdp.shape[1])
    root3.compress(1, eps)

    matrixes_permutated_compressed.append((root1,root2,root3))
```

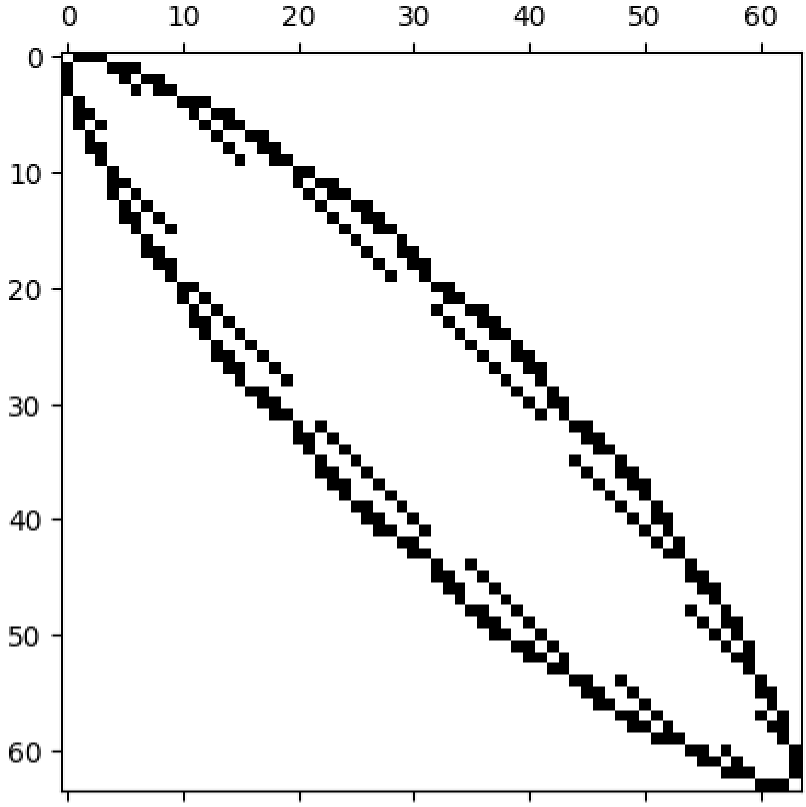
Rysowanie

In [ ]:
```
for i in range(len(matrixes)):
    # draw sparsity pattern
    fig, ax = plt.subplots()
    ax.spy(matrixes[i])
    ax.set_title(f"Sparsity pattern for k={i+2}")
    plt.show()
    # draw tree of compressed matrix
    fig, ax = plt.subplots()
    ax.imshow(create_tree_image(matrixes_compressed[i]))
    ax.set_title(f"Tree of compressed matrix for k={i+2}")
    plt.show()

    # draw sparsity pattern for permutated matrix
    for j in range(3):
        fig, ax = plt.subplots()
        ax.spy(matrixes_permutated[i][j])
        ax.set_title(f"Sparsity pattern for k={i+2} with {['cuthill mckee
        plt.show()
        # draw tree of compressed matrix
        fig, ax = plt.subplots()
        ax.imshow(create_tree_image(matrixes_permutated_compressed[i][j])
        ax.set_title(f"Tree of compressed matrix for k={i+2} with {['cuth
        plt.show()
```

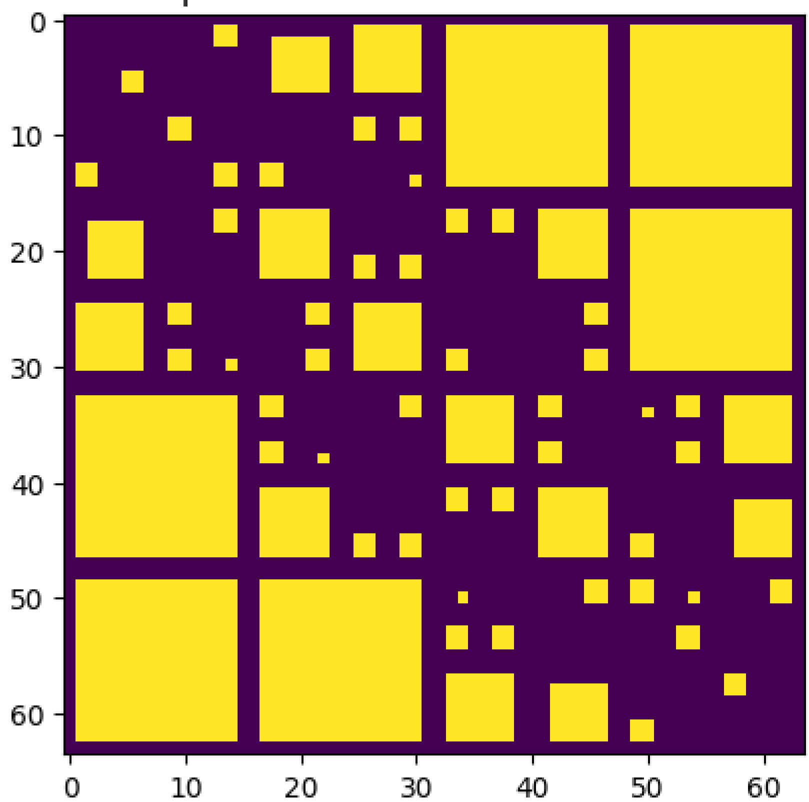

Sparsity pattern for k=2
0
10
20
30
40
50
60


Tree of compressed matrix for k=2
0
10
20
30
40
50
60

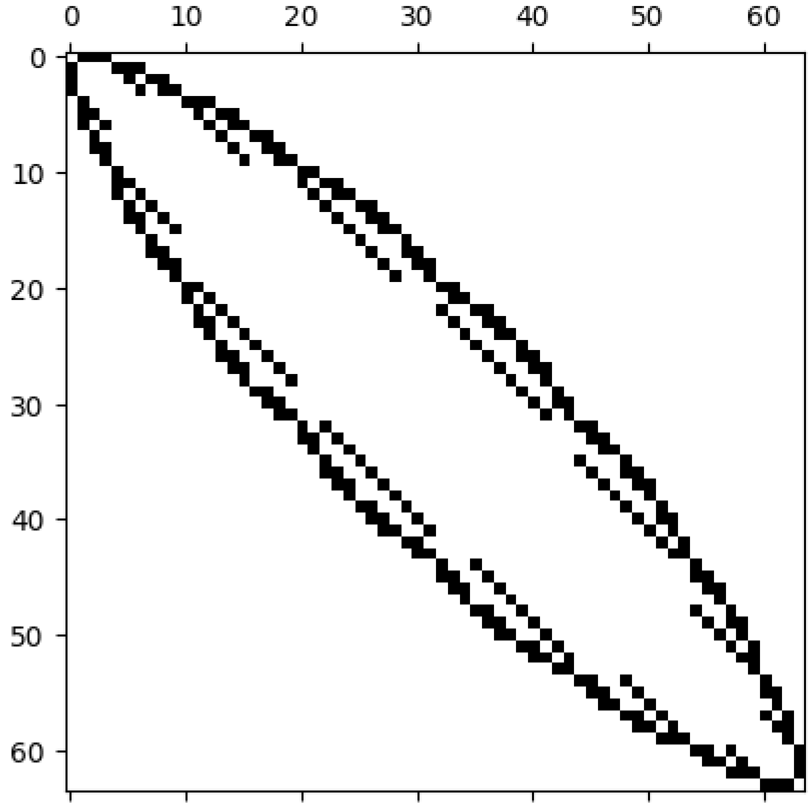
Sparsity pattern for k=2 with cuthill mckee



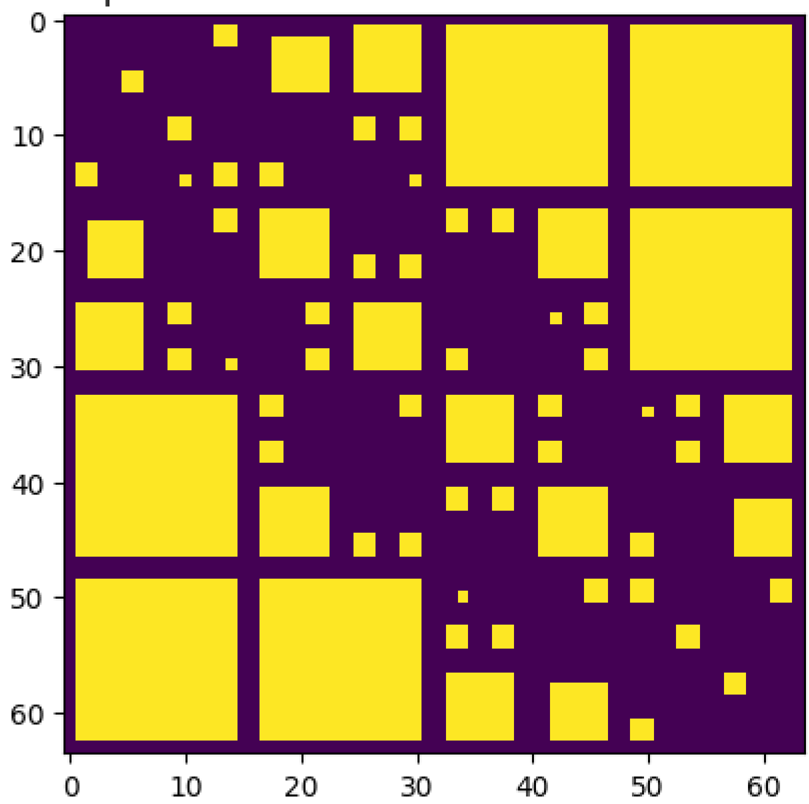
Tree of compressed matrix for k=2 with cuthill mckee

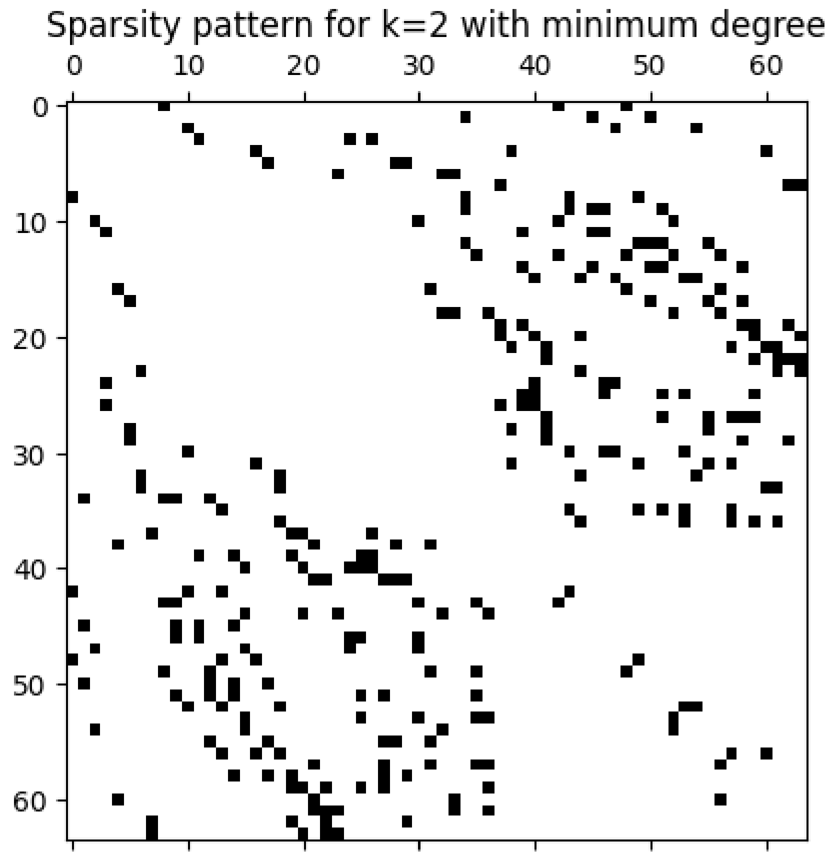


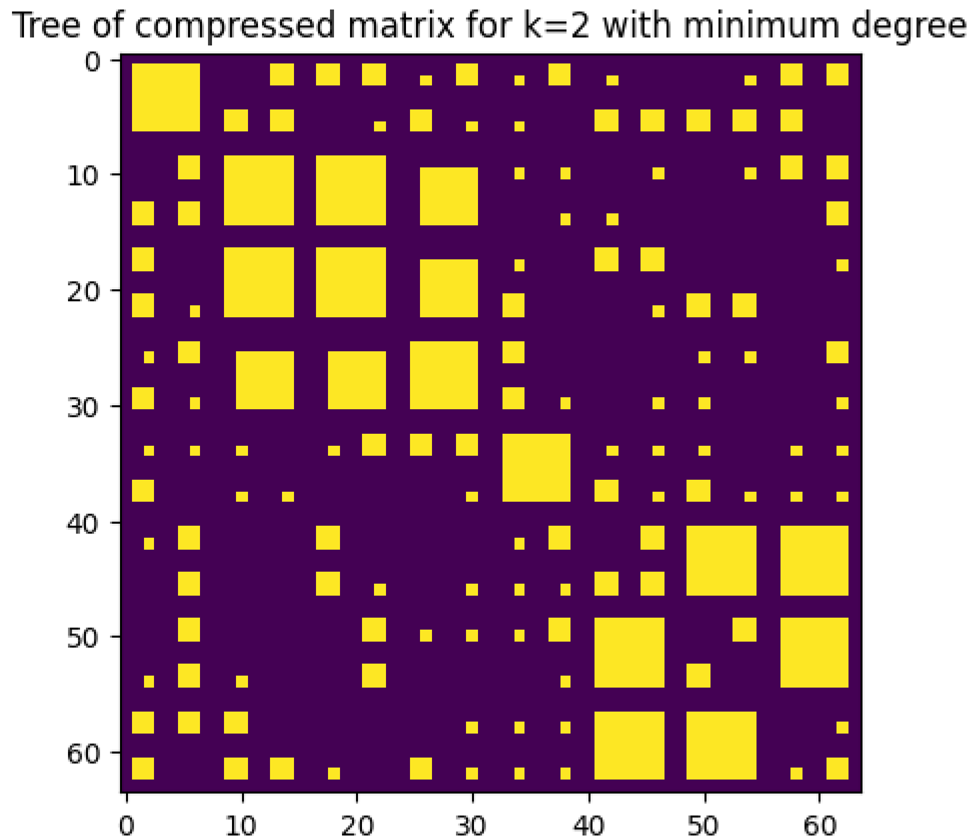
Sparsity pattern for k=2 with reversed cuthill mckee

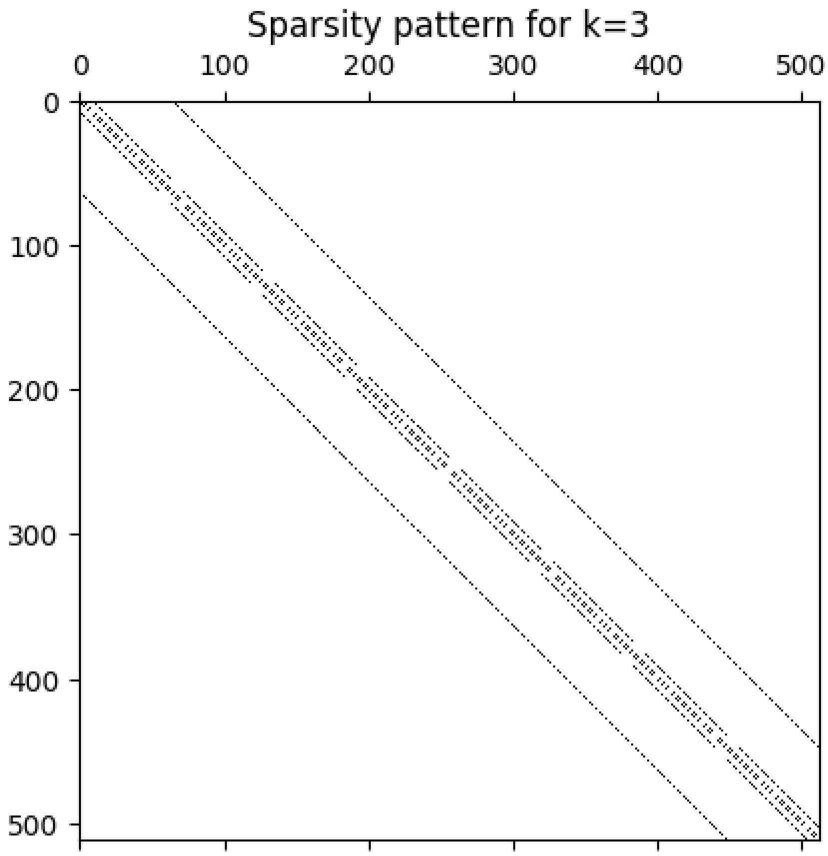


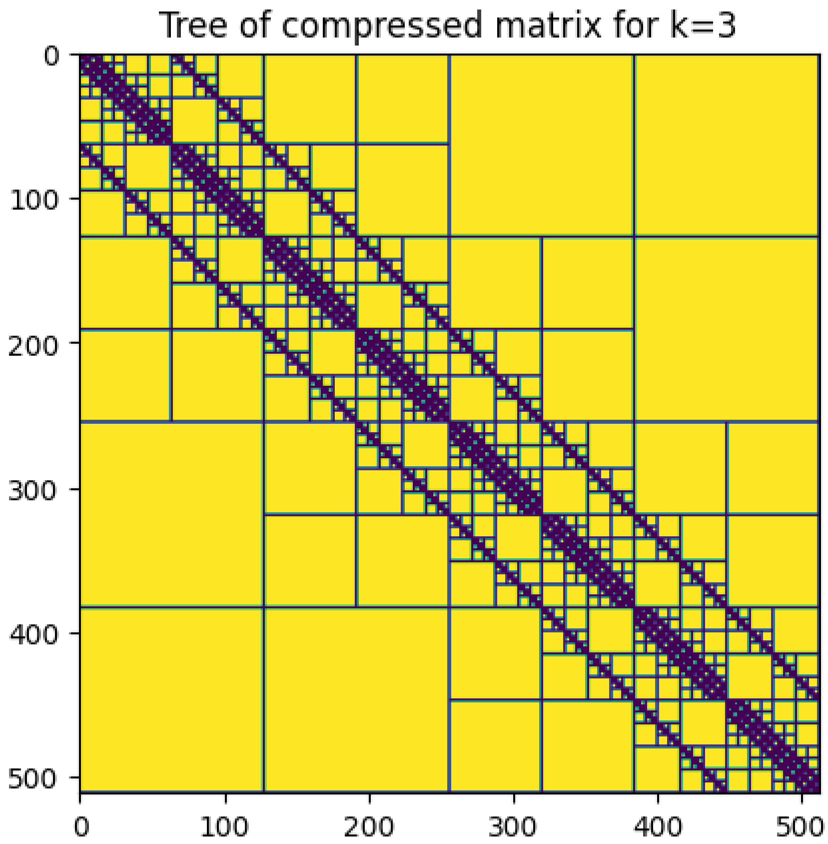
Tree of compressed matrix for k=2 with reversed cuthill mckee

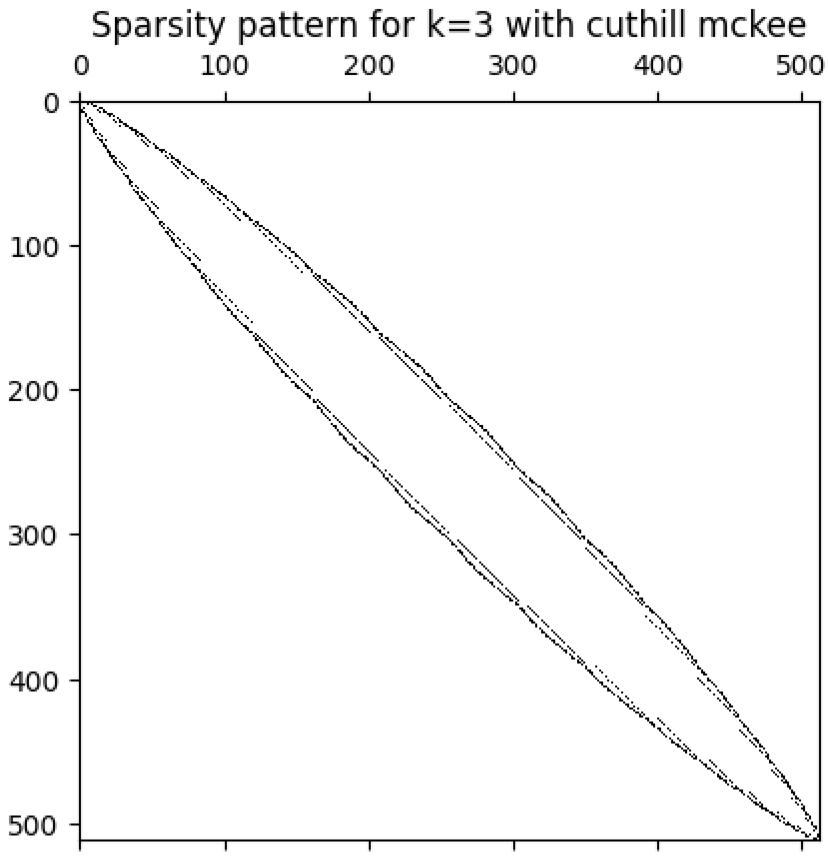


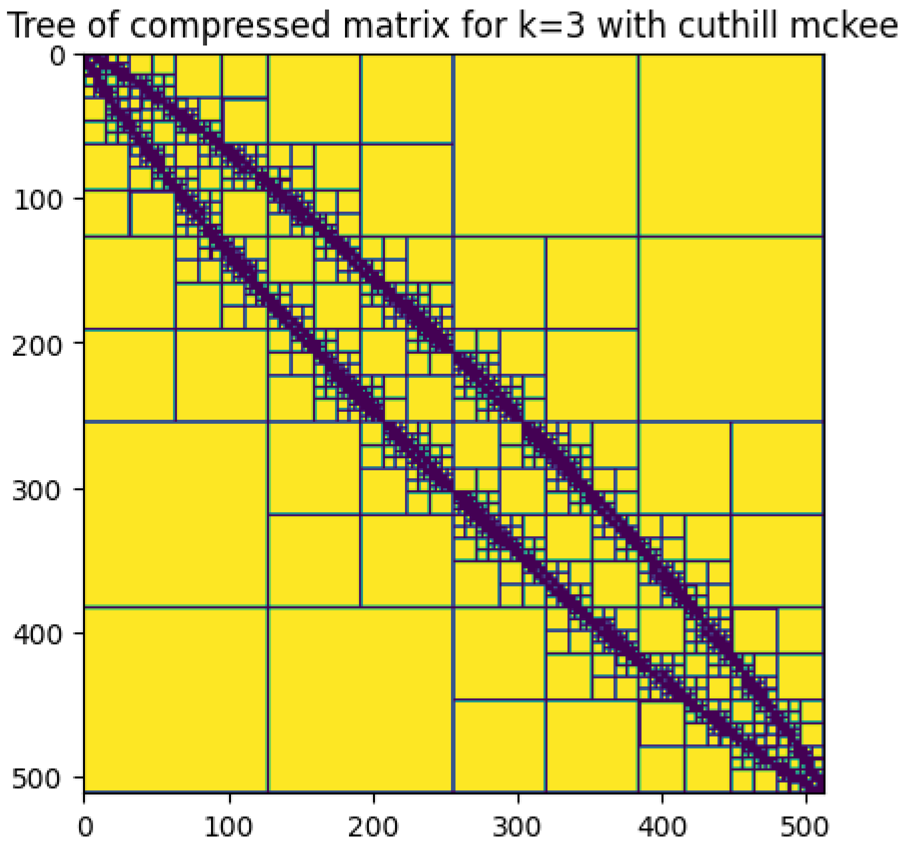

Sparsity pattern for k=2 with minimum degree
0
10
20
30
40
50
60


Tree of compressed matrix for k=2 with minimum degree
0
10
20
30
40
50
60

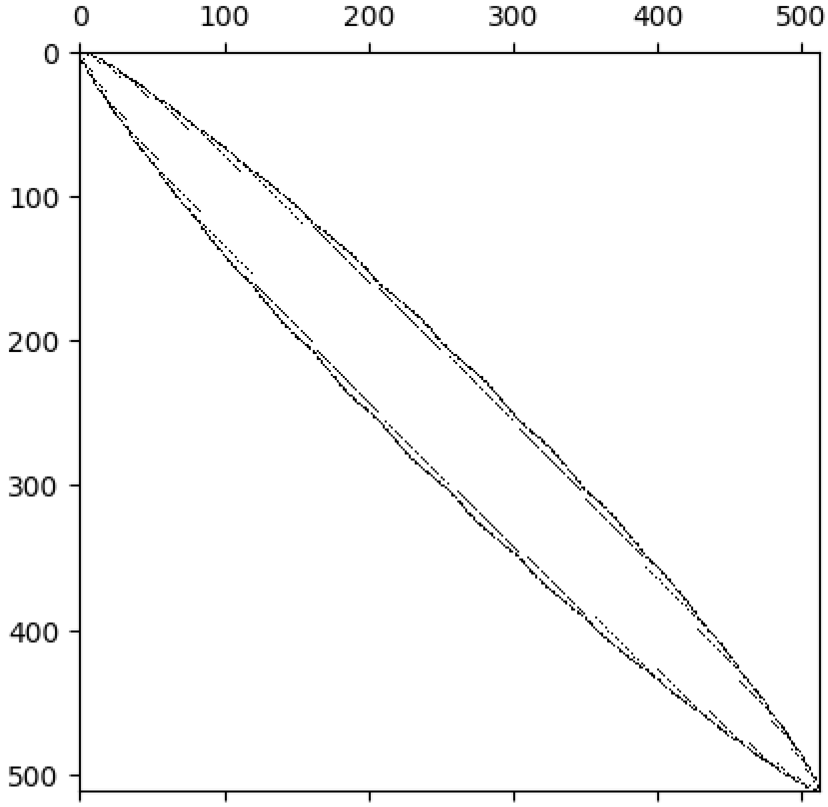

Sparsity pattern for k=3
0
100
200
300
400
500


Tree of compressed matrix for k=3
0
100
200
300
400
500

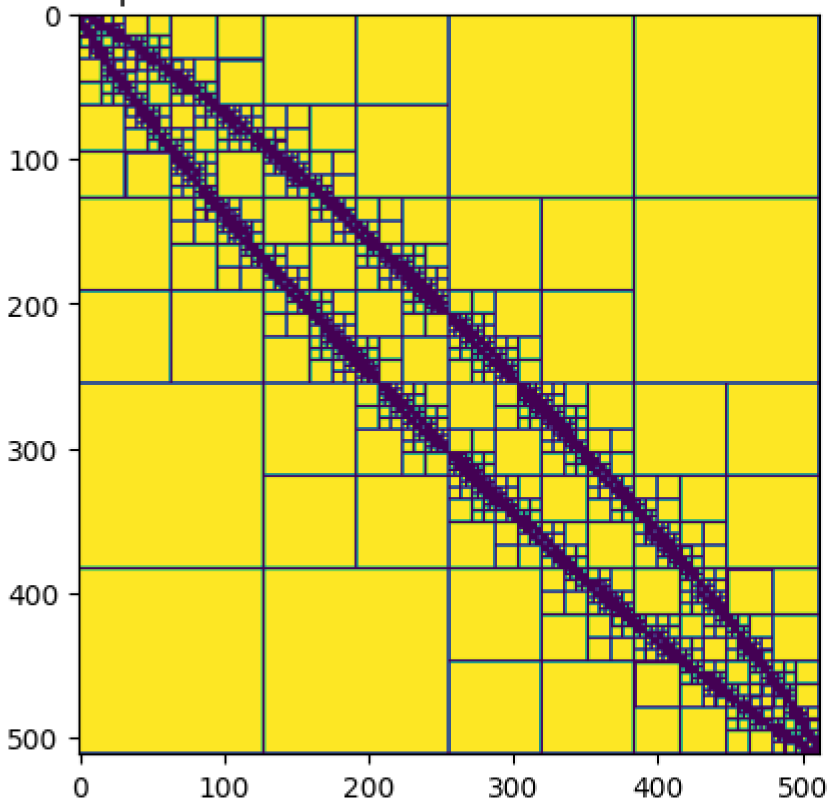

Sparsity pattern for k=3 with cuthill mckee
0
100
200
300
400
500

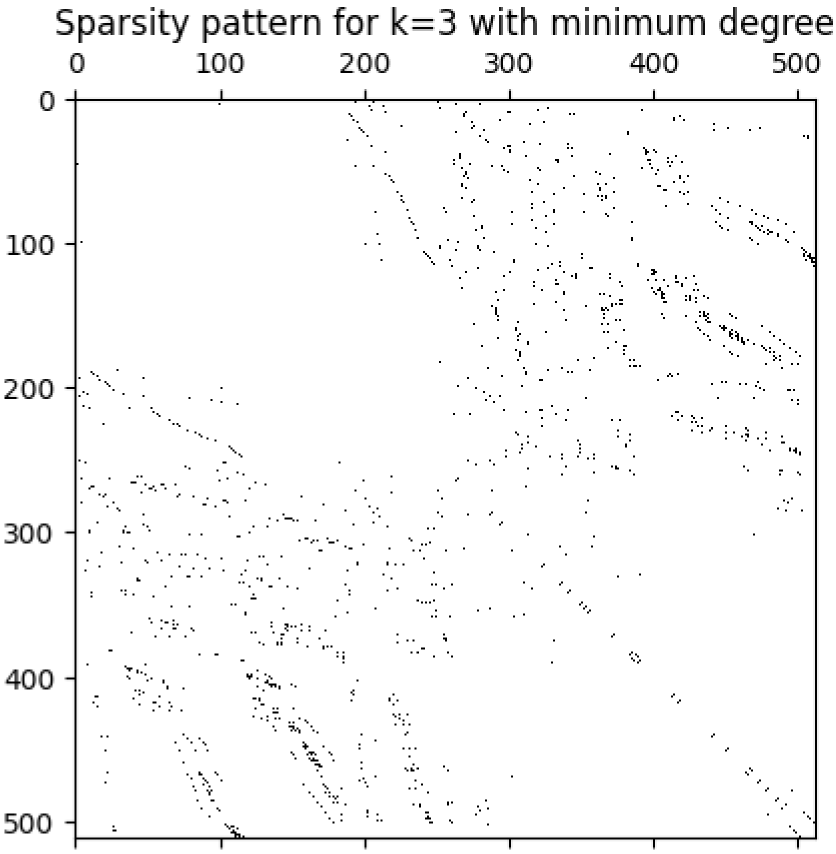

Tree of compressed matrix for k=3 with cuthill mckee
0
100
200
300
400
500

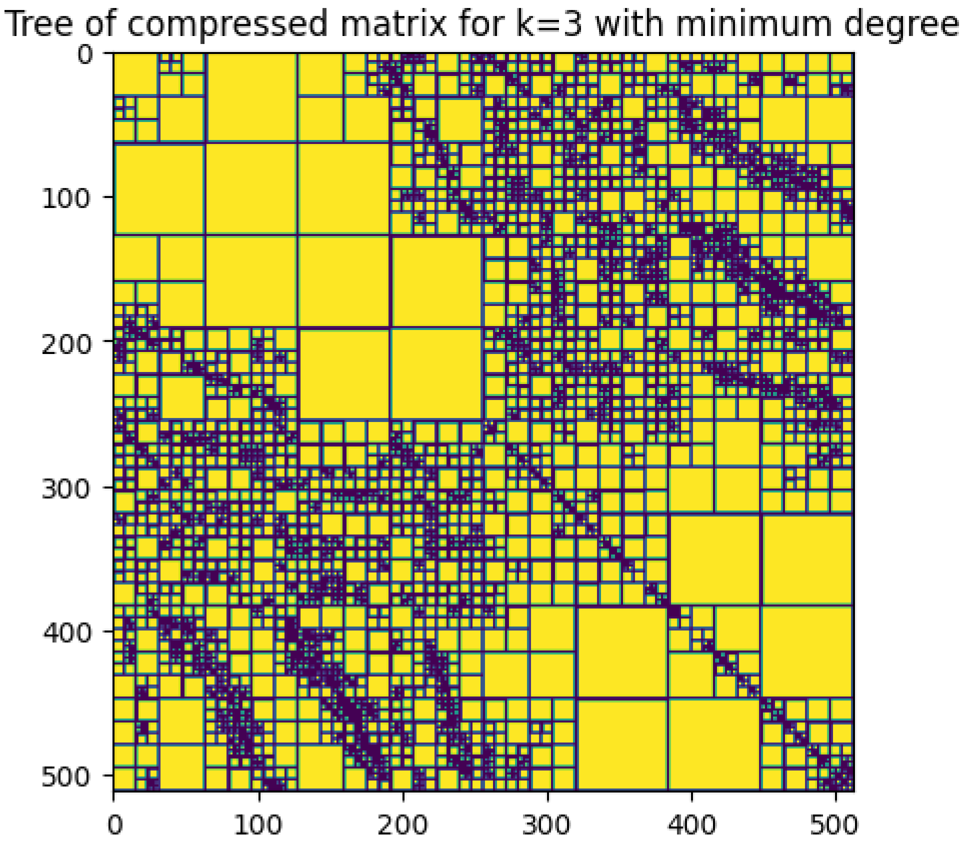
Sparsity pattern for k=3 with reversed cuthill mckee

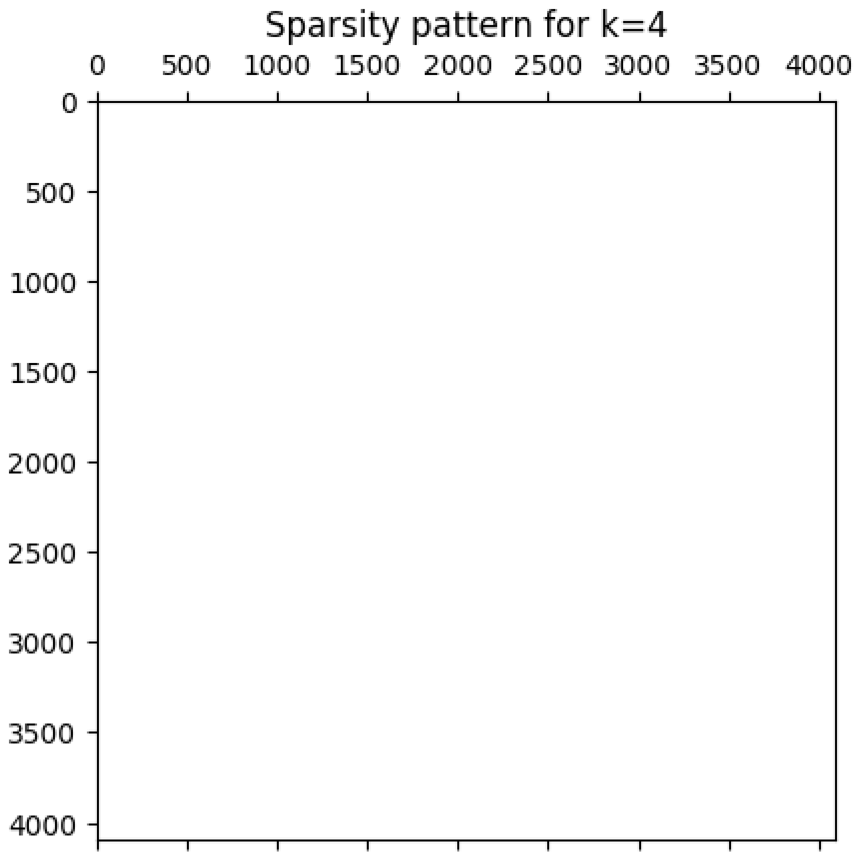


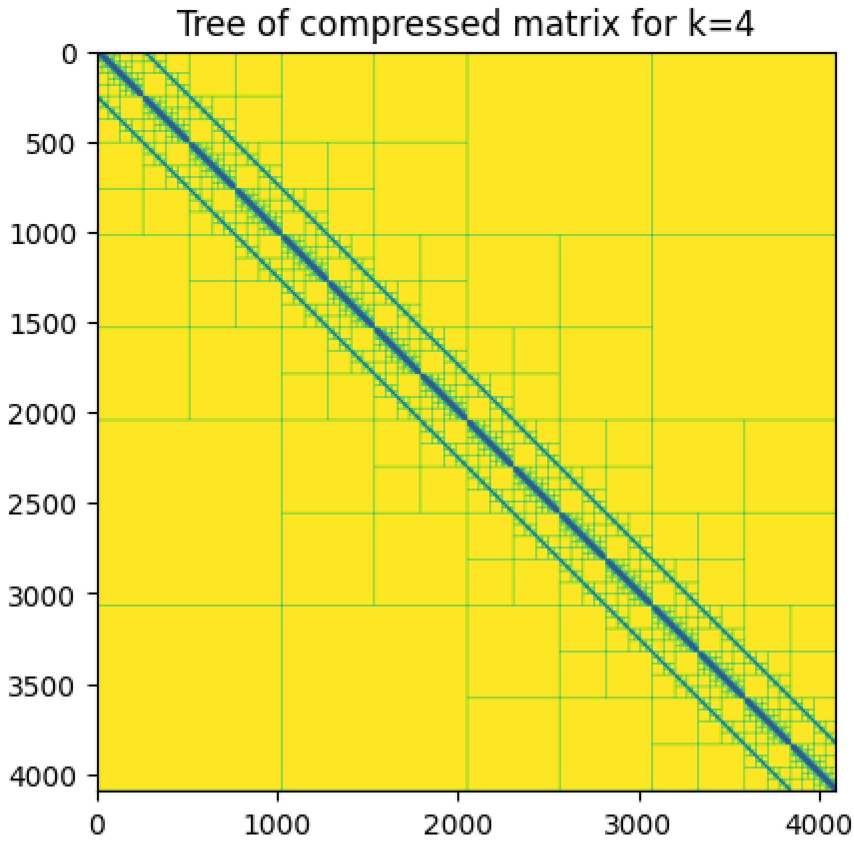
Tree of compressed matrix for k=3 with reversed cuthill mckee

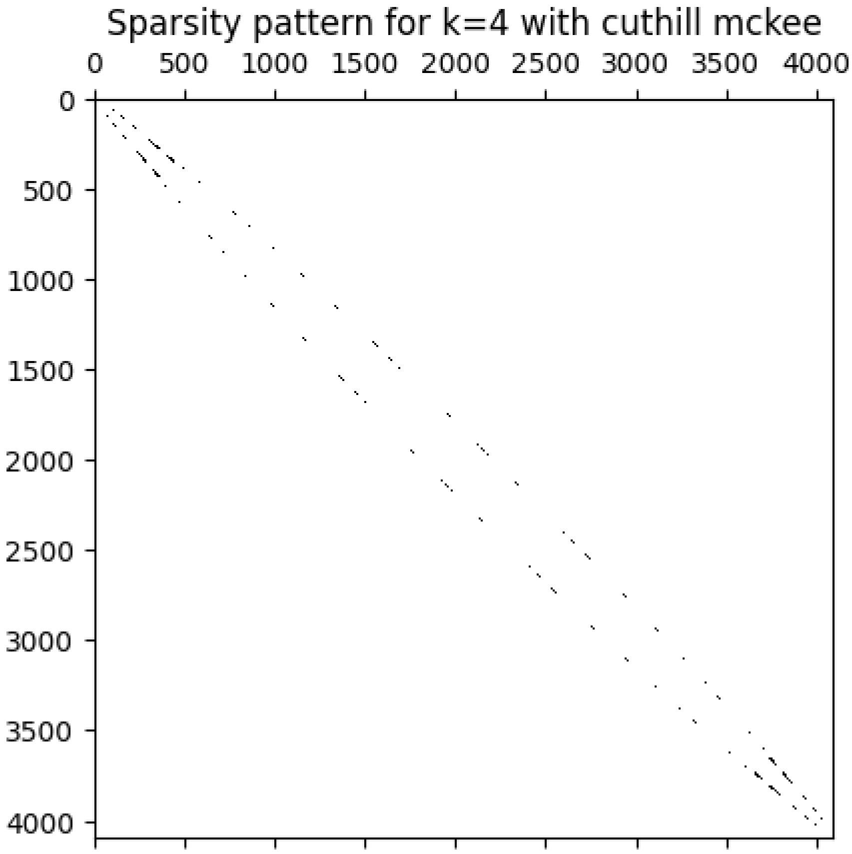


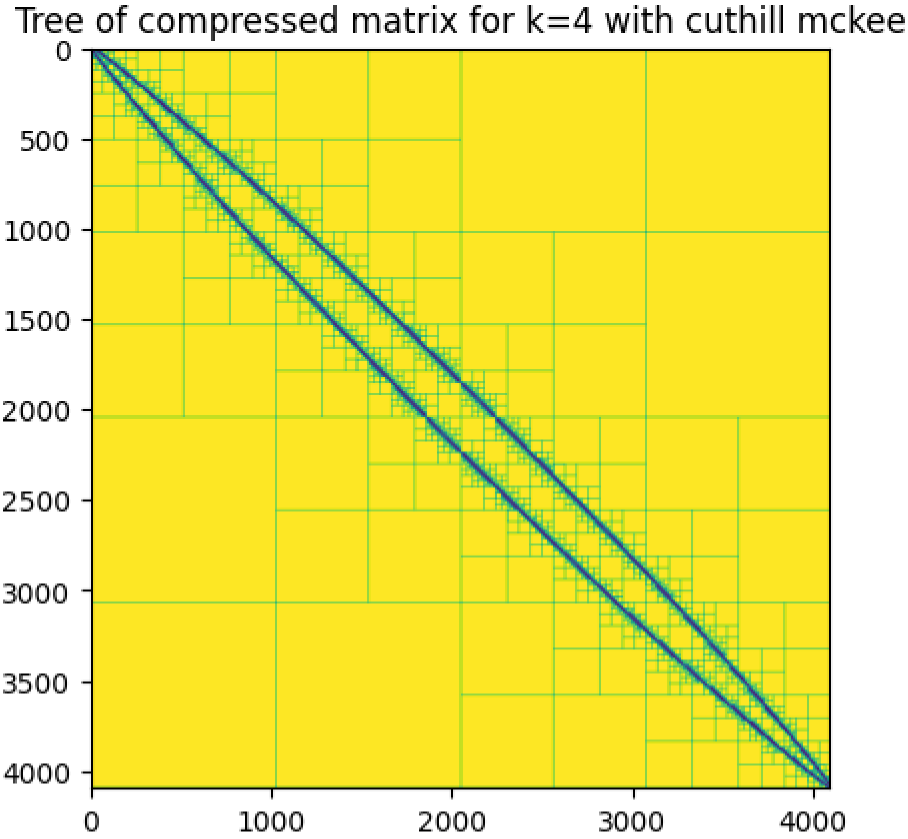

Sparsity pattern for k=3 with minimum degree
0
100
200
300
400
500


Tree of compressed matrix for k=3 with minimum degree
0
100
200
300
400
500

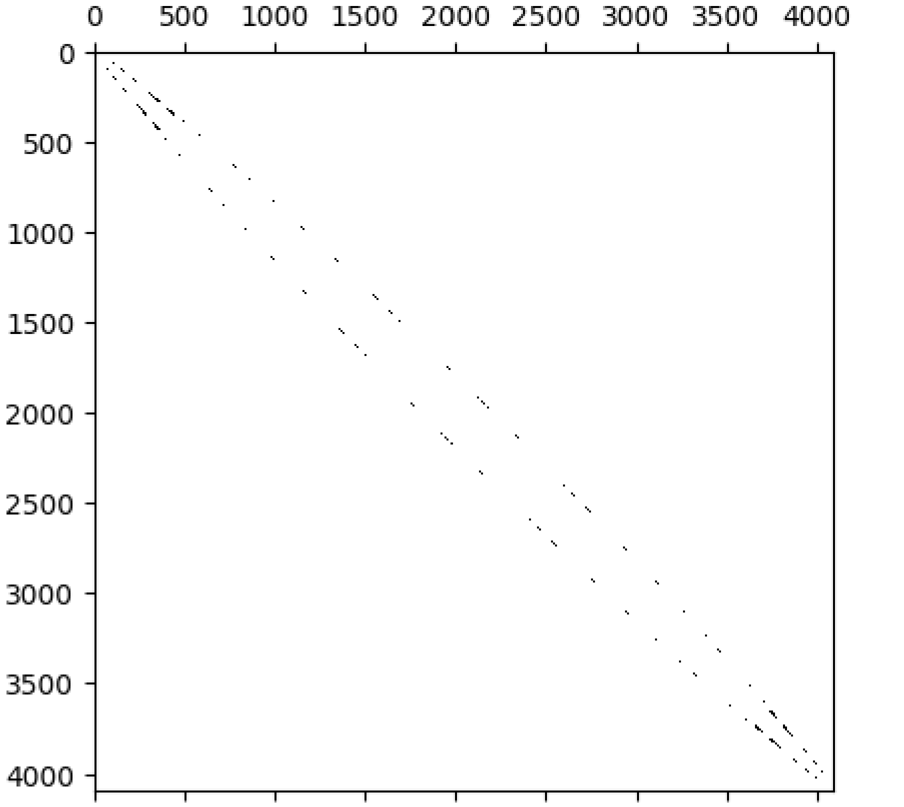

Sparsity pattern for k=4
0
500
1000
1500
2000
2500
3000
3500
4000


Tree of compressed matrix for k=4
0
500
1000
1500
2000
2500
3000
3500
4000

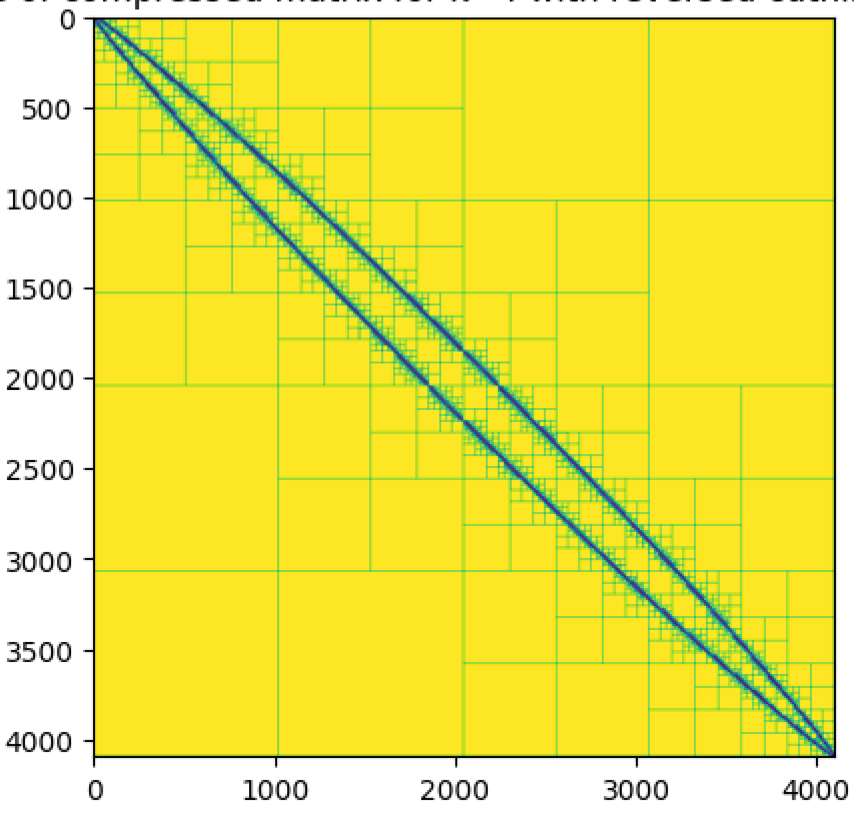

Sparsity pattern for k=4 with cuthill mckee
0
500
1000
1500
2000
2500
3000
3500
4000

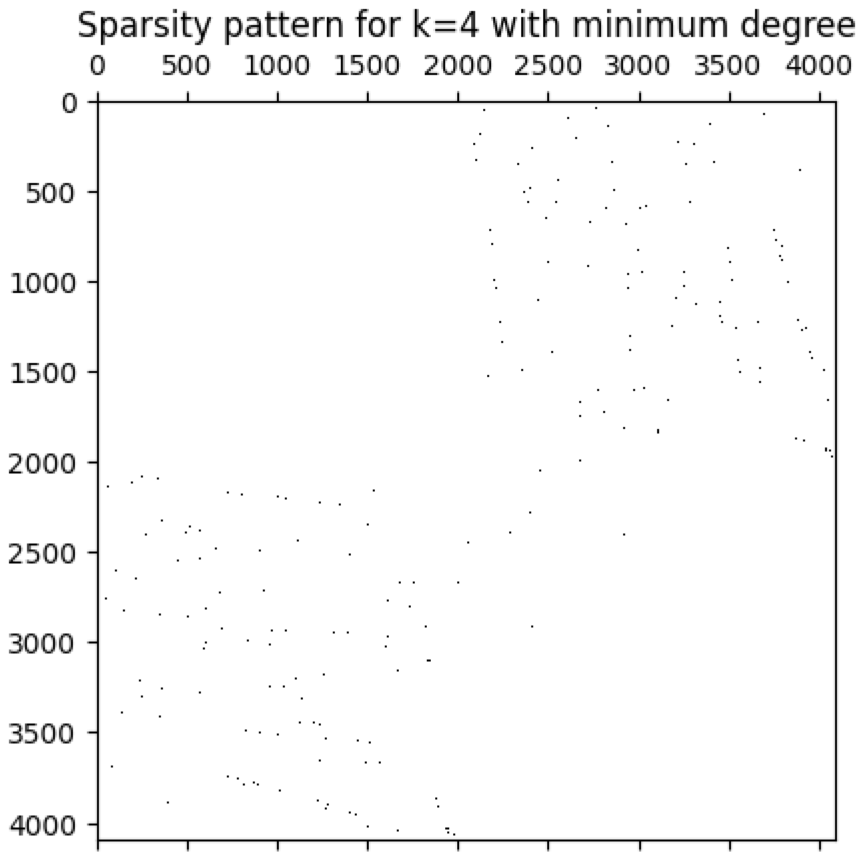

Tree of compressed matrix for k=4 with cuthill mckee
0
500
1000
1500
2000
2500
3000
3500
4000
0
1000
2000
3000
4000

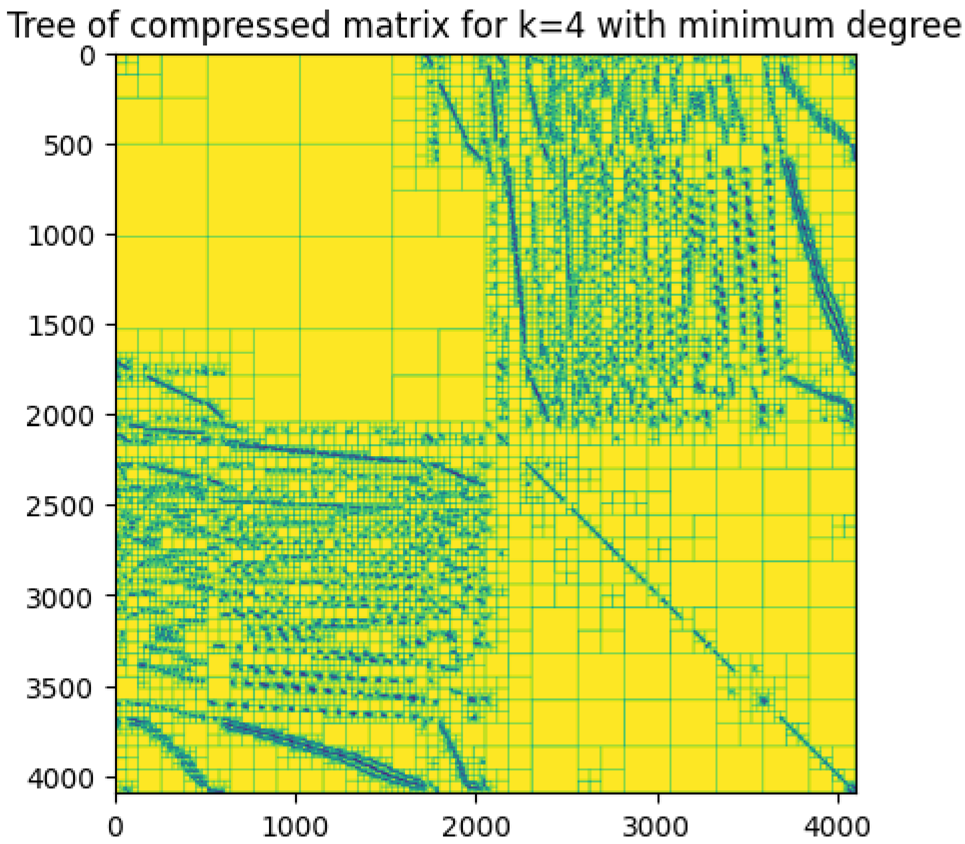
Sparsity pattern for k=4 with reversed cuthill mckee



Tree of compressed matrix for k=4 with reversed cuthill mckee







Porównanie stopnia kompresji macierzy

In [ ]:
```
import pandas as pd

df = pd.DataFrame(columns=['k','no permutation', 'cuthill mckee', 'revers
for i in range(len(matrixes_compressed)):
```

```
    new_row = pd.DataFrame({'k': [i+2],
                            'no permutation': [matrixes_compressed[i].com
                            'cuthill mckee': [matrixes_permutated_compres
                            'reversed cuthill mckee': [matrixes_permutate
                            'minimum degree': [matrixes_permutated_compre
    df = pd.concat([df, new_row], ignore_index=True)


df.style.set_caption("Compression ratio for different permutations")
df.style.hide(axis='index')
```

Out[ ]:

Compression ratio for different permutations

	k	no permutation	cuthill mckee	reversed cuthill mckee	minimum degree
0	2	3.723636	2.580970	2.580970	1.570552
1	3	22.260870	15.784200	15.784200	5.427187
2	4	146.285714	109.271485	109.271485	27.580224

Wnioski

Każdy wzorzec rzadkości dla każdej macierzy przed kompresją dla każdego algorytmu permutacji zdaje się odpowiadać skompresowanej macierzy hierarchicznej. Oba algorytmy zdają się działać poprawnie. Minimal degree te wartości niezerowe rozrzuca bardziej po macierzy i dane są dalej od przekątnej niż normalnie. Cuthill mckee oraz jego wersja reversed układa je w formie elipsy/wstęgi, co jest poprawne. Kompresja mocno zależy od algorytmu, aczkolwiek wyszło, że najlepszy stopień udało się uzyskać przy braku permutacji, najgorzej dla algorytmu minimum degree, a tak średnio (gdzie oba wyszły tak samo) dla cuthull mckee oraz jego wersji reversed