

Mathcounts / AMC 8 (Week 7)

- (1) _____ What is the arithmetic mean of all of the positive two-digit integers with the property that the integer is equal to the sum of its first digit plus its second digit plus the product of its two digits?
- (2) _____ How many two-digit prime numbers have a units digit of 7?
- (3) _____ Diane wants to arrange her collection of 96 bottle caps into a rectangular grid rather than keep them in the line of 96 bottle caps she has now. How many distinct, rectangular-grid arrangements could she use instead of the 1 by 96 grid of bottle caps she has now? One such arrangement is 3 by 32, which is considered the same as 32 by 3.
- (4) _____ The difference between two prime numbers is 17. What is their sum?
- (5) _____ What is the only integer value of n for which $\frac{n+1}{13-n}$ is a positive prime number?
- (6) _____ One gear turns $33\frac{1}{3}$ times in a minute. Another gear turns 45 times in a minute. Initially, a mark on each gear is pointing due north. After how many seconds will the two gears next have both their marks pointing due north?
- (7) _____ A segment has endpoints $P(1, 1)$ and $Q(x, y)$. The coordinates of the midpoint of segment PQ are positive integers with a product of 36. What is the maximum possible value of x ?

~~(8)~~ _____ The letters of the alphabet are given numeric values based on the two conditions below.

- Only the numeric values of -2 , -1 , 0 , 1 and 2 are used.
- Starting with A and going through Z, a numeric value is assigned to each letter according to the following pattern:

1, 2, 1, 0, -1 , -2 , -1 , 0, 1, 2, 1, 0, -1 , -2 , -1 , 0, ...

Two complete cycles of the pattern are shown above. The letter A has a value of 1, B has a value of 2, F has a value of -2 and Z has a value of 2. What is the sum of the numeric values of the letters in the word "numeric"?

- 8 ~~(9)~~ _____ How many integers between 100 and 300 have both 11 and 8 as factors?
- 9 ~~(10)~~ _____ What is the digit in the tens place when 7^{2005} is expressed in decimal notation?
- 10 ~~(11)~~ _____ Zach has three bags and a bunch of pencils to be placed into the bags. He is told to place the greatest number of pencils possible into each of the three bags while also keeping the number of pencils in each bag the same. What is the greatest number of pencils he could have left over?
- 11 ~~(12)~~ _____ What is the sum of all the distinct positive two-digit factors of 144?
- 12 ~~(13)~~ _____ A number is called perfect if the sum of its divisors, except itself, is equal to the original number. What is the sum of the two perfect numbers between 2 and 30?
- 13 ~~(14)~~ _____ If $x + 5 < 8$ and x is a prime number, what is the value of x ?
- 14 ~~(15)~~ _____ How many perfect squares less than 1000 have a ones digit of 2, 3 or 4?
- 15 ~~(16)~~ _____ The positive difference of the cube of an integer and the square of the same integer is 100. What is the integer?
- 16 ~~(17)~~ _____ What is the least positive multiple of 72 that has exactly 16 positive factors?
- 17 ~~(18)~~ _____ The members of a band are arranged in a rectangular formation. When they are arranged in 8 rows, there are 2 positions unoccupied in the formation. When they are arranged in 9 rows, there are 3 positions unoccupied. How many members are in the band if the membership is between 100 and 200?
- 18 ~~(19)~~ _____ How many positive whole-number divisors does 196 have?

19 ~~(20)~~ _____

We know the following to be true:

- 1. Z and K are integers with $500 < Z < 1000$ and $K > 1$;
- 2. $Z = K \times K^2$.

What is the value of K for which Z is a perfect square?

20 ~~(21)~~ _____

The number 24 can be made by multiplying together four prime numbers: 2, 2, 2 and 3. How many primes must be multiplied to make 2400?

Answer Sheet

Number	Answer	Problem ID
1	59	5342
2	5 numbers	2031
3	5 arrangements	0AA
4	21	51B1
5	12	D3112
6	36 seconds	1A1
7	71	034B
8	1	031
8 9	2 integers	3B2C
9 10	0	2531
10 11	2 pencils	0031
11 12	226	2242
12 13	34	D0B
13 14	2	3D31
14 15	6 squares	C2D5
15 16	5	4242
16 17	216	AD31
17 18	150 members	24112
18 19	9 divisors	43112
19 20	9	2A33
20 21	8 primes	CAC

Solutions

(1) **59** ID: [5342]

Let AB be a two-digit integer with the property that AB is equal to the sum of its first digit plus its second digit plus the product of its two digits. Thus we have

$10A + B = A + B + AB \Leftrightarrow 9A = AB$. Now since AB is a two-digit integer, $A \neq 0$, so we can divide both sides by A to get $9 = B$. Therefore, 19, 29, 39, 49, 59, 69, 79, 89, and 99 all have the property. Their arithmetic mean is $\frac{9(19+99)}{2} = \frac{19+99}{2} = \boxed{59}$.

(2) **5 numbers** ID: [2031]

The tens digit cannot be 2, 5, or 8 because otherwise, the sum of the digits, and thus the number itself, is divisible by 3. The remaining possibilities are 17, 37, 47, 67, 77, and 97. Only $77 = 7 \cdot 11$ is composite, so there are $\boxed{5}$ two-digit prime numbers that have a units digit of 7.

(3) **5 arrangements** ID: [0AA]

No solution is available at this time.

(4) **21** ID: [51B1]

All primes besides 2 are odd. If we subtract two odd numbers, our result will always be even. Hence, one of our two primes is 2. If x is the other prime number, we have that $x - 2 = 17$, meaning that $x + 2 = 17 + 2 \cdot 2 = \boxed{21}$.

(5) **12** ID: [D3112]

The expression $\frac{n+1}{13-n}$ is negative if n is less than -1 or greater than 13 , so n must be between 0 and 12 inclusive. Also, since $\frac{n+1}{13-n}$ is prime, it must be greater than or equal to 2 . Solving

$$\begin{aligned}\frac{n+1}{13-n} &\geq 2 \\ n+1 &\geq 26-2n \\ 3n &\geq 25 \\ n &\geq 8\frac{1}{3},\end{aligned}$$

we only have to check 9 , 10 , 11 , and 12 . (The multiplication by $13-n$ is justified since we have already know that n is less than 13). We find that when $n = \boxed{12}$, the expression equals 13 .

(6) **36 seconds** ID: [1A1]

One gear turns $33\frac{1}{3} = 100/3$ times in 60 seconds, so it turns $5/9$ times in one second, or 5 times in 9 seconds. The other gear turns 45 times in 60 seconds, so it turns $3/4$ times in one second, or 3 times in 4 seconds. To find out after how many seconds the two gears next have both their marks pointing due north, we have to find the least common multiple of $4 = 2^2$ and $9 = 3^2$, which is $2^2 \cdot 3^2 = 36$. Therefore, the two gears next have both their marks pointing due north after $\boxed{36}$ seconds. (One gear turns exactly $5 \times 4 = 20$ times, and the other gear turns exactly $3 \times 9 = 27$ times.)

(7) **71** ID: [034B]

Using the midpoint formula, we can see that the coordinates of the midpoint of segment PQ will be $(\frac{x+1}{2}, \frac{y+1}{2})$. The product of these coordinates must equal 36 , so we have:

$$\begin{aligned}\frac{(x+1)(y+1)}{2 \cdot 2} &= 36 \\ (x+1)(y+1) &= 144\end{aligned}$$

In order to maximize x , we let the term $(x+1)$ equal the largest factor of 144 , so $(x+1) = 144$, and $(y+1) = 1$. However, this implies that $y = 0$ which is not a positive integer. Thus, we let $(x+1)$ equal the next largest factor of 144 , which is 72 . We then have:

$$\begin{aligned}(x+1) &= 72 \\ (y+1) &= 2 \\ x &= 71, y = 1\end{aligned}$$

Thus, the largest possible value of x is $\boxed{71}$.

~~(8)~~ 1 ID: [031]

The cycle has length 8. So the numeric value of a letter is determined by its position within the alphabet, modulo 8. So we determine the positions of all the letters in the word and use them to find the values:

n is the 14th letter. $14 \pmod{8} = 6$, so its value is -2 .

u is the 21st letter. $21 \pmod{8} = 5$, so its value is -1 .

m is the 13th letter. $13 \pmod{8} = 5$, so its value is -1 .

e is the 5th letter. $5 \pmod{8} = 5$, so its value is -1 .

r is the 18th letter. $18 \pmod{8} = 2$, so its value is 2 .

i is the 9th letter. $9 \pmod{8} = 1$, so its value is 1 .

c is the 3rd letter. $3 \pmod{8} = 3$, so its value is 1 .

The sum is $(-2) + (-1) + (-1) + (-1) + 2 + 1 + 1 = \boxed{-1}$.

8 ~~(9)~~ 2 integers ID: [3B2C]

The only numbers that have 11 and 8 as a factor are multiples of 88. If we list the first few multiples of 88:

88, 176, 264, 352, ...

we can see that there are exactly $\boxed{2}$ between 100 and 300.

9 ~~(10)~~ 0 ID: [2531]

Let's find the cycle of the last two digits of 7^n , starting with $n = 1$:

07, 49, 43, 01, 07, 49, 43, 01, The cycle of the last two digits of 7^n is 4 numbers long:

07, 49, 43, 01. Thus, to find the tens digit of 7^n for any positive n , we must find the remainder, R , when n is divided by 4 ($R = 0$ or 1 corresponds to the tens digit 0, and

$R = 2$ or 3 corresponds to the tens digit 4). Since $2005 \div 4 = 501R1$, the tens digit of 7^{2005} is $\boxed{0}$.

10 ~~(11)~~ 2 pencils ID: [0031]

If Zach has three or more pencils left over, then he can add another pencil to each bag.

Therefore, Zach can have at most $\boxed{2}$ pencils left over.

11 ~~(12)~~ 226 ID: [2242]

Prime factorize $144 = 2^4 \cdot 3^2$. The sum of the positive two-digit factors of 144 is

$2^4 + 2 \cdot 3^2 + 2^2 \cdot 3 + 2^2 \cdot 3^2 + 2^3 \cdot 3 + 2^3 \cdot 3^2 + 2^4 \cdot 3 = \boxed{226}$.

12 ~~(13)~~ 34 ID: [D0B]

No solution is available at this time.

13 ~~(14)~~ 2 ID: [3D31]

$x + 5 < 8 \Leftrightarrow x < 3$. x is prime, so $x = \boxed{2}$.

14 ~~(15)~~ 6 squares ID: [C2D5]

Checking the squares from 1^2 to 10^2 , we see that no squares end in 2 or 3, while a square ends in 4 if its square root ends in 2 or 8. Since $31^2 < 1000 < 32^2$, we see that the squares less than 1000 ending in 4 are 2, 8, 12, 18, 22, 28. Thus the desired answer is $\boxed{6}$.

15 ~~(16)~~ 5 ID: [4242]

Let n be the positive integer such that $n^3 - n^2 = 100$. Factoring, we have $n^2(n - 1) = 100 = 10^2$. From this we see that $n - 1$ must be a perfect square. $n - 1 = 1^2$ is too small, but $n - 1 = 2^2 = 4$ works: $(4 + 1)^2 \times 4 = 25 \times 4 = 100$. Thus, $n = \boxed{5}$. (Clearly, bigger perfect square values for $n - 1$ do not work, and if we assume that n is negative, we get no solutions.)

16 ~~(17)~~ 216 ID: [AD31]

Let M be the least positive multiple of 72 that has exactly 16 positive factors. M is divisible by $72 = 2^3 \cdot 3^2$, which has $(3 + 1)(2 + 1) = 12$ positive factors. M has no prime factors other than 2 and 3 because if M were divisible by a third prime, p , then M would have at least $12(1 + 1) > 16$ positive factors. Thus, $M = 2^3 \cdot 3^3 = \boxed{216}$.

17 ~~(18)~~ 150 members ID: [24112]

The number of members in the band leaves a remainder of 6 when divided by 8 and a remainder of 6 when divided by 9. Therefore, the number of members is 6 more than a multiple of 72. The only such number between 100 and 200 is $72 \cdot 2 + 6 = 150$, so there are $\boxed{150}$ members.

18 ~~(19)~~ 9 divisors ID: [43112]

First prime factorize $196 = 2^2 \cdot 7^2$. The prime factorization of any divisor of 196 cannot include any primes other than 2 and 7. We are free to choose either 0, 1, or 2 as the exponent of 2 in the prime factorization of a divisor of 196. Similarly, we may choose 0, 1, or 2 as the exponent of 7. In total, there are $3 \times 3 = 9$ possibilities for the prime factorization of a divisor of 196. Distinct prime factorizations correspond to distinct integers, so there are $\boxed{9}$ divisors of 196.

19 (20) 9 ID: [2A33]

From the second fact, we know that $Z = K^3$. Z is a perfect square if K^3 is a perfect square, so Z is the sixth power of some integer. Since $500 < Z < 1000$, the only value of Z that works is $Z = 3^6 = 729$. Thus, $K = \sqrt[3]{729} = \boxed{9}$.

20(21) 8 primes ID: [CAC]

$2400 = 2^5 \cdot 3^1 \cdot 5^2$, so $5 + 1 + 2 = \boxed{8}$ primes must be multiplied to make 2400.