Mathcounts / AMC 8 (Week 11)

(1) ________ A jar contains two red marbles, three green marbles, ten white marbles and no other marbles. Two marbles are randomly drawn from this jar without replacement. What is the probability that these two marbles drawn will both be red? Express your answer as a common fraction.

(2) ________ On a given day, the probability of rain is 60%. What are the odds against rain on that day? Express your answer as ~~$a : b$ in simplest form.~~ a common fraction.

(3) ________ Five balls are numbered 1 through 5 and placed in a bowl. Josh will randomly choose a ball from the bowl, look at its number and then put it back into the bowl. Then Josh will again randomly choose a ball from the bowl and look at its number. What is the probability that the product of the two numbers will be even and greater than 10? Express your answer as a common fraction.

(4) ________ A fair, twenty-faced die has 19 of its faces numbered from 1 through 19 and has one blank face. Another fair, twenty-faced die has 19 of its faces numbered from 1 through 8 and 10 through 20 and has one blank face. When the two dice are rolled, what is the probability that the sum of the two numbers facing up will be 24? Express your answer as a common fraction.

(5) ________ A customer ordered 15 pieces of gourmet chocolate. The order can be packaged in small boxes that contain 1, 2 or 4 pieces of chocolate. Any box that is used must be full. How many different combinations of boxes can be used for the customer's 15 chocolate pieces? One such combination to be included is to use seven 2-piece boxes and one 1-piece box.

(6) ________ A play has two different male roles, two different female roles and two different roles that can be either gender. Only a man can be assigned to a male role, and only a woman can be assigned to a female role. If five men and six women audition, in how many ways can the six roles be assigned?

(7) ________ In a school of 250 students, everyone takes one English class and one history class each year. Today, 15 total students were absent from their English class and ten total students were absent from their history class. Five of the students were absent from both classes. If a student is chosen at random from this school, what is the probability that s/he was not absent from either class? Express your answer as a percent.

(8) ________ Four couples are at a party. Four people of the eight are randomly selected to win a prize. No person can win more than one prize. What is the probability that both members of at least one couple win a prize? Express your answer as a common fraction.

(9) ________ Ten distinct points are identified on the circumference of a circle. How many different convex quadrilaterals can be formed if each vertex must be one of these 10 points?

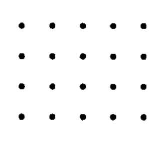
(10) ________ In an algebra class, half of the students are boys. One-third of the students are wearing glasses. Half the boys are wearing glasses. What fraction of the girls is wearing glasses? Express your answer as a common fraction.

(11) ________ Suelyn counts up from 1 to 9, and then immediately counts down again to 1, and then back up to 9, and so on, alternately counting up and down

$$(1, 2, 3, 4, 5, 6, 7, 8, 9, 8, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4, \ldots).$$

What is the 1000^{th} integer in her list?

(12) ________ Set R is a set of rectangles such that (1) only the grid points shown here are used as vertices, (2) all sides are vertical or horizontal and (3) no two rectangles in the set are congruent. If R contains the maximum possible number of rectangles given these conditions, what fraction of the rectangles in set R are squares? Express your answer as a common fraction.



(13) __________ How many non-congruent triangles are there with sides of integer length having at least one side of length five units and having no side longer than five units?

(14) __________ The n^{th} triangular number is the number $1+2+3+4+\ldots+n$. Thus, the third triangular number is $1+2+3=6$. What is the sum of the fourth and fifth triangular numbers?

(15) __________ During a party, a total of 78 handshakes occurred. If each person shook hands once with each of the other people, how many people were at the party?

(16) __________ How many ordered pairs (x, y) satisfy BOTH conditions below?

- • Condition I: $x=1$ or $y=0$ or $y=2$
- • Condition II: $x=0$ or $x=2$ or $y=1$

(17) __________ A reference book lists a set of annual calendars. For any given year, there is a calendar in the set that corresponds to it. How many annual calendars must be included in the set in order to have a corresponding calendar for every possible year?

(18) __________ John, Kevin, Larry, Mary and Naomi all volunteered to do some math tutoring. If their teacher randomly chooses two of the five students, what is the probability of selecting the two girls? Express your answer as a common fraction.

(19) __________ Erika, who is 14 years old, flips a fair coin whose sides are labeled 10 and 20, and then she adds the number on the top of the flipped coin to the number she rolls on a standard die. What is the probability that the sum equals her age in years? Express your answer as a common fraction.

(20) __________ Chun is playing a game in which he can score 0, 2, 6, or 10 points in each round. After four rounds the sum of his scores is 16. How many different scoring sequences could have produced this sum? (One such sequence to include is 0, 6, 10, 0.)

(21) __________ Nathan will roll two six-sided dice. What is the probability that he will roll a number less than three on the first die and a number greater than three on the second die? Express your answer as a common fraction.

(22) ________ Thad has an unlimited supply of 3-cent and 4-cent stamps. If he has to put exactly 37 cents of postage on a letter, how many different combinations of 3-cent and/or 4-cent stamps could Thad use?

(23) ________ Two different integers are randomly chosen from the set

$$\{-5, -8, 7, 4, -2\}.$$

What is the probability that their product is negative? Express your answer as a common fraction.

Answer Sheet

Number	Answer	Problem ID
1	1/105	525B
2	~~2:3~~ 2/3	55A
3	1/5	3D1
4	3/80	025B
5	20 combinations	45D
6	25,200 ways	4342
7	92 percent	5531
8	27/35	4DD
9	210 quadrilaterals	D531
10	1/6	02B
11	8	B5D
12	2/5	44
13	9 triangles	25D
14	25	1131
15	13 people	D2B
16	5 ordered pairs	A242
17	14 calendars	C241
18	1/10	BD31
19	1/12	50B
20	22 sequences	C3112
21	1/6	D031
22	3 combos	A531
23	3/5	3431

Solutions

(1) **1/105** **ID: [525B]**
The total number of marbles is $2+3+10=15$. The probability that the first marble drawn will be red is $2/15$. Then, there will be one red left, out of 14. Therefore, the probability of drawing out two red marbles will be:

$$\frac{2}{15}\cdot\frac{1}{14}=\boxed{\frac{1}{105}}$$

(2) **2:3** **ID: [55A]**
The odds against an event are calculated by taking the ratio of the probability of the event not happening to the probability of the event happening. In this case, that's $40\% : 60\% = \boxed{2:3}$.

(3) **1/5** **ID: [3D1]**
There are a total of $5\times 5=25$ possibilities. Since $2\times 5=10$, which is not GREATER than 10, we know that Josh must draw a 4. Thus, his possibilities are:
$(3,4); (4,3); (4,4); (4,5); (5,4)$, making for 5 possibilities, and a probability of $\frac{5}{25}=\boxed{\frac{1}{5}}$.

(4) **3/80 ID: [025B]**

If both dice were numbered 1 through 20, we could get a sum of 24 in the following ways:

$$\begin{gathered} 4+20 \\ 5+19 \\ 6+18 \\ \vdots \\ 18+6 \\ 19+5 \\ 20+4 \end{gathered}$$

This is a total of $20-4+1=17$ ways. However, the first die does not have a face with 20, so we must remove the possibility of rolling $20+4$. Also, the second die does not have a face with 9, so we must remove the possibility of rolling $15+9$. This leaves $17-2=15$ possible ways to roll 24. There are a total of $20 \cdot 20 = 400$ possible rolls, so the final probability is:

$$\frac{15}{400} = \boxed{\frac{3}{80}}.$$

(5) **20 combinations ID: [45D]**

Note that at least one of the boxes must be a 1-piece box, because the number of chocolates ordered was odd. The problem is now to determine how many ways 14 pieces can be assembled using the 1, 2, and 4-piece boxes. If we begin with all 1-piece boxes, there's one way to do that. There are seven ways to have a mix of 1 and 2-piece boxes (one 2-piece, two 2-pieces, etc. all the way through seven 2-pieces). Now, each pair of 2-piece boxes can be replaced by a 4-piece box. If there is one 4-piece box, there are six ways to box the remaining ten pieces of chocolate with 1 and 2-piece boxes (no 2-pieces, one 2-piece, etc. all the way through five 2-pieces). If there are two 4-piece boxes, there are four ways to box the remaining six pieces of chocolate (zero through three 2-piece boxes). Finally, if there are three 4-piece boxes, there are two ways to box the remaining two pieces of chocolate (either no 2-piece boxes or one 2-piece box). Thus, there are a total of $1+7+6+4+2=\boxed{20}$ combinations of boxes possible.

(6) **25,200 ways ID: [4342]**

We start with the most restrictive conditions: the roles that are not open to both men and women. First, we will fill the two male roles. There are 5 men, so there are $5 \cdot 4 = 20$ ways to assign 2 of the 5 men to the distinct male roles. Doing the same for the female roles, there are $6 \cdot 5 = 30$ ways. Finally, of the $5 + 6 - 2 - 2 = 7$ remaining actors, there are $7 \cdot 6 = 42$ ways to assign the leftovers to the either-gender roles. Multiplying, there are $20 \cdot 30 \cdot 42 = \boxed{25200}$ ways of assigning the six roles.

(7) **92 percent ID: [5531]**

No solution is available at this time.

(8) **27/35 ID: [4DD]**

The only way to not have both members of at least one couple win a prize is to have one member of each couple win a prize. There are $2 \cdot 2 \cdot 2 \cdot 2 = 16$ ways to pick one member from each couple, and there are $\binom{8}{4} = 70$ ways to pick any four people out of the eight, so the probability of not having both members of at least one couple win a prize is $\frac{16}{70} = \frac{8}{35}$. Therefore, the probability that both members of at least one couple win a prize is $1 - \frac{8}{35} = \boxed{\dfrac{27}{35}}$.

(9) **210 quadrilaterals ID: [D531]**

With the ten points on the circumference of a circle, any set of 4 of them will form a convex (indeed, cyclic) quadrilateral. So, with ten points, and we can choose any 4 of them to form a distinct quadrilateral, we get $\binom{10}{4} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2} = 10 \cdot 3 \cdot 7 = \boxed{210}$ quadrilaterals.

(10) **1/6 ID: [02B]**

Call the total number of students x. If half of the students are boys, and half are wearing glasses, than $\frac{x}{4}$ is the number of boys wearing glasses. Thus, the number of girls wearing glasses is $\frac{x}{3} - \frac{x}{4}$, since one third of the students are wearing glasses. So, $\frac{x}{12}$ is the number of girls wearing glasses. Since $\frac{x}{2}$ of the students are girls, the fraction of girls wearing glasses is $\boxed{\dfrac{1}{6}}$.

(11) **8** **ID: [B5D]**

We can treat this list as a sequence with a repetitive pattern. We see the sequence repeats itself every 16 elements (from 1 to 9 then back to 2). Because 1000 divided by 16 is 62 with a remainder of 8, to get 1000 terms in this list, we repeat the block 62 times, and then go 8 more elements. This means that the 1000^{th} integer is the same as the 8^{th} integer, which is $\boxed{8}$.

(12) **2/5** **ID: [44]**

If we list out the possible sizes of the rectangles, we have:

$$\begin{aligned}&1\times 1,\ 1\times 2,\ 1\times 3,\ 1\times 4,\\&2\times 2,\ 2\times 3,\ 2\times 4,\\&3\times 3,\ 3\times 4,\ \text{and}\\&4\times 4.\end{aligned}$$

Thus, there are ten possible sizes, each of which must be represented by one rectangle in set R. Four of these sizes are squares, so the fraction in R that are squares is $\frac{4}{10} = \boxed{\frac{2}{5}}$.

(13) **9 triangles** **ID: [25D]**

At least one side must be exactly 5 units long. Let a and b be the other two sides, and without loss of generality, assume that $a \leq b$. If $a = 1$, then b must be 5. If $a = 2$, then b can be 4 or 5. If $a = 3$, then b can be 3, 4, or 5. If $a = 4$, then b can be 4 or 5. If $a = 5$, then b must be 5. In total, there are $\boxed{9}$ non-congruent triangles that meet the conditions.

(14) **25** **ID: [1131]**

The fourth triangular number is the third plus 4, which is $6 + 4 = 10$. Similarly, the fifth triangular number is the fourth plus 5, which is $10 + 5 = 15$. Thus, the sum of the fourth and fifth triangular numbers is $10 + 15 = \boxed{25}$.

(15) **13 people** **ID: [D2B]**

Since each person shakes hands with each other person, every pair of people will shake hands once. So, 78 represents the number of pairs, which we can count as $\binom{n}{2}$ where n is the number of people at the party. So, $n(n-1) = 2 \cdot 78 = 2 \cdot 6 \cdot 13 = 12 \cdot 13$. So, $n = 13$ gives us $\boxed{13}$ people at the party.

(16) **5 ordered pairs** **ID: [A242]**

Proceed case-by-case in condition I. If $x = 1$, then by condition II, $y = 1$ since the first two possibilities are excluded. If $y = 0$, then either $x = 0$ or $x = 2$. If $y = 2$, then likewise, either $x = 0$ or $x = 2$. This gives $\boxed{5}$ possible ordered pairs.

(17) **14 calendars** **ID: [C241]**

No solution is available at this time.

(18) **1/10** **ID: [BD31]**

No solution is available at this time.

(19) **1/12** **ID: [50B]**

The only way for the sum to be a 14 is for her coin flip to be a 10 and for her roll to be a 4. This can only occur in $\frac{1}{2} \cdot \frac{1}{6} = \boxed{\frac{1}{12}}$.

(20) **22 sequences** **ID: [C3112]**

First we find the sets of four scores which produce a sum of 16 (regardless of order). They are $\{10, 6, 0, 0\}$, $\{6, 6, 2, 2\}$, and $\{10, 2, 2, 2\}$. There are $\frac{4!}{2!} = 12$ sequences corresponding to the set $\{10, 6, 0, 0\}$, $\frac{4!}{2!2!} = 6$ sequences corresponding to $\{6, 6, 2, 2\}$, and $\frac{4!}{3!} = 4$ sequences corresponding to $\{10, 2, 2, 2\}$. In total, there are $12 + 6 + 4 = \boxed{22}$ sequences resulting in a sum of 16.

(21) **1/6** **ID: [D031]**

For the first die to be less than three, it must be a 1 or a 2, which occurs with probability $\frac{1}{3}$. For the second die to be greater than 3, it must be a 4 or a 5 or a 6, which occurs with probability $\frac{1}{2}$. The probability of both of these events occuring, as they are independent, is $\frac{1}{3} \cdot \frac{1}{2} = \boxed{\frac{1}{6}}$.

(22) **3 combos** **ID: [A531]**

Thad must use at least one 3-cent stamp, because there is no way to make 37 cents from 4-cent stamps. So, the problem is now to make 34 cents from 3-cent and 4-cent stamps.

If we use one 4-cent stamp, we are left with 30 cents. This can be satisfied with ten 3-cent stamps.

If we use two 4-cent stamps, we are left with 26 cents. If we use three, we are left with 22 cents. If we use four, we are left with 18 cents. This can be satisfied with six 3-cent stamps.

If we use five 4-cent stamps, we are left with 14 cents. If we use six, we are left with 10 cents. If we use seven, we are left with 6 cents, which can only be satisfied with two 3-cent stamps.

Thus, there are $\boxed{3}$ different combinations.

(23) **3/5** **ID: [3431]**

We don't consider the order that we choose integers in this problem, meaning that choosing a -5 and then a -8 is the same as choosing a -8 and then a -5. The product of two integers is negative if one integer is positive and the other is negative. There are three ways to choose a negative integer and two ways to choose a positive integer, for a total of $3 \cdot 2 = 6$ ways to choose both integers. There are

$$\binom{5}{2} = \frac{5!}{2!3!} = 10$$

ways to choose any two different integers, so the probability that the two integers have a negative product is $6/10 = \boxed{3/5}$.