Matrix Inverse with Minors and Cofactor Matrices [*]

Procedure

Probably the very first method for obtaining the inverse of some matrix A introduced to students is the following: Let's have some matrix A:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then, its inverse is given by:

$$\boxed{\mathbf{A^{-1}} = \frac{1}{det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

The goal here is to show where the blue matrix on the right comes from and then apply the technique to invert a 3x3 matrix. This method could be used as an alternative to the often tedious Gaussian Elimination. To derive the blue matrix, we need a couple of tools, the **minors** of a matrix and its **cofactors**.

Minors of a Matrix [1]

A minor M_{ij} is the reduced determinant of a determinant expansion that is formed by omitting the i_{th} row and j_{th} column of a matrix A. Below are shown a couple of examples of the minors at positions $(i, j) = (1, 1)$ and $(i, j) = (1, 2)$.

Figure 1: Matrix minors at positions $(i, j) = (1, 1)$ and $(i, j) = (1, 2)$

$$M_{11} = det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \qquad M_{12} = det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

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[1] Source: Wolfram MathWorld

Cofactor Matrix The cofactor matrix contains the minors of the matrix as its elements. In addition, we need to keep track of signs of each element which depend on its position.

$$C_{ij} \equiv (-1)^{i+j} M_{ij}$$

Figure 2: Signs associated with the cofactor matrix entries

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

To illustrate this, let's find the cofactor matrix of the symbolic matrix A:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Find the minors of A .
This is quite easy as this is just a 2x2 matrix:

$$\boxed{M_{11} = d,\ M_{12} = c,\ M_{21} = b,\ M_{22} = a}$$

Cofactor elements

$$\begin{aligned} C_{11} &= (-1)^{1+1}\, M_{11} &\rightarrow&\quad C_{11} = d \\ C_{12} &= (-1)^{1+2}\, M_{12} &\rightarrow&\quad C_{12} = -c \\ C_{21} &= (-1)^{2+1}\, M_{21} &\rightarrow&\quad C_{21} = -b \\ C_{22} &= (-1)^{2+2}\, M_{22} &\rightarrow&\quad C_{22} = a \end{aligned}$$

Hence, the cofactor matrix is given by:

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

Transpose the Cofactor Matrix .
If we transpose the cofactor matrix, we will obtain the blue matrix shown above. This is the so called **classical adjoint**, also known as **adjugate**. Hence,

$$C^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = adj(A)$$

Now, we can write the matrix inverse more consicely as:

$$\boxed{\mathbf{A^{-1}} = \frac{1}{det(A)} adj(A)}$$

Computational Example: 3x3 Matrix Inverse

Find A^{-1}, given:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ -1 & 3 & -1 \end{bmatrix}$$

Step 1 Find det(A)

I have already used this matrix to illustrate Cramer's rule. There we found $det(A) = 4$.

Step 2 Find the Minors of A

Matrix A has 9 elements, so it will have 9 minors as well.

$$M_{11} = det\begin{bmatrix} -1 & 1 \\ 3 & -1 \end{bmatrix} = 1 - 3 = -2, \quad M_{12} = det\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = -2 - (-1) = -1$$

$$M_{13} = det\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = 6 - (1) = 5, \quad M_{21} = det\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} = -1 - 3 = -4$$

$$M_{22} = det\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = -1 - (-1) = 0, \quad M_{23} = det\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} = 3 - (-1) = 4$$

In a similar manner we find the values for the 3rd row elements M_{31}, M_{32}, M_{33}. In this case, they are given by:

$$M_{31} = 2, \quad M_{32} = -1, \quad \text{and} \quad M_{33} = -3$$

To summarize, we have found the following minors of matrix A:

$$\begin{cases} M_{11} = -2, & M_{12} = -1, & M_{13} = 5 \\ M_{21} = -4, & M_{22} = 0, & M_{23} = 4 \\ M_{31} = 2, & M_{32} = -1, & M_{13} = -3 \end{cases}$$

Step 3 Construct the Cofactors Matrix

Recall the formula we used earlier to obtain the cofactor elements:

$$C_{ij} \equiv (-1)^{i+j} M_{ij}$$

Applying it to the minors of matrix A, we get:

$$C_{11} \equiv (-1)^{1+1}\, M_{11} \quad \rightarrow \quad C_{11} \equiv -2$$

$$C_{12} \equiv (-1)^{1+2}\, M_{12} \quad \rightarrow \quad C_{12} \equiv (-1)(-1) = 1$$

$$C_{13} \equiv (-1)^{1+3}\, M_{13} \quad \rightarrow \quad C_{13} \equiv 5$$

$$M_{11} = det \begin{bmatrix} \cancel{1} & \cancel{1} & \cancel{1} \\ \cancel{2} & -1 & 1 \\ \cancel{-1} & 3 & -1 \end{bmatrix} \quad M_{12} = det \begin{bmatrix} \cancel{1} & \cancel{1} & \cancel{1} \\ 2 & \cancel{-1} & 1 \\ -1 & \cancel{3} & -1 \end{bmatrix} \quad M_{13} = det \begin{bmatrix} \cancel{1} & \cancel{1} & \cancel{1} \\ 2 & -1 & \cancel{1} \\ -1 & 3 & \cancel{-1} \end{bmatrix}$$

Figure 3: Matrix Minors at positions $(i, j) = (1, 1)$, $(i, j) = (1, 2)$, and $(i, j) = (1, 3)$

Essentially the only difference here is that the elements whose row and column indices add up to an odd number, need a negative sign in front(see Fig. 2 above). Following the outlined procedure, the other elements of the cofactor elements are:

$$C_{21} \equiv 4, \quad C_{22} \equiv 0, \quad C_{23} \equiv -4$$

$$C_{31} \equiv 2, \quad C_{32} \equiv 1, \quad C_{33} \equiv -3$$

Since we now have all 9 elements, we can construct the cofactors matrix:

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} -2 & 1 & 5 \\ 4 & 0 & -4 \\ 2 & 1 & -3 \end{bmatrix}$$

Step 4 Adjugate of matrix **A**

Transpose the cofactors matrix **C** to obtain the adjugate of **A**:

$$C^T = \begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 1 \\ 5 & -4 & -3 \end{bmatrix} = adj(A)$$

Step 5 Find $\mathbf{A^{-1}}$

Now we have all necessary information to find the inverse of matrix A.

$$A^{-1} = \frac{1}{det(A)} adj(A)$$

Here, this translates to :

$$A^{-1} = \frac{1}{4} \begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 1 \\ 5 & -4 & -3 \end{bmatrix}$$