Quantum Field Theory II

(MPhys Thesis)

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29th April, 2013

MPhys Theoretical Physics (Honours)

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Abstract

This report is an attempt to develop and explore the principles underlying the venerable theory that is formulated through the amalgamation of special relativity and quantum theory known as quantum field theory, which is by far the most successful theory that provides the best description of the world as we know it. A thorough quantitative investigation of the canonical quantization approach has been taken into account, in particular with regard to scalar, Dirac and interacting field theories within a Minkowski space-time. The most profound consequences and subtleties upon quantization for the respective theories are delineated and discussed. In addition, we discuss an application for the machinery that is implemented by field theories, in the context of the early universe. We study the history and development of the inflationary paradigm. Furthermore, we attempt to understand the link between elementary particle theory and cosmology, through the phenomenon of particle production by understanding the behaviour of an oscillating inflaton field within a non-expanding background.

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Chapter 1: Introduction.

Electromagnetism, the weak force, the strong force and gravitation are the four fundamental forces of nature as we know them today. The quest for unifying them has been a tremendous area of research and interest for physicists, dating back to the late 19th century when Maxwell [1] combined the seemingly distinct forces of electricity and magnetism. In the 1960s, Weinberg, Glashow and Salam, made their ingenious contribution by proposing the idea for a unified theory of the weak and electromagnetic forces [2-4], which was later, along with all the other Yang-Mills theories, made renomalizable [5-11], during 1971-1973. With the development of the strong force, through the context of quantum chromodynamics (QCD), the unification of the electroweak and strong interactions via the symmetry group, $U(1) \otimes SU(2) \otimes SU(3)$ was established [12] and through the framework of quantum field theory (QFT) this was soon referred to as the Standard Model (SM) [13, 14]. Over the years, many of the predictions that have been made by the SM have been verified by experiment, thus acclaiming to its success. For instance, with regard to the measured value of the magnetic dipole moment of the muon, $\left(g_\mu - 2\right)_{\text{exp.}} = 233\ 184\ 600\ (1680)\text{x}\ 10^{-11}$, compared with the theoretical predicted value of $\left(g_\mu - 2\right)_{\text{theor.}} = 233\ 183\ 6478\ (308)\text{x}\ 10^{-11}$, [15] and accurate predictions for phenomena such as particle scattering (although not discussed in the context of this report we did investigate the phenomenon of nucleon scattering in the Lab book [16]), absorption spectra and particle decay processes (as we shall see in Chapter 3, with regard to the decay of a muon process).

However, the SM is by no means perfect, and there are several areas that suggest it to be incomplete. The model requires the input of 19 parameters (with particular interest of their properties such as particles' charges and masses). Keeping the notion of unification in mind, the motivation for having the SM symmetry group embedded in some *higher* group but with only *one* coupling continued to be plausible, considering when broken would result into the constituents of the SM. In 1974, the first of these unified *gauge theories* (which is their classification) of the weak, strong and electromagnetic forces with a (simple) *higher* symmetry group were founded, are more commonly known as grand unified theories (GUTs) [17]. More recent studies have extended this idea of unification to such an extent that one cannot help be in awe of the incredible intuition, in particular with the profound developments of the Kaluza-Klein theories [18-21] and superstring theory [22-24], which embodies the fact that our four-dimensional space-time is a consequence due to the spontaneous compactification of a higher-dimensional space-time, are some of the first theories to unify **all** the fundamental forces of nature.

The incredible rapid development of particle theories over the three decades has rendered theoretical physicists to be in a very unusual situation. The premise behind any science is that with the development of such exotic and complex theories, to say that 'validation is an absolute requirement' is an understatement and somehow overlooked. In this context, typical elementary particle energies for testing the validity of GUTs, required for experiment, are of the order $\sim 10^{15}$ GeV, while the testing for Kaluza-Klein theories and superstring theory would require an even higher order of magnitude of energy $\sim 10^{19}$ GeV. Given our current technology, the highest energy beams that current particle accelerators have been capable of producing is $\sim 10^{4}$ GeV, where the ATLAS experiment at CERN was finally successful in discovering a particle that is consistent with the Higgs Boson [25, 26].

As Zeldovich once wrote, "the universe is the poor man's accelerator: experiments don't need to be funded, and all we have to do is collect the experimental data and interpret them properly" [27]. In fact, due to further studies of the early universe cosmology, it turns out that, the *only* laboratory in which particles with energies of $\sim 10^{15} - 10^{19}$ GeV, could not only *exist* but actually *interact* with one another is our own universe in the earliest stages of its evolution.

However, one maybe sceptical to rely on 'new' theories and the respective experiments, based on events that occurred more than thirteen billion years ago. However, taking the various subtleties from cosmology into account, it has been shown that, in fact it is possible and very successful for cosmology to be a prime platform to investigate these elementary particle theories. In particular, the simplest scenario that describes the evolution of the universe, the so-called *inflationary universe scenario* [28-40] has been the most successful in linking the two theories together (cosmology and particle theory). The crux behind the inflationary universe scenario is that at the earliest stage of evolution, the universe, was in some unstable vacuum-like state that then expanded exponentially (and according to the model in consideration, whether it is Guth's original model of Inflation [33] or Linde's *new* model of Inflation [34] or the more general model known as Chaotic inflation [37]; taking the various subtleties each model presents into account, this is developed further in Chapter 5) is known as the stage of inflation. The vacuum-like state decayed, causing the universe to heat up and then the evolution of the universe is then explained by the traditional hot universe theory or the Big Bang theory [41]. Guth [33] suggested by *limiting* the amount of time during the rapid phase of expansion, the physical motivation behind such a constraint was to enable him to produce a scenario that corresponds to our own universe. Finally, by using ideas from particle physics and studying classes of GUTs, he established how particle production came about during the early expansion of the universe. This notion of using the formalism describing the universe to describe the phenomenology of particle physics; simply is a testament to one of the greatest insight in modern theoretical physics [41-43].

Adopting the general formalism, that was developed in the literature review [44] where we explored the foundations of field theory, in particular for free fields both in the classical and quantum realm. In this report we will be extending the discussion of the quantum theory to give a more through insight into the canonical quantization procedure for field theory. In Chapter 2, we will be revisiting the scalar field namely through the Klein-Gordon (KG) example where we shall take the approach of quantizing the simple harmonic oscillator (SHO), generalizing to a many body SHO problem and determine the spectrum of its Hamiltonian.

In Chapter 3, we introduce the formalisms of normal and time ordering, that not only resolves the infinity problem when calculating particle states of the Hamiltonian, but also when we discuss the concept of interacting field theories. We talk about the three pictures that are used to describe field theories; the Heisenberg, Schrödinger and Interaction pictures, to introduce the language by which interactions are described in QFT. Using the example of the meson decay process, we discuss two very challenging concepts within interacting fields, the S-matrix and Wick's theorem, which are used to describe scattering and decay processes [45-47].

In Chapter 4, we move away from scalar fields and talk about the field formalism that is responsible for describing all the particles in the SM (except for the Higgs Boson, which is a scalar); particles with spin $\frac{1}{2}$ known as fermions. We show that not only do these particles require a different Lorentz representation compared to that of scalar fields (representing bosons) but also we show that if we naively proceed under the same canonical commutation relations that we used in [43] and in Chapter 2, to quantize scalar fields, we end up with a wrong formalism for quantizing spin $\frac{1}{2}$ fields, which are also known as Dirac fields. Thus, we show that for the Dirac field, representing fermions requires anti-commutation relations to allow for excitations within the field, giving rise to spin $\frac{1}{2}$ particles. In addition, we constantly compare the two types of fields; scalar and Dirac, and discuss the subtleties unique to each theory that distinguishes the manner by which we describe the two classes of particles we known to represent the entire universe, bosons and fermions [48-53].

In Chapter 5, we attempt to qualitatively study a phenomenon of field theory: particle production, in the context of cosmology. We start by giving an overview to why Inflation was necessary to an already successful Big Bang theory, and then proceed to the three models within the inflationary scenario, that can be explained in the context particle physics. We then conclude the report by exploring the behaviour of an oscillating inflaton field within a non-expanding background [41-43, 54-59].

Chapter 2: Canonical quantization of the scalar field.

2.1 The transition revisited.

When we considered the SHO problem in non-relativistic quantum mechanics (QM), we illustrated how it is described by the operators in the Heisenberg picture, $x_i(t)$: position and $p_j(t)$: momentum, at a particular time, *t*, related by the commutation relation:

$$\left[x_i(t), p_j(t)\right] = i\delta_{ij} \tag{1}$$

The transition from QM to QFT was achieved by replacing the discrete quantities in the Heisenberg uncertainty relation by continuous parameters in the form of $\phi(\boldsymbol{x}, t)$: field and $\pi(\boldsymbol{x}, t)$: conjugate momenta. The *discrete quantities* in the form of indices (i, j) have been replaced by $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$: 3 *continuous parameters*.

The *fundamental* commutation relation obeyed by the operators in QFT is:

$$[\phi(\boldsymbol{x}, t), \pi(\boldsymbol{y}, t)] = i\delta(\boldsymbol{x} - \boldsymbol{y}) \tag{2}$$

where the δ_{ij}: Kronecker delta function (QM) is replaced by the $\delta(\boldsymbol{x} - \boldsymbol{y})$: Dirac delta function (QFT). So, by using the commutation relation (2), we can now proceed to canonically quantizing the theory [45].

2.2 Quantizing the Klein-Gordon field.

The Lagrangian density for a real scalar field, $\phi(\boldsymbol{x}, t)$ with mass, *m* is defined as:

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \tag{3}$$

Using the Euler-Lagrange equation:

$$\frac{\partial L}{\partial\phi} - \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\phi)}\right) = 0 \tag{4}$$

where,

$$\frac{\partial L}{\partial \phi} = -m^2\phi \qquad \text{and} \qquad \frac{\partial L}{\partial(\partial_\mu \phi)} = \partial^\mu \phi \tag{5}$$

we attain the equation of motion for the Lagrangian (3), and is known as the *KG equation*, where the $\phi(\boldsymbol{x}, t)$ represents the KG field:

$$\partial_\mu \phi \partial^\mu \phi + m^2 \phi = 0 \tag{6}$$

It should be stated that as the action for the KG field is invariant under Lorentz transformation, the equation of motion is covariant under Lorentz transformation. Furthermore as (6) only has a second order derivative term, this implies locality [46].

The Hamiltonian of the system, H is defined as:

$$H = \int d^3 Ƕ = \int d^3 x \left[\frac{\partial L}{\partial(\partial_\mu \phi)} \cdot \partial_0 \phi - L\right] \tag{7}$$

where,

$$\frac{\partial L}{\partial(\partial_\mu \phi)} = \dot{\phi} \qquad \Rightarrow \qquad Ƕ = \dot{\phi}^2 - L \tag{8}$$

giving us the definition of the Hamiltonian density, Ƕ. Using (8) and (3), the Hamiltonian for the KG equation is defined as:

$$H = \int d^3 x \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\phi + \frac{1}{2}m^2\phi^2\right) \tag{9}$$

By converting the ϕs in the integrand to be operators, we will be able to write the Hamiltonian as a sum of normal modes which will then give us the spectrum for the system.

The equation of motion (6) allows plane wave solutions in the form:

$$\phi(\boldsymbol{x}, t) \sim \exp(-ik \cdot x) \tag{10}$$

$$\text{where} \quad k \cdot x = k_\mu k^\mu = k_0 x_0 - k \cdot x \qquad \text{provided} \quad \omega = \mathrm{k}_0 = \sqrt{\mathrm{k}^2 + \mathrm{m}^2} \tag{11}$$

As $\phi(\boldsymbol{x}, t)$ is a real field, the more general solution to the KG equation of motion will be a superposition of the $\phi(\boldsymbol{x}, t)$ plane wave solutions (10) and its complex conjugate, thus giving us:

$$\phi(\boldsymbol{x}, t) = \int \frac{d^3 k}{(2\pi)^3 2\omega}\left[a(\boldsymbol{k}) \exp(-ik \cdot x) + a^\dagger(\boldsymbol{k}) \exp(ik \cdot x)\right] \tag{12}$$

and by using the expression,

$$\pi(\boldsymbol{x}, t) = \frac{\partial L}{\partial(\partial_0 \phi)} = \partial_0 \phi \tag{13}$$

$$= \int \frac{d^3 k}{(2\pi)^3 2\omega}(-i\omega)\left[a(\boldsymbol{k}) \exp(-ik \cdot x) - a^\dagger(\boldsymbol{k}) \exp(-ik \cdot x)\right] \tag{14}$$

The factor 2ω is put in for the sake of convenience, allowing the integral expressions to be Lorentz invariant. Upon quantization, as the $\phi(\boldsymbol{x}, t)$ and $\pi(\boldsymbol{y}, t)$ are operators, this would imply that $a(\boldsymbol{k})$ and $a^\dagger(\boldsymbol{k})$ integral expressions (12) and (14) are also operators, and by inversing the above integral expressions, and expressing $a(\boldsymbol{k})$and $a^{\dagger}(\boldsymbol{k})$ in terms of the $\phi(\boldsymbol{x},t)$ and $\pi(\boldsymbol{y},t)$ operators, we can attain a commutation relation similar to that of (2). Therefore,

Inversing the expressions, we attain:

$$a(\boldsymbol{k}) = \int d^3x \, \{\exp(ik\cdot x)\,[i\pi(x) + \omega\phi(x)]\} \tag{15}$$

$$a^{\dagger}(\boldsymbol{k}) = \int d^3x \, \{\exp(-ik\cdot x)\,[-i\pi(x) + \omega\phi(x)]\} \tag{16}$$

Similar to the fundamental commutation relation, we attain:

$$\left[a(\boldsymbol{k}), a^{\dagger}(\boldsymbol{k}')\right] = (2\pi)^3 2\omega \cdot \delta(\boldsymbol{k}-\boldsymbol{k}') \tag{17}$$

2.3 Spectrum of the Hamiltonian.

We can now express the Hamiltonian in (9) in terms of $a(\boldsymbol{k})$ and $a^{\dagger}(\boldsymbol{k})$,

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega} \cdot \omega \, \left\{a^{\dagger}(\boldsymbol{k})a(\boldsymbol{k}) + \frac{1}{2}\left[a(\boldsymbol{k}), a^{\dagger}(\boldsymbol{k})\right]\right\} \tag{18}$$

and due to the inclusion of the quantity $\left(\frac{d^3k}{(2\pi)^3 2\omega}\right)$, the *H* is manifestly Lorentz invariant. We can ignore the commutator term in (18) as it a simply a (commutator number) C-number, but if we find the spectrum for the $a^{\dagger}(\boldsymbol{k})a(\boldsymbol{k})$ term, we find the spectrum for the total Hamiltonian, *H*, which is what we are after!

Considering the <u>first term</u> of (18):

$$\text{Let} \quad H_n = \int \frac{d^3k}{(2\pi)^3 2\omega} \cdot \omega \, a^{\dagger}(\boldsymbol{k})a(\boldsymbol{k}) \tag{19}$$

and assume that H_n obeys the eigenvalue equation, $H_n|E\rangle = E|E\rangle$, such that $|E\rangle$: energy eigenstate and E: (corresponding) energy eigenvalue. We now investigate what happens when H_n acts on the eigenstate $a(\boldsymbol{k})|E\rangle$.

$$H_n\{a(\boldsymbol{k})|E\rangle\} = \int \frac{d^3k'}{(2\pi)^3 2\omega'} \cdot \omega' \, a^{\dagger}(\boldsymbol{k}')a(\boldsymbol{k}')a(\boldsymbol{k})|E\rangle \tag{20}$$

As $[a(\boldsymbol{k}), a(\boldsymbol{k}')] = 0$,
$$= \int \frac{d^3k'}{(2\pi)^3 2\omega'} \cdot \omega' \, a^{\dagger}(\boldsymbol{k}')a(\boldsymbol{k})a(\boldsymbol{k}')|E\rangle$$

$$= a(\boldsymbol{k}) \int \frac{d^3k'}{(2\pi)^3 2\omega'} \cdot \omega' \, a^{\dagger}(\boldsymbol{k}')a(\boldsymbol{k}')a(\boldsymbol{k}')|E\rangle - \omega a(\boldsymbol{k})|E\rangle$$

$$= a(\boldsymbol{k})H_n|E\rangle - \omega a(\boldsymbol{k})|E\rangle$$

Using $H_n|E\rangle = E|E\rangle$
$$= a(\boldsymbol{k})E|E\rangle - \omega a(\boldsymbol{k})|E\rangle$$

$$\therefore \quad H_n\{a(\boldsymbol{k})|E\rangle\} = (E-\omega)a(\boldsymbol{k})|E\rangle \tag{21}$$

So, if $|E\rangle$ is an eigenstate of H_n, with eigenvalue E, then $a(\boldsymbol{k})|E\rangle$ is <u>also</u> an eigenstate of H_n, with eigenvalue $(E-\omega)$. Hence, $a(\boldsymbol{k})$ is known as an *annihilation operator*, due to the fact it annihilates a quantum of energy (ω) from the eigenstate $|E\rangle$ [16].

Similarly, under the same principles,

$$H_n\{a^\dagger(\boldsymbol{k})|E\rangle\} = (E+\omega)a^\dagger(\boldsymbol{k})|E\rangle \tag{22}$$

$a^\dagger(\boldsymbol{k})$ is known as a creation operator, due to the fact it creates a quantum of energy (ω) in the $|E\rangle$ eigenstate.

The first statement we can make regarding the spectrum of the Hamiltonian is the fact the energy density, Ƕ(x) has a lower bound. This is confirmed by taking the expectation value of H_n,

$$\langle\Psi|H_n|\Psi\rangle \geq 0 \tag{23}$$

implying that as H_n involves the operators $a^\dagger(\boldsymbol{k})a(\boldsymbol{k})$, all the terms of Ƕ$(x)$ are positive definite. But as the annihilation operator lowers the energy by ω, there <u>must</u> exist a state where:

$$a(\boldsymbol{k})|0\rangle = 0 \qquad \text{where } |0\rangle\text{: ground state of the system (vacuum)} \tag{24}$$

Whereas, when the creation operator acts on the ground state of the system, we attain:

$$a^\dagger(\boldsymbol{k})|0\rangle = |\boldsymbol{k}\rangle \qquad \text{where } |\boldsymbol{k}\rangle\text{: one particle state} \tag{25}$$

When considering the momentum operator for the KG field, as discussed in [44]:

$$P^i = \int d^3x\, T^{0i} \tag{26}$$

$$= \int d^3x\, \pi(x)\partial^i\phi$$

Simplifies to,

$$\Rightarrow P^i = \int \frac{d^3k}{(2\pi)^3 2\omega}\cdot k^i\,\left(a^\dagger(\boldsymbol{k})a(\boldsymbol{k})\right) \tag{27}$$

Again, using (25):

$$a^\dagger(\boldsymbol{k})|0\rangle = |\boldsymbol{k}\rangle, \qquad H_n|\boldsymbol{k}\rangle = \omega|\boldsymbol{k}\rangle$$

$$\therefore P^i|\boldsymbol{k}\rangle = k^i|\boldsymbol{k}\rangle \tag{28}$$

Thus, $|\boldsymbol{k}\rangle$ is an eigenstate of H_n and P^i, with eigenvalues ω and P^i respectfully. Also, from the relation, $\omega(\boldsymbol{k}) = \sqrt{\boldsymbol{k}^2+m^2}$ stated in (11), it is valid to use $|\boldsymbol{k}\rangle$ as the state in consideration for both the energy <u>and</u> momentum operators.

The above formalism can be extended, involving two creation operators:

$$\left(a^\dagger(\boldsymbol{k_1})a^\dagger(\boldsymbol{k_2})\right)|0\rangle = |\boldsymbol{k_1},\boldsymbol{k_2}\rangle \tag{29}$$

where the state $|\boldsymbol{k_1},\boldsymbol{k_2}\rangle$ is a representation of a two particle state, with a creation of two quanta of energy $(\omega_1+\omega_2)$ and two momenta (k_1+k_2). In fact, the formalism can be generalised to describe a multi-particle system:

$$\left(a^\dagger(\boldsymbol{k_1})a^\dagger(\boldsymbol{k_2})\ldots\ldots a^\dagger(\boldsymbol{k_{n-1}})a^\dagger(\boldsymbol{k_n})\right)|0\rangle = |\boldsymbol{k_1},\boldsymbol{k_2}\ldots\ldots\boldsymbol{k_{n-1}},\boldsymbol{k_n}\rangle \tag{30}$$

creating n-number quanta of energy $(\omega_1+\omega_2+\ \ldots\ldots+\omega_{n-1}+\omega_n)$ and n-number of momenta $(k_1+k_2+\ \ldots\ldots+k_{n-1}+k_n)$.

Finally, returning to the two particle state defined in (29), consider the ordering:

$$|\boldsymbol{k_1},\boldsymbol{k_2}\rangle = \left(a^\dagger(\boldsymbol{k_1})a^\dagger(\boldsymbol{k_2})\right)|0\rangle = \left(a^\dagger(\boldsymbol{k_2})a^\dagger(\boldsymbol{k_1})\right)|0\rangle = |\boldsymbol{k_2},\boldsymbol{k_1}\rangle \tag{31}$$

The interchange of $|\boldsymbol{k_1},\boldsymbol{k_2}\rangle = |\boldsymbol{k_2},\boldsymbol{k_1}\rangle$, leaves the state invariant. These states obey Bose-Einstein statistics for particles called *Bosons*. In fact, within the KG theory, particle excitations due the KG field are all Bosons!

When dealing with a multi-particle system, it is especially useful to introduce a number operator $N(\boldsymbol{k})$, which is defined as:

$$N(\boldsymbol{k}) = a(\boldsymbol{k})a^\dagger(\boldsymbol{k}) \tag{32}$$

such that when it acts upon the various states of the Hamiltonian, the eigenvalues of the $N(\boldsymbol{k})$ operator denotes the occupation number of the particle states. For instance,

$N(\boldsymbol{k})|0\rangle = 0|\boldsymbol{k}\rangle = 0,$ implying that the vacuum state has no particles.

$N(\boldsymbol{k})|\boldsymbol{k}\rangle = 1|\boldsymbol{k}\rangle,$ implying that the one-particle state has one particle.

$N(\boldsymbol{k})|\boldsymbol{k_1},\boldsymbol{k_2}\dots\dots\boldsymbol{k_{n-1}},\boldsymbol{k_n}\rangle = N|\boldsymbol{k_1},\boldsymbol{k_2}\dots\dots\boldsymbol{k_{n-1}},\boldsymbol{k_n}\rangle$ implying the n-particle state has N particles.

2.4 From infinity to normal ordering.

The full Hamiltonian of the KG field, has the extra term involving the commutation relation of the $a^\dagger(\boldsymbol{k})$ and $a(\boldsymbol{k})$ operators,

$$H = H_n - \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega}\cdot \omega\left[a(\boldsymbol{k}), a^\dagger(\boldsymbol{k})\right] \tag{33}$$

by implementing the commutation relation of (17) but with respect to the $\boldsymbol{k}$ parameter, equation (22) simplifies to,

$$H = H_n - \frac{1}{2}\int d^3k\left(\omega\cdot\delta(0)\right) \tag{34}$$

Equation (33) has a similar foundation to the construction of P^i [16]. But the second term in (33) has the term ω instead of k^i, and thus, the integral is non-vanishing but is divergent to infinity. This is a nonsensical expression and poses a problem, as we do not have particle states that have infinite energies! The important point is that when conducting measurements, it is the *relativistic energies* of particles that are of interest <u>not</u> the *absolute energies*! This plays a key part in our theory as, it allows us to ignore the second term of (33), leaving the total Hamiltonian to simply be:

$$\boldsymbol{H} = \boldsymbol{H_n}. \tag{35}$$

When we are operating in the classical regime, the structure of the Hamiltonian is well known as discussed in [44]. However, upon quantization, the fields within the Hamiltonian are no longer functions but they are <u>operators</u> with non-trivial commutation relations. And so, the ambiguity in the theory is that there is no *definite* order for these operators to begin with. Simply put, there is <u>no</u> fundamental reason, why the operators in (32) are ordered in that way.

To rectify this problem, the construction of a *new* ordering system is required; in particular it enforces the non-existence of a term within the Hamiltonian that can be interpreted as particles having infinite energies within a system. The procedure is known as *Normal ordering*, which is simply to order the operators in such a way that the creation operators $a^\dagger(\boldsymbol{k})$ are to the left of the annihilation operators $a(\boldsymbol{k})$. This formalism ensures the non-existence of the term, $\frac{1}{2}\int d^3k\,\big(\omega\cdot\delta(0)\big)$ in the Hamiltonian [45, 46].

Using the expansion [16], we do not need to rearrange the Hamiltonian so as to get the commutation relation, $\left[a(\boldsymbol{k}), a^\dagger(\boldsymbol{k})\right]$ in (18), instead due to normal ordering *N*,

$$H_n = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega}\cdot\omega\, N\{a(\boldsymbol{k})a^\dagger(\boldsymbol{k}) + a^\dagger(\boldsymbol{k})a(\boldsymbol{k})\} \tag{36}$$

$$= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega}\cdot\omega\, \{a^\dagger(\boldsymbol{k})a(\boldsymbol{k}) + a^\dagger(\boldsymbol{k})a(\boldsymbol{k})\}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega}\cdot\omega\, \{a^\dagger(\boldsymbol{k})a(\boldsymbol{k})\} \tag{37}$$

Just as we introduced in (19), where H_n: normal ordered Hamiltonian and the spectrum of H_n, all the eigenstates have finite eigenvalues. Thus, to resolve any ambiguities in the theory, all physical observables such as the momentum and Hamiltonian operators are all normal ordered! However, this is not successful for every theory in Physics, as Padmanabhan points out [60, 61] the formalism of normal ordering does not work for Hamiltonians in a theory of gravity.

Chapter 3: Interacting Field Theory.

3.1 From causality to time ordering.

Considering two events x and y, that are separated by space-like distances. In the context of QFT, the two events are represented by operators, $O_1 = \phi(x)$ and $O_2 = \phi(y)$ respectivtely and a measurement of the two events is denoted as,

$$[\phi(x)\,,\phi(y)] = 0 \tag{38}$$

which states that a measurement of one of the operators does not affect the measurement of the other operator, only if x and y are space-like separated with respect to each other.

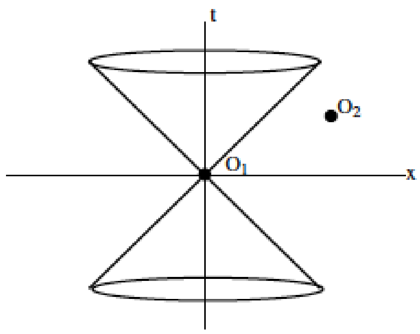

Figure 1: Two events separated by space-like distance, in Minkowski space-time.

If we recall the most general form of the $\phi(x)$ as denoted in (12), and consider the time dependence of the creation and annihilation operators, we can see that,

$$a(\boldsymbol{k})\exp(-ik\cdot x) \rightarrow \exp(-i\omega t) \tag{39}$$

$$a^\dagger(\boldsymbol{k})\exp(ik\cdot x) \rightarrow \exp(i\omega t) \tag{40}$$

Thus, we can see that the annihilation operator involves *positive* frequency modes and the creation operator involves *negative* frequency modes.

Let, $\phi^+(x) = a(\boldsymbol{k})\exp(-ik\cdot x)$ <u>and</u> $\phi^-(x) = a^\dagger(\boldsymbol{k})\exp(ik\cdot x)$, (41)

such that (12) can be expressed in terms of the operators denoted in (41),

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega}[\phi^+(x) + \phi^-(x)] \tag{42}$$

Now we can express (38) in terms of operators defined in (41), $\phi^+(x)$ and $\phi^-(x)$; $\phi^+(y)$ and $\phi^-(y)$.

$$\therefore [\phi(x)\,,\phi(y)] = [\phi^+(x)\,,\phi^-(y)] - [\phi^-(x)\,,\phi^+(y)] \tag{43}$$

$$\Rightarrow [\phi^+(x)\,,\phi^-(y)] = \int \frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}\exp(-ik\cdot x)\exp(ik'\cdot y)\big[a(\boldsymbol{k})\,,a^\dagger(\boldsymbol{k}')\big]$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega}\exp -ik(x-y) \equiv i\Delta^+(x-y) \tag{44}$$

where, $$i\Delta^+(x) \equiv -i\int \frac{d^3k}{(2\pi)^3 2\omega}\exp(-ik\cdot x) \tag{45}$$

Similarly, $$[\phi^-(x)\,,\phi^+(y)] = -[\phi^+(x)\,,\phi^-(y)] \tag{46}$$

$$= -i\Delta^+(y-x) \equiv i\Delta^-(x-y) \tag{47}$$

$$\therefore \quad \Rightarrow [\phi(x)\,,\phi(y)] = i\Delta^+(x-y) + i\Delta^-(x-y) \equiv i\Delta(x-y) \tag{48}$$

The $\Delta(x-y)$ is an odd function and is a Lorentz invariant quantity. Now if two events x and y are *space-like* separated, and then under a Lorentz transformation we can make the two events be *simultaneous*, implying, $\Delta(x-y) = 0$, then the commutation relation in expression (48) will equal to 0. Furthermore, by exploiting the principle of Lorentz invariance, if in one frame of reference the $[\phi(x)\,,\phi(y)] = 0$, then it is so in <u>all</u> frames of reference. This is what is known as *causality* [45, 46].

An example of a theory that is causal is the one that we discussed in Chapter 2, the KG theory. If two events are space-like separated with respect to each other, then the measurement of one of the observables does not affect the measurement of the other observable occurring at another space-time event. When applied to the KG theory, the $\Delta(x-y)$ function is known as the *Green function* of the KG operator.

Now, considering the two points, where a particle propagates form $y = (y^0, \boldsymbol{y})$ to $x = (x^0, \boldsymbol{x})$. One could ask, what is the *amplitude* of this particle when propagating from a point $\boldsymbol{y}$ at time y^0 to a point $\boldsymbol{x}$ at time x^0, such that $x^0 > y^0$? To do this, we would have to compute the vacuum expectation value of $\phi(x)\,\phi(y)$, simply put $\langle 0|\,\phi(x)\,\phi(y)|0\rangle$ also known as the *retarded* Green's function for the KG operator [45, 46].

As $\phi(y)$ has both a positive and negative frequency components as shown in (42), when acted upon the vacuum

$$\phi(y)|0\rangle = |y\rangle \quad \text{, say.} \tag{50}$$

where the positive frequency component containing the annihilation operator will annihilate the vacuum, $|0\rangle = 0$ and the negative frequency component containing the creation operator creates the single particle state, $\phi(y)|0\rangle = |y\rangle$, as stated in (50). But if, $x^0 < y^0$, this would imply that the propagation of the particle goes from $(x^0, \boldsymbol{x}) \rightarrow (y^0, \boldsymbol{y})$, thus we would have to compute the vacuum expectation value of $\langle 0|\phi(y)\,\phi(x)\,|0\rangle$.

Instead of talking about whether $x^0 > y^0$ <u>or</u> $x^0 < y^0$, we could combine both expectation values together via the formalism of *time ordering*.

For two observables, $\phi(x)$ and $\phi(y)$,

$$T(\phi(x)\,\phi(y)) = \begin{cases} \phi(x)\,\phi(y) & \text{if } x^0 > y^0 \\ \phi(y)\,\phi(x) & \text{if } x^0 < y^0 \end{cases} \tag{51}$$

$$= \Theta(x^0 - y^0)\phi(x)\phi(y) + \Theta(y^0 - x^0)\phi(y)\,\phi(x) \tag{52}$$

where the Θ functions are known as step functions.

Thus, if $x^0 > y^0$, then the $\Theta(x^0 - y^0)\phi(x)\phi(y)$ term <u>only</u> survives and so,

$$T(\phi(x)\,\phi(y)) = \phi(x)\,\phi(y) \tag{53}$$

and so, if $x^0 < y^0$, then the $\Theta(y^0 - x^0)\phi(y)\,\phi(x)$ term <u>only</u> survives and so,

$$T(\phi(x)\,\phi(y)) = \phi(y)\,\phi(x) \tag{54}$$

3.2 Interacting theory.

In particle physics, physical processes such as scattering events, absorption and particle decay to name a few, cannot be described by *free* field theory, which is what we have discussed so far in the literature. What *is* of considerable interest, in fact, the formalism of QFT that is most applicable to the real world, *interacting* field theory. Examples stemming from classical field theory, interactions of an electromagnetic field with a charged scalar field, say, that led to fundamental concepts such as Gauge transformations, forms an integral part of interacting field theories. But in this report we will consider a simpler theory to represent interactions, namely the ϕ^4 theory, representing a real scalar field [48, 49].

The Lagrangian for such as theory is defined as:

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4 \tag{55}$$

and the respective equation of motion,

$$\partial_\mu\phi\partial^\mu\phi + m^2\phi = \lambda\phi^3 \tag{54}$$

Unlike the equation of motion (6) for free fields, (54) is no longer *linear* for the field ϕ, as the $\lambda\phi^3$ term comes from the $\frac{\lambda}{4}\phi^4$: interaction term. In addition, the ϕ is no longer a free field, but a self-interacting field.

In general, the Lagrangian for a system is broken down into,

$$L = L_0 + L' \tag{55}$$

where the L_0: free Lagrangian term and L': interacting Lagrangian term. Correspondingly, the Hamiltonian for the system is also broken down as,

$$H = H_0 + H' \tag{56}$$

where the H_0: free Hamiltonian term and H': interacting Hamiltonian term.

Before we develop the formalism of quantizing such a theory, we need to find the respective spectrum and determining the evolution of a $|\Psi(t)\rangle$ state in the presence of the interacting Hamiltonian, H', it would be prudent to briefly discuss the different pictures used in quantization [45].

3.3 The three pictures.

Heisenberg Picture:

So far the free fields that we have discussed, namely the KG field, we assumed the observables or operators are time dependent, $\phi(\boldsymbol{x}, t)$ while the states are time independent, for instance $|0\rangle$, $a^{\dagger}(\boldsymbol{k})|0\rangle$ etc. This is known as the Heisenberg Picture (HP). where the time evolution of the operators are governed by the *Heisenberg equation of motion*, where if $A(t)$ is an operator:

$$\text{then,} \qquad \frac{dA(t)}{dx} = -i[A(t), H] \tag{57}$$

Schrodinger Picture:

Quite the opposite to the HP, in the Schrodinger Picture (SP), the observables/operators are time independent while the states are time dependent. Thus the time evolution of the states is governed by the Schrödinger Equation, where if the $|\Psi(t)\rangle$ is a state vector:

$$\text{then,} \quad H|\Psi(t)\rangle = i\frac{d|\Psi(t)\rangle}{dx} \qquad \xrightarrow{\text{where}} \qquad |\Psi(t)\rangle = exp(-iHt)|\Psi(0)\rangle \tag{58}$$

And so if we were to relate the HP with the SP, the evolution of the operators in the HP:

$$A(t) = exp(iHt)A(0)exp(-iHt) \tag{59}$$

Interaction Picture:

Considered as a hybrid of the HP and SP, the Hamiltonian for the Interaction picture (IP) is split just as in (56), where the H_0 governs the time dependence of operators: $A_I(t)$ and recall, as discussed in Chapter 2, where the H' term did not appear, the resulting total Hamiltonian: $H = H_0$, implying that we are operating in the HP. Whereas the H' term governs the time dependence of states: $|\Psi(t)\rangle_I$. To show that this is consistent with our theory,

Let, the operators in the IP be defined as:

$$A_I(t) = exp(iH_0t)A(t)exp(-iH_0t) \tag{60}$$

and by defining the state vector, $|\Psi(t)\rangle_I$ obeying the Schrodinger equation of motion, where H: full Hamiltonian,

$$H|\Psi(t)\rangle = i\frac{d|\Psi(t)\rangle}{dt} \qquad \xrightarrow{\text{where}} \qquad |\Psi(t)\rangle_I = exp(-iH_0t)|\Psi(t)\rangle \tag{61}$$

$$\therefore \quad i\frac{d|\Psi(t)\rangle}{dt} = exp(iH_0t)\left(-H_0|\Psi(t)\rangle + i\frac{d}{dt}|\Psi(t)\rangle\right)$$

$$\Rightarrow \; i\frac{d|\Psi(t)\rangle_I}{dt} = exp(iH_0t)(H - H_0)|\Psi(t)\rangle$$

As $H = H_0 + H'$ and let the Identity operator: $I = exp(-iH_0 t)exp(iH_0 t)$

$$= exp(iH_0 t)H' exp(-iH_0 t)exp(iH_0 t)|\Psi(t)\rangle$$

$$\therefore \quad i\frac{d|\Psi(t)\rangle_I}{dt} = H'_I(t)|\Psi(t)\rangle_I \tag{62}$$

which is the equation of motion in the IP, governing the evolution of the $|\Psi(t)\rangle_I$: state in the IP. But considering the commutation relations for a system described by (55), where $\phi(\boldsymbol{x}, t)$ and $\pi_\phi(\boldsymbol{x}, t)$ are operators in the HP and the commutation relation in the SP, with regard to the operators as stated in (2). However in general, this does not hold in the IP, but by making the assumption that L' does not contain any time derivatives of $\phi(\boldsymbol{x}, t)$, i.e. L' does not depend on $\partial_0\phi$. Resulting to (2),

$$[\phi(\boldsymbol{x}, t), \pi_\phi(\boldsymbol{y}, t)] = i\delta(\boldsymbol{x} - \boldsymbol{y}) \tag{63}$$

to be valid for <u>both</u> free and interacting theories. Similar to (60) we define the more interested relation between IP and HP for operators,

$$A(t) = exp(iHt)A(0)exp(-iHt) \tag{64}$$

where H: full Hamiltonian, $A(t)$: operators in the HP and $A(0)$: operators in the SP. By inverting (64) and substituting (65) into (60),

$$A_I(t) = exp(iH_0 t)exp(-iHt)A(t)exp(iHt)exp(-iH_0 t) \tag{65}$$

Introducing the unitary operator, such that we can re-write (65):

$$U(t) = exp(iH_0 t)exp(-iHt) \tag{66}$$

$$\Rightarrow \ A_I(t) = U(t)A(t)U^{-1}(t) \tag{67}$$

3.4 The Evolution operator.

Recall the Lagrangian for the interacting KG field (55),

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4 \tag{55}$$

where according to (56), the interaction Hamiltonian for the ϕ^4 theory,

$$H' = -\int d^3x \frac{\lambda}{4!}\phi^4 \tag{68}$$

and the corresponding equation of motion,

$$\partial_\mu\phi\partial^\mu\phi + m^2\phi = \frac{\lambda}{4!}\cdot 4\phi^3 \tag{69}$$

comparing with the equation of motion, that does not take interactions into account, as stated in (6), but this time we are operating in the IP and so,

$$\partial_\mu\phi\partial^\mu\phi_I(t) + m^2\phi_I(t) = 0 \tag{70}$$

which is equivalent to Heisenberg's equation of motion and is obeyed by the well know general solution:

$$\phi_I(t) = \int \frac{d^3k}{(2\pi)^3 2\omega}\left[a_I(\boldsymbol{k})\exp(-ik\cdot x) + a_I^\dagger(\boldsymbol{k})\exp(ik\cdot x)\right] \tag{71}$$

By using the operator relation (67), we will be able to determine the $\phi(t)$: field in the HP, given we know the expression for $U(t)$, and we need to represent it as a functional of the $\phi_I(t)$ in the IP.

Let,

$$i\frac{\partial U(t)}{\partial t} = i\frac{\partial}{\partial t}\left(exp(iH_0t)exp(-iHt)\right) \tag{72}$$

$$i\frac{\partial U(t)}{\partial t} = exp(iH_0t)H'exp(-iHt)$$

Knowing that, $H_I(t) = exp(iH_0t)H'exp(-iH_0t)$,

$$\Rightarrow i\frac{\partial U(t)}{\partial t} = H_I(t)U(t) \tag{73}$$

By solving this we express $H_I(t)$ and hence $U(t)$ as functionals of $\phi_I(t)$.

Thus, by using (66) and letting $U(0) = 1$, consider:

$$U(t) = U(0) - \int_0^t idt_1 H_I(t_1)\,U(t_1) + \ \ldots$$

$$= 1 - \int_0^t idt_1 H_I(t_1)\,U(t_1) + \ \ldots \tag{74}$$

Bearing in mind that if (74) is differentiated, we get back to (73)! Now, applying the same formalism to the $U(t_1)$ in (74),

$$U(t_1) \ = 1 - \int_0^t idt_2 H_I(t_2)\,U(t_2) \tag{75}$$

Combining expressions (74) and (75) together, we attain,

$$\therefore \quad U(t) = U(0) + (-i)\int_0^t dt_1 H_I(t_1) + (-i)^2\int_0^t dt_1\int_0^{t_1} dt_2 H_I(t_1)H_I(t_2)U(t_2) + \ \ldots \tag{76}$$

which can be expressed as an infinite series,

$$\Rightarrow \quad U(t) = \sum_{n=0}^{\infty}(-i)^n\int_0^t dt_1\int_0^{t_1} dt_2 \ldots \int_0^{t_n} dt_{n+1} H_I(t_1)\ldots\ldots H_I(t_{n+1}) \tag{77}$$

To make the series expansion of $U(t)$ in a more compact form, we consider the <u>third</u> term in the expansion (76) in more detail,

$$U^{(2)}(t) = (-i)^2\int_0^t dt_1\int_0^{t_1} dt_2 H_I(t_1)H_I(t_2) \tag{78}$$

where the t_1 and t_2 are considered to be dummy variables and so if we introduce a step function $\Theta(t_1 - t_2)$ we can change the limit of integration for the integral involving dt_2, instead of $(0 \to t_1)$, the second integral is being integrated $(0 \to t)$. Thus we have,

$$U^{(2)}(t) = (-i)^2\int_0^t dt_1\int_0^t dt_2 \Theta(t_1 - t_2)H_I(t_1)H_I(t_2) \tag{79}$$

As we mentioned earlier, that t_1 and t_2 were dummy variables and so if we make a substitution, $\widetilde{t_1} = t_2$ and $\widetilde{t_2} = t_1$, we could re-write (78) implementing the dummy variables $\widetilde{t_1}$ and $\widetilde{t_2}$, such that the overall the variables of the integrand have swapped,

$$\therefore\ U^{(2)}(t) = (-i)^2 \int_0^t dt_2 \int_0^{t_2} dt_1 H_I(t_2) H_I(t_1) \tag{80}$$

$$\Rightarrow\ U^{(2)}(t) = (-i)^2 \int_0^t dt_2 \int_0^t dt_1 \Theta(t_2 - t_1) H_I(t_2) H_I(t_1) \tag{81}$$

By implementing the same formalism as earlier, we attain a similar expression. Now by adding the two expressions (79) and (81) we attain,

$$\Rightarrow\ \frac{(-i)^2}{2} \int_0^t \int_0^t dt_1 dt_2 [dt_2 \Theta(t_1 - t_2) H_I(t_1) H_I(t_2) + \Theta(t_2 - t_1) H_I(t_2) H_I(t_1)] \tag{82}$$

The integrand in (82) is simply the *time ordering formalism* that was introduced in section 3.1. In particular,

$$\text{should } t_1 > t_2 \xrightarrow{\text{results to}} \text{the expression above is ordered as } H_I(t_1) H_I(t_2)$$
$$\text{should } t_2 > t_1 \xrightarrow{\text{results to}} \text{the expression above is ordered as } H_I(t_2) H_I(t_1)$$

and thus (82) is more compactly written as a time ordered product,

$$\Rightarrow\ \frac{(-i)^2}{2} \int_0^t \int_0^t dt_1 dt_2 T[H_I(t_1) H_I(t_2)] \tag{83}$$

More generally, the $U(t)$ expressed in terms of an infinite series (77), can be better expressed as,

$$U(t) = \sum_{n=0}^{\infty} U^n(t) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_0^t \dots \int_0^t dt_1 \dots dt_n\, T[H_I(t_1) \dots H_I(t_n)] \tag{84}$$

Had there been no time ordered product, then the above summation would be reduced to a more general expression, an exponential in the form of,

$$U(t, t_0) = T\left\{exp\left(-i \int_{t_0}^t dt' H_I(t')\right)\right\} \tag{85}$$

where T operator expands the exponential in terms of the power series and for each term we take the time ordered product of the operators. The above expression is known as *Dyson's formula* [45, 46, 50].

Discussion

Applying what we have developed so far in the context of a scattering theory, namely an example of *Yukawa theory*, one that involves the interaction of both a real and complex field. Using the formalism denoted by Tong [45]:

Let,
$$\phi \sim a + a^\dagger \tag{86}$$
$$\psi \sim b + c^\dagger \tag{87}$$
$$\psi^\dagger \sim c + b^\dagger \tag{88}$$

where (86) represents the operator ϕ, that creates/destroys ϕ particles: *mesons*; (87) represents the ψ operator, that destroys ψ particles and creates $\psi^\dagger$ particles, both of which we shall call *nucleons*; (88) represents the $\psi^\dagger$ operator, that creates ψ particles and destroys $\psi^\dagger$ particles, also called *nucleons*.

In particular, the interaction Hamiltonian for the Yukawa theory in consideration is,

$$H_I(t) = H' = g\psi^\dagger\psi\phi \tag{89}$$

although so far we have only been discussing the field structure and formalisms for the particle excitations in a KG field called bosons, the ψ in (89) would be representing Dirac particles also known as fermions, which are the particle excitations in a Dirac field (which will be discussed in Chapter 4). If this particular Yukawa theory is applied in the context of the Standard Model, then the ϕ in (89) would be representing the Higgs Boson.

The unitary evolution operator in (85), is also used to represent interacting Hamiltonians such as the one described in (89). Upon expansion, using the calculation in (76), we can put matters into perspective as the first order expansion of $H_I(t)$ could be used to describe processes such as a meson decay (eg. $\phi \to \psi\overline{\psi}$) and the second order expansion of $H_I(t)$ could be used to describe scattering theories such as nucleon scattering (eg. $\psi\overline{\psi} \to \phi \to \psi\overline{\psi}$).

3.5 The S-matrix.

In the interacting theory, we define the interacting Hamiltonian density, $\mathcal{H}_I(x)$ as:

$$H_I(t) = \int d^3x\, \mathcal{H}_I(x) \qquad \text{where} \qquad \mathcal{H}_I(x) = \mathcal{H}_I(\boldsymbol{x}, t) \tag{90}$$

where recall the $H_I(t)$ is the interaction Hamiltonian in the IP, thus by using the expressions stated in (90) we can express (85) as,

$$U(t) = T\{exp(-i \int d^4x\, \mathcal{H}_I(x))\} \tag{91}$$

Extending the same formalism for the evolution of $U(t)$, to the evolution of the state vectors in the IP, $|\Psi(t)\rangle_I$ (62) such that,

$$|\Psi(t, t_0)\rangle_I = |i\rangle - i \int_{t_0}^{t} dt_1 H_I(t')\, |\Psi(t')\rangle_I \tag{92}$$

where, the initial state: $|i\rangle = |\Psi(t)\rangle_I \Big|_{t\,=\,t_0}$ <u>and</u> $|\Psi(t')\rangle_I = T\left\{exp\left(-i \int_{t_0}^{t} dt' H_I(t')\right)\right\}|i\rangle$ (93)

By iteration we ultimately would be able to determine, for instance when we have scattering, preparing the $|i\rangle$ at $t \to -\infty$ (i.e. the particles were essentially non-interacting $\therefore$ $H_I(t) = 0$ as $t \to -\infty$. After interaction, they scatter, such that at $t \to \infty$, there is some final state, $|f\rangle$. The logical question would be, what is the probability of $|i\rangle$ at $t \to -\infty$ to become $|f\rangle$ at $t \to \infty$, in the presence of interactions?

Since, $|i\rangle = \lim_{t\to-\infty} |\Psi(t)\rangle_I$ <u>and</u> $|f\rangle = \lim_{t\to\infty} |\Psi(t)\rangle_I$, thus using (93) we have:

$$|f\rangle = T\{exp(-i \int_{-\infty}^{\infty} dt' H_I(t'))\}|i\rangle = S|i\rangle \tag{94}$$

Allowing us to deduce, that the information concerning the $|f\rangle$ state is all contained in the S-matrix and is so defined as [46],

$$S = T\left\{exp\left(-i \int_{t_0}^{t} dt' H_I(t')\right)\right\} \tag{95}$$

Discussion

Consider the meson decay process $\left(\phi \to \psi\overline{\psi}\right)$, using the formalism we introduced (86-8) we are now able to compute the quantum amplitude for a ϕ: meson in the $|i\rangle$ state to decay into a $\psi\overline{\psi}$: nucleon-anti-nucleon pair in the $|f\rangle$ state.

Let, $$|i\rangle = \sqrt{2E_{\boldsymbol{p}}}\, a_{\boldsymbol{p}}^{\dagger}|0\rangle \tag{96}$$

$$|f\rangle = \sqrt{4E_{\boldsymbol{q_1}}E_{\boldsymbol{q_2}}}\, b_{\boldsymbol{q_1}}^{\dagger}c_{\boldsymbol{q_2}}^{\dagger}|0\rangle \tag{97}$$

where $\boldsymbol{p}, \boldsymbol{q_1}, \boldsymbol{q_2}$ are the respective momenta the particles have. The $a_{\boldsymbol{p}}^{\dagger}$ term in (96) indicates the creation of a single *meson* from the vacuum state, $|0\rangle$. While, $b_{\boldsymbol{q_1}}^{\dagger}$ and $c_{\boldsymbol{q_2}}^{\dagger}$ represents the creation of a nucleon and anti-nucleon, respectively. The $\sqrt{2E_{\boldsymbol{p}}}$ and $\sqrt{4E_{\boldsymbol{q_1}}E_{\boldsymbol{q_2}}}$ are the relativistic normalization factors for the respective particles. In order to attain the quantum amplitude, we will have to compute the triple scalar product of $\langle f|S|i\rangle$.

First, by replacing the *S* by the unitary evolution operator of (85),

$$S \equiv U(t, t_0) = T\left\{exp\left(-i\int_{t_0}^{t} dt' H_I(t')\right)\right\} \tag{98}$$

As described by the infinite Taylor series expansion of the $U(t, t_0)$ in (76), we can see that for this $\langle f|S|i\rangle$, since the first term is equal to 1, implies that there is no overlap of the $|i\rangle$ state with the $|f\rangle$ state. As for the second term is concerned,

$$-ig\langle f|\int d^4x\psi^{\dagger}(x)\psi(x)\phi(x)\,|i\rangle \tag{99}$$

Although the total Hamiltonian is integrated over all space, but the reason for integrating over d^4x, is due to the fact the S-matrix in consideration, represents a unitary evolution of states from $t \to -\infty$ to $t \to \infty$ and so there is a component integrating with respect to time. There is only a single power of g in (99), but further terms of the $U(t, t_0)$ operator do exist and so there will be other powers of g as well. However, they will be smaller than the g in the second term, and thus we can ignore them. So acting from the right,

$$\Rightarrow -ig\,\langle f|\int d^4x\psi^{\dagger}(x)\psi(x)\int \frac{d^3k}{(2\pi)^3}\frac{\sqrt{2E_p}}{\sqrt{2E_{\boldsymbol{K}}}}a_{\boldsymbol{K}}a_{\boldsymbol{p}}^{\dagger}\exp\left(-ik\cdot x\right)|0\rangle \tag{100}$$

According to (86), the $a_{\boldsymbol{p}}^{\dagger}$ term acts on the $|i\rangle$ state, creating another meson, but there is a 0 overlap with the $|f\rangle$ state. Thus, the only non-zero overlap between the $|i\rangle$ and $|f\rangle$ states f we leave the $a_{\boldsymbol{K}}$ term in the integral.

$$\therefore -ig\langle f|\int d^4x\psi^{\dagger}(x)\psi(x)\exp\left(-ip\cdot x\right)|0\rangle \tag{101}$$

Acting from the left, we attain:

$$\Rightarrow\ -ig\,\langle f|\int d^4x\int \frac{d^3k_1d^3k_2}{(2\pi)^6\sqrt{4E_{\boldsymbol{k_1}}E_{\boldsymbol{k_2}}}}b_{\boldsymbol{q_1}}^{\dagger}c_{\boldsymbol{q_2}}^{\dagger}\exp\left(i[k_1+k_2-p]\cdot x\right)|0\rangle \tag{102}$$

$$\Rightarrow -ig\langle 0|\int d^4x\{\exp\left(i[k_1+k_2-p]\cdot x\right)\}\,|0\rangle$$

$$\therefore\ \langle f|S|i\rangle = -ig(2\pi)^4\delta^{(4)}(q_1+q_2-p) \tag{103}$$

This is the quantum amplitude for a meson decay process. The delta function ensures that there are no violations of any conservation laws and in fact this is just the kinematic delta function (from special relativity) which states that the 4-momentum of the $|f\rangle$ particles should equal to the 4-momentum of the $|i\rangle$ particle, and thus the energy and momentum conservation laws are not violated [45]!

3.6 Wick's theorem.

According to Dyson's formula (85), $U(t,t_0)$ is a unitary operator that leads to the time evolution in the interaction picture, in particular to the time ordered exponential of $H_I(x)$, which is the Schrödinger equation in the interaction Hamiltonian in the interaction picture. It is the Time ordered operator (T) that makes Dyson's formula *formal*! But this makes the computation of integrals explicitly difficult (typically impossible in QFT). The resolution to this problem is that we'd have to compute by Taylor expanding the unitary operator. So, computing:

$$\langle f|T\{H_I(x_1)\dots\dots H_I(x_n)\}|i\rangle \tag{104}$$

Typically, just as with the calculation of the Meson decay in the previous section. Each of the $H_I(x)$ contains fields. By expanding the fields in the mode expansion in terms of creation and annihilation operators and hope that some of the terms *vanish*, as the annihilation operators destroy the $|i\rangle$ state (which was done for the Meson decay case, where there were 8 possibilities, but only one where there was a non-zero overlap as the rest of the $H_I(x)$ annihilated the $|i\rangle$ and $|f\rangle$ states. What makes computing (104) so tedious is the fact that the $c^\dagger$ and $b^\dagger$ from the $H_I(x)$ term would have to pass through all the $H_I(x_1)\dots\dots H_I(x_n)$, picking up the delta functions from the commutations relations between all the operators.

What we require is an algorithm to formalise and show how the ordering of these operators takes place, by recalling the vacuum energy example when we introduced the retarded green's function in section 3.1, and thus consider re-writing the time ordered operators as normal ordered operators. It is here where Wick's theorem captures the technicality and formalism of such a structure.

Recall the notation introduced by Pauli and Heisenberg (41-8) and the time ordered product (51-4), we can establish that the difference between the time ordered product and the normal ordered product when $x^0 > y^0$, is in the commutator $[\phi^+(x)\,,\phi^-(y)]$, such that:

$$\begin{aligned} &\text{For } x^0 > y^0 \quad \rightarrow \quad T\phi(x)\phi(y) = :\phi(x)\phi(y): + \Delta^+(x-y) \\ &\text{For } y^0 > x^0 \quad \rightarrow \quad T\phi(x)\phi(y) = :\phi(x)\phi(y): + \Delta^-(x-y) \end{aligned} \tag{105}$$

(105) is defined as the Feynman propagator, depending on the ordering of x^0 and y^0, expressed in terms of both time and normal ordering:

$$T\phi(x)\phi(y) =: \phi(x)\phi(y): + \Delta(x-y) \tag{106}$$

Its application in Wick's theorem is where the usefulness is exhibited! As from Dyson's formula, evaluation of the evolution of states we attain expressions such as $T\phi(x)\phi(y)$ which are time ordered. By determining what the expression is, we need to move all the annihilation operators to the right and annihilate the $|i\rangle$ state and impose normal ordering upon the operators, $:\phi(x)\phi(y):$ and due to the commutation relation, we pick up $\Delta(x-y)$: the Feynman propagator.

Defining: *Contraction* of a pair of fields in a string of operators,

$$\dots\dots\overbrace{\phi(x_1)\dots\phi(x_2)}\dots\dots \tag{107}$$

to mean replacing the operators with the Feynman propagator, such that $\overbrace{\phi(x_1)\phi(x_2)} = \Delta(x_1 - x_2)$ and as $\Delta(x_1 - x_2)$ is a function, it does not matter where the function is placed, i.e. where the operators have been contracted. The purpose of Wick's theorem is to formalize the calculations in scattering for QFT [46, 50]. So in this report we shall just state a generalization to the 4th order,

$$\begin{aligned}
&T(\phi_1\phi_2\phi_3\phi_4) = \\
&:\phi_1\phi_2\phi_3\phi_4: + :\overbrace{\phi_1\phi_2}\phi_3\phi_4: + :\overbrace{\phi_1\phi_2\phi_3}\phi_4: + :\overbrace{\phi_1\phi_2\phi_3\phi_4}: + :\phi_1\overbrace{\phi_2\phi_3}\phi_4: + :\phi_1\overbrace{\phi_2\phi_3\phi_4}: + \\
&\quad + :\overbrace{\phi_1\phi_2}\,\overbrace{\phi_3\phi_4}: + :\overbrace{\phi_1\,\overbrace{\phi_2\phi_3}\,\phi_4}: + \text{similar terms.}
\end{aligned} \tag{108}$$

where the contraction of two fields ϕ_1 and ϕ_2 is equivalent to the vacuum expectation value of the time-ordered product of ϕ_1 and ϕ_2

$$\overbrace{\phi_1\phi_2} = \langle 0|T(\phi_1\phi_2)|0\rangle \tag{109}$$

Chapter 4: The Dirac Field.

4.1 Spinor representation of the Lorentz group.

So far, we have discussed and developed the formalisms and structure to describe fields for spin 0 particles, bosons. In this chapter, we will put forward the motivation to study spin $\frac{1}{2}$ fields and the manner by which it transforms under Lorentz transformations.

In general, a field can transform in the following way,

$$\phi^a(x) \to D^a{}_b(\Lambda)\phi^b(\Lambda^{-1}x) \tag{110}$$

where a set of fields $\phi^a(x)$ that sums from $a = 1, \dots, n$, under a Lorentz transformation, can transform into each other by means of the matrix $D^a{}_b(\Lambda)$, where Λ denotes what type of Lorentz transformation is being considered, which has its own properties, such that mathematically they are representation of the Lorentz group. Thus, $D^a{}_b(\Lambda)$ is a representation of the Lorentz group, reflecting all the multiplication properties of Lorentz transformations,

$$D(\Lambda_1)D(\Lambda_2) = D(\Lambda_1\Lambda_2) \tag{111}$$

When we considered how the scalar field transformed under a *trivial* (active) Lorentz transformation [44],

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x) \tag{112}$$

The new field $\phi'(x)$ is simply the old field evaluated at a point in space where it was before the active Lorentz transformation $\phi(\Lambda^{-1}x)$ took place. But as we know, fundamental particles are <u>not</u> described by scalar fields, in fact the Higgs particle is the <u>only</u> scalar particle in the Standard Model [53]. Everything we know of that is fundamental has the extra property of intrinsic angular momentum known as spin.

Thus (112) gives rise to spin 0 particles, and so to describe particles with spin, we will have to consider fields that have non-trivial transformations under the Lorentz group. For instance, the gauge field for electromagnetism (EM) [45, 51] is an example for a field with a *non-trivial* Lorentz transformation,

$$A^{\mu}(x) \rightarrow A^{\mu'}(x) = \Lambda^{\mu}{}_{\nu} A^{\nu}(\Lambda^{-1}x) \tag{113}$$

which behaves just as the scalar field with respect to position (x), denoted by $A^{\nu}(\Lambda^{-1}x)$. But, the field could also be rotated (say, the $\boldsymbol{E_x} \rightarrow \boldsymbol{E_y}$), and is denoted by $\Lambda^{\mu}{}_{\nu}$. It should be stated that upon quantization, EM fields comes out with its own interesting and difficult subtleties [45], which does provide an interesting opportunity for future research. We shall now discuss briefly the important components of the Lorentz group, which lays out the foundation when we develop necessary representation for spin $\frac{1}{2}$ fields.

The Lorentz group is an example of a Lie group that is subject to its own Lie algebra. This implies that the group depends on a continuous number of transformations rather than a discrete number. Furthermore, the standard technology to understand all the representations for any Lie group is to look at *the effect of the generator of infinitesimal transformations*. Everything that is needed to know about Lorentz transformations can be summed up by two statements. *Firstly*, all the elements of the Lorentz group can be written as an exponential of some linear combination of the infinitesimal generators,

$$\Lambda = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}\mathrm{M}^{\rho\sigma}\right) \tag{114}$$

where by each of those elements represent a particular Lorentz transformation. The $\Omega_{\rho\sigma}$ represents six numbers, indicating which of the six Lorentz transformations (three boosts and three rotations) is in action, while $\mathrm{M}^{\rho\sigma}$ is a six (4x4) matrix linear combination representing the mathematical formalism for those transformations. Both $\Omega_{\rho\sigma}$ and $\mathrm{M}^{\rho\sigma}$ are labelled by a pair of anti-symmetric indices where $\rho, \sigma = 0,1,2,3$. *Secondly*, the entire structure of the Lorentz group is based on how the Lorentz transformations multiply together; this is elegantly encoded in the following commutation relation,

$$[\mathrm{M}^{\rho\sigma}, \mathrm{M}^{\tau\nu}] = \eta^{\sigma\tau}\mathrm{M}^{\rho\nu} - \eta^{\rho\tau}\mathrm{M}^{\sigma\nu} + \eta^{\rho\nu}\mathrm{M}^{\sigma\tau} - \eta^{\sigma\nu}\mathrm{M}^{\rho\tau} \tag{115}$$

The motivation for the requirement of a new representation of the Lorentz group is realised when we considered arbitrary fields that were described by (110) being able to consistently transform in that manner. But perhaps there is some way, the internal indices of ϕ^{a} being able to transform into each other in a *consistent* manner under Lorentz transformation. In other words, if there is a μ index, representing a vector that rotates, *consistency* implies $D^{a}{}_{b}(\Lambda)$ obeys the same product structure of the Lorentz group (111). Furthermore with regard to the commutation relations (115), they must be classified according to the spin of the particle and so for the remainder of this chapter not only will we demonstrate that there is a necessity for having separate representations for both spin 0 and spin $\frac{1}{2}$ fields, but especially when quantizing the Dirac field using the canonical commutation quantization approach for *scalar fields*, leads to an incorrect theory [45].

Clifford algebra

The Spinor representation is an example of another type of representation of the Lorentz group, obeying the same rules as stated (114) and (115), defined by the anti-commutation relations,

$$\{\gamma^{\mu}, \gamma^{\nu}\} = \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\eta^{\mu\nu}1 \tag{116}$$

where $\eta^{\mu\nu}$ signifies a coefficient of $+1$ or -1 depending on the indices of μ and ν, and since the γ^{μ} and γ^{ν} matrices are fixed it is convention to put the unit matrix, 1, in the above expression. Moreover, these $\gamma^{\mu,\nu}$ matricies obey the relations,

$$(\gamma^0)^2 = 1 \qquad \underline{\text{and}} \qquad (\gamma^i)^2 = -1 \qquad \text{where } i = 1,2,3 \tag{117}$$

The relations (116) and (117) are known as the Clifford algebra. We should note that the indices for Clifford algebra could be defined in any dimension. For instance, although here we are operating $\mu, \nu = 0,1,2,3$ but should we be considering some d-dimensional space, the (116) relation should still stand, regardless if we were operating in some Euclidean signature or different signature of space-time. In fact, although the solutions are well known, the representation of Clifford algebra is very interesting when implemented in higher dimensions, leading to innovative mathematical formalisms such as Hopf algebra in String theory and Supersymmetry [62-64], an exciting area for future research.

The simplest representation obeying Clifford algebra, is in terms of the (4x4) matricies,

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{118}$$

known as the Chiral representation, where σ^i are the Pauli matricies. The relation with the Lorentz group is established when we consider the matrices denoted by,

$$S^{\rho\sigma} = \frac{1}{4}[\gamma^{\rho}, \gamma^{\sigma}] \tag{119}$$

an analogous expression to that of $\mathrm{M}^{\rho\sigma}$ in (114). This is quite interesting as we note the anti-commutation relations obeyed by the Chiral representation but in (119) we investigate the commutation relations of the γ matricies. The six anti-symmetric matrices (119) have the special property of,

$$[S^{\mu\nu}, S^{\rho\sigma}] = S^{\mu\sigma}\eta^{\nu\rho} - S^{\nu\sigma}\eta^{\rho\mu} + S^{\rho\mu}\eta^{\nu\sigma} - S^{\rho\nu}\eta^{\sigma\mu} \tag{120}$$

which is the exact same commutation relation as expressed in (115). Thus, allowing us to conclude that $S^{\rho\sigma}$ does provide a representation of the Lorentz lie algebra. We now define the complex valued object known as the Dirac spinor, $\psi^a(x)$, such that under Lorentz transformation,

$$\psi^a(x) \to S(\Lambda)^a{}_b \psi^b(\Lambda^{-1}x) \tag{121}$$

where Λ is the some transformation defined in (114) and so we can define $S(\Lambda)$ as being,

$$S(\Lambda) = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right) \tag{122}$$

where $S^{\rho\sigma}$ is the *new* infinitesimal generator of the Lorentz group and the $S(\Lambda)$ matricies in turn form a representation of the Lorentz group obeying the same mathematical structure as defined by (111). We have further shown in [16] that the two representations, $S(\Lambda)$ and Λ, have different implications in the context of rotations and boosts, and are not just a re-statement of each other.

4.2 The Dirac Equation.

When constructing the action, it follows the same formalism as the one in for the scalar field as discussed in [44], it has to be *real* and it must be a *Lorentz invariant quantity*. In particular, we are interested in equations of motion covariant under Lorentz transformations for which the Dirac spinor obeys as $\psi^{a}(x)$ themselves transform interestingly under Lorentz transformations, as shown by Tong [45]. So, we can construct a Lorentz scalar by introducing the *Dirac conjugate*,

$$\bar{\psi}(x) = \psi^{\dagger}(x)\gamma^{0} \tag{123}$$

such that we have a Lorentz invariant quantity, $\bar{\psi}(x)\psi(x)$, under a Lorentz transformation, $\bar{\psi}(x)\psi(x) \rightarrow \bar{\psi}(\Lambda^{-1}x)\psi(\Lambda^{-1}x)$ only the arguments for (x) changes not the spinor indices. Similarly, a Lorentz vector can be constructed $\bar{\psi}\gamma^{\mu}\psi$, such that when acted on by a Lorentz transformation, γ^{μ} transforms like a vector field,

$$\bar{\psi}(x)\gamma^{\mu}\psi(x) \rightarrow \Lambda^{\mu}{}_{\nu}\bar{\psi}(\Lambda^{-1}x)\gamma^{\mu}\psi(\Lambda^{-1}x) \tag{124}$$

After the construction of the two Lorentz invariant quantities comprised of the Dirac spinors and considering the Spinor representation of the Lorentz group, $S(\Lambda)$, the following action is invariant under Lorentz transformation,

$$S = \int d^{4}x\, \bar{\psi}(x)\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi(x) \tag{125}$$

attained via the contraction of the vector γ^{μ} with the Lorentz invariant quantity, ∂_{μ}. The corresponding equation of motion that is attained by considering $\psi(x)$ and $\bar{\psi}(x)$ to be independent of each other given that $\psi(x)$ is a complex field and then varying them with respect to one another,

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi = 0 \tag{126}$$

This is a matrix equation, where γ^{μ} is a (4x4) matrix, the mass: m comes with an implicit (4x4) unit matrix and the Dirac spinor being a four component complex column vector.

This matrix equation is known as the Dirac equation and according to Tong [45], "This is the most beautifully, amazing equation that was ever written down in the history of human civilization." This statement is substantiated by the massive implications it has had for physics, for instance Dirac immediately predicting that the magnetic moment of the electron was valued at 2. He then showed how this equation predicted the existence of anti-mater but most importantly, this equation is the heart and soul of the Standard Model, simply because all the fermions that we know of (quarks, leptons) are all described by the Dirac equation (or variances of it).

Mathematicians in the 1960s, using the Dirac equation, revolutionized areas of Mathematics. For example, Atiyah and Singer [65-70] developed the *Index theorem* using Dirac's formalisms. In fact, most of theoretical physics is built upon the un-told consequences of the $\bar{\psi}(x)\gamma^{\mu}\psi(x)$ Lorentz vector; in particular the existence of fermions is due to consequences of this quantity, which in turn leads to various phenomena in condensed matter physics, biology etc, whilst bearing in mind the abstract nature of the γ^{μ} matricies.

Although, we briefly mentioned the impact of the Dirac equation in Physical sciences, the most surprising fact about this equation is that such an equation exists at all! Dirac managed to write an equation that is *linear* in derivatives but is nonetheless *Lorentz invariant*. The key physics behind this equation lies in the properties of the γ^{μ} matrix.

To show the brilliance behind the equation, let ψ be an arbitrary scalar field (without any indices) and if we were to construct an equation of motion that was *linear in derivatives* and *Lorentz invariant*, then there would have to be a ∂_μ term in the equation, but there would have to be a term that also had a μ-index. However there is nothing natural with a μ-index! So, if we were to consider a four vector with a μ-index that points somewhere in the universe, the resulting equation of motion would not be Lorentz invariant, it would be depending on the 'thing' that the four vector was pointing at. Hence, in some sense this was Dirac's greatest discovery, in the context of natural vectors floating around in the universe, the abstract nature of the γ^μ matricies in (126) rotate *naturally* like a vector under Lorentz transformations.

As far as the association with the KG theory is concerned, which was discussed in the early chapters of this report, the $\psi(x)$ is in fact a solution to the KG equation. However, it should be stressed that the Lorentz invariance of the Dirac equation does not arise due to the relation it has with the KG theory but due to the properties of the γ^μ matricies.

Finally, with regard to dimensionality, when working in which ever dimension a theory could be thought of to be operational, the γ matricies must still satisfy the (116) relations. In particular, due to the nature of the anti-commutation relations, they show very interesting properties when operating in ten dimensions according to [63, 64] plays a key role in the mathematical consistency of Superstring theory, this certainly points to very exciting area of further research: investigating Dirac formalisms in higher dimensions!

4.3 The Weyl spinor and Plane wave solutions.

Before discussing the plane wave solutions to the Dirac Field, we would briefly like to introduce the concept of the *Weyl spinor* and its respective equation of motion. This will form a integral part in the final section of the Dirac field, when we discuss quantising the field. Picking up from where we left off in section 4.1, we proved that the *original* and *Spinor* representations of the Lorentz group have different consequences, in the examples of rotations and boosts, by using the Chiral representation (118). The Dirac spinor as we noted earlier, is a four component complex valued field. It can be represented as,

$$\psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \tag{127}$$

where $u_\pm$ are called Weyl spinors. Each is a two component complex spinor and are the constituents of the Dirac spinor. The Chiral representation are an irreducible representation of the Clifford algebra, but from the γ matricies, we were able to construct a representation of the Lorentz group that turned out to be a reducible one, the $S(\Lambda)$ when acted upon by a rotation or a boost we show that,

$$S(\Lambda_{rot}) = \begin{pmatrix} exp\left(i\boldsymbol{\varphi}\cdot \boldsymbol{\sigma}/_2\right) & 0 \\ 0 & exp\left(i\boldsymbol{\varphi}\cdot \boldsymbol{\sigma}/_2\right) \end{pmatrix} \quad \text{and} \quad S(\Lambda_{boost}) = \begin{pmatrix} exp\left(\boldsymbol{\kappa}\cdot \boldsymbol{\sigma}/_2\right) & 0 \\ 0 & exp\left(-\boldsymbol{\kappa}\cdot \boldsymbol{\sigma}/_2\right) \end{pmatrix} \tag{128}$$

as the spinor representations were block diagonal, this implied that an irreducible representation of the Lorentz group was <u>not</u> attained. So relating to the Weyl spinors we can establish,

$$\begin{aligned} \text{Under rotations:} \quad & u_\pm \to exp\left(i\boldsymbol{\varphi}\cdot \boldsymbol{\sigma}/_{\mathbf{2}}\right)u_\pm \\ \text{Under boosts:} \quad & u_\pm \to exp\left(\pm\boldsymbol{\kappa}\cdot \boldsymbol{\sigma}/_{\mathbf{2}}\right)u_\pm \end{aligned} \tag{129}$$

If we consider the Dirac action (126) and express it in terms of the $u_\pm$ we attain,

$$\bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi = \begin{pmatrix} u_+^\dagger & u_-^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left[i \begin{pmatrix} 0 & \partial_0 + \sigma^i \partial_i \\ \partial_0 - \sigma^i \partial_i & 0 \end{pmatrix} - m1 \right] \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$$

$$= iu_-^\dagger \sigma^\mu \partial_\mu u_- + iu_+^\dagger \bar{\sigma}^\mu \partial_\mu u_+ - m\left(u_+^\dagger u_- + u_-^\dagger u_+\right) \tag{130}$$

noting the difference in the indices for the Pauli matricies, where $\mu = 0,1,2,3$ as opposed to the i index in (118) where $i = 1,2,3$. At the first insight we can see how the Dirac spinor is <u>more</u> useful than the Weyl spinor. This is because the Weyl spinor is a (2x2) object, that have *separate* kinetic terms for both u_- and u_+, which are the first two terms in the (130) expression. But if we are dealing with a massive fermion with mass m (the third term in the above expression), we need the expressions for both u_+ and u_- due to the fact they couple with each other. However, for a massless fermion that would arise from a massless spinor field, knowing a <u>single</u> Weyl spinor would be suffice. Each of the u_+ and u_- is described by the respective Weyl equations of motion,

$$\begin{aligned} &\text{For } u_+ \quad \rightarrow \quad i\bar{\sigma}^\mu \partial_\mu u_+ = 0 \\ &\text{For } u_- \quad \rightarrow \quad i\sigma^\mu \partial_\mu u_- = 0 \end{aligned} \tag{131}$$

We shall later find out that upon quantization of the Dirac field, would give rise to (massive) fermions described by a Dirac spinor. Regarding massless fermions, the Dirac spinor decomposes into the Weyl spinor ($u_+ \; or \; u_-$) and at the free level, they <u>do not</u> interact with each other. This suggests that particles in nature are described by one of these objects <u>but</u> not the other. In fact, <u>**all**</u> particles in the SM are described by the Weyl equation of motion (131) and <u>not</u> the Dirac spinor. We have reached a paradox, since none of the particles, including neutrinos, are massless! And so how can they be described by Weyl spinors?

All particles that are described by the Lagrangian within the SM are massless! There are never any mass terms, $m\left(u_+^\dagger u_- + u_-^\dagger u_+\right)$ of this type. The particles within the SM only get their mass through the Higgs mechanism and because of the weak force, all the fermions within the SM are described by the Weyl spinors. In fact for all fermions, they come in <u>pairs</u> as described by the unique properties of the two Weyl equations of motion (131).

Physically, when we discuss the quantization section, we should keep in mind that there will be particles and anti-particles that arise as a result of quantization, furthermore these particles will also have 'extra' internal degrees of freedom which we call as *spin* and there is the possibility for a particle to have spin-up and spin-down. So, overall a much more complex structure to that of scalar fields.

<u>Plane wave solution</u>

We end this section by briefly discussing the simple solutions to the Dirac equation. Just as we had plane wave solutions for the KG equation, the Dirac equation in the Chiral representation has two sets of solutions: *positive* and *negative* frequency solutions.

→ *Positive frequency solutions* are defined by,

$$\psi = u_{\boldsymbol{k}} exp(-ik \cdot x) \tag{132}$$

where $u_{\boldsymbol{k}}$ signifies a particular four spinor being fixed according to the solution that is found in the form of $exp(-ik \cdot x)$. Now according to this anzatz, $u_{\boldsymbol{k}}$ is represented via,

$$u_{\boldsymbol{k}} = \begin{pmatrix} \sqrt{k \cdot \sigma^{\mu}} & \xi \\ \sqrt{k \cdot \sigma^{-\mu}} & \xi \end{pmatrix} \tag{133}$$

such that the ξ is an arbitrary two component complex spinor and k is the 4-momentum that is particular to the anzatz, such that $\boldsymbol{k}^0 = +\sqrt{\boldsymbol{k}^2 + m^2}$, which will turn out to be the 4-momentum upon quantization. The dot product $k \cdot \sigma^{\mu}$ shows that the 4-momentum is coupled with the σ^{μ}, such that $\boldsymbol{k}^0$ couples with σ^0(: unit matrix); and $\boldsymbol{k}^i$ couples with σ^i(: Pauli matricies).

It will be useful to introduce a basis of two component spinors (ξ), in particular when determining the spin of the fermions as a result from quantizing single particle states. The basis for these ξ spinors is one such that, ${\xi^r}^{\dagger}\xi^s = \delta^{rs}$, where $\xi^{s=1,2}$ where *s* does not signify the spinor index, but just indicating which spinor index is taken into account.

→ *Negative frequency solutions* are defined by,

$$\psi = v_{\boldsymbol{k}} exp(ik \cdot x) \tag{134}$$

such that the k for this case is the same as for the positive frequency solution, except the anzatz for this case is one that is represented via,

$$v_{\boldsymbol{k}} = \begin{pmatrix} \sqrt{k \cdot \sigma} & \eta \\ -\sqrt{k \cdot \overline{\sigma}} & \eta \end{pmatrix} \tag{135}$$

It so happens that both the positive and negative solutions to the Dirac equation are also solutions to the classical field theory [1] and have positive energy that is calculated from the stress-energy tensor $T^{\mu\nu}$ in particular T^{00} where we have $E = \int d^3x\, T^{00}$ [45, 46].

4.4 Quantizing the Dirac field.

So far we have formulated a Lagrangian and hence an equation of motion that captures the dynamics the Dirac field but we are yet to establish how the excitations of this field give rise to fermions. This is achieved through the quantization procedure. Recall, upon quantizing the scalar field gave rise to bosons [44, Chapter 2]. Naively, we could simply implement the exact same canonical method, which should give rise to the elementary particles attained from the Dirac field. However, we will briefly show why it is wrong to do so and conclude this chapter by illustrating that due to the subtle consequences of spin statistics in QFT, by imposing anti-commutation relations we will be able to get a we defined theory and quantize the field in order to attain fermions.

Imposing canonical commutation relations: 'the *wrong* approach'

Considering the action (125) for the classical theory and simply considering the conjugate momenta of ψ as in (13),

$$\pi = i\psi^{\dagger} \tag{136}$$

However, we can immediately see how the Dirac formalism differs from the scalar field theory, where π is not proportional to a velocity component due to the fact the Dirac equation (126) is a *first* order equation (as opposed to the KG equation (6) which is a *second* order equation). Since the Dirac field is a function of space and time and the momentum which is also a function of space and time, if we impose the *canonical commutation relations* between position and momentum, just as we did in the scalar theory (2) operating in the Schrödinger picture,

$$\left[\psi_\alpha(\boldsymbol{x}), \psi_\beta(\boldsymbol{y})\right] = \left[{\psi^\dagger}_\alpha(\boldsymbol{x}), {\psi^\dagger}_\beta(\boldsymbol{y})\right] = 0 \quad \underline{\text{and}} \quad \left[\psi_\alpha(\boldsymbol{x}), {\psi^\dagger}_\beta(\boldsymbol{y})\right] = \delta_{\alpha\beta}(\boldsymbol{x}-\boldsymbol{y}) \tag{137}$$

Recall, when we did a mode expansion for the $\phi(\boldsymbol{x}, t)$ and $\pi(\boldsymbol{x}, t)$ fields in the scalar theory (12, 14) we expressed it in terms of *creation* and *annihilation* operators. For the mode expansion of the Dirac field, we express it in terms of the spinors,

$$\psi(\boldsymbol{x}) = \sum_{s=1}^{2} \int \frac{d^3k}{(2\pi)^3 2\omega} \left[b_{\boldsymbol{k}}^s u_{\boldsymbol{k}}^s exp(ik \cdot x) + {c_{\boldsymbol{k}}^s}^\dagger v_{\boldsymbol{k}}^s exp(-ik \cdot x)\right] \tag{138}$$

$$\psi^\dagger(\boldsymbol{x}) = \sum_{s=1}^{2} \int \frac{d^3k}{(2\pi)^3 2\omega} \left[{b_{\boldsymbol{k}}^s}^\dagger u_{\boldsymbol{k}}^s exp(-ik \cdot x) + c_{\boldsymbol{k}}^s v_{\boldsymbol{k}}^s exp(ik \cdot x)\right] \tag{139}$$

where the $\psi(\boldsymbol{x})$ and $\psi^\dagger(\boldsymbol{x})$ are spinor operators, with the index α is suppressed. Both $u_{\boldsymbol{k}}^s$ and $v_{\boldsymbol{k}}^s$ are the spinors (133, 135) that soak up the spinor index in the $\psi(\boldsymbol{x})$ and $\psi^\dagger(\boldsymbol{x})$. The s index, represents the basis for the two component object ξ^s and η^s which are $\begin{pmatrix}0\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\end{pmatrix}$, for the $u_{\boldsymbol{k}}^s$ and $v_{\boldsymbol{k}}^s$ spinors respectively.

Just as in the scalar field, the $\left(b_{\boldsymbol{k}}^s, {b_{\boldsymbol{k}}^s}^\dagger\right)$ and $\left(c_{\boldsymbol{k}}^s, {c_{\boldsymbol{k}}^s}^\dagger\right)$ represents the annihilation and creation operators for particles in the Fock space. As we stated that there are two operators within each of the four operators described for the $\psi(\boldsymbol{x})$ and $\psi^\dagger(\boldsymbol{x})$ fields. Taking the ${c_{\boldsymbol{k}}^s}^\dagger$ for instance, this operator has two creation operator within itself so as to create a spin-up and spin-down particle.

Just as with the scalar field, there are commutation relations for the $\psi(\boldsymbol{x})$ and $\psi^\dagger(\boldsymbol{x})$ which in turn implies commutation relations between the four operators within the two fields. Thus, we can establish similar field commutation relations as in (137) which essentially boils down to,

$$\left[b_{\boldsymbol{k_1}}^r, {b_{\boldsymbol{k_2}}^s}^\dagger\right] = (2\pi)^3 \delta^{rs} \delta(\boldsymbol{k_1} - \boldsymbol{k_2}) \quad \text{and} \quad \left[c_{\boldsymbol{k_1}}^r, {c_{\boldsymbol{k_2}}^s}^\dagger\right] = -(2\pi)^3 \delta^{rs} \delta(\boldsymbol{k_1} - \boldsymbol{k_2}) \tag{140}$$

From the definition of the Hamiltonian (8) we can establish a similar expression for the Dirac field, such that,

$$\text{Ƕ} = \pi\dot{\psi} - L = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi \tag{141}$$

The time ordering disappears completely from Ƕ due to the fact the Dirac formalism is a first order structure. By implementing the definition of (7), we turn the Hamiltonian into an operator and after normal ordering, we attain the expression,

$$H = \sum_{s=1}^{2} \int \frac{d^3k}{(2\pi)^3} \omega \cdot \left({b_{\boldsymbol{k}}^s}^\dagger b_{\boldsymbol{k}}^s - {c_{\boldsymbol{k}}^s}^\dagger c_{\boldsymbol{k}}^s\right) \tag{142}$$

which illustrates why imposing the canonical commutation relations is the incorrect method to quantize the Dirac field. It all lies in the minus sign, due to the fact the theory has no stable ground state. The vacuum state is annihilated by $b_{\boldsymbol{k}}^s$ and $c_{\boldsymbol{k}}^s$, but when acted on by ${c_{\boldsymbol{k}}^s}^\dagger$ we attain a state with lower energy! The theory <u>does not</u> make any sense. In fact, we cannot make sense of the spinor fields being quantized this way by imposing commutation relations (137). Here we get the first consequence due to the spin statistics in QFT, as the $\psi(\boldsymbol{x})$ and $\psi^\dagger(\boldsymbol{x})$ must be treated as fermionic (spinor) fields as opposed to bosonic fields, $\phi(\boldsymbol{x})$ and $\pi(\boldsymbol{x})$, in order to attain sensible solutions in the form of having energies that are bounded below.

Imposing anti-commutation relations: 'the *right* approach'

Instead of having the commutation relations (137), the spinor fields satisfies the *anti*-commutation relations of,

$$\{\psi_\alpha(\boldsymbol{x}), \psi_\beta(\boldsymbol{y})\} = \{\psi^\dagger{}_\alpha(\boldsymbol{x}), \psi^\dagger{}_\beta(\boldsymbol{y})\} = 0 \quad \underline{\text{and}} \quad \{\psi_\alpha(\boldsymbol{x}), \psi^\dagger{}_\beta(\boldsymbol{y})\} = \delta_{\alpha\beta}(\boldsymbol{x}-\boldsymbol{y}) \tag{143}$$

Using the same mode expansions (138, 139) but this time implementing the anti-commutation relations of (143) we attain the relationship for the creation and annihilation operators,

$$\left\{b^r_{\boldsymbol{k_1}}, b^s_{\boldsymbol{k_2}}{}^\dagger\right\} = (2\pi)^3\delta^{rs}\delta(\boldsymbol{k_1}-\boldsymbol{k_2}) \quad \text{and} \quad \left\{c^r_{\boldsymbol{k_1}}, c^s_{\boldsymbol{k_2}}{}^\dagger\right\} = (2\pi)^3\delta^{rs}\delta(\boldsymbol{k_1}-\boldsymbol{k_2}) \tag{144}$$

establishing the crucial relationship without any minus signs floating around as we saw in (140). Applying the same procedure as before, after normal ordering we can compute the Hamiltonian, similar to that of (142), except with the much desired amendment of,

$$H = \sum_{s=1}^{2}\int\frac{d^3k}{(2\pi)^3}\omega\cdot\left[b^{s\dagger}_{\boldsymbol{k}}b^s_{\boldsymbol{k}} + c^{s\dagger}_{\boldsymbol{k}}c^s_{\boldsymbol{k}} - (2\pi)^3\delta(0)\right] \tag{142}$$

The infinity that comes with this H (142) is with a minus sign as opposed the infinity for the bosonic Hamiltonian which comes with a plus sign. With regard to free theories, this can always be normal ordered, and so not really relevant in the end. As an aside, had this Hamiltonian been coupled with gravity in the bosonic case, the infinity term, should contribute to the cosmological constant [45], resulting to a much larger value than what we see. Earlier when we mentioned Supersymmetry being an interesting extension for future research with regard to the Dirac formalism, dealing with these infinities also plays a key role. The contribution from bosons $[+(2\pi)^3\delta(0)]$ cancels the contribution from fermions $[-(2\pi)^3\delta(0)]$, this essentially leads to founding ideas behind Supersymmetry: exact cancellations occur due to the contributions from bosons and fermions due to the existence of equal numbers of both fundamental particles and their related interactions.

We now formally highlight the morals within the spin statistics in QFT. There is no choice with regard to which statistics is applied to the spin of a particle. As we have seen earlier in this subsection, if the quantization of spinors is done the same manner as spin 0 particles, it led to an inconsistent theory. The resolution to this problem is to quantize integer spin objects using bosonic statistics and quantizing objects with half integer spin using fermionic statistics.

With regard to the Fock space, the vacuum is defined as, $b^s_{\boldsymbol{k}}|0\rangle = c^s_{\boldsymbol{k}}|0\rangle = 0$ for all $\boldsymbol{k}$ and $s = 1, 2$. Bearing in mind that, each of the annihilation operators has two further operators within them representing spin-up and spin-down particles. For the single particle states, $b^{s\dagger}_{\boldsymbol{k}}|0\rangle$ and $c^{s\dagger}_{\boldsymbol{k}}|0\rangle$, where the creation operator creates particle and anti-particle states respectively, also taking into account the additional consequences due to the spinor index, s. To gain some perspective regarding the Dirac Hamiltonian and the respective energy eigenvalues, due to the creation and annihilation operators (stated above), we can exploit the fact that the Hamiltonian is typically an *even* operator and so are the creation operators, while the annihilation operators are odd. Hence, we can establish the commutation relations to reveal the energy spectrum,

$$[H, b_{\boldsymbol{k}}^s] = -E_{\boldsymbol{k}} b_{\boldsymbol{k}}^s \quad \left[H, b_{\boldsymbol{k}}^{s\dagger}\right] = E_{\boldsymbol{k}} b_{\boldsymbol{k}}^{s\dagger}$$
$$[H, c_{\boldsymbol{k}}^s] = -E_{\boldsymbol{k}} c_{\boldsymbol{k}}^s \quad \left[H, c_{\boldsymbol{k}}^{s\dagger}\right] = E_{\boldsymbol{k}} c_{\boldsymbol{k}}^{s\dagger} \tag{143}$$

Although the s indicates the spin of the particle, but regard to its projection, that information is stored in the mode expansions (138, 139) of $\psi(\boldsymbol{x})$ and $\psi^\dagger(\boldsymbol{x})$, in particular with the nature of respective spinors, $u_{\boldsymbol{k}}^s$ and $v_{\boldsymbol{k}}^s$. If we consider for the $u_{\boldsymbol{k}}^s$ spinor, say a basis for $\xi^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\xi^2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, as we introduced earlier, if ξ^1 and ξ^2 are eigenstates of the third Pauli matrix (133) then as $s = 1, 2$, this would correspond to the spin-up and spin-down particles being projected in the z-direction only. Finally, we conclude this chapter, by stating that not only do excitations of the Dirac field gives rise to fermions, but also the unique set of statistics they obey known as Fermi-Dirac statistics, through the construction of a two particle state, highlighting the *anti-symmetric* nature of $b_{\boldsymbol{k_1}}^{s_1\dagger}$ and $b_{\boldsymbol{k_2}}^{s_2\dagger}$ operators [45, 46, 49, 50],

$$b_{\boldsymbol{k_1}}^{s_1\dagger} b_{\boldsymbol{k_2}}^{s_2\dagger} |0\rangle = |\boldsymbol{k_1}, s_1;\ \boldsymbol{k_2}, s_2\rangle = -|\boldsymbol{k_2}, s_2;\ \boldsymbol{k_1}, s_1\rangle \tag{144}$$

Chapter 5: Cosmology.

5.1 Why inflation?

Since its conception, the standard version of the hot universe model [71, 72] (or the Big Bang model) has been successful in describing the universe we live in today. For instance, as Liddle [43] and Watson [42] highlights, the existence of Cosmic Background Radiation (CBR); the fact that there are copious amount of elements from the lighter side in the periodic table (such as trace quantities of Lithium and Beryllium, $\sim 25\%$ Helium and $\sim 75\%$ Hydrogen); taking into account the fact the phenomenon known as nucleosynthesis (variations such as Big Bang nucleosynthesis and Cosmic ray spallation) takes place and the fact that the universe in continually expanding, in accordance with the Hubble Law. However, just as with the SM of elementary particles being a beautiful rigid scheme of all the particles that we know of today and as Wilczek [16] points out, is marvellously precise with what it attempts to describe, the model still has mysteries within it that we wish to solve and understand.

Analogous to the situation the SM is in, within the Big Bang model (BBM) are anomalies that it fails to describe, for instance the *Horizon problem* [43] which questions the validity for the universe to be homogeneous and isotropic for large scales. Keeping the concepts of General relativity in mind, photons not being able to convey their information (specifically with regard to their temperature) from one side of the universe to the other is inconsistent with the data received by the Wilkinson Microwave Anisotropy Probe (WMAP) and the Cosmic Background Explorer (COBE), which states that radiation from opposite sides of the universe appear to be uniform in temperature. A contrasting problem, is the *large-scale structure of the universe*, which states that while the Big Bang model does not predict inhomogeneity, we can see that galaxies are shown to be clustering locally. Liddle [43] and Watson [42] called this the *fine tuning problem*, where not only did BBM predict the formation of homogeneous structures within the universe but when going right back to the point of inception, the 'explosion' that occurred (as cosmologists call it, the moment when we consider the beginning of time) had to take place in just the right way so as to avoid collapse. We can clearly see that despite the many successes (mentioned in the beginning) the BBM has; it is rigid when we consider the constraints that the model is subject to.

Finally, we mention the *Monopole problem* described by Peacock [73]. We will be discussing Guth's [33] original model for inflation in the next section, but as a consequence to his amazing intuition to implement particle theory as a means of describing the early universe evolution, his theory suggests the creation of topological defects. Subject to dimensionality, there are many variations of such defects occurring, but (quite literally) as the nature of the problem suggests, we will be considering point like defects known as monopoles. These are field points that point radially outward from the defects, where each of these defects has their own unique mass. Unfortunately, the unusual situation here is the fact there has still up to this date, been no observation of an actual magnetic monopole. These set of problems that we have discussed are collectively known as the initial value problems for the BBM [41, 43].

As Bergström and Goobar [58] points out, the anomalies summarized by Liddle [43] and Watson [42], for the (hot) BBM presented the opportunity for cosmologists to investigate and think of new approaches to tackle these mysteries. Eventually, as we will discuss in the following section, the concept of inflation was successful (and in some cases, trivially as shown more quantitatively by Watson [42] and Liddle [43]) in tackling those issues.

5.2 The Inflationary scenario.

The concept of the inflation paradigm being referred to as a *scenario* as opposed to being just 'a model' simply stems from the fact that over the past few years, there has been a massive interest in the realm of the early universe, which ultimately resulted to the suggestion of so many models, each with their own subtleties and unique parameterization. Despite the luxury of the many approaches to tackle the mysteries within the BBM, the common feature that is shared throughout is a period of rapid expansion subject to the dynamics and evolution of the scale factor, $a(t)$. It is this rapid phase of evolution of the universe, as Watson [42] simply puts it, where $a(t) \sim t^n$ for $n > 1$ is what is known as inflation.

The first attempt at introducing the concept of inflation was done through the DeSitter model [58]. However the major flaw as Linde [41] pointed out was the fact if the evolution of the universe under this model was taken into account, under its parameters, the resulting universe would be one that is empty with essentially no radiation or matter. Furthermore, since the DeSitter model was based on Einstein's equations, it turns out that there was no connection between particle theory and cosmology, which was the one of the primary reasons to investigate the early expansion of the universe, this ultimately resulted in the DeSitter model being rejected and forgotten [42].

We will now give an overview of the three models that has been most significant especially in the context of formulating a link between cosmology and elementary particle theory.

5.2.1 Original Toy model for inflation (Guth [33])

Guth's [33] premise behind the (original) toy model for inflation was to allow only a *limited* amount of time for rapid expansion of the universe (unlike the DeSitter Model), which was represented in the context of an inflaton field, ϕ (which is a scalar field that generates the accelerated expansion of space). The motivation behind modelling the space as a *field* will be discussed in the final section of this Chapter, which finally links the importance of understanding the dynamics of field theory, which is thoroughly discussed throughout the bulk of this report (Chapters 2-4).

Guth was initially trying to solve the initial value problems of BBM, in the context of a class of GUTs (namely the SU(5) symmetry group), which then resulted to an abundance in topological defects. In particular, a large number of monopole creation in SU(5) GUT.

Now, according to his model, Guth proposed the ϕ-field, was situated at a local minimum, trapped within the false vacuum state. Recall, that unlike DeSitter's Model (which was based upon Einstein's Field equations) Guth wanted to explain the rapid phase of evolution of the inflaton field in the context of particle physics. So, this vacuum state of the ϕ-field, simply represented, the lowest energy state that is available to the system. This analogous to the SHO problem we discussed in Chapter 2 and [44], where the vacuum state represented the minimum energy $\left(\frac{1}{2}\hbar\omega\right)$that is available for the harmonic oscillator system.

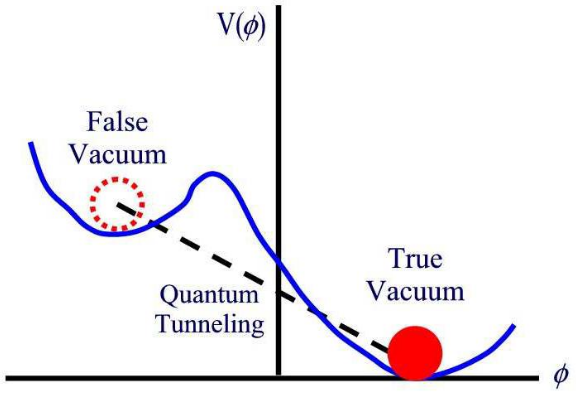

Fig. 2. Alan Guth's original Toy model for inflation [42]

With the ϕ-field being in the lowest available energy state, within the false vacuum, the only means of escape is via the quantum tunnelling effect, after a given period of time. This quantum phenomenon is only possible due to the fact that the universe is in a phase of expansion. As a consequence of the universe inflating, the ϕ-field releases *latent heat* that is trapped within. As shown in Fig. 2, when the inflaton field was at the false vacuum state, the respective potential for the field is much higher compared to when the ϕ-field is in the (lower and more favourable) true vacuum state, thus showing the physical manifestation of the release of this latent heat.

Guth deemed that this quantum tunnelling effect that is responsible for the latent heat to be released as a first order phase transition, to be consistent in explaining the physical mechanism behind such an inflationary period of the early universe. However, in his zeal to use particle physics as his basis for explanation, the release of latent heat corresponded as expanding domains in the inflation model, which also expanded at an exponential rate. The formation of more expanding domains eventually collided with each other and the more frequent this process took place it was conclusive that this was the phenomenon responsible for the formation of the large number of topological defects (which was the large number of monopoles for the SU(5) GUT). The success for Guth's model was in a precarious position, if the expanding domains continued to inflate at a very rapid pace which would result to an observable universe that was described by the DeSitter Model (one that would be empty and lacking in any form of structure). This was known as the *graceful exit problem* [42, 43]

5.2.2 'New' Toy model for inflation and the (general) Chaotic model of inflation (Linde [41])

Linde's [41] attempt to solve the *graceful exit problem* was to change some of the initial assumptions that Guth implemented for his 'original' Toy model. Instead of an explosive phase of expansion of the early universe, Linde assumed that the ϕ-field evolved *very slowly* from its initial state and as a result, as opposed to a first order phase transition taking place, the slow nature of the evolution of the inflaton field warranted a *second order phase transition*. Linde [41] insisted that the slow rate of expansion nevertheless provided sufficient inflation that was needed to solve the initial value problem.

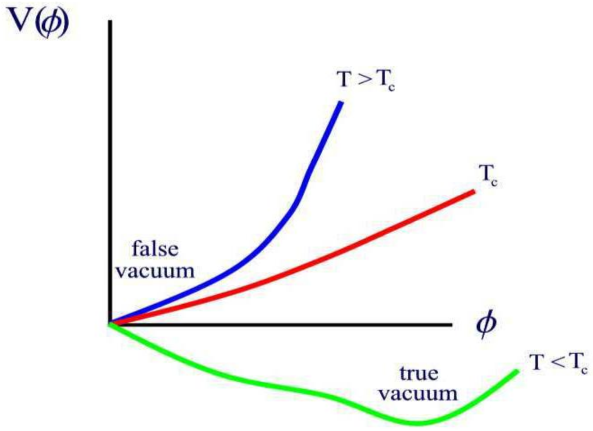

Fig. 3. Andre Linde's 'new' Toy model for inflation [42].

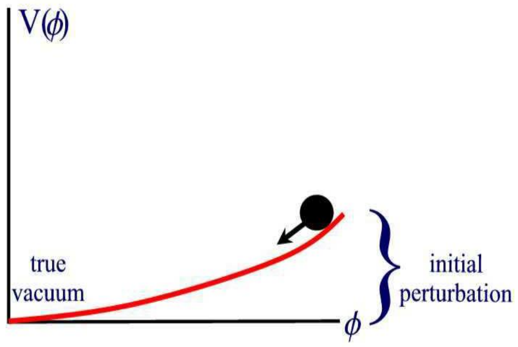

Fig. 4. Simplified version of Andre Linde's (general) Chaotic inflation model [42].

Starting from when the universe is at a temperature, T, that is greater than the critical temperature, T_c, the inflaton field slowly rolls down a potential, consistent with Guth's model in the sense that just before the ϕ-field commences the 'transition' to a more favourable state (true vacuum), the universe has to cool down to the T_c, as shown in Fig 3. It should be stated that, Linde's incredible intuition of proposing that due to the slow nature of evolution, the phase transition from the (less desirable) false vacuum to the (more favourable) true vacuum, is of second order which implies that there is no need for a phenomenon of quantum tunnelling. Watson [42] points out that, this not only gives a more desirable description of the early universe but also, avoids the fine tuning problem that physicists try to avoid, in the sense of a classical picture predicting the evolution of the inflaton field is characterised by classical parameters. Linde's 'new' Toy model for inflation was later independently corroborated by the efforts of Albrecht and Steinhardt [35], thus further acclaiming its success and validation.

However, Linde wanted to do one better. It was not satisfactory that although he did propose a valid solution to the *graceful exit problem*, as we have just seen, which turned out to be a more promising description of our universe, there had to be a more general version to the inflationary model. It must be reaffirmed that Guth's motivation for his 'original' model was due to the fact he was attempting to solve the initial value problem in the context of the SU(5) GUT symmetry group. Linde [41] then proposed the more ambitious chaotic inflationary model, which differed from both the 'old' and 'new' Toy models, simply due to the fact, there were **no** phase transitions taking place at all! As shown in Fig. 4, there is inflation that has been displaced (due to some quantum or thermal fluctuations) from the state that we want the ϕ-field to be within; the true vacuum state. Using the similar assumption to his previous 'new' toy model, Linde [41] stated that the inflaton field *slowly* rolls down the potential {typically, the simplest potential to consider as represented by Fig. 4, $V(\phi) = \frac{1}{2}m\phi^2$ where inflation is proposed to occur at $\phi \geq M_p$(: Plank mass) [57]} curve to the true vacuum, indicating that the early universe is in a phase of expansion. Watson [42] highlights the major advantage of this model not having to deal with any of the subtleties the T_c parameter contains, which is also in accordance with other inflationary models.

The crux for this model lies in the dynamics of the ϕ-field, when it is in the state of the true vacuum. The important so-called reheating phase [54, 55, 57], where the inflaton field has the possibility of coupling to other particle or gauge fields. Incredibly, after the phase of reheating, the description for the evolution of the universe is one that is described by the BBM! So, Linde not only was able to formulate a scenario that describes the evolution of the early universe, but constructed as Liddle [43] puts it “a bolt on accessory”, that links to the very description (BBM) that best describes the current state of our universe. Finally, it must be stated, that in order to remain consistent (as with all inflationary models), the reason for their existence is to solve the initial value problems for the BBM. Linde’s (general) chaotic model of inflation has achieved this, through the displacement (although not exactly specified by an amount, but suffice to say a ‘minimal’ amount) for the inflaton field.

5.3 The evolution of the ϕ-field: Mechanism for particle production.

Finally, we conclude this report by briefly stating that while Watson [42] first implemented the machinery from field theory (which we developed in Chapter 2) to model the inflaton field as a fluid and obtain the energy density describing the ϕ-field, we show just as [54, 55, 57] that an equivalent Lagrangian in structure to (3) could also be constructed due to the evolution of the Friedmann equation [57],

$$H^2 = \frac{8\pi}{3{M_p}^2}\left(\frac{1}{2}\dot{\phi} + V(\phi)\right) \qquad \Rightarrow \qquad L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{145}$$

where the Hubble parameter , $H = \frac{\dot{a}}{a}$; the scale factor, $a(t)$ and the simplest possible potential for the single scalar field as stated in Section 5.2.2, $V(\phi) = \frac{1}{2}m\phi^2$.

Due to the same formalism in describing the KG equation (Chapter 2), we attain a similar equation to describe the evolution of the inflaton field,

$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi) \tag{146}$$

where $V'(\phi) = \frac{dV(\phi)}{d\phi}$. For large initial values of $\phi > M_p$, the term $3H\dot{\phi}$ is known to be the friction term that dominates over the $\ddot{\phi}$ term in (146). Moreover, the friction term also dominates over,

$$\frac{8\pi}{3{M_p}^2}\left(\frac{1}{2}\dot{\phi}^2\right) = \frac{4\pi\dot{\phi}^2}{3{M_p}^2} \tag{147}$$

where the H^2 from (145) represents the kinetic energy term in the Lagrangian. This is what is known as the inflationary stage [57], where as the universe expands quasi-exponentially [42, 43, 57]. A more quantitative approach has been taken and analysed in [16], including with regard to the solutions to the equation of motion (146). The following two diagrams (Fig. 5 and Fig. 6) illustrate the evolution of the inflaton field. In particular, we replicated the investigation that was conducted by Linde [57], to illustrate the phenomenon of particle creation, as the inflaton field oscillates, say if the ϕ-field is coupled to a scalar field, the oscillations will result to bosons being produced [55, 57]. For future research, a more through quantitative investigation would seal the understanding of the author with regard to how field theories are implemented in the realm of early universe cosmology, and in particular, we would like to investigate the behaviour of the inflator field in an expanding background for re-heating scenarios [57] as well as looking at a quantitative description of the behaviour of stochastic expansion. Finally, with reference to all the exotic theories [55] showed an application of supersymmetry being included within the early universe cosmology.

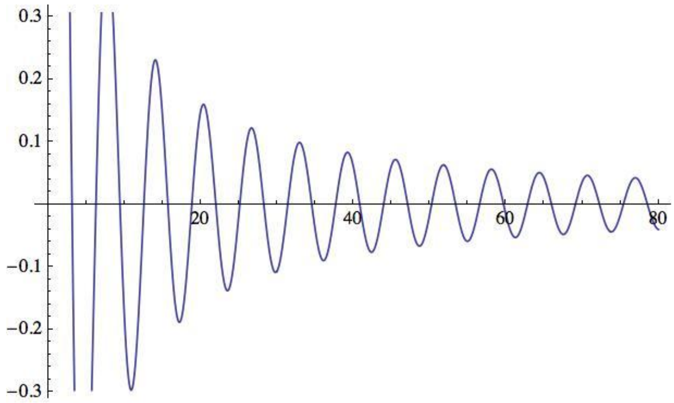

Fig. 5. Oscillating inflaton field, giving rise to particle production, depending on the matter field it is coupled to.

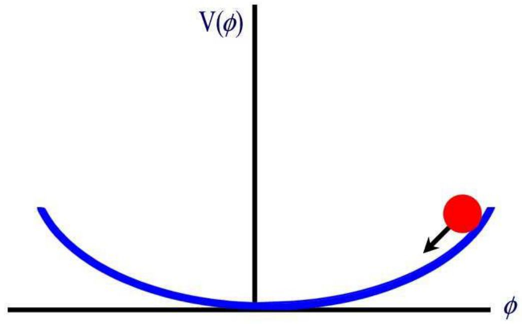

Fig. 6. Evolution of the inflaton field, subject to a simple quadratic potential [42].

Chapter 6: Conclusion.

In this report we began by revisiting the relationship between QM and QFT and by using the fundamental commutation relation to quantize the KG field, we attained expressions for the creation $a^{\dagger}(\boldsymbol{k})$, and annihilation $a(\boldsymbol{k})$ operators responsible for the creation and annihilation, respectfully, of particle excitations for the KG field, known as bosons. However, upon quantizing the Hamiltonian and expressing it in term of $a^{\dagger}(\boldsymbol{k})$ and $a(\boldsymbol{k})$, we came across the ambiguity where the Dirac delta function due to the commutation relation was being integrated for the value of $\delta(0)$, implying the existence of particle states with infinite energies. This contradicted the spectrum of the Hamiltonian that was constructed, insisting the particle states to have positive and definite values of energy. The infinity problem was rectified by means of *normal ordering* insisting the arrangement of the $a^{\dagger}(\boldsymbol{k})$s' to placed to the left of $a(\boldsymbol{k})$s' resulting to the construction of a normal ordered Hamiltonian for the KG theory, H_n, and we found the appropriate eigenstates and eigenvalues, generalizing to a multi-particle system.

In Chapter 3, we introduced the notion of causality and formulated the interaction picture which is used to describe interacting field theories. Using the example of a meson decay, we thoroughly studied the mechanisms known as the S-Matrix and Wick's theorem, which essentially describes processes such as scattering and decay events. Through the notion of time ordering we further showed how the behaviour of the Hamiltonian for more complicated systems respond, as opposed to the simple free Hamiltonian.

In Chapter 4, we looked at a comprehensive overall picture of how spin $\frac{1}{2}$ fields are described in the context of QFT. Due to the consequences of the spin statistics theorem, we have illustrated that one cannot naively implement the same formalisms and structure that was used to describe scalar fields (Chapter 2 and 3). Starting from their own representation of the Lorentz group, Spinor representation and its respective (Lie) Clifford's algebra, we constructed an Action which by solving the Euler-Lagrange equation, we attained the Dirac equation which describes the dynamics responsible for fermions. Furthermore, variations for the Dirac equation, mainly in the context of the Weyl equation, we discuss the equation of motion that is responsible for describing all the particles within the SM, except for the Higgs Boson. Finally, using the solutions to the Dirac equation, we discussed the quantization procedure for the Dirac field, giving rise to fermions and the statistics they obey called Fermi-Dirac statistics.

Finally, in Chapter 5, we looked at an example to where the machinery of field theory that was developed throughout this report, In this chapter, we began by giving a brief qualitative overview of the main features behind the inflationary scenario, first by considering the physical motivation behind the existence of the phenomenon known as inflation. We focused the attention towards the three primary models of inflation and explored the phenomenon of particle creation due to parametric resonance within a non-expanding background.

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