Assignment 1: CS 203 (Fall 2021)

1. (25 marks) There's restaurant named 'Treat'. In this after every meal, the manager picks one from each of the five well shuffled pack of cards having cards labeled A-T. If the picked cards form the restaurant's name (in any order) the meal is on the house.
   (a) What is the probability that a person will enjoy a free meal?
   (b) The manager wants to change the structure of game so that the probability of winning would be less for an expensive meal while it will be more for a cheaper meal. Suggest a way to change the 'winning word' (according to charge of meal), to satisfy manager's needs.

   Solution: □

2. (20 marks) Consider n coding machines M_1, M_2, … M_n producing binary codes 0 and 1. The machine M_1 produces codes 0 and 1 with respective probabilities $\frac{1}{4}$ and $\frac{3}{4}$. The code produced by machine M_k is fed into machine M_{k+1} which may either leave the received code unchanged or may change it. Suppose that each of the subsequent machines change the code with probability $\frac{3}{4}$.
   Given that final produced code is 1. What is the probability that the machine M_1 produced code 0.

   Solution: □

3. (25 marks) Let $\{A_i\}_{i=1}^{n}$ be a family of sets indexed from 1 to n. Let $I \subseteq [n]$ be a subset of index set. Let B be the event when only and all A_i's from I have happened, i.e.,

$$B = \left(\cap_{i\in I} A_i\right) \cap \left(\cap_{i\notin I} A_i^c\right).$$

   Notice that B is subset of $\cap_{i\in I} A_i$, but need not be equal to it. Show that,

$$P(B) = \sum_{J\supseteq I} (-1)^{|J|-|I|} P(\cap_{i\in J} A_i).$$

   Hint: can you define some new sets in terms of A_i, such that, this problem looks like inclusion-exclusion?

   Solution: □

4. (30 marks) We play a game with a randomly shuffled deck of 52 regular playing cards. Cards are placed face down on table. You have two options, either "take" or "skip" the top card. The skipped card is revealed and game is continued. If only one card is left in deck it is automatically taken. Game stops when you take the top card; you win if the card taken is black, otherwise you lose.
   Prove that no other strategy is better than taking the top card. *Strategy is considered better than the other, if the probability of winning is higher.*
   Hint: Consider the general scenario, suppose there are x black and y red cards, show that whatever you do, the probability of winning is bounded by $\frac{x}{x+y}$.

   Solution: □

– **1. Solution:**

  – Letter T has to come from 2 decks. Number of Ways of choosing the deck is ${}^5C_2$. 3! possible permutations for E,A and R. Total possibilities: 20^5.
    Probability$=({}^5C_2 * 3!)/20^5$

  – For cheaper meals change one of T's to any letter except E, A and R. The probability of winning becomes twice. For expensive meals change one or more of E, A or R to T.

$\square$

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– **3. Solution:**

For every s in I^C, define

$$L_s = (\cap_{i\in I} A_i) \cap A_s$$

Let $D = \bigcup_{s\in I^C} L_s$
Consider two elements s_1, $s_2 \in I^C$. For, $L_{s_1} \cup L_{s_2}$ using $(A \cap B) \cup (A \cap C) = A \cap (B \cup C)$ we get

$$L_{s_1} \cup L_{s_2} = (\cap_{i\in I} A_i) \cap (A_{s_1} \cup A_{s_2})$$

Using this constructively, we get

$$\bigcup_{s\in I^C} L_s = (\cap_{i\in I} A_i) \cap (\cup_{i\in I^C} A_i)$$

Now, using $\cup_{i\in I^C} A_i = (\cap_{i\in I^C} A_i^c)^C$ and $P(L \cap M^c) = P(L) - P(L \cap M)$, we have

$$P(D) = P(\cap_{i\in I} A_i) - P(B)$$

$$\implies P(B) = P(\cap_{i\in I} A_i) - P(D)$$

Using Principle of Inclusion and exclusion on D, we get the required result.

$\square$

2. **Solution**
Event M_1^0 : Machine 1 produces code is 0
Event M_1^1 : Machine 1 produces code is 1
Event R_1 : Final code produced is 1

For some integer i,j:
Event E_i^0 : $2i$ unique machines change code
Event E_j^1 : $2j-1$ unique machines change code

As machines act independently, each permutation of some k machines changing code is equally likely.
Since we have $\binom{n-1}{k}$ ways of choosing k machines out of $(n-1)$ machines:
Probability of event E_i^0 : $\text{P}[E_i^0] = \binom{n-1}{2i}(\frac{3}{4})^{2i}(\frac{1}{4})^{n-1-2i}$
Probability of event E_j^1 : $\text{P}[E_j^1] = \binom{n-1}{2j-1}(\frac{3}{4})^{2j-1}(\frac{1}{4})^{n-2j}$

Event E^0 : Resulting code is same as code produced by M_1, i.e. even machines change code. $E^0 = \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} E_i^0$
Event E^1 : Resulting code is opposite of code produced by M_1, i.e. odd machines change code. $E^1 = \sum_{j=1}^{\lceil\frac{n-1}{2}\rceil} E_j^1$

$$P[M_1^1|R_1] = \frac{P[M_1^1 \cap R_1]}{P[R_1]}$$

$$P[M_1^1|R_1] = \frac{P[M_1^1] * P[E_0]}{P[M_1^1] * P[E_0] + P[M_1^0] * P[E_1]}$$

$$P[M_1^1|R_1] = \frac{(\frac{3}{4}) * \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} \binom{n-1}{2i}(\frac{3}{4})^{2i}(\frac{1}{4})^{n-1-2i}}{(\frac{3}{4}) * \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} \binom{n-1}{2i}(\frac{3}{4})^{2i}(\frac{1}{4})^{n-1-2i} + (\frac{1}{4}) * \sum_{j=1}^{\lceil\frac{n-1}{2}\rceil} \binom{n-1}{2j-1}(\frac{3}{4})^{2j-1}(\frac{1}{4})^{n-2j}}$$

4 **Solution**

We will use induction on total number of cards to show that there is no better strategy than taking the top card.

Let us define $p_{x,y}$ as probability of winning with some strategy when we have x red cards and y black cards.

Induction Hypothesis: $p_{x,y} \leq \frac{x}{x+y}$

Observe that our strategy of taking top card also gives the probability of winning as $\frac{x}{x+y}$. So, if above hypothesis holds for all x and y, then we can say that taking the top card is always the best strategy.

Base Cases :

- *Boundary Case-1:* x = 0
  If no black card is left in deck, then for any strategy $p_{0,y} = 0$, i.e. you cannot win this game.
- *Boundary Case-2:* y = 0
  If only black cards are left in deck, then for any strategy $p_{x,0} = 1$, i.e. you always win this game.
- *Base Case:* $x + y = 2$ and $x = y = 1$
  If our new strategy picks top card with probability ***k***, and skips card with probability ***(1-k)***
  Probabilty of winning = P[picking top card]*P[top card is black] + P[skipping top card]*P[top card is red]
  $p_{1,1} = k * \frac{1}{2} + (1-k) * \frac{1}{2} = \frac{1}{2} \leq \frac{1}{2}$

Induction Step: Assume that hypothesis holds for $(x + y) < k$, prove that it holds for $(x + y) = k$ also.
Proof:
We have two possible choices at any step.

(a) *Skip the card*
If we skip the card, then the probability of it being red and black card are $\frac{x}{x+y}$ and $\frac{y}{x+y}$ respectively.

$$\begin{aligned}
\text{Now, total probability of winning } &= \frac{x}{x+y} \times p_{x-1,y} + \frac{y}{x+y} \times p_{x,y-1} \\
&\leq \frac{x}{x+y} \times \frac{x-1}{x+y-1} + \frac{y}{x+y} \times \frac{x}{x+y-1} \\
&\textit{(follows from induction hypothesis)} \\
&\leq \frac{x^2 - x + xy}{(x+y)(x+y-1)} \\
&\leq \frac{x(x+y-1)}{(x+y)(x+y-1)} \\
&\leq \frac{x}{x+y}
\end{aligned} \tag{1}$$

(b) *Take the card*
If we take the card, then we win if it is black card.

$$\text{Probability} = \frac{x}{x+y} \tag{2}$$

Since both probabilities are $\leq \frac{x}{x+y}$, any strategy will not be better than taking the top card.

Note: As you may have noticed, if we change our hypothesis as $p_{x,y} = \frac{x}{x+y}$, you may show that every strategy is equally as good as picking top card (not better).
This may seem confusing, but our odds are not changed if we condition our action on previous decisions on skipping cards.