Number Theory 3A3

Name ____________________

(1) ________ What is the remainder when the sum of the first 100 positive integers is divided by 9?

(2) ________ What is the smallest integer greater than 2 that will have a remainder of 2 when divided by any member of the set $\{3, 4, 5, 6, 8\}$?

(3) ________ How many digits are in the value of the following expression: $2^{2001} \times 5^{1950} \div 4^{27}$?

(4) ________ A *palindrome* is a number which reads the same forward as backward. For example, 343 and 1221 are palindromes. What is the least natural number that can be added to 40,305 to create a palindrome?

(5) ________ What is the largest perfect square factor of 1512?

(6) ________ What is the remainder when 10! is divided by 2^7? (Reminder: If n is a positive integer, then $n!$ stands for the product $1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$.)

(7) ________ What is the sum of the last two digits of this portion of the Fibonacci Factorial Series: $1! + 1! + 2! + 3! + 5! + 8! + 13! + 21! + 34! + 55! + 89!$?

(8) ________ What is the 453rd digit to the right of the decimal point in the decimal expansion of $\frac{6}{13}$?

(9) ________ What is the number of positive factors of 648?

(10) ________ Let m and n denote the greatest and least positive three-digit multiples of 7, respectively. What is the value of $m+n$?

(11) ________ One year ago, the number of years in Jane's age was a perfect square, and one year from now, her age will be a perfect cube. How many years old is Jane?

(12) ________ The number 839 can be written as $19q+r$ where q and r are positive integers. What is the greatest possible value of $q-r$?

(13) ________ The total area of four squares, each with whole-number side measurements, is 23 square inches. In inches, what is the positive difference between the perimeter of the largest square and the perimeter of the smallest square?

(14) ________ In any month with five Mondays, a school declares the fifth Monday a Marvelous Monday. School begins on Monday, September 4. What is the date of the first Marvelous Monday after school starts?

(15) ________ What is the base two representation of the sum of the binary numbers 1011_2 and 111_2?

(16) ________ What is the units digit of $(133^{13})^3$?

(17) ________ A positive multiple of 45 less than 1000 is randomly selected. What is the probability that it is a two-digit integer? Express your answer as a common fraction.

(18) ________ What is the sum of the three distinct prime factors of 47,432?

(19) ________ How many odd perfect square factors does $2^4 \times 3^6 \times 5^{10} \times 7^9$ have?

(20) ________ How many prime positive integers are divisors of 555?

(21) __________ John, Joe and James go fishing. At the end of the day, John comes to collect his third of the fish. However, there is one too many fish to make equal thirds, so John throws it out, takes his third and leaves. Joe comes to get his fish without realizing John has already taken his third. He notices there is one too many fish to make equal thirds so he throws one fish out, takes his third and leaves. James notice that there is one too many fish to make equal thirds so he throws one out, takes his fish and leaves. Assuming no fish are divided into pieces, what is the minimum possible number of fish before John threw out the first fish?

(22) __________ If $2^x \cdot 9^y$ is equal to the four-digit number $2x9y$ where x is the hundreds digit and y is the units digit, what is the product of x and y?

(23) __________ What is the sum of the tens digit and the units digit in the decimal representation of 9^{2004}?

(24) __________ What is the tens digit of the product of the first six prime numbers?

(25) __________ Suppose a and b are different prime numbers greater than 2. How many whole-number divisors are there for the integer $a(2a+b) - 2a^2 + ab$?

(26) __________ The greatest common divisor of positive integers m and n is 6. The least common multiple of m and n is 126. What is the least possible value of $m + n$?

(27) __________ What is the base five product of the numbers 121_5 and 11_5?

(28) __________ The natural number n has exactly two natural-number factors. How many natural number factors does n^5 have?

(29) __________ Cards are numbered from 1 to 100. One card is removed and the values on the other 99 are added. The resulting sum is a multiple of 77. What number was on the card that was removed?

(30) ________ What is the remainder when $13^{13} + 5$ is divided by 6?

Answer Sheet

Number	Answer	Problem ID
1	1	53001
2	122	BB13
3	1950	ABC3
4	99	BB42
5	36	4A03
6	0	DCB3
7	5	0CA5
8	1	5BC3
9	20	35001
10	1099	1002
11	26	DAC3
12	41	2AA5
13	8	CA03
14	October 30	1DC2
15	10010	0BA5
16	7	C5B31
17	1/11	4002
18	20	AD001
19	120	2B03
20	3	DB55
21	25	B013
22	10	B5B31
23	7	5002
24	3	1CC3
25	8	ABB3
26	60	0A13
27	1331	2C55
28	6	DBA5
29	45	BBB3
30	0	BA001

Solutions

(1) **1** **ID: [53001]**
No solution is available at this time.

(2) **122** **ID: [BB13]**
No solution is available at this time.

(3) **1950** **ID: [ABC3]**
We have

$$\begin{aligned}2^{2001} \times 5^{1950} \div 4^{27} &= 2^{2001} \div 2^{54} \times 5^{1950} \\ &= 2^{1947} \times 5^{1950} \\ &= (2 \times 5)^{1947} \times 5^3 \\ &= 125 \times 10^{1947}\end{aligned}$$

Since 125×10^{1947} has three non-zero digits followed by 1947 zeros, it has a total of $\boxed{1950}$ digits.

(4) **99** **ID: [BB42]**
We are asked to find the positive difference between 40305 and the least palindrome greater than 40305. The only five-digit palindrome beginning with 403 is 40304, which is less than 40305. The next smallest possibility for the first three digits is 404, which gives the palindrome 40404. The difference between 40404 and 40305 is $\boxed{99}$.

(5) **36** **ID: [4A03]**
Let's find the prime factorization of 1512: $1512 = 2^3 \cdot 189 = 2^3 \cdot 3^3 \cdot 7$. The only two squares of primes that divide 1512 are $2^2 = 4$ and $3^2 = 9$. Therefore, the largest perfect square factor of 1512 is $2^2 \cdot 3^2 = (2 \cdot 3)^2 = \boxed{36}$.

(6) 0 **ID: [DCB3]**
10! is divisible by 2, $4 = 2^2$, $6 = 2 \cdot 3$, $8 = 2^3$, and $10{=}2 \cdot 5$, so 10! is divisible by 2^8. Therefore, the remainder when 10! is divided by 2^7 is $\boxed{0}$.

(7) **5 ID: [0CA5]**

This expression $n!$, is the number you get by multiplying n by $(n-1)$ by $(n-2)$ by $(n-3)$ and so on, all the way down to 1. So $5! = (5)(4)(3)(2)(1) = 120$. Notice that 5! ends in a 0 since it has a factor of 10 (there is a 5 and a 2 in it list of factors) and that 10! has to end in two zeroes since it has a factor of 10, 5 and 2 which is really a factor of 100. Since any factorial greater than 10 (such as 13! or 21!) includes all of the factors of 10!, the last two digits of 13!, 21!, and so on are zeroes. These terms, therefore will not affect the last two digits of the sum of the Fibonacci factorial series.

To find the last two digits, you only need to find the last two digits of each of the terms of $1! + 1! + 2! + 3! + 5! + 8!$. We do not need to calculate 8!, only to find its last two digits. Starting with 5!, we can work our way to 8!, using only the last two digits of each value along the way. We know $5! = 120$, so use 20 when finding 6!, which will bring us to $6(2) = 120$ or 20. Therefore, the last two digits of 7! are from $7(2) = 140$ or 40. Finally 8! is $8(40) = 320$ or finally 20. The last two digits of the entire series will come from $1 + 1 + 2 + 6 + 20 + 20 = 50$. Therefore, the sum of the last two digits is $5 + 0 = \boxed{5}$.

(8) **1 ID: [5BC3]**

The decimal representation of $\frac{6}{13}$ is $0.\overline{461538}$, which repeats every 6 digits. Since 453 divided by 6 has a remainder of 3, the 453rd digit is the same as the third digit after the decimal point, which is $\boxed{1}$.

(9) **20 ID: [35001]**

No solution is available at this time.

(10) **1099 ID: [1002]**

Since $98 = 7 \cdot 14$, we know that $98 + 7 = 105$ is the least three-digit multiple of 7. Furthermore, 980 is a multiple of 7, as are 987, 994, and 1001. The greatest three-digit multiple is 994. The sum of these values is

$$994 + 105 = \boxed{1099}.$$

(11) **26 ID: [DAC3]**

Let's consider a few small perfect cubes to see two less which is a perfect square: $2^3 - 2 = 6$, not a perfect square; $3^3 - 2 = 25 = 5^2$. Thus Jane is $27 - 1 = \boxed{26}$ years old.

(12) 41 **ID: [2AA5]**

In order to get the greatest possible $q - r$, we want to maximize q and minimize r. We divide 839 by 19 to find the maximum q. The quotient q is 44 and the remainder r is 3, and we can check that $839 = 19(44) + 3$. So the greatest possible value of $q - r = 44 - 3 = \boxed{41}$.

(13) **8 ID: [CA03]**

No solution is available at this time.

(14) **October 30 ID: [1DC2]**

September has 30 days. September 4 is a Monday, so September 9 is a Saturday. Since September 30 is exactly 21 days later (or 3 weeks), September 30 is also a Saturday.

Then October 1 is a Sunday, and October 2 is a Monday. Then October 2, 9, 16, 23, and 30 are all Mondays, so the first Marvelous Monday is $\boxed{\text{October 30}}$.

(15) **10010 ID: [0BA5]**

No solution is available at this time.

(16) **7 ID: [C5B31]**

No solution is available at this time.

(17) **1/11 ID: [4002]**

The positive multiples of 45 are

$$45, 90, 135, \ldots, 990 = 1 \cdot 45, 2 \cdot 45, 3 \cdot 45, \ldots, 22 \cdot 45.$$

There are 22 multiples on this list. Every positive multiple of 45 less than 1000 is either a two-digit integer or a three-digit integer. Out of the $99 - 10 + 1 = 90$ two-digit integers, 45 and 90 are multiples of 45. Therefore, the probability that the selected multiple of 45 has two digits is $2/22 = \boxed{1/11}$.

(18) **20 ID: [AD001]**

No solution is available at this time.

(19) **120 ID: [2B03]**

No solution is available at this time.

(20) 3 **ID: [DB55]**
When we find the prime factorization of 555, we end up with $3 \cdot 5 \cdot 37$, which means we have $\boxed{3}$ prime positive divisors.

(21) **25** **ID: [B013]**
No solution is available at this time.

(22) **10** **ID: [B5B31]**
No solution is available at this time.

(23) **7** **ID: [5002]**
Write 9 as $10-1$ and consider raising 9 to the 2004 power by multiplying out the expression

$$\overbrace{(10-1)(10-1)(10-1)\cdots(10-1)}^{\text{2004 factors}}$$

There will be 2^{2004} terms in this expansion (one for each way to choose either 10 or -1 for each of the 2004 factors of $(10-1)$), but most of them will not affect the tens or units digit because they will have two or more factors of 10 and therefore will be divisible by 100. Only the 2004 terms of -10 which come from choosing -1 in 2003 of the factors and 10 in the remaining one as well as the term $(-1)^{2004} = 1$ remain. Let N represent the sum of all of the terms with more than 1 factor of 10. We have

$$\begin{aligned}(10-1)^{2004} &= N + 2004(-10) + 1 \\ &= N - 20{,}040 + 1 \\ &= (N - 20{,}000) - 40 + 1 \\ &= (N - 20{,}000) - 39.\end{aligned}$$

So 9^{2004} is 39 less than a multiple of 100 and therefore ends in 61. The sum of 6 and 1 is $\boxed{7}$.

(24) **3** **ID: [1CC3]**
No solution is available at this time.

(25) **8** **ID: [ABB3]**
Distributing and combining like terms, we have
$a(2a+b) - 2a^2 + ab = 2a^2 + ab - 2a^2 + ab = 2ab$. Now a and b are different prime numbers greater than 2, so $2ab = 2^1 \cdot a^1 \cdot b^1$ has $(1+1)(1+1)(1+1) = \boxed{8}$ divisors.

(26) **60** **ID: [0A13]**

Since the GCD of m and n is 6, $m = 6x$ and $n = 6y$ for some integers x and y. Note that minimizing $m + n = 6x + 6y = 6(x + y)$ is equivalent to minimizing $x + y$.

The LCM of m and n is $126 = 2 \cdot 3^2 \cdot 7 = 6 \cdot 3 \cdot 7$, so one of x and y is divisible by 3 and one is divisible by 7. Then we can minimize $x + y$ by setting x and y to be 3 and 7 in some order. Therefore, the least possible value of $m + n$ is $6(3 + 7) = \boxed{60}$.

(27) **1331** **ID: [2C55]**

Notice that $121_5 \times 11_5 = 121_5 \times (10_5 + 1_5) = 1210_5 + 121_5 = \boxed{1331}_5$.

(28) 6 **ID: [DBA5]**

No solution is available at this time.

(29) 45 **ID: [BBB3]**

The sum of the numbers from 1 to 100 is

$$1 + 2 + \cdots + 100 = \frac{100 \cdot 101}{2} = 5050.$$

When this number is divided by 77, the remainder is 45. Therefore, the number that was removed must be congruent to 45 modulo 77.

However, among the numbers 1, 2, . . . , 100, only the number $\boxed{45}$ itself is congruent to 45 modulo 77. Therefore, this was the number of the card that was removed.

(30) **0** **ID: [BA001]**

Start off by looking for a pattern. $(13^1 + 5)/6$ leaves no remainder; $(13^2 + 5)/6$ leaves no remainder, ..., $(13^k + 5)/6$ always leaves no remainder. This is true because 13 is 1 more than a multiple of 6, so any power of 13 will also be 1 more than a multiple of 6. When 5 is added to a number that is 1 more than a multiple of 6, the result is a multiple of 6, so the remainder is $\boxed{0}$.