Number Theory 3A2

Name ____________________

(1) __________ Find the integer n such that $n \times 3^4 \times 7^5 = 21^6$.

(2) __________ What is the probability that the square root of a randomly selected two-digit whole number is less than eight? Express your answer as a common fraction.

(3) __________ What is the least natural number that has exactly four distinct positive factors?

(4) __________ For what value of n is the five-digit number $\underline{7n933}$ divisible by 33? (Note: the underlining is meant to indicate that the number should be interpreted as a five-digit number whose ten thousands digit is 7, whose thousands digit is n, and so on).

(5) __________ What is the least four-digit positive integer, with all different digits, that is divisible by each of its digits?

(6) __________ Ten days from Thursday, it will be Sunday. What day of the week will it be in 1,000,000 days from Thursday?

(7) __________ Tim and Kurt are playing a game in which players are awarded either 3 points of 7 points for a correct answer. What is the greatest score that cannot be attained?

(8) __________ How many positive perfect squares less than 300 are multiples of 9?

(9) ________ Express the next term in the sequence as a decimal:

$$0, 0.5, 0.\overline{6}, 0.75 \ldots$$

(10) ________ What perfect-square integer is closest to 273?

(11) ________ Brianna was having a party for 95 guests. Hot dogs are sold in package of eight; buns are sold in packages of ten. If she purchased the minimum number of packages of each to guarantee at least one hot dog and one bun for each guest, how many more buns than hot dogs did she buy?

(12) ________ One number is chosen from the first three prime numbers, and a second number is chosen from the first three positive composite numbers. What is the probability that their sum is greater than or equal to 9? Express your answer as a common fraction.

(13) ________ July 4, 1903, was a Thursday. On what day of the week was July 4, 1904?

(14) ________ When its digits are reversed, a particular positive two-digit integer is increased by 20%. What is the original number?

(15) ________ What is the least positive integer with exactly 10 factors?

(16) ________ The length of the year on the planet Mars is exactly 697 days. If Mars has a calendar with a 12-day week, and year 0 begins on the first day of the week, what is the next year which will begin on the first day of the week?

(17) ________ What is the minimum number of United States coins Samantha needs (pennies, nickels, dimes, quarters, half-dollars) to ensure she is capable of making change for any amount of money from one cent to 99 cents?

(18) ________ Three consecutive positive prime numbers have a sum that is a multiple of 7. What is the least possible sum?

(19) ________ The product of positive integers x, y and z equals 2004. What is the minimum possible value of the sum $x + y + z$?

(20) ________ Let A = 1, B = 2, C = 3, ..., Z = 26. The product value of a word is equal to the product of the values of its letters. For example, CAB has a product value of 3 × 1 × 2 = 6. What common English word has a product value of 715?

(21) ________ What is the least natural number that will have a remainder of 3 when divided by any of the numbers 4, 5, 6, 8 or 10?

(22) ________ What is the units digit of $1! + 3! + 5! + 7! + 9! + 11!$?

(23) ________ What is the sum of all positive integer values of n such that $\frac{n+18}{n}$ is an integer?

(24) ________ The number 100 can be written as the sum of a 1-digit prime number and a 2-digit prime number. What is the product of these prime numbers?

(25) ________ *Bertrand's Postulate* states that there is at least one prime number between any counting number and its double. How many prime numbers are there between 25 and 50?

(26) ________ What is the sum of the whole-number factors of 24?

(27) ________ The integer 6 has four positive divisors: 1, 2, 3 and 6. What is the smallest positive integer with exactly five positive divisors?

(28) __________ A Chicago subway card is worth \$11. For each ride on the subway, either \$1.50 or \$1.80 is deducted from the value remaining on the card. What is the least number of cents that could be left on the card after any number of rides?

(29) __________ A group of N students, where $N < 50$, is on a field trip. If their teacher puts them in groups of 8, the last group has 5 students. If their teacher instead puts them in groups of 6, the last group has 3 students. What is the sum of all possible values of N?

(30) __________ The product of the base seven numbers 24_7 and 30_7 is expressed in base seven. What is the base seven sum of the digits of this product?

Answer Sheet

Number	Answer	Problem ID
1	63	44AC
2	3/5	A3A2
3	6	1455
4	5	2BCC
5	1236	35D3
6	Friday	43C3
7	11	A4AC
8	5 perfect squares	53A2
9	0.8	42C3
10	289	4A42
11	4	B0001
12	2/3	51001
13	Saturday	C3AC
14	45	C422
15	48	D4C3
16	12	24A2
17	9 coins	C2D2
18	49	2422
19	174	C34C
20	MAKE	344C
21	123	DDD3
22	7	D3C2
23	39	5DC3
24	291	54C3
25	6	02001
26	60	C1B3
27	16	C343
28	20	C5B3
29	66	0BD3
30	6	DBD3

Solutions

(1) **63 ID: [44AC]**
No solution is available at this time.

(2) **3/5 ID: [A3A2]**
There are 90 choices for a two-digit positive integer. Of these, all of the integers $n < 64$ satisfy $\sqrt{n} < 8$. So, n can be chosen from the set $\{10, 11, 12, \ldots, 63\}$ which has 54 members. So the probability is $\frac{54}{90} = \boxed{\frac{3}{5}}$.

(3) 6 **ID: [1455]**
No solution is available at this time.

(4) **5 ID: [2BCC]**
Divisibility by 33 requires that a number be divisible by both 11 and 3. If a five-digit number is divisible by 11, the difference between the sum of the units, hundreds and ten-thousands digits and the sum of the tens and thousands digits must be divisible by 11. Thus $(7 + 9 + 3) - (n + 3) = 16 - n$ must be divisible by 11. The only digit which can replace n for the number to be divisible by 11, then, is $n = 5$. Furthermore, if a number is $7 + 5 + 9 + 3 + 3 = 27$, so the number is divisible by 3. Hence, $n = \boxed{5}$.

(5) **1236 ID: [35D3]**
Since the problem asks for the least possible number, you should start with the lowest number (0) and work your way up (and across the number.) Nothing is divisible by zero, so zero cannot be one of the digits in the four-digit number. Every whole number is divisible by 1, so the digit 1 should be in the thousands place to create the smallest number. The digits must be different, so put a 2 in the hundreds place. Now, you have to make sure the number is even. You can put a 3 in the tens place, but you cannot use 4 for the ones place since 1234 is not divisible by 3 or 4. 1235 is not even, so it is not divisible by 2 (or for that matter, by 3). $\boxed{1236}$ is divisible by all of its own digits.

(6) **Friday ID: [43C3]**
No solution is available at this time.

(7) **11 ID: [A4AC]**
No solution is available at this time.

(8) **5 perfect squares ID: [53A2]**
Since $9 = 3^2$, any perfect square that is a multiple of 9 must be in the form $9n^2$ for some positive integer n. For $1 \leq n \leq 5$, $9n^2$ is less than 300 and for $n \geq 6$, $9n^2$ is greater than 300. Therefore there are $\boxed{5}$ positive perfect squares less than 300 that are multiples of 9.

(9) **0.8 ID: [42C3]**
To find the pattern of the sequence, we begin by converting each of the decimal values into a common fraction. The first term 0 is equal to $\frac{0}{1}$. The next term, 0.5, can be written as $\frac{5}{10} = \frac{1}{2}$. To express $0.\overline{6}$ as a common fraction, we call it x and subtract it from $10x$:

$$\begin{array}{rrcl} & 10x & = & 6.66666\ldots \\ - & x & = & 0.66666\ldots \\ \hline & 9x & = & 6 \end{array}$$

This shows that $0.\overline{6} = \frac{6}{9} = \frac{2}{3}$. The fourth term in the series, 0.75, becomes $\frac{75}{100} = \frac{3}{4}$. Thus, when we write fractions instead of decimals, our sequence is:

$$\frac{0}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$$

By observing this sequence, we realize that the first term of the sequence is $\frac{0}{1}$ and each successive term is found by adding 1 to both the numerator and denominator of the previous term. Thus, the next term in the sequence is $\frac{3+1}{4+1} = \frac{4}{5} = \boxed{0.8}$.

(10) **289 ID: [4A42]**
Note that $16^2 = 256 < 273$ while $17^2 = 289 > 273$. Since all other perfect squares are farther away from 273, our answer is $\boxed{289}$.

(11) **4 ID: [B0001]**
The minimum number of packages of hot dogs Brianna can buy is 12, because $95 \div 8$ is 11 with a remainder of 7. This means if she only buys 11 packages, she will be short 7 hot dogs, so she should buy another package. This means there are $12 \times 8 = 96$ total hot dogs. Likewise, Brianna should buy 10 packages of buns, which will give her 100 total buns, leaving 5 extra. If there are 100 buns and 96 hot dogs, there are $\boxed{4}$ more buns than hot dogs.

(12) **2/3** **ID: [51001]**
No solution is available at this time.

(13) **Saturday** **ID: [C3AC]**
No solution is available at this time.

(14) **45** **ID: [C422]**
Let AB be the integer with the property that when its digits are reversed, it is increased by 20%. Thus we have $1.2(10A + B) = 10B + A \Leftrightarrow 12A + 1.2B = 10B + A \Leftrightarrow 11A = 8.8B \Leftrightarrow 110A = 88B \Leftrightarrow 5A = 4B$. Keeping in mind that $0 < A, B \leq 9$, we look for a multiple of 5 that is also a multiple of 4. Since $(5, 4) = 1$, we are looking for a multiple of $5 \times 4 = 20$. Taking 40 gives $B = 10$, which is not a digit, so the only possibility is $5A = 4B = 20$, which gives $A = 4$ and $B = 5$. Thus, the original number is $AB = \boxed{45}$.

(15) **48** **ID: [D4C3]**
We need to find the smallest integer, k, that has exactly 10 factors. $10 = 5 \cdot 2 = 10 \cdot 1$, so k must be in one of two forms:

- (1) $k = p_1^4 \cdot p_2^1$ for distinct primes p_1 and p_2. The smallest such k is attained when $p_1 = 2$ and $p_2 = 3$, which gives $k = 2^4 \cdot 3 = 48$.
- (2) $k = p^9$ for some prime p. The smallest such k is attained when $p = 2$, which gives $k = 2^9 > 48$.

Thus, the least positive integer with exactly 10 factors is $\boxed{48}$.

(16) **12** **ID: [24A2]**
Since $697 = 12 \cdot 58 + 1$, each Martian year consists of 58 weeks and a day. Therefore, for every year that passes, the first day of the year shifts to the next day of the week. Since year 0 begins on the first day, year 1 begins on the second day, then year 2 begins on the third day, and so on. A week consists of 12 days, so the next year that begins on the first day again will be year $\boxed{12}$.

(17) **9 coins** **ID: [C2D2]**
No solution is available at this time.

(18) **49** **ID: [2422]**

We are interested in the remainders when prime numbers are divided by 7. The first ten primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29. The remainders when these prime are divided by 7 are 2, 3, 5, 0, 4, 6, 3, 5, 2, 1, respectively. Starting with the first triple, add the remainders to see whether the sum is a multiple of 7, in which case the corresponding prime numbers have a sum that is a multiple of 7. We see that $6+3+5=14$. Thus, the least possible sum is $13+17+19=\boxed{49}$.

(19) **174** **ID: [C34C]**

Prime factorize $2004 = 2^2 \cdot 3 \cdot 167$. One of the summands x, y, or z should be 167, for otherwise the summand which has 167 as a prime factor is at least $2 \cdot 167$. The other two summands multiply to give 12, and the minimum sum of two positive integers that multiply to give 12 is $4+3=7$. Therefore, the minimum value of $x+y+z$ is $167+4+3=\boxed{174}$.

(20) **MAKE** **ID: [344C]**

Prime factorize 715 to find $715 = 5 \cdot 11 \cdot 13$. The only ways to write 715 as a product of positive integers greater than 1 are the distinct ways of grouping the prime factors:

$$\begin{aligned}(5)\cdot(11)\cdot(13) &= 5\cdot 11\cdot 13\\ (5\cdot 11)\cdot 13 &= 55\cdot 13\\ 5\cdot(11\cdot 13) &= 5\cdot 143\\ (5\cdot 13)\cdot 11 &= 65\cdot 11, \text{ and}\\ (5\cdot 11\cdot 13) &= 715,\end{aligned}$$

where the last one is a product with only one factor. Since the letters cannot represent numbers greater than 26, only $5 \cdot 11 \cdot 13$ could come from calculating the product value of a word. The 5th, 11th, and 13th letters of the alphabet are E, K, and M. Since E, K, and M do not form a word, we introduce the letter A (which doesn't affect the product since its value is 1) to form the word $\boxed{\text{MAKE}}$.

(21) 123 **ID: [DDD3]**

No solution is available at this time.

(22) **7** **ID: [D3C2]**

We observe that for all $n > 5$, $n!$ has a units digit of 0, because 5! has a factor of 5 and 2, which become a factor of 10. So, the terms in the sum, 5!, 7!, 9!, and 11! all have a 0 for the units digit. And, $1! + 3! = 1 + 6 = \boxed{7}$ is the units digit of the sum.

(23) **39** **ID: [5DC3]**
$\frac{n+18}{n} = 1 + \frac{18}{n}$. Thus, $\frac{n+18}{n}$ is an integer if and only if $n|18$. The positive factors of 18 are 1, 18, 2, 9, 3, and 6. Their sum is $\boxed{39}$.

(24) **291** **ID: [54C3]**
The 1-digit primes are 2, 3, 5, and 7. Let's check each:

- • $100 - 2 = 98$ is a composite.
- • $100 - 3 = 97$ is a prime.
- • $100 - 5 = 95$, is a composite.
- • $100 - 7 = 93$ is a composite.

(Check primes less than $\sqrt{100} = 10$ as potential divisors.) Thus $100 = 3 + 97$. Our answer is $3 \times 97 = \boxed{291}$.

(25) **6** **ID: [02001]**
No solution is available at this time.

(26) 60 **ID: [C1B3]**
The sum of the whole-number factors of 24 is $1 + 24 + 2 + 12 + 3 + 8 + 4 + 6 = \boxed{60}$.

(27) **16** **ID: [C343]**
No solution is available at this time.

(28) 20 **ID: [C5B3]**
No solution is available at this time.

(29) 66 **ID: [0BD3]**
We are given that $N \equiv 5 \pmod{8}$ and $N \equiv 3 \pmod{6}$. We begin checking numbers which are 5 more than a multiple of 8, and we find that 5 and 13 are not 3 more than a multiple of 6, but 21 is 3 more than a multiple of 6. Thus 21 is one possible value of N. By the Chinese Remainder Theorem, the integers x satisfying $x \equiv 5 \pmod{8}$ and $x \equiv 3 \pmod{6}$ are those of the form $x = 21 + \text{lcm}(6, 8)k = 21 + 24k$, where k is an integer. Thus the 2 solutions less than 50 are 21 and $21 + 24(1) = 45$, and their sum is $21 + 45 = \boxed{66}$.

(30) **6** **ID: [DBD3]**

We can ignore the 0 digit for now, and find the product of $24_7 \times 3_7$. First, we need to multiply the units digit: $4_7 \times 3_7 = 12_{10} = 15_7$. Hence, we write down a 5 and carry-over the 1. Evaluating the next digit, we need to multiply $2_7 \times 3_7 + 1_7 = 7_{10} = 10_7$. Thus, the next digit is a 0 and 1 is carried over. Writing this out:

$$\begin{array}{r} \overset{1}{2}4_7 \\ \times \quad 3_7 \\ \hline 105_7 \end{array}$$

We can ignore the 0 in 30_7, since it does not contribute to the sum. Thus, the answer is $1 + 0 + 5 = \boxed{6}$.

Notice that the base seven sum of the digits of a number leaves the same remainder upon division by 6 as the number itself.