1.13. Prove that $F(n)$ is the closest int to $\frac{\varphi^n}{\sqrt{5}}$!

Def 1. $F(n) = \begin{cases} \text{if } n=0 & 0 \\ \phantom{\text{if }} n=1 & 1 \\ \text{else} & F(n-1)+F(n-2) \end{cases}$ $\quad n \in \mathbb{N}$

Def 2. $\varphi = \frac{1+\sqrt{5}}{2}$

Def 3. $\psi = \frac{1-\sqrt{5}}{2}$

Lemma 1 $F(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}}$

Proof by induction

→ Base case $n=0$

LHS $F(0)=0$

RHS $\frac{\varphi^0-\psi^0}{\sqrt{5}} = 0$

→ Base case $n=1$

LHS $F(1)=1$

RHS $\frac{\varphi-\psi}{\sqrt{5}} = \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} = 1$

→ Inductive step

Assume $F(n-2) = \frac{\varphi^{n-2}-\psi^{n-2}}{\sqrt{5}}$

Assume $F(n-1) = \frac{\varphi^{n-1}-\psi^{n-1}}{\sqrt{5}}$

$$F(n) = F(n-1) + F(n-2)$$

$$F(n) = \frac{\varphi^{n-1}-\psi^{n-1}}{\sqrt{5}} + \frac{\varphi^{n-2}-\psi^{n-2}}{\sqrt{5}}$$

$$F(n) = \frac{\varphi^{n-1}+\varphi^{n-2}-\psi^{n-1}-\psi^{n-2}}{\sqrt{5}}$$

$$F(n) = \frac{\varphi^n\left(\frac{1+\varphi}{\varphi^2}\right) - \psi^n\left(\frac{1+\psi}{\psi^2}\right)}{\sqrt{5}}$$

$$F(n) = \frac{\varphi^n\left(\frac{1+\frac{1+\sqrt{5}}{2}}{\frac{6+2\sqrt{5}}{4}}\right) - \psi^n\left(\frac{1+\frac{1-\sqrt{5}}{2}}{\frac{6-2\sqrt{5}}{4}}\right)}{\sqrt{5}}$$

$$F(n) = \frac{\varphi^n\left(\frac{6+2\sqrt{5}}{6+2\sqrt{5}}\right) - \psi^n\left(\frac{6-2\sqrt{5}}{6-2\sqrt{5}}\right)}{\sqrt{5}}$$

$$F(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}}$$

Using Lemma 1 prove that $F(n)$ is the closest int to $\frac{\varphi^n}{\sqrt{5}}$!

<u>To prove</u> $\left| F(n) - \frac{\varphi^n}{\sqrt{5}} \right| \leq \frac{1}{2}$

Substitute by Lemma 1

$$\left| \frac{\varphi^n - \psi^n}{\sqrt{5}} - \frac{\varphi^n}{\sqrt{5}} \right| \leq \frac{1}{2}$$

$$\left| -\frac{\psi^n}{\sqrt{5}} \right| \leq \frac{1}{2}$$

$$\frac{|-\psi^n|}{|\sqrt{5}|} \leq \frac{1}{2}$$

$$|\psi^n| \leq \frac{\sqrt{5}}{2}$$ we introduced a stricter constraint given $1 < \frac{\sqrt{5}}{2}$

$$|\psi^n| \leq 1 < \frac{\sqrt{5}}{2}$$

$|\psi| \leq 1$ holds therefore for all $n \geq 0$ $|\psi^n| \leq 1$