Number Theory 4A3

Name ____________________

(1) __________ How many combinations of two positive two-digit integers have 429 as the product?

(2) __________ What is the units digit of 1998^{1998}?

(3) __________ How many positive even three-digit integers are divisible by 3?

(4) __________ A positive integer less than 100 is a multiple of 3. Determine the probability that it also leaves a remainder of 1 when divided by 7. Express your answer as a common fraction.

(5) __________ If three different odd positive integers less than 50 are randomly selected, what is the probability none of the three are prime?

(6) __________ The decimal representation of a fraction ends in $0.\overline{3}$. When the decimal is changed to a common fraction and reduced to lowest terms, what is the denominator of the fraction?

(7) __________ The digital root of a number is found by computing the sum of its digits, computing the sum of the digits of that sum, and continuing to compute the sum of the digits until a one-digit number is obtained. What is the digital root of 2^{100} ?

(8) __________ The product of the digits of a three-digit number is 63. What is the greatest possible value of the number?

(9) ________ Prime pairs are pairs of prime numbers that differ by 2, such as 5 and 7. What is the sum of the lesser member of the first prime pair in which both members are greater than 60 and the greater member of the first prime pair in which both members are greater than 100?

(10) ________ The number of students in Teresa's graduating class is more than 50 and fewer than 100 and is 1 less than a multiple of 3, 2 less than a multiple of 4, and 3 less than a multiple of 5. How many students are in Teresa's graduating class?

(11) ________ When an integer is divided by 15, the remainder is 7. Find the sum of the remainders when the same integer is divided by 3 and by 5.

(12) ________ Select two different numbers from the set $\{4, 9, 12, 16, 18\}$. Of all possible pairs, what is the least possible value for the sum of the greatest common factor and least common multiple of the two numbers?

(13) ________ What is the smallest positive integer that has twelve distinct factors?

(14) ________ Given that both p and $p+1$ are prime numbers, what is the least positive composite number that is not divisible by p nor $p+1$?

(15) ________ What is the length of the longest sequence of consecutive positive composite numbers, each number less than 101?

(16) ________ The number 3465 is divisible by 11 and by 5. Given that you change the position of two or more of these four digits, how many of the resulting numbers are divisible by 11 but not by 5?

(17) ________ The product of a set of distinct positive integers is 84. What is the least possible sum of these integers?

(18) ________ Find the value of $A + B$ in the multiplication table below:

$\times$	?	?	?
?	B	12	20
?		21	
?	18	27	A

(19) ________ a is randomly selected from the set $2, 3, 4, 5$, replaced, and then b is randomly selected from the same set. What is the probability that the fraction $\frac{a}{b}$ is already in reduced form?

(20) ________ What is the smallest positive integer greater than 5 with the property that the number of odd primes less than or equal to it equals the number of odd non-primes less than or equal to it?

(21) ________ Find the smallest positive integer x so that the fraction below represents a fraction whose decimal equivalent terminates.

$$\frac{1}{10+x}$$

(22) ________ In how many zeroes does 20! end?

(23) ________ How many positive two-digit integers are increased by exactly nine when the digits are reversed?

(24) ________ How many positive integers are factors of 21^{75}?

(25) ________ A proposal will make years that end in double zeroes a leap year only if the year leaves a remainder of 200 or 600 when divided by 900. Under this proposal, how many leap years will there be that end in double zeroes between 1996 and 4096?

(26) ________ Given that 3^n divides 15!, what is the greatest possible integral value of n?

(27) __________ Natasha has more than \$1 but less than \$10 worth of dimes. When she puts her dimes in stacks of 3, she has 1 left over. When she puts them in stacks of 4, she has 1 left over. When she puts them in stacks of 5, she also has 1 left over. How many dimes does Natasha have?

(28) __________ What is the least positive integer with exactly ten factors?

(29) __________ When two different numbers are divided by 7, remainders of 2 and 3, respectively, are left. What is the greatest possible three-digit product of these two numbers?

(30) __________ What is the least integer $n \geq 2$ such that $2^n - 1$ is a composite number?

Answer Sheet

Number	Answer	Problem ID
1	2 combinations	33B21
2	4	13A11
3	150	C1AA
4	$\frac{5}{33}$	03AA
5	$\frac{33}{460}$	12AA
6	3	C4AB
7	7	A1AA
8	971	2C3D
9	174	22111
10	62	4D331
11	3	BB011
12	16	42AA
13	60	104D
14	25	2D331
15	7 (90,91,...)	0C011
16	6	034D
17	14	A5DD
18	53	AC111
19	$\frac{5}{8}$	5B011
20	87	D24D
21	6	C2B21
22	4	1ADD
23	8	0D331
24	5776 integers	03B21
25	5	424D
26	6	A34D
27	61	CDDD
28	48	11AA
29	993	A2AA
30	4	A4AA

Solutions

(1) **2 combinations** **ID: [33B21]**
No solution is available at this time.

(2) **4** **ID: [13A11]**
No solution is available at this time.

(3) **150** **ID: [C1AA]**
No solution is available at this time.

(4) $\frac{5}{33}$ **ID: [03AA]**
No solution is available at this time.

(5) $\frac{33}{460}$ **ID: [12AA]**
No solution is available at this time.

(6) **3** **ID: [C4AB]**
Let $S = 0.\overline{3}$. Then $10S = 3.\overline{3}$. Subtracting the second equation from the first we obtain $9S = 3$, so $S = \frac{1}{3}$. The desired denominator is $\boxed{3}$.

(7) **7** **ID: [A1AA]**
No solution is available at this time.

(8) **971** **ID: [2C3D]**
No solution is available at this time.

(9) **174** **ID: [22111]**
No solution is available at this time.

(10) **62** **ID: [4D331]**

Let the number of students in Teresa's class be a. Then

$$a \equiv -1 \equiv 2 \pmod 3,$$
$$a \equiv -2 \equiv 2 \pmod 4,$$
$$a \equiv -3 \equiv 2 \pmod 5.$$

Since $\gcd(3,4) = \gcd(4,5) = \gcd(3,5) = 1$, we have

$$a \equiv 2 \pmod{3 \cdot 4 \cdot 5},$$

that is, $a \equiv 2 \pmod{60}$. Since a is then of the form $a = 2 + 60n$, the only such number in the range $50 < a < 100$ is $\boxed{62}$.

(11) **3** **ID: [BB011]**

We let our integer be n. the first sentence tells us that

$$n \equiv 7 \pmod{15}.$$

Since 3 and 5 are both factors of 15 we deduce

$$n \equiv 7 \equiv 1 \pmod 3$$
$$n \equiv 7 \equiv 2 \pmod 5.$$

Therefore the remainders in question are 1 and 2, and their sum is $\boxed{3}$.

(12) **16** **ID: [42AA]**

No solution is available at this time.

(13) **60** **ID: [104D]**

No solution is available at this time.

(14) **25** **ID: [2D331]**

No solution is available at this time.

(15) **7 (90,91,...)** **ID: [0C011]**

No solution is available at this time.

(16) **6** **ID: [034D]**

No solution is available at this time.

(17) **14** **ID: [A5DD]**

We know that the prime factors of the set of numbers must equal the prime factors of 84, which are $2^2 \cdot 3 \cdot 7$. The set with the smallest sum would be the factors themselves - 2, 2, 3, and 7. However, the set can't have two 2's since the integers must be distinct, but it can have a 4, 3, and 7 instead. The sum of those numbers is $\boxed{14}$. We could also have paired one of the 2's with the 3, to have 2, 6, and 7, but these have sum 15. Grouping the extra 2 with 7 gives 2, 3, and 14 (which sum to 19), and any other grouping clearly gives a sum higher than 14.

(18) **53** **ID: [AC111]**

No solution is available at this time.

(19) $\frac{5}{8}$ **ID: [5B011]**

No solution is available at this time.

(20) **87** **ID: [D24D]**

No solution is available at this time.

(21) **6** **ID: [C2B21]**

Recall that the decimal representation of a simplified fraction terminates if and only if the denominator of the fraction is not divisible by any prime other than 2 and 5. To see this, first note that any terminating decimal can be written as an integer divided by a power of 10. When simplified, the prime factorization of the denominator of the fraction will contain only 2s and 5s. Conversely, if the denominator of a fraction is of the form 2^a5^b for nonnegative integers a and b, then the numerator and denominator of the fraction can be multiplied by a suitable power of 2 or 5 to make the denominator a power of 10. This implies that the decimal representation of the fraction terminates.

Applying this observation to the question at hand, we conclude by checking the prime factorizations of 11, 12, 13, etc. that the least integer greater than 10 which is of the form 2^a5^b is 16, which corresponds to $x = \boxed{6}$.

(22) **4** **ID: [1ADD]**

No solution is available at this time.

(23) **8** **ID: [0D331]**

No solution is available at this time.

(24) **5776 integers** **ID: [03B21]**
No solution is available at this time.

(25) **5** **ID: [424D]**
We start with 1800, a multiple of 900, and add 200 to get 2000. So 2000 has a remainder of 200 when divided by 900. The next year with a remainder of 200 when divided by 900 is $2000 + 900 = 2900$. The year after that is $2900 + 900 = 3800$. Adding another 900 would result in a year greater than 4096.

Now we add 600 to 1800 and get 2400, which has a remainder of 600 when divided by 900. The next year with a remainder of 600 is $2400 + 900 = 3300$. Adding another 900 would result in a year greater than 4096.

So the years with remainders of 200 or 600 are 2000, 2900, 3800, 2400, and 3300. All of them end in double zeroes, so they are all leap years. We have a total of $\boxed{5}$ leap years.

OR

We can create inequalities. A leap year is equal to either $900a + 200$ or $900b + 600$, where a and b are positive integers. We solve for how many possible values of a and b we have.

$$1996 < 900a + 200 < 4096 \qquad \Rightarrow 1796 < 900a < 3896$$

So the value of a can be 2, 3, or 4, giving us 3 different leap years.

$$1996 < 900a + 600 < 4096 \qquad \Rightarrow 1396 < 900b < 3496$$

So the value of b can be 2 or 3, giving us 2 different leap years. In total we have $\boxed{5}$ leap years.

OR

We will end up with leap years when we add 200 or 600 to a multiple of 900. With 1800, we can add 200 or 600 to get two leap years. With 2700, we can add 200 or 600 to get two leap years. With 3600, we only get one leap year since $3600 + 600 = 4200$ is after 4096. We get a total of $\boxed{5}$ leap years.

(26) **6** **ID: [A34D]**
No solution is available at this time.

(27) **61** **ID: [CDDD]**

Let n be the number of dimes Natasha has. We know that $10 < n < 100$. The stacking data can be rephrased as

$$\begin{aligned} n &\equiv 1 \pmod 3 \\ n &\equiv 1 \pmod 4 \\ n &\equiv 1 \pmod 5. \end{aligned}$$

Notice that any number n such that $n \equiv 1 \pmod{60}$ solves this system. (The Chinese Remainder Theorem tells us that 1 is the only residue class modulo 60 that solves all of these equivalences.) Therefore $n = \boxed{61}$ is between 10 and 100 and solves this system.

(28) **48** **ID: [11AA]**

No solution is available at this time.

(29) **993** **ID: [A2AA]**

No solution is available at this time.

(30) **4** **ID: [A4AA]**

No solution is available at this time.