Assignment 1: CS 203 (Fall 2021)
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1. (25 marks) There's restaurant named 'Treat'. In this after every meal, the manager picks one from each of the five well shuffled pack of cards having cards labeled A-T. If the picked cards form the restaurant's name (in any order) the meal is on the house.
   (a) What is the probability that a person will enjoy a free meal?
   (b) The manager wants to change the structure of game so that the probability of winning would be less for an expensive meal while it will be more for a cheaper meal. Suggest a way to change the 'winning word' (according to charge of meal), to satisfy manager's needs.

   Solution:

   (a) We need two card to contain T, one for r, one for e and one for a. Therefore the probability will be:
   $$\binom{5}{2} * \frac{1}{20^2} * \binom{3}{1} * \frac{1}{20} * \binom{2}{1} * \frac{1}{20} * \binom{1}{1} * \frac{1}{20} = \frac{60}{20^5} \quad (1)$$
   (b) One way could be to make the 5-letter word contain all different letters for the cheap meals, or the word contain three or less distinct letters for expensive meals. The manager could also have a letter from T-Z for expensive meals to make the probability 0 for a free meal in case of an expensive meal.

   □

2. (20 marks) Consider n coding machines M_1, M_2, ... M_n producing binary codes 0 and 1. The machine M_1 produces codes 0 and 1 with respective probabilities $\frac{1}{4}$ and $\frac{3}{4}$. The code produced by machine M_k is fed into machine M_{k+1} which may either leave the received code unchanged or may change it. Suppose that each of the subsequent machines change the code with probability $\frac{3}{4}$.
   Given that final produced code is 1. What is the probability that the machine M_1 produced code 0.

   Solution: After the first machine, we have n-1 machines left. Suppose that we get 0 by first machine, then, there should be odd number of changes to get final code as 1. Similarly, if we get 1 by first machine, then, there should be even number of changes to get final code as 1.
   The probability of i changes throughout the n-1 machines is $\binom{n-1}{i} * (1/4)^i * (3/4)^{n-1-i}$. For probability of even changes, we have to take sum of terms with even i, from above equation. Using the binomial theorem, we can easily see that
   $$\frac{(1/4+3/4)^{n-1} + (1/4-3/4)^{n-1}}{2} = \binom{n-1}{0} * (1/4)^0 * (3/4)^{n-1-0} + \binom{n-1}{2} * (1/4)^2 * (3/4)^{n-1-2} + ... \quad (2)$$
   Therefore, sum of even terms will be $\frac{1+(-\frac{1}{2})^{n-1}}{2}$. Similarly, we can find sum of odd terms using binomial theorem and it comes out to be $\frac{1-(-\frac{1}{2})^{n-1}}{2}$.
   Here, M_i means first machine produced i. By bayes theorem, we can say that,
   $$P(M_0|1) = \frac{P(M_0) * P(oddchanges))}{P(M_0) * P(oddchanges)) + P(M_1) * P(evenchanges)} \quad (3)$$
   which gives
   $$P(M_0|1) = \frac{(1/2) * (1/4) * (1 - (-\frac{1}{2})^{n-1})}{(1/2) * (1/4) * (1 - (-\frac{1}{2})^{n-1}) + (1/2) * (3/4) * (1 + (-\frac{1}{2})^{n-1})} = \frac{1 - (\frac{-1}{2})^{n-1}}{2 * (2 + (\frac{-1}{2})^{n-1})} \quad (4)$$

   □

3. (25 marks) Let $\{A_i\}_{i=1}^n$ be a family of sets indexed from 1 to n. Let $I \subseteq [n]$ be a subset of index set. Let B be the event when only and all A_i's from I have happened, i.e.,

$$B = (\cap_{i\in I} A_i) \cap (\cap_{i\notin I} A_i^c).$$

Notice that B is subset of $\cap_{i\in I} A_i$, but need not be equal to it. Show that,

$$P(B) = \sum_{J\supseteq I} (-1)^{|J|-|I|} P(\cap_{i\in J} A_i).$$

Hint: can you define some new sets in terms of A_i, such that, this problem looks like inclusion-exclusion?

Solution: Let X and Y be two sets such that $X = (\cap_{i\in I} A_i)$ and $Y = (\cap_{i\notin I} A_i)$.
We can say that $B = X \cap Y^c$. We can also say that

$$P(B) = P(X) - P(X \cap Y) \implies P(B) = P(X \cup Y) - P(Y) \tag{5}$$

Let $i_1, i_2, ..., i_m$ be indexes not in I which means that $Y = \cup_{k=1}^m A_{i_k}$.
Inserting Y in 5 gives

$$P(B) = P(X \cup (\cup_{k=1}^m A_{i_k})) - P(\cup_{k=1}^m A_{i_k}) \tag{6}$$

Principle of Inclusion-Exclusion states that:

$$P(\cup_{i=1}^n A_i) = \sum_{k=1}^n (-1)^{k+1} (\Sigma_{i_1 \leq i_2 \leq ... \leq i_k} P(A_{i_1} \cap A_{i_2} ... \cap A_{i_k})) \tag{7}$$

If we insert 7 in 6, components of first term which have no intersection with X will get cancelled and we will be left with terms which are union of X and $\mathrm{A_i}$ where i can range in the set J which will be a subset of [n] and super set of I, i.e.

$$P(B) = \sum_{J\supseteq I} (-1)^{|J|-|I|} P(\cap_{i\in J} A_i)$$

□

4. (30 marks) We play a game with a randomly shuffled deck of 52 regular playing cards. Cards are placed face down on table. You have two options, either "take" or "skip" the top card. The skipped card is revealed and game is continued. If only one card is left in deck it is automatically taken. Game stops when you take the top card; you win if the card taken is black, otherwise you lose.
Prove that no other strategy is better than taking the top card. *Strategy is considered better than the other, if the probability of winning is higher.*
Hint: Consider the general scenario, suppose there are x black and y red cards, show that whatever you do, the probability of winning is bounded by $\frac{x}{x+y}$.

Solution: When we have not picked up any card, the probability of getting a black card will be $\frac{1}{2}$. Suppose that we have picked up i-1 cards, and now we are left with x black cards and y red cards. We can say that $P(i) = \frac{x}{x+y}$. Now,if we can say that P(i+1) = P(Black at i+1|Black at i) + P(Black at i+1|Red at i). Therefore,

$$P(i+1) = \frac{x-1}{x+y-1} * \frac{x}{x+y} + \frac{x}{x+y-1} * \frac{y}{x+y}$$

$$\implies P(i+1) = \frac{x(x+y-1)}{(x+y)(x+y-1)}$$

$$\implies P(i+1) = \frac{x}{x+y}$$

Therefore, We can see that the probability of getting black never changes. So, we should take the first card on top of deck. □