Olympiad Problems

2009-2010

Division M

WITH
ANSWERS AND SOLUTIONS

MATH
OLYMPIADS
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Mathematical Olympiads for Elementary and Middle Schools

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Mathematical Olympiads — November 18, 2009

1A *Time: 3 minutes*
Express as a single number:

$$125 \times 25 \times 5 \times 2 \times 4 \times 8$$

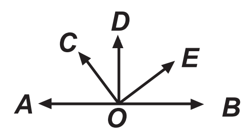
1B *Time: 3 minutes*
Find the least whole number N so that $123 + N$ is a perfect square.

1C *Time: 5 minutes*
How many numbers between 19 and 79 are the product of two even numbers?

1D *Time: 6 minutes*
Points A, B, C, and D lie on a straight line in the given order. $AC = 25$ cm and $BD = 46$ cm. The ratio of length CD to length AB is $\frac{5}{2}$. Find the length of line segment BC in cm.

1E *Time: 7 minutes*
A bookseller has 15 different novels: 4 are in German, 5 are in Spanish, and 6 are in French. Emma buys two novels. They are written in two different languages. In how many ways can this be done? Ignore the order in which she buys them.

2A *Time: 3 minutes*
As shown, $\overleftrightarrow{AOB}$ is a straight line; $\overrightarrow{OC}$, $\overrightarrow{OD}$, and $\overrightarrow{OE}$ are rays. $\angle$COE and $\angle$DOB each contain 90°. $\angle$COB contains 130°. Find the number of degrees in $\angle$DOE.



2B *Time: 5 minutes*
Find the value of the following:

$$\frac{2.3 \times 2.01 + 3.7 \times 2.01}{0.3 \times 4.02}$$

2C *Time: 6 minutes*
Mr. Alvarez gives each of his students 4 sheets of paper and 16 sheets are left over. But if two students were absent, each of the remaining students would receive 5 sheets, with only 3 sheets left over. How many sheets of paper does Mr. Alvarez have?

2D *Time: 5 minutes*
The sum $1 + 3 + 5 + \cdots + 21 + 23 + 25$ is 169.
Find the sum $1 + 5 + 9 + \cdots + 41 + 45 + 49$, in which each successive term, after the first, is 4 greater than the previous term.

2E *Time: 5 minutes*
Jess runs an outdoor stand at City Stadium. When it rains, Jess earns \$1500 selling umbrellas. But when it doesn't rain, she earns \$400 selling sunglasses. On any given day, the chance of rain is 40%. On the average, how much can Jess expect to earn daily?

3A *Time: 3 minutes*

Find the number of digits to the left of the decimal point when 500 million is divided by one hundred seventy thousand.

3B *Time: 5 minutes*

Kim multiplies all the counting numbers from 30 through 2 inclusive:

$$30 \times 29 \times 28 \times 27 \times \cdots \times 4 \times 3 \times 2.$$

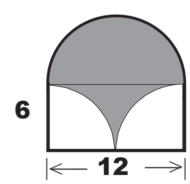
If this expression is rewritten as the product of prime numbers, how many times will 7 be used as a factor?

3C *Time: 4 minutes*

Chloe and Jack play 3 games. The probability that Chloe wins any game is $\frac{3}{5}$. What is the probability that Chloe wins for the first time in the third game?

3D *Time: 5 minutes*

A semicircle rests atop a 12 cm by 6 cm rectangle. Two quarter-circles, each of radius 6 cm are removed from the bottom corners of the rectangle. Find the number of square cm in the area of the shaded region thus formed.



3E *Time: 7 minutes*

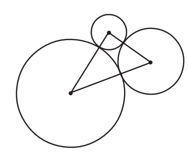
Find whole numbers a, b, and c so that

$$a + \cfrac{1}{b + \cfrac{1}{c}} = \frac{45}{7}$$

4A ***Time: 3 minutes***
Suppose $52 \times 50 \times N = 40 \times 13 \times 35$.
Find the whole number N.

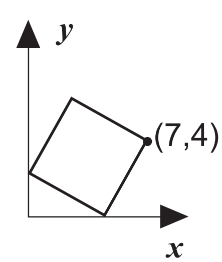
4B ***Time: 5 minutes***
Two consecutive positive integers are each less than 100. One integer is divisible by 17 and the other integer is divisible by 21. Find the greater of the two integers.

4C ***Time: 5 minutes***
Three circles are externally tangent as shown. Their areas are 9π, 25π, and 100π sq cm. A triangle is formed by connecting the centers of the three circles. Find the perimeter of the triangle, in cm.



4D ***Time: 6 minutes***
The four-digit whole number 3▮11 is exactly divisible by 13.
Find the missing digit ▮.

4E ***Time: 7 minutes***
A square is positioned in quadrant I on graph paper so that two vertices lie on the axes, while a third vertex lies at the point (7,4).
Find the area of the square.



5A ***Time: 3 minutes***
For any two numbers ***a*** and ***b***, define the value of ***a*** ✷ ***b*** as ***a*** + 3 × ***b***. For example, 4✷5 means 4 + 3 × 5 = 19. If 2 ✷ 6 and *N* ✷ 4 represent the same number, what is the value of *N*?

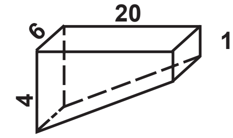
5B ***Time: 5 minutes***
Express the product as a fraction in simplest terms.

$$\frac{1}{3} \times \frac{2}{4} \times \frac{3}{5} \times \frac{4}{6} \times \frac{5}{7} \times \frac{6}{8} \times \frac{7}{9} \times \frac{8}{10}.$$

5C ***Time: 5 minutes***
The sum of the integers from –10 through *N*, inclusive, equals 50. Find *N*.

5D ***Time: 5 minutes***
The Pumas lost 7 of their first 9 games. By winning 75% of their remaining games, they ended with victories in exactly $\frac{2}{3}$ of all their games. In all, how many games did they win?

5E ***Time: 7 minutes***
The rectangular top of an in-ground swimming pool is 20 m by 6 m. The pool is 4 m deep at one end and 1 m deep at the other. How many cubic meters of water can the pool hold?



Not drawn to scale.
All measures in meters.

Note: Number in parentheses indicates percent of all competitors with a correct answer.

OLYMPIAD 1 NOVEMBER 18, 2009

Answers: [1A] **1,000,000** [1B] **21** [1C] **15** [1D] **11** [1E] **74**

61% correct

1A *Strategy: Rearrange the factors in a more convenient order.*

$$125 \times 25 \times 5 \times 2 \times 4 \times 8$$
$$= (5\times2) \times (25\times4) \times (125\times8)$$
$$= 10 \times 100 \times 1000$$
$$= \mathbf{1{,}000{,}000}.$$

59%

1B *Strategy: Look for perfect squares near the given number.*
Since $11^2 = 121$ and $12^2 = 144$, the least whole number value for N is the value for which $123 + N = 144$. **The least whole number *N* is 21.**

36%

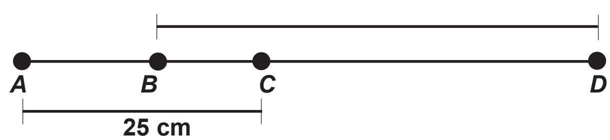
1C **METHOD 1:** *Strategy: Use the definition of even number.*
Because each of the two even numbers has a factor of 2, their product is a multiple of 4. Conversely, every multiple of 4 can be written as the product of two even numbers. The products we seek are the multiples of 4 between 19 and 79, ranging from 20 ($= 4 \times 5$) to 76 ($= 4 \times 19$), inclusive. The possible products are all but the first four of the 19 multiples of 4, ending with 76. **Thus, 15 numbers between 19 and 79 are the product of two even numbers.**

METHOD 2: *Factor each even number between 19 and 79.*
List the even numbers in the interval: 20, 22, 24, …, 74, 76, 78. Try factoring each into the product of 2 even numbers. Eliminate those that cannot be so factored. The remaining numbers are 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, 60, 64, 68, 72, and 76. In all, there are 15 possible products between 19 and 79.

FOLLOW-UP: *How many multiples of 3 that are less than 500 are not the product of two multiples of 3?* [111]

20%

1D **METHOD 1:** *Strategy: First find the length* AB.



Not drawn to scale

$AB + BC = 25$ cm and $BC + CD = 46$ cm. Hence CD is 21 cm longer than AB. But $CD = 2.5 \times AB$. Then $(2.5 \times AB) - AB = 21$. Thus, $1.5 \times AB = 21$ and the length $AB = 21 \div 1.5 = 14$. Finally, **the length of $\overline{BC}$ is** $25 - 14 =$ **11 cm.**

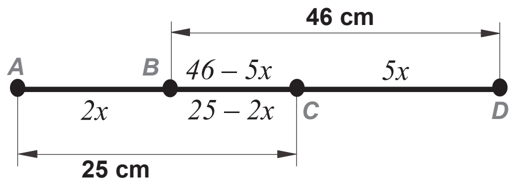
METHOD 2: *<u>Strategy</u>: Use possible lengths AB and CD to find BC.*

The table shows possible lengths *CD* and *AB*, each pair in the ratio of 5 to 2, and calculates the length *BC* in two ways. Only when *BC* is 35 and *AB* is 14 do both computations give the same value of *BC*. The length of $\overline{BC}$ is 11 cm.

CD	20	25	30	**35**	40
AB	8	10	12	**14**	16
BC* = 46 – *CD	26	21	16	**11**	6
BC* = 25 – *AB	17	15	13	**11**	9

METHOD 3: *<u>Strategy</u>: Use algebra.*

Because *CD*:*AB* = 5:2, represent the length *CD* by $5x$ and the length *AB* by $2x$. Then the length *BC* can be represented two ways, by $25 - 2x$ and by $46 - 5x$.



Not drawn to scale

Thus $25 - 2x = 46 - 5x$. Solving, $3x = 21$ and $x = 7$. Then *BC* = 46 – 5(7) = 11. Checking, *BC* = 25 – 2(7) = 11.

FOLLOW-UP: *Points A, B, C, D, and E lie on a straight line in the given order. The ratios of lengths are as follows — BC:AB = 4:3 and BC:CD = 2:1. If AE is 48 units and DE = BA, what is BC?* [16 units]

22%

1E METHOD 1: *<u>Strategy</u>: Use counting principles.*

Emma buys a German novel and a Spanish novel, or a German novel and a French novel, or a Spanish novel and a French novel. She has 4 choices for the German novel and 5 for the Spanish novel, so she can buy novels in those two languages in 4 × 5 = 20 ways. Likewise, she can buy a German novel and a French novel in 4 × 6 = 24 ways, and she can buy a Spanish novel and a French novel in 5 × 6 = 30 ways. In all **Emma can purchase two novels in two languages in** 20 + 24 + 30 = **74 ways.**

METHOD 2: *<u>Strategy</u>: Make an organized list.*

Denote the 4 German novels as G_1, G_2, G_3, G_4, and similarly for the others.

If G_1 is one of the books, the other book can be S_1, S_2, S_3, S_4, S_5, *or* F_1, F_2, F_3, F_4, F_5, F_6 — that is 11 ways. Likewise, there are 11 ways if G_2 is chosen, 11 with G_3, and 11 with G_4 for a total of 44 ways.

If no German novel is chosen, one book must be in Spanish and one in French.

If S_1 is one book, F_1, F_2, …, F_6 is the other: 6 ways. Continuing, there are 6 with S_2, etc., for a total of 30 ways. In all Emma can purchase the two novels in 44 + 30 = 74 ways.

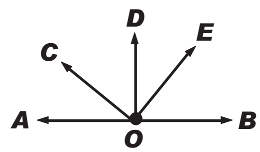
FOLLOW-UP: *(1) In how many ways can Emma purchase, in any order, two novels in the same language?* [31] *(2) In how many ways can she purchase, in any order, two novels in one language and one in each of the other two languages?*[720]

OLYMPIAD 2 DECEMBER 16, 2009

Answers: [2A] **50** [2B] **10** [2C] **108** [2D] **325** [2E] **840**

47% correct

2A METHOD 1: *<u>Strategy</u>: First find m∠COD.*



(1) $m\angle COB = 130$
$-\ m\angle DOB = 90$
$m\angle COD = 40$

(2) $m\angle COE = 90$
$-\ m\angle COD = 40$
$\mathbf{m\angle DOE = 50}$

METHOD 2: *<u>Strategy</u>: First find m∠AOC.*

(1) $m\angle AOB = 180$
$-\ m\angle COB = 130$
$m\angle AOC = 50$

(2) $m\angle AOD = 90$
$-\ m\angle AOC = 50$
$m\angle COD = 40$

(3) $m\angle COE = 90$
$-\ m\angle COD = 40$
$m\angle DOE = 50$

38%

2B *<u>Strategy</u>: Use the distributive property.*

$$\frac{2.3 \times 2.01 + 3.7 \times 2.01}{0.3 \times 4.02} = \frac{(2.3 + 3.7) \times 2.01}{0.3 \times 4.02} = \frac{6.0 \times 2.01}{0.3 \times 4.02} = \frac{6.0}{0.3} \times \frac{2.01}{4.02} = 20 \times \frac{1}{2} = \mathbf{10}.$$

<u>Variation</u>: Notice that 4.02 is twice 2.01. Rewrite the denominator as 0.6 × 2.01.

Then $\frac{6.0 \times 2.01}{0.3 \times 4.02} = \frac{6.0 \times \cancel{2.01}}{\mathbf{0.6 \times \cancel{2.01}}} = \frac{6.0}{0.6} = \frac{60}{6} = 10.$

25%

2C METHOD 1: *<u>Strategy</u>: Combine the two cases.*

Suppose 2 students are absent and Mr. Alvarez still gives each student 4 sheets. He will have the original 16 sheets left over and in addition the 4 sheets that he would have given to each of the absentees. This total of 24 sheets is enough to give each of the students who are present 1 additional sheet, with 3 left over. Then there are 24 – 3 = 21 students present. **Mr. Alvarez has** 5 × 21 + 3 = **108 sheets of paper.**

METHOD 2: *<u>Strategy</u>: Use algebra.*

Let N = the number of students in the class.

The number of sheets is the same whether or not the 2 students are absent:

$$4N + 16 = 5(N - 2) + 3$$

Solving, $N = 23$.

Then Mr. Alvarez has 4 × 23 + 16 = 108 sheets of paper.

FOLLOW-UP: *(1) Find the least whole number* N *such that* N *is 1 more than a multiple of 3,* N *– 3 is 2 more than a multiple of 5, and* N *– 6 is 3 more than a multiple of 7.* [100; Hint: Find a number near *N* that is divisible by 3, 5, and 7.]

43%

2D METHOD 1: *<u>Strategy</u>: Compare terms.*

The terms in Series A increase by 2. The terms in Series B increase by 4. If Series A is multiplied by 2 (row 2), its terms will also increase by 4.

Series A:	1 + 3 + 5 + 7 + … + 21 + 23 + 25 = 169
2 × Series A:	2 + 6 + 10 + 14 + ⋯ + 42 + 46 + 50 = 338
Series B:	1 + 5 + 9 + 13 + ⋯ + 41 + 45 + 49 = ?

Each of the 13 terms in row 2 is 1 greater than the corresponding term in row 3. Therefore **the sum 1 + 5 + 9 + ⋯ + 41 + 45 + 49** = 338 – 13 **is 325.**

METHOD 2: *Strategy: Use "Gaussian Addition".*
In the series 1 + 5 + ⋯ + 49, we get from 1 to 49 by adding 4 twelve times, so the series has 13 terms. Pair these terms as follows, working from the outside inward.
(1 + 49) + (5 + 45) + (9 + 41) , and so on. The sum of each pair is 50 and there are 6 pairs. The number without a pair is 25, the middle number. The sum is then 6 × 50 + 25 = 325.

METHOD 3: *Strategy: Look for a pattern in the partial sums.*
The table at the right examines the sums of the first few terms. In each case, the sum is the product of the number of terms by the middle term (or the average of the 2 middle terms). Since the series has 13 terms, the sum we are looking for is 1+5+9+…+41+45+49 = 13 × 25 = 325.

Series	Sum	Sum, Factored
1	1	1 × 1
1+5	6	2 × 3
1+5+9	15	3 × 5
1+5+9+13	28	4 × 7
⋮	⋮	⋮

Other approaches are possible. How many can you find?.

Follow-Ups: *(1) In the series 1 + 5 + 9 + …, find the formula for the value of the n^{th} term.* [$4n - 3$] *(2) What is the formula for calculating the sum of this series?* [$n(2n - 1)$; see table] *(3) If $x^3 = 3 \times 6 \times 12 \times 24 \times 48 \times 96$, what is the value of x?* [288]

31%

2E **METHOD 1:** *Strategy: Use a frequency definition of probability.*
Consider a large and convenient number of days, say 100. Rain is expected for 40 days and fair weather for 60 days. Jess would expect to earn a total of (40 × \$1500) + (60 × \$400) = \$84,000. During the 100 day period, **Jess expects to earn** 60,000 + 24,000 = \$84,000, which is **an average of \$840 daily.**

METHOD 2: *Strategy: Pretend the average weather actually happens one day..*
Consider an "average" day. Assume it rains 40% of that day. During that time Jess earns 40% of \$1500, which is \$600 that day. It is fair the other 60% of the day, so Jess earns 60% of \$400, which is another \$240. Thus, on that "average" day, Jess expects to earn \$840.

OLYMPIAD 3 JANUARY 13, 2010

Answers: [3A] **4** [3B] **4** [3C] $\mathbf{\frac{12}{125}}$ [3D] **72** [3E] (a,b,c =) **6,2,3**

49% correct

3A **METHOD 1:** *Strategy: Write the numbers in standard form.*
500,00$\cancel{0}$,$\cancel{0}\cancel{0}\cancel{0}$ ÷ 17$\cancel{0}$,$\cancel{0}\cancel{0}\cancel{0}$ = 50,000 ÷ 17, which is between 2000 and 3000. **There are 4 digits to the left of the decimal point.**

METHOD 2: *Strategy: Use scientific notation.*
500,000,000 ÷ 170.000 = $(5 \times 10^8) \div (1.7 \times 10^5)$ = 2.■■ × 10^3. Multiplying a one-digit number by 1000 produces 4 digits to the left of the decimal point.

36%

3B <u>Strategy</u>: *Count the number of multiples of 7.*
Only multiples of 7 contain factors of 7. There are four multiples of 7 less than 30 and each contains 7 as a factor exactly once. Thus, **7 appears as a factor of the product 4 times.**

> ***Follow-Up:*** *(1) The product is often written as 30! ("30 factorial") How many factors of 3 does 30! have?* [14] *(2) Why is the answer 14 instead of 10 or 13?* [9 and 18 each contain 3^2 and 27 contains 3^3] *(3) 30! has how many factors of 10?* [7] *(2) Why is the answer 7 instead of 3?* [10 is not prime. Look for the number of times 5 appears as a factor.]

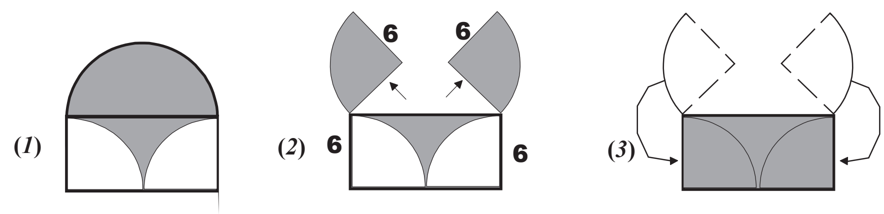
15%

3C <u>Strategy</u>: *Consider all three games.*
To win for the first time in the third game, Chloe must lose the first two games and then win the third. Since the probability that Chloe wins a game is $\frac{3}{5}$, then the probability that she does *not* win a game is $\frac{2}{5}$. By the multiplication principle, **the probability that Chloe loses the first two games and then wins the third is** $\frac{2}{5} \times \frac{2}{5} \times \frac{3}{5} = \mathbf{\frac{12}{125}}$.

> ***Follow-Ups:*** *(1) Using the same information, what is the probability that Chloe wins for the second time in the third game?* $\left[\frac{36}{125}\right]$ *(2) What is the probability that Chloe wins at least 2 games?* $\left[\frac{81}{125}\right]$

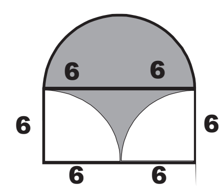
42%

3D **METHOD 1:** <u>Strategy</u>: *Rearrange the regions more conveniently.*



The area of the shaded region is the same as the area of the rectangle, which is 6 × 12 = **72 sq cm.**

METHOD 2: <u>Strategy</u>: *Add and subtract areas.*
The radius of the given semicircle is 6 cm.



From the sum of the areas of the given semicircle and the rectangle, subtract the areas of the two quarter circles:

<u>The area of the given semicircle</u>:
[$\pi 6^2 \div 2$] sq cm + [12 × 6] sq cm – [($\pi 6^2 \div 4$) × 2] sq cm =
18π sq cm + 72 sq cm – 18π sq cm =
72 sq cm.

The area of the shaded region is 72 sq cm.

15%

3E <u>Strategy</u>: *Write the fraction as a mixed number.*

$a + \dfrac{1}{b + \frac{1}{c}} = \dfrac{45}{7} = 6 + \dfrac{3}{7}$, so $\boldsymbol{a} = \mathbf{6}$.

Now write $\frac{3}{7}$ so that its numerator is 1.

$$\frac{1}{b+\frac{1}{c}} = \frac{3}{7} = \frac{1}{\frac{7}{3}}$$

Therefore, $b + \frac{1}{c} = \frac{7}{3} = 2\frac{1}{3}$.

Thus, $b + \frac{1}{c} = 2 + \frac{1}{3}$. Then $\boldsymbol{b}$ **= 2** and $\boldsymbol{c}$ **= 3.**

OLYMPIAD 4 — FEBRUARY 10, 2010

Answers: [4A] **7** [4B] **85** [4C] **36** [4D] **2** [4E] **25**

69% correct

4A **METHOD 1:** *<u>Strategy</u>: Express each side as the product of factors.*
Rewrite each side of the equation as the product of common factors.

$52 \times 50 \times N = 40 \times 13 \times 35$
$4 \times 13 \times 5 \times 10 \times N = 4 \times 10 \times 13 \times 5 \times 7$
$= 4 \times 13 \times 5 \times 10 \times 7$

Notice the common factors of 4, 13, 5 and 10 on each side. ***N* is 7.**

METHOD 2: *<u>Strategy</u>: Use algebra.*
Do the multiplication to get 2600N = 18,200. Solving, N = 7.

FOLLOW-UP: *Find the least common multiple of 520 and 280.* [3640]

62%

4B **METHOD 1:** *<u>Strategy</u>: List the multiples of each.*
Write two lists of multiples between 0 and 100, as shown below.

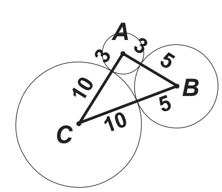
Multiples of 17: 17, 34, 51, 68, **85**, …
Multiples of 21: 21, 42, 63, **84**, …

The only pair of consecutive integers is 84 and 85. Therefore, **the greater of the two consecutive integers is 85.**

FOLLOW-UPS: *(1) Without continuing the lists, how could you find other pairs of consecutive integers such that one is a multiple of 17 and the other of 21?* [Add 17×21 to both 84 and 85 as often as desired.] *(2) Is it possible for two multiples of 15 and 21 to be consecutive integers? Explain.*[No. With a common factor of 3, every pair must differ by a multiple of 3.]

39%

4C *<u>Strategy</u>: Find the radii of the circles.*
The area of a circle is given by the formula A = πr^2, so circles with areas of 9π, 25π, and 100π have radii of 3, 5, and 10 respectively. The lengths of the sides of the triangle are found by adding pairs of radii together. Therefore, AB = 8 cm, BC = 15 cm, and CA = 13 cm. **The perimeter of the triangle is** 8 + 15 + 13 = **36 cm.**



57%

4D METHOD 1: *Strategy: Use the multiplication algorithm.*
Start with the multiplication on the left. Each partial product contains either 2 or 3 digits. Since q is 1, C = 7. Then pq represents 91. In the tens column, (9 + s) ends in 1. Then s is 2, so B = 4 and rs represents 52. This yields the multiplication on the right.

13	13
x ABC	x A47
pq	91
rs	52
tu	tu
3□11	3□11

To get a thousands digit of 3 in the product, t is 2 or 3. Then A is 2 or 3. If A is 3, tu represents 39 and the final product is **4**511. So A is 2, tu is 26 and the final product is 3211. **The missing digit is 2**.

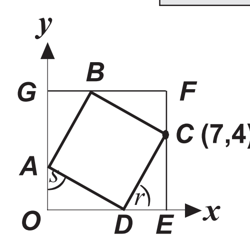
METHOD 2: *Strategy: Start with an arbitrary value.*
Divide 3011 by 13 to get a quotient of 231 and a remainder of 8. Since 100 divided by 13 leaves a remainder of 9, each increase of 100 to 3011 adds 9 to the remainder value of 8. Determine how many 9's must be added to 8 to get a value that is a multiple of 13. Since 8 + 2×(9) = 26 is a multiple of 13, increasing 3011 by 200 will leave no remainder. The number is 3211 and the missing digit is 2.

METHOD 3: *Strategy: Use a divisibility test.*
There are several tests for divisibility by 13. Here are two:

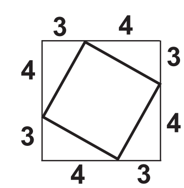
1) Start at the right and split the digits into groups of 3. Alternately add and subtract the groups until a 3 digit number remains. The original number is divisible by 13 if and only if the final 3 digit number is. Here, 3□11 is divisible by 13 only if □11 – 3 = □08 is. Use Method 1 to find that the missing digit is 2.
2) Multiply the units digit by 4 and add it to the original number without the units digit. Continue in this fashion until a 2 digit number remains. The original number is divisible by 13 if and only if the final 2 digit number is. Here 3□11 → 3□1 +4(1) = 3□5. 3□ + 4(5) = 5□. Only if the missing digit is 2 is 5□ a multiple of 13.

21%

4E *Strategy: Wrap the square in a box.*
Draw Δ*DEC* congruent to Δ*AOD* as shown. ∠*r* is congruent to ∠*s*, so the sum of the three angles at *D* (180º) equals the sum of the three angles in a triangle. Thus $\overline{ODE}$ is a straight line and *E* is on the *x*-axis. Construct 2 more triangles congruent to Δ*AOD*, as shown, to produce square *OEFG*.



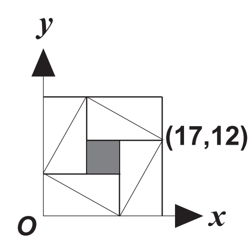
Because *C* is at (7,4), *OE* = 7 and *CE* = 4. Then *OD* = 4 and *DE* = 3. Since the four triangles are congruent, the legs of lengths in each are 3 and 4.



METHOD 1: *Strategy: Use known areas.*
OEFG is a square of side 7, so its area is 49. The area of each of the four right triangles is ½(3)(4) = 6. **The area of the square *ABCD* is** then 49 – 4 × 6 = **25.**

METHOD 2: *Strategy: Use the Pythagorean Theorem.*
In right Δ*CDE*, $CD^2 = 3^2 + 4^2$. So *CD* = 5, and the area of the square *ADCB* = 5^2 = 25.

***Follow-Up:** Find the area of the shaded square in the center of the largest square in this picture.* [49]



OLYMPIAD 5 MARCH 10, 2010

Answers: [5A] **8** [5B] $\frac{1}{45}$ [5C] **14** [5D] **38** [5E] **300**

59% correct

5A *Strategy: Use the definition of the operation.*
2 ✵ 6 = 2 + 3 × 6 and N ✵ 4 = N + 3 × 4.
Then 20 = N + 12 and **the value of N is 8.**

44%

5B **METHOD 1:** *Strategy: Extend the process of "cancellation".*
"Cancel" identical numerators and denominators with each other (that is, divide out each common factor greater than 1). This can be done 6 times. Then we are left with $\frac{2}{90}$. **In simplest terms, the product is $\frac{1}{45}$.**

METHOD 2: *Strategy: Multiply all fractions and then simplify.*
The product of all the numerators is 40,320.
The product of all the denominators is 1,814,400.
1,814,400 ÷ 40,320 = 45. In simplest terms, the product is $\frac{1}{45}$.

45%

5C *Strategy: Simplify the sum.*
The sum $^-10 + {}^-9 + {}^-8 + \cdots + 8 + 9 + 10 = 0$. Continue to add integers starting with 11 until the desired sum is obtained. Because 11 + 12 + 13 + 14 = 50, **N = 14.**

12%

5D **METHOD 1a:** *Strategy: Consider the number of wins.*
The following table shows that the Pumas won 2 of their first 9 games and then 3 of every 4 games on average. They ended with 2 wins in every 3 games.

Games won	2	5	8	…	35	**38**
Games played	9	13	17	…	53	**57**

Since only $\frac{38}{57}$ simplifies to $\frac{2}{3}$, **the Pumas won 38 games in all.**

METHOD 1b: *Strategy: Consider the number of losses.*
The following table shows that the Pumas lost 2 of their first 9 games and then 1 of every 4 games on average. They ended with1 loss in every 3 games.

Games won	7	8	9	…	18	**19**
Games played	9	13	17	…	53	**57**

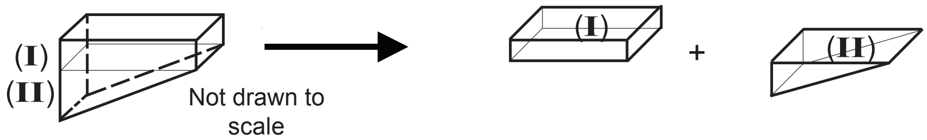
Since they lost a total of 19 games, the Pumas won 38 games in all.

METHOD 2: *Strategy: Use algebra.*
Let w = ratio factor. Then they won $3w$ and lost $4w$ of the remaining games.
In all the Pumas won $2 + 3w$ games and played $9 + 4w$ games.

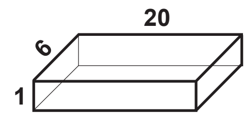
$$\frac{2 + 3w}{9 + 4w} = \frac{2}{3}.$$

Cross multiply: $3(2 + 3w) = 2(9 + 4w)$
Then $6 + 9w = 18 + 8w$
$w = 12$ and $2 + 3w = 38$. The Pumas won 38 games in all.

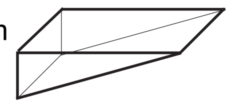
5E METHOD 1: *Strategy: Split the figure into more familiar shapes and add.*



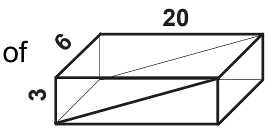
(I) Volume (I) = 6 × 20 × 1 = 120 cu m



(II) Volume (II) = ½ × 6 × 20 × 3 = 180 cu m

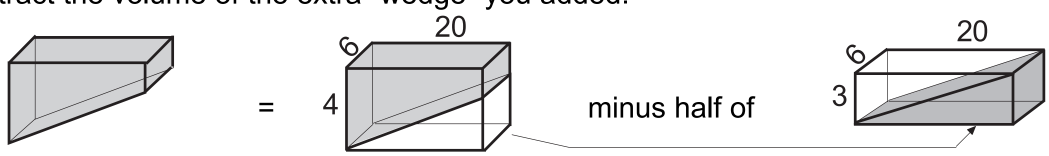


= half of



The pool can hold 120 + 180 = **300 cubic meters of water.**

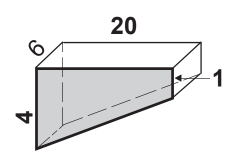
METHOD 2: *Strategy: Embed the figure in a more familiar shape and subtract.*
Box in the pool and compute the volume of the resulting rectangular solid. Next, find and subtract the volume of the extra "wedge" you added.



(Not drawn to scale)

The volume of water in the pool = (4 × 6 × 20) – ½ × (3 × 6 × 20)
= 480 – 180
= 300 cu m

METHOD 3: *Strategy: Use formulas.*
In the diagram, consider the nearest side of the pool as the base of a prism. The volume of the prism is $V = Bh$, where B is the area of the base and h is the height of the prism.



The base is a trapezoid, and its area is given by $\mathbf{A = \frac{1}{2} h(b_1 + b_2)}$, where h is the height of the trapezoid and b_1 and b_2 are the lengths of its bases. The area of the trapezoid is A = ½ (20) (4 + 1) = 50 sq m, and the volume of the pool is 50(6). The pool can hold 300 cu m of water.

***FOLLOW-UP:** Find the area of a triangle whose vertices are A(0,0), B (4,5) and C(2,6).*
[7 sq units; wrap the triangle in a rectangle and subtract areas.]