Cosets

Let $(G, *)$ be the group and $(H, *)$ be the subgroup of G, for any $a \in G$ the set

$a * H = \{ a * h \mid h \in H \mid a \in G \}$ [Left coset of H Determined by a.]

$H * a = \{ h * a \mid h \in H \}$ [Right coset of H determined by a on G]

* If operation is addition then
  $a + H = \{ a + h \mid h \in H \}$ = (Left coset)

  $H + a = \{ h + a \mid h \in H \}$ = (Right coset)

* If operation is multiplication then
  $a \cdot H = \{ a \cdot h \mid h \in H \}$ - (Left coset)

  $H \cdot a = \{ h \cdot a \mid h \in H \}$ = (Right coset)

Example $G = (Z_4, +_4)$ and $H = \{0, 2\}$ is subgroup of $(Z_4, +_4)$ then find Left cosets & Right cosets of H in G.

Solution $G = (Z_4, +_4)$ $\quad Z_4 = \{0, 1, 2, 3\}$ $\quad H = \{0, 2\}$

for $a \in G$ (Left coset)

$a (+_4) H = \{ a +_4 h / h \in H \}$

(1) for $a = 0$

$a +_4 H = \{ a +_4 h / h \in H \}$ $\to$ 0, 2

$= \{ 0 + 0, 0 + 2 \} = \{0, 2\}$

for $a = 1$

$a +_4 H = \{a +_4 h \mid h \in H\} = \{1 +_4 h \mid h \in H\}$

$= \{(1+0), (1+2)\} = \{1, 3\}$

Same coset as $a = 0$

4, but $+_4 = 0$

for $a = 2$

$a +_4 H = \{2 +_4 h \mid h \in H\} = \{(2 +_4 0), (2 +_4 2)\}$

$\neq \quad = (2, 0) = (0 +_4 H)$

Same coset as ($a = 1$)

for $a = 3$

$a +_4 H = \{3 +_4 h \mid h \in H\} = \{(3+0), (3+2)\}$

$= \{3, 1\} = (1 +_4 H)$

①	②
0	1
2	3

$\rightarrow Z_4$

$\rightarrow 1 (+_4) H$

$3 (+_4) H$

$(a +_4 H)$
$=$
$(2 +_4 H)$

→ Intersection of two coset is 0

→ Union of two coset is whole Group G.

② To find Right coset

1) for $a = 0$ $\quad H +_4 a = \{h +_4 0\}$
$= \{0, 2\}$

2) for $a = 1$ $\quad H +_4 1 = \{0+1, 2+1\}$
$= \{1, 3\}$

3) for $a = 2$ $\quad H +_4 2 = \{0+2, 2+2\}$
$= \{2, 0\}$

4) for $a = 3$ $\quad H +_4 3 = \{0+3, 2+3\}$
$= \{3, 1\}$

Here Right coset = Left coset.
because G is abelian Group. + is commutati[ve]

For Every abelian Group right coset = left cosets.

Example: $G = \{1, -1, i, -i\}$ is Group under multiplication and $H = \{1, -1\}$ is subgroup of G under multiplication then find right coset and left coset.

Solution: $G = \{1, -1, i, -i\}$ $H = \{1, -1\}$

Left coset.

$\therefore$ 2 distinct coset

$a \times H = \{a \times h / h \in H\}$

for $a = 1$

$1 \cdot H = \{(1 \times 1), (1 \times -1)\} = \{1, -1\}$

same

for $a = -1$

$-1 \cdot H = \{(-1 \times 1), (-1 \times -1)\} = \{-1, 1\}$

for $a = i$

$i \cdot H = \{(i \times 1), (i \times -1)\} = \{i, -i\}$

same.

for $a = -i$

$-i \cdot H = \{(-i \times 1), (-i \times -1)\} = \{-i, i\}$

$\therefore$ for left coset Union = G

Intersection = Empty

Right cosets

for $a=1$ $H \times 1 = \{(1 \times 1), (-1 \times 1)\}$

$= \{1, -1\}$

for $a=-1$ $H \times -1 = \{(1 \times -1), (-1 \times -1)\}$

$= \{-1, 1\}$

for $a=i$ $H \times i = \{(1 \times i), (-1 \times i)\}$

$= \{i, -i\}$

for $a=-i$ $H \times -i = \{-i, i\}$

→ ## Normal Subgroup

A subgroup H of G is said to be normal sub group if for every $a \in G$, $a * H = H * a$ [left coset = Right coset]

Example $G = (Z_4, +_4)$ $H = \{0, 2\}$
for every $a \in G$ $a (+_4) H = H (+_4) a$
= left coset = Right cosed.

$\therefore$ H is normal subgroup of G

* **Normal Subgroup denoted by "N"**

2) $G = \{1, -1, i, -i\}$ and $H = \{1, -1\}$ is subgroup of G under multiplication. is it N?
→ Yes

3) $G = (z, +)$ be Group and $H = (2z, +)$ Subgroup of G. Is H is normal Subgroup of G? Yes.

for $a = 1$ $(1 +_4 H) \rightarrow \{(1+2), (1+4), (1+8) ----- \infty\}$
it will be infinite

* **Every subgroup of abelian group is Normal Subgroup**

Factor / Quotient Group (G/N → Notation)

Let N be normal subgroup of G, then G/N is a set of all cosets of N in G.
G/N is called as ~~Quotient Group~~. Quotient Group.

Example $G = (Z, +)$ $H = (2Z, +)$ is Normal Subgroup of G
then $Z/2Z$ is Quotient Group.

Example $G = (Z_4, +_4)$ $H = \{0, 2\}$

G/H = set of all coset = $\{ 0(+_4)H,\ 1(+_4)H,\ 2(+_4)H,\ 3(+_4)H, \}$

coset for element 0 = $\{0, 2\}$
coset for element 1 = $\{1, 3\}$
coset for elem 2 = $\{0, 2\}$
coset for elem 3 = $\{1, 3\}$

= $\{ (0,2),\ (1,3),\ (0,2),\ (1,3) \}$

we cannot repeat element in set so remove them.

$G/H = \{ (0,2) ; (1,3) \}$

no. of element in G/H is 2.

Imp

Lagrenge's Theorem [only applicable for finite group]

a divides b i.e a/b = $b = c\ a$
$\hookrightarrow$ integer.

i.e $\frac{b}{a} = c \rightarrow$ integer

Then we say a divides b.

→ For any finite group G and H is subgroup of G, then order of H divides the order of G. [G/H]
order = no. of elements in G or Group.

for example Find the order
$G = (Z_4 +_4)$ = order = 4 $G = \{0, 1, 2, 3\}$
$G/H = \{(0,2)\ (1,3)\}$ = order = 2
$G = (Z, +)$ = order = infinite.

Order of H divides order of G = $|H| / |G|$

But converse of this theorem need not be true.
if $|H| / |G|$
then H need not be subgroup of G.
But it's contrapositive.
i.e $|H| \nmid |G|$ the H is not subgroup of G.

Example $G = \{1, -1, i, -i\}$ be group under multiplication.

(1) Is $H = \{1, -1, i\}$ is subgroup of G?
Solve using Lagrange's Theorem.

Solution $|H| = 3$ $|G| = 4$ $3 \nmid 4$ *

i.e Order of H doesnot divide order of G, therefore by using contrapositive. H is not a subgroup.

(2) If $H = \{-1, i\}$ is subgroup of G using Lagrange's Theorem

Solution $|H| = 2$ $|G| = 4$ $2/4 =$ Integre.

$\therefore$ $|H| \mid |G|$ but H is not subgroup of G because identity element is not present $e = 1$, $e \notin H$

(3) If $H = \{1, i\}$ is Subgroup of G. ? No.

Soln This not subgroup of G because inverse is not present for $i \notin H$.

(4) If $H = \{1, -1\}$ is subgroup of G? It is a Sub Group.

→ If $|H| \mid |G|$ = integre, but then also check for the condition of subgroup for H.