Mathcounts / AMC 8 (Week 7)

(1) ________ What is the arithmetic mean of all of the positive two-digit integers with the property that the integer is equal to the sum of its first digit plus its second digit plus the product of its two digits?

(2) ________ How many two-digit prime numbers have a units digit of 7?

(3) ________ Diane wants to arrange her collection of 96 bottle caps into a rectangular grid rather than keep them in the line of 96 bottle caps she has now. How many distinct, rectangular-grid arrangements could she use instead of the 1 by 96 grid of bottle caps she has now? One such arrangement is 3 by 32, which is considered the same as 32 by 3.

(4) ________ The difference between two prime numbers is 17. What is their sum?

(5) ________ What is the only integer value of n for which $\frac{n+1}{13-n}$ is a positive prime number?

(6) ________ One gear turns $33\frac{1}{3}$ times in a minute. Another gear turns 45 times in a minute. Initially, a mark on each gear is pointing due north. After how many seconds will the two gears next have both their marks pointing due north?

(7) ________ A segment has endpoints P(1, 1) and Q(x, y). The coordinates of the midpoint of segment PQ are positive integers with a product of 36. What is the maximum possible value of x?

(8) ________ The letters of the alphabet are given numeric values based on the two conditions below.

- • Only the numeric values of −2, −1, 0, 1 and 2 are used.
- • Starting with A and going through Z, a numeric value is assigned to each letter according to the following pattern:

$$1, 2, 1, 0, -1, -2, -1, 0, 1, 2, 1, 0, -1, -2, -1, 0, \ldots$$

Two complete cycles of the pattern are shown above. The letter A has a value of 1, B has a value of 2, F has a value of −2 and Z has a value of 2. What is the sum of the numeric values of the letters in the word "numeric"?

8 ~~(9)~~ ________ How many integers between 100 and 300 have both 11 and 8 as factors?

9 ~~(10)~~ ________ What is the digit in the tens place when 7^{2005} is expressed in decimal notation?

10 ~~(11)~~ ________ Zach has three bags and a bunch of pencils to be placed into the bags. He is told to place the greatest number of pencils possible into each of the three bags while also keeping the number of pencils in each bag the same. What is the greatest number of pencils he could have left over?

11 ~~(12)~~ ________ What is the sum of all the distinct positive two-digit factors of 144?

12 ~~(13)~~ ________ A number is called perfect if the sum of its divisors, except itself, is equal to the original number. What is the sum of the two perfect numbers between 2 and 30?

13 ~~(14)~~ ________ If $x + 5 < 8$ and x is a prime number, what is the value of x?

14 ~~(15)~~ ________ How many perfect squares less than 1000 have a ones digit of 2, 3 or 4?

15 ~~(16)~~ ________ The positive difference of the cube of an integer and the square of the same integer is 100. What is the integer?

16 ~~(17)~~ ________ What is the least positive multiple of 72 that has exactly 16 positive factors?

17 ~~(18)~~ ________ The members of a band are arranged in a rectangular formation. When they are arranged in 8 rows, there are 2 positions unoccupied in the formation. When they are arranged in 9 rows, there are 3 positions unoccupied. How many members are in the band if the membership is between 100 and 200?

18 ~~(19)~~ ________ How many positive whole-number divisors does 196 have?

19 ~~(20)~~ ________ We know the following to be true:

- 1. Z and K are integers with $500 < Z < 1000$ and $K > 1$;
- 2. $Z = K \times K^2$.

What is the value of K for which Z is a perfect square?

20 ~~(21)~~ ________ The number 24 can be made by multiplying together four prime numbers: 2, 2, 2 and 3. How many primes must be multiplied to make 2400?

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21 ~~(1)~~ ________ What is the units digit of 2013^{2013}?

22 ~~(2)~~ ________ The product of the digits of positive integer n is 20, and the sum of the digits is 13. What is the smallest possible value of n?

23 ~~(3)~~ ________ Hector spent a total of exactly \$200 on shirts and pants. The shirts cost \$15 each and the pants cost \$22 each. How many shirts did Hector buy?

24 ~~(4)~~ ________ For what base, b, is $14_b + 24_b = 41_b$ true?

25 ~~(5)~~ ________ M and N are both perfect squares less than 100. If $M - N = 27$, what is the value of $\sqrt{M} + \sqrt{N}$?

26 ~~(6)~~ ________ The integer m is between 30 and 80 and is a multiple of 6. When m is divided by 8, the remainder is 2. Similarly, when m is divided by 5, the remainder is 2. What is the value of m?

27 ~~(7)~~ ________ Cammie has some pennies, nickels, dimes, and quarters. What is the least number of coins that she can use to make 93 cents?

28 ~~(8)~~ ________ What is the sum of the units digits of all the multiples of 3 between 0 and 50?

Answer Sheet

Number	Answer	Problem ID
1	59	5342
2	5 numbers	2031
3	5 arrangements	0AA
4	21	51B1
5	12	D3112
6	36 seconds	1A1
7	71	034B
~~8~~	~~1~~	~~031~~
8 ~~9~~	2 integers	3B2C
9 ~~10~~	0	2531
10 ~~11~~	2 pencils	0031
11 ~~12~~	226	2242
12 ~~13~~	34	D0B
13 ~~14~~	2	3D31
14 ~~15~~	6 squares	C2D5
15 ~~16~~	5	4242
16 ~~17~~	216	AD31
17 ~~18~~	150 members	24112
18 ~~19~~	9 divisors	43112
19 ~~20~~	9	2A33
20 ~~21~~	8 primes	CAC

Answer Sheet

Number	Answer	Problem ID
21 ~~1~~	3	A1B34
22 ~~2~~	111145	CCB24
23	6 shirts	0C403
24	7	340D3
25	9	AB3D3
26	42	BB403
27	8 coins	4A3D3
28	78	3B403

Solutions

(1) **59 ID: [5342]**
Let AB be a two-digit integer with the property that AB is equal to the sum of its first digit plus its second digit plus the product of its two digits. Thus we have $10A + B = A + B + AB \Leftrightarrow 9A = AB$. Now since AB is a two-digit integer, $A \neq 0$, so we can divide both sides by A to get $9 = B$. Therefore, 19, 29, 39, 49, 59, 69, 79, 89, and 99 all have the property. Their arithmetic mean is $\frac{\frac{9(19+99)}{2}}{9} = \frac{19+99}{2} = \boxed{59}$.

(2) **5 numbers ID: [2031]**
The tens digit cannot be 2, 5, or 8 because otherwise, the sum of the digits, and thus the number itself, is divisible by 3. The remaining possibilities are 17, 37, 47, 67, 77, and 97. Only $77 = 7 \cdot 11$ is composite, so there are $\boxed{5}$ two-digit prime numbers that have a units digit of 7.

(3) **5 arrangements ID: [0AA]**
No solution is available at this time.

(4) **21 ID: [51B1]**
All primes besides 2 are odd. If we subtract two odd numbers, our result will always be even. Hence, one of our two primes is 2. If x is the other prime number, we have that $x - 2 = 17$, meaning that $x + 2 = 17 + 2 \cdot 2 = \boxed{21}$.

(5) **12** **ID: [D3112]**

The expression $\frac{n+1}{13-n}$ is negative if n is less than -1 or greater than 13, so n must be between 0 and 12 inclusive. Also, since $\frac{n+1}{13-n}$ is prime, it must be greater than or equal to 2. Solving

$$\begin{aligned} \frac{n+1}{13-n} &\geq 2 \\ n+1 &\geq 26-2n \\ 3n &\geq 25 \\ n &\geq 8\frac{1}{3}, \end{aligned}$$

we only have to check 9, 10, 11, and 12. (The multiplication by $13-n$ is justified since we have already know that n is less than 13). We find that when $n = \boxed{12}$, the expression equals 13.

(6) **36 seconds** **ID: [1A1]**

One gear turns $33\frac{1}{3} = 100/3$ times in 60 seconds, so it turns 5/9 times in one second, or 5 times in 9 seconds. The other gear turns 45 times in 60 seconds, so it turns 3/4 times in one second, or 3 times in 4 seconds. To find out after how many seconds the two gears next have both their marks pointing due north, we have to find the least common multiple of $4 = 2^2$ and $9 = 3^2$, which is $2^2 \cdot 3^2 = 36$. Therefore, the two gears next have both their marks pointing due north after $\boxed{36}$ seconds. (One gear turns exactly $5 \times 4 = 20$ times, and the other gear turns exactly $3 \times 9 = 27$ times.)

(7) **71** **ID: [034B]**

Using the midpoint formula, we can see that the coordinates of the midpoint of segment PQ will be $\left(\frac{x+1}{2}, \frac{y+1}{2}\right)$. The product of these coordinates must equal 36, so we have:

$$\begin{aligned} \frac{(x+1)(y+1)}{2 \cdot 2} &= 36 \\ (x+1)(y+1) &= 144 \end{aligned}$$

In order to maximize x, we let the term $(x+1)$ equal the largest factor of 144, so $(x+1) = 144$, and $(y+1) = 1$. However, this implies that $y = 0$ which is not a positive integer. Thus, we let $(x+1)$ equal the next largest factor of 144, which is 72. We then have:

$$\begin{aligned} (x+1) &= 72 \\ (y+1) &= 2 \\ x = 71, y &= 1 \end{aligned}$$

Thus, the largest possible value of x is $\boxed{71}$.

~~(8)~~ **-1 ID: [031]**

The cycle has length 8. So the numeric value of a letter is determined by its position within the alphabet, modulo 8. So we determine the positions of all the letters in the word and use them to find the values:

n is the 14th letter. $14 \pmod 8 = 6$, so its value is -2.
u is the 21st letter. $21 \pmod 8 = 5$, so its value is -1.
m is the 13th letter. $13 \pmod 8 = 5$, so its value is -1.
e is the 5th letter. $5 \pmod 8 = 5$, so its value is -1.
r is the 18th letter. $18 \pmod 8 = 2$, so its value is 2.
i is the 9th letter. $9 \pmod 8 = 1$, so its value is 1.
c is the 3rd letter. $3 \pmod 8 = 3$, so its value is 1.
The sum is $(-2)+(-1)+(-1)+(-1)+2+1+1=\boxed{-1}$.

8 ~~(9)~~ **2 integers ID: [3B2C]**

The only numbers that have 11 and 8 as a factor are multiples of 88. If we list the first few multiples of 88:

$$88, 176, 264, 352, \ldots$$

we can see that there are exactly $\boxed{2}$ between 100 and 300.

9 ~~(10)~~ **0 ID: [2531]**

Let's find the cycle of the last two digits of 7^n, starting with $n = 1$:
$07, 49, 43, 01, 07, 49, 43, 01, \ldots$. The cycle of the last two digits of 7^n is 4 numbers long: 07, 49, 43, 01. Thus, to find the tens digit of 7^n for any positive n, we must find the remainder, R, when n is divided by 4 ($R = 0$ or 1 corresponds to the tens digit 0, and $R = 2$ or 3 corresponds to the units digit 4). Since $2005 \div 4 = 501R1$, the tens digit of 7^{2005} is $\boxed{0}$.

10 ~~(11)~~ **2 pencils ID: [0031]**

If Zach has three or more pencils left over, then he can add another pencil to each bag. Therefore, Zach can have at most $\boxed{2}$ pencils left over.

11 ~~(12)~~ **226 ID: [2242]**

Prime factorize $144 = 2^4 \cdot 3^2$. The sum of the positive two-digit factors of 144 is $2^4 + 2 \cdot 3^2 + 2^2 \cdot 3 + 2^2 \cdot 3^2 + 2^3 \cdot 3 + 2^3 \cdot 3^2 + 2^4 \cdot 3 = \boxed{226}$.

12 ~~(13)~~ **34 ID: [D0B]**

No solution is available at this time.

13 ~~(14)~~ **2** **ID: [3D31]**

$x+5<8 \Leftrightarrow x<3$. x is prime, so $x = \boxed{2}$.

14 ~~(15)~~ **6 squares** **ID: [C2D5]**

Checking the squares from 1^2 to 10^2, we see that no squares end in 2 or 3, while a square ends in 4 if its square root ends in 2 or 8. Since $31^2 < 1000 < 32^2$, we see that the squares less than 1000 ending in 4 are $2, 8, 12, 18, 22, 28$. Thus the desired answer is $\boxed{6}$.

15 ~~(16)~~ **5** **ID: [4242]**

Let n be the positive integer such that $n^3 - n^2 = 100$. Factoring, we have $n^2(n-1) = 100 = 10^2$. From this we see that $n-1$ must be a perfect square. $n-1 = 1^2$ is too small, but $n-1 = 2^2 = 4$ works: $(4+1)^2 \times 4 = 25 \times 4 = 100$. Thus, $n = \boxed{5}$. (Clearly, bigger perfect square values for $n-1$ do not work, and if we assume that n is negative, we get no solutions.)

16 ~~(17)~~ **216** **ID: [AD31]**

Let M be the least positive multiple of 72 that has exactly 16 positive factors. M is divisible by $72 = 2^3 \cdot 3^2$, which has $(3+1)(2+1) = 12$ positive factors. M has no prime factors other than 2 and 3 because if M were divisible by a third prime, p, then M would have at least $12(1+1) > 16$ positive factors. Thus, $M = 2^3 \cdot 3^3 = \boxed{216}$.

17 ~~(18)~~ **150 members** **ID: [24112]**

The number of members in the band leaves a remainder of 6 when divided by 8 and a remainder of 6 when divided by 9. Therefore, the number of members is 6 more than a multiple of 72. The only such number between 100 and 200 is $72 \cdot 2 + 6 = 150$, so there are $\boxed{150}$ members.

18 ~~(19)~~ **9 divisors** **ID: [43112]**

First prime factorize $196 = 2^2 \cdot 7^2$. The prime factorization of any divisor of 196 cannot include any primes other than 2 and 7. We are free to choose either 0, 1, or 2 as the exponent of 2 in the prime factorization of a divisor of 196. Similarly, we may choose 0, 1, or 2 as the exponent of 7. In total, there are $3 \times 3 = 9$ possibilities for the prime factorization of a divisor of 196. Distinct prime factorizations correspond to distinct integers, so there are $\boxed{9}$ divisors of 196.

19 ~~(20)~~ **9 ID: [2A33]**

From the second fact, we know that $Z = K^3$. Z is a perfect square if K^3 is a perfect square, so Z is the sixth power of some integer. Since $500 < Z < 1000$, the only value of Z that works is $Z = 3^6 = 729$. Thus, $K = \sqrt[3]{729} = \boxed{9}$.

20 ~~(21)~~ **8 primes ID: [CAC]**

$2400 = 2^5 \cdot 3^1 \cdot 5^2$, so $5 + 1 + 2 = \boxed{8}$ primes must be multiplied to make 2400.

Solutions

2(1) **3** **ID: [A1B34]**
No solution is available at this time.

2(2) **111145** **ID: [CCB24]**
No solution is available at this time.

2(3) **6 shirts** **ID: [0C403]**
Let m be the number of shirts that Hector bought, and n be the number of pants he bought. It follows that $15m + 22n = 200$. Re-arranging, $22n = 200 - 15m = 5(40 - 3m)$. Thus, the right-hand side is divisible by 5, so the left-hand side must also be. Since 22 is not divisible by 5, it follows that n is divisible by 5. If $n \geq 10$, then $22 \times n \geq 220 > 200$, so $n = 0, 5$. Also, $n \neq 0$ since 15 does not divide into 200. Thus, $n = 5$, and $15m + 22 \cdot 5 = 15m + 110 = 200 \Longrightarrow 15m = 90 \Longrightarrow m = \boxed{6}$ shirts.

2(4) **7** **ID: [340D3]**
Translating $14_b + 24_b = 41_b$ into base 10 yields

$$\begin{aligned} 4b + 1 &= (b+4) + (2b+4) \\ &= 3b + 8 \\ \Rightarrow \quad b + 1 &= 8 \\ \Rightarrow \quad b &= \boxed{7}. \end{aligned}$$

2(5) **9** **ID: [AB3D3]**
Because M and N are perfect squares, let $M = a^2$ and $N = b^2$ where a and b are non-negative integers. Because M and N are less than 100, a and b are less than 10. Substituting our equations for M and N into the given information, we get $a^2 - b^2 = 27$ and we want to solve for $a + b$.

Seeing a difference of squares, we can factor to get $(a+b)(a-b) = 27$. Because a and b are both non-negative integers, $a + b \geq a - b$ and both expressions must be factors of 27. There are only 2 ways to factor 27: $27 \cdot 1$ and $9 \cdot 3$. Thus we have two cases: $a - b = 1$ and $a - b = 3$. If $a - b = 1$, then $a + b = 27$. However, a and b are both less than 10, so this is impossible. Thus, $a - b = 3$ and $a + b = \boxed{9}$.

Note: the values of a and b that satisfy this is $a = 6$ and $b = 3$.

2(6) **42** **ID: [BB403]**

According to the problem statement, we have the system of linear congruences

$$m \equiv 0 \pmod{6}$$
$$m \equiv 2 \pmod{8}$$
$$m \equiv 2 \pmod{5}.$$

It follows by the Chinese Remainder Theorem that $m \equiv 2 \pmod{40}$. The only number that satisfies this criterion for $30 \leq m \leq 80$ is $m = \boxed{42}$, which is indeed divisible by 6.

2(7) **8 coins** **ID: [4A3D3]**

If we want to use as few coins as possible, we will want to use as many large coins as possible. The largest coins is a quarter, worth 25 cents. The most quarters Cammie can have is 3 because $93 \div 25 = 3R18$. Thus, after 3 quarters, we still have 18 cents left. The next largest coin is a dime, worth 10 cents. Because $18 \div 10 = 1R8$, Cammie can have 1 dime. Thus, we have 8 cents left. Nickels are next, worth 5 cents each. Because $8 \div 5 = 1R3$, Cammie can have 1 nickel. Finally, we have 3 cents remaining. The last coin is a penny, worth 1 cent each. Because $3 \div 1 = 3R0$, Cammie has 3 pennies, and all of the cents are accounted for. Thus, the minimal combination of coins is 3 quarters, 1 dime, 1 nickel, and 3 pennies, for a total of $\boxed{8}$ coins.

2(8) **78** **ID: [3B403]**

We start by computing the sum of the units digits of all multiples of 3 between 0 and 30. Excluding 0, every possible digit appears exactly once as a unit digit of a multiple of 3: the set of multiples of 3 between 0 and 30 consists of the numbers $0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30$. Thus, the sum of their units digits is equal to

$$1+2+3+4+5+6+7+8+9 = \frac{9 \cdot 10}{2} = 45.$$

We must sum the units digits of the multiples of 3 between 31 and 50. The relevant multiples of 3 are $33, 36, 39, 42, 45, 48$, and the sum of their units digits is $3+6+9+2+5+8 = 33$. Thus, the answer is $45 + 33 = \boxed{78}$.