Name ______________________

Mathcounts / AMC 8

1. ________ What is the smallest integer greater than 2 that will have a remainder of 2 when divided by any member of the set $\{3, 4, 5, 6, 8\}$?

2. ________ The total area of four squares, each with whole-number side measurements, is 23 square inches. In inches, what is the positive difference between the perimeter of the largest square and the perimeter of the smallest square?

3. ________ John, Joe and James go fishing. At the end of the day, John comes to collect his third of the fish. However, there is one too many fish to make equal thirds, so John throws it out, takes his third and leaves. Joe comes to get his fish without realizing John has already taken his third. He notices there is one too many fish to make equal thirds so he throws one fish out, takes his third and leaves. James notice that there is one too many fish to make equal thirds so he throws one out, takes his fish and leaves. Assuming no fish are divided into pieces, what is the minimum possible number of fish before John threw out the first fish?

4. ________ What is the tens digit of the product of the first six prime numbers?

5. ________ What is the sum of the three distinct prime factors of 47,432?

6. ________ Cards are numbered from 1 to 100. One card is removed and the values on the other 99 are added. The resulting sum is a multiple of 77. What number was on the card that was removed?

7. ________ What is the sum of the last two digits of this portion of the Fibonacci Factorial Series: $1! + 1! + 2! + 3! + 5! + 8! + 13! + 21! + 34! + 55! + 89!$?

8. ________ What is the remainder when the sum of the first 100 positive integers is divided by 9?

9. ________ What is the base two representation of the sum of the binary numbers 1011_2 and 111_2?

10. __________ The number 839 can be written as $19q + r$ where q and r are positive integers. What is the greatest possible value of $q - r$?

11. __________ What is the 453rd digit to the right of the decimal point in the decimal expansion of $\frac{6}{13}$?

12. __________ What is the sum of the tens digit and the units digit in the decimal representation of 9^{2004}?

13. __________ What is the largest perfect square factor of 1512?

14. __________ What is the number of positive factors of 648?

15. __________ What is the remainder when 10! is divided by 2^7? (Reminder: If n is a positive integer, then $n!$ stands for the product $1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$.)

16. __________ How many odd perfect square factors does $2^4 \times 3^6 \times 5^{10} \times 7^9$ have?

17. __________ Let m and n denote the greatest and least positive three-digit multiples of 7, respectively. What is the value of $m + n$?

~~18.~~ __________ In any month with five Mondays, a school declares the fifth Monday a Marvelous Monday. School begins on Monday, September 4. What is the date of the first Marvelous Monday after school starts?

18 ~~19.~~ __________ The natural number n has exactly two natural-number factors. How many natural number factors does n^5 have?

19 ~~20.~~ ________ A magic square is an array of numbers in which the sum of the numbers in each row, in each column, and along the two main diagonals are equal. The numbers in the magic square shown are not written in base 10. For what base will this be a magic square?

13	1	11
3	10	12
4	14	2

20 ~~21.~~ ________ What is the units digit of $(133^{13})^3$?

21 ~~22.~~ ________ If $2^x \cdot 9^y$ is equal to the four-digit number $2x9y$ where x is the hundreds digit and y is the units digit, what is the product of x and y?

22 ~~23.~~ ________ The whole numbers are written consecutively in the pattern below. In which row (A, B, C, D or E) will the number 500 be written?

Row E: 1

Row D: 2

Row C: 3

Row B: 4

Row A: 5, 6, 7

Row B: 8

Row C: 9

Row D: 10

Row E: 11, 12, 13

Row D: 14

Row C: 15

Row B: 16

Row A: 17,18,19

23 ~~24.~~ ________ Suppose a and b are different prime numbers greater than 2. How many whole-number divisors are there for the integer $a(2a + b) - 2a^2 + ab$?

24 ~~25.~~ ________ How many prime positive integers are divisors of 555?

25 ~~26~~. __________ A positive multiple of 45 less than 1000 is randomly selected. What is the probability that it is a two-digit integer? Express your answer as a common fraction.

26 ~~27~~. __________ What is the base five product of the numbers 121_5 and 11_5?

Answer Sheet

Number	Answer	Problem ID
1	122	BB13
2	8	CA03
3	25	B013
4	3	1CC3
5	20	AD001
6	45	BBB3
7	5	0CA5
8	1	53001
9	10010	0BA5
10	41	2AA5
11	1	5BC3
12	7	5002
13	36	4A03
14	20	35001
15	0	DCB3
16	120	2B03
17	1099	1002
~~18~~	~~October 30~~	~~1DC2~~
18 ~~19~~	6	DBA5
19 ~~20~~	5	1A001
20 ~~21~~	7	C5B31
21 ~~22~~	10	B5B31
22 ~~23~~	B	2DC2
23 ~~24~~	8	ABB3
24 ~~25~~	3	DB55
25 ~~26~~	1/11	4002
26 ~~27~~	1331	2C55

Solutions

(1) **122** **ID: [BB13]**
No solution is available at this time.

(2) **8** **ID: [CA03]**
No solution is available at this time.

(3) **25** **ID: [B013]**
No solution is available at this time.

(4) **3** **ID: [1CC3]**
No solution is available at this time.

(5) **20** **ID: [AD001]**
No solution is available at this time.

(6) 45 **ID: [BBB3]**
The sum of the numbers from 1 to 100 is

$$1 + 2 + \cdots + 100 = \frac{100 \cdot 101}{2} = 5050.$$

When this number is divided by 77, the remainder is 45. Therefore, the number that was removed must be congruent to 45 modulo 77.

However, among the numbers 1, 2, ..., 100, only the number $\boxed{45}$ itself is congruent to 45 modulo 77. Therefore, this was the number of the card that was removed.

(7) **5** **ID: [0CA5]**

This expression $n!$, is the number you get by multiplying n by $(n-1)$ by $(n-2)$ by $(n-3)$ and so on, all the way down to 1. So $5! = (5)(4)(3)(2)(1) = 120$. Notice that 5! ends in a 0 since it has a factor of 10 (there is a 5 and a 2 in it list of factors) and that 10! has to end in two zeroes since it has a factor of 10, 5 and 2 which is really a factor of 100. Since any factorial greater than 10 (such as 13! or 21!) includes all of the factors of 10!, the last two digits of 13!, 21!, and so on are zeroes. These terms, therefore will not affect the last two digits of the sum of the Fibonacci factorial series.

To find the last two digits, you only need to find the last two digits of each of the terms of $1!+1!+2!+3!+5!+8!$. We do not need to calculate 8!, only to find its last two digits. Starting with 5!, we can work our way to 8!, using only the last two digits of each value along the way. We know $5! = 120$, so use 20 when finding 6!, which will bring us to $6(2) = 120$ or 20. Therefore, the last two digits of 7! are from $7(2) = 140$ or 40. Finally 8! is $8(40) = 320$ or finally 20. The last two digits of the entire series will come from $1 + 1 + 2 + 6 + 20 + 20 = 50$. Therefore, the sum of the last two digits is $5 + 0 = \boxed{5}$.

(8) **1** **ID: [53001]**

No solution is available at this time.

(9) **10010** **ID: [0BA5]**

No solution is available at this time.

(10) 41 **ID: [2AA5]**

In order to get the greatest possible $q - r$, we want to maximize q and minimize r. We divide 839 by 19 to find the maximum q. The quotient q is 44 and the remainder r is 3, and we can check that $839 = 19(44) + 3$. So the greatest possible value of $q - r = 44 - 3 = \boxed{41}$.

(11) **1** **ID: [5BC3]**

The decimal representation of $\frac{6}{13}$ is $0.\overline{461538}$, which repeats every 6 digits. Since 453 divided by 6 has a remainder of 3, the 453rd digit is the same as the third digit after the decimal point, which is $\boxed{1}$.

(12) **7** **ID: [5002]**

Write 9 as $10-1$ and consider raising 9 to the 2004 power by multiplying out the expression

$$\overbrace{(10-1)(10-1)(10-1)\cdots(10-1)}^{2004 \text{ factors}}$$

There will be 2^{2004} terms in this expansion (one for each way to choose either 10 or -1 for each of the 2004 factors of $(10-1)$), but most of them will not affect the tens or units digit because they will have two or more factors of 10 and therefore will be divisible by 100. Only the 2004 terms of -10 which come from choosing -1 in 2003 of the factors and 10 in the remaining one as well as the term $(-1)^{2004} = 1$ remain. Let N represent the sum of all of the terms with more than 1 factor of 10. We have

$$\begin{aligned}(10-1)^{2004} &= N + 2004(-10) + 1 \\ &= N - 20{,}040 + 1 \\ &= (N - 20{,}000) - 40 + 1 \\ &= (N - 20{,}000) - 39.\end{aligned}$$

So 9^{2004} is 39 less than a multiple of 100 and therefore ends in 61. The sum of 6 and 1 is $\boxed{7}$.

(13) **36** **ID: [4A03]**

Let's find the prime factorization of 1512: $1512 = 2^3 \cdot 189 = 2^3 \cdot 3^3 \cdot 7$. The only two squares of primes that divide 1512 are $2^2 = 4$ and $3^2 = 9$. Therefore, the largest perfect square factor of 1512 is $2^2 \cdot 3^2 = (2 \cdot 3)^2 = \boxed{36}$.

(14) **20** **ID: [35001]**

No solution is available at this time.

(15) 0 **ID: [DCB3]**

10! is divisible by 2, $4 = 2^2$, $6 = 2 \cdot 3$, $8 = 2^3$, and $10 = 2 \cdot 5$, so 10! is divisible by 2^8. Therefore, the remainder when 10! is divided by 2^7 is $\boxed{0}$.

(16) **120** **ID: [2B03]**

No solution is available at this time.

(17) **1099** **ID: [1002]**

Since $98 = 7 \cdot 14$, we know that $98 + 7 = 105$ is the least three-digit multiple of 7. Furthermore, 980 is a multiple of 7, as are 987, 994, and 1001. The greatest three-digit multiple is 994. The sum of these values is

$$994 + 105 = \boxed{1099}.$$

(18) **October 30** **ID: [1DC2]**
September has 30 days. September 4 is a Monday, so September 9 is a Saturday. Since September 30 is exactly 21 days later (or 3 weeks), September 30 is also a Saturday.

Then October 1 is a Sunday, and October 2 is a Monday. Then October 2, 9, 16, 23, and 30 are all Mondays, so the first Marvelous Monday is $\boxed{\text{October 30}}$.

(19) 6 **ID: [DBA5]**
No solution is available at this time.

(20) **5** **ID: [1A001]**
Let b be the base in which the numbers in the square are expressed. The first row and the first column must have the same sum, which implies that $1 + 11_b = 4 + 3$. Writing 11_b as $b + 1$, we find that $1 + b + 1 = 7$, which implies $b = \boxed{5}$.

(21) **7** **ID: [C5B31]**
No solution is available at this time.

(22) **10** **ID: [B5B31]**
No solution is available at this time.

(23) **B** **ID: [2DC2]**
No solution is available at this time.

(24) **8** **ID: [ABB3]**
Distributing and combining like terms, we have $a(2a + b) - 2a^2 + ab = 2a^2 + ab - 2a^2 + ab = 2ab$. Now a and b are different prime numbers greater than 2, so $2ab = 2^1 \cdot a^1 \cdot b^1$ has $(1 + 1)(1 + 1)(1 + 1) = \boxed{8}$ divisors.

(25) 3 **ID: [DB55]**
When we find the prime factorization of 555, we end up with $3 \cdot 5 \cdot 37$, which means we have $\boxed{3}$ prime positive divisors.

(26) **1/11** **ID: [4002]**
The positive multiples of 45 are

$$45, 90, 135, \ldots, 990 = 1 \cdot 45, 2 \cdot 45, 3 \cdot 45, \ldots, 22 \cdot 45.$$

There are 22 multiples on this list. Every positive multiple of 45 less than 1000 is either a two-digit integer or a three-digit integer. Out of the $99 - 10 + 1 = 90$ two-digit integers, 45 and 90 are multiples of 45. Therefore, the probability that the selected multiple of 45 has two digits is $2/22 = \boxed{1/11}$.

(27) **1331** **ID: [2C55]**
Notice that $121_5 \times 11_5 = 121_5 \times (10_5 + 1_5) = 1210_5 + 121_5 = \boxed{1331}_5$.