Number Theory 5B3

(1) ________ What percent of all three digit numbers contain the digit 5 at least once?

(2) ________ Express the ratio of $\frac{2}{7}$ to $1.\overline{09}$ as a common fraction.

(3) ________ The sum of the proper divisors of 18 is 21. What is the sum of the proper divisors of 198?

(4) ________ What is the 43rd digit after the decimal point in the decimal representation of $\frac{1}{13}$?

(5) ________ A bag of candy can be divided in equal shares among 2, 3, 4, 5, or 6 friends. What is the least number of pieces of candy that the bag could contain?

(6) ________ The arithmetic mean of an odd number of consecutive odd integers is y. Find the sum of the smallest and the largest of the integers.

(7) ________ Find the product of the square root of 1764 and the largest prime factor of 1764.

(8) ________ Two different prime numbers are selected at random from among the first ten prime numbers. What is the probability that the sum of the two primes is 24? Express your answer as a common fraction.

(9) ________ What two whole numbers, neither of which contains zeros, when multiplied together equal exactly 1,000,000,000?

(10) ________ Given that the repetend in the decimal representation of $\frac{1}{19}$ contains 18 digits, find the 39th digit in the decimal representation.

(11) ________ Any even whole number greater than 7 can be written as the sum of two distinct primes. What is the largest product for two such primes whose sum is 150?

(12) ________ What is the greatest three-digit number divisible by both 7 and 8?

(13) ________ What number is less than 35^2, greater than 34^2, divisible by 19, and a multiple of 9?

(14) ________ Find a positive integer that is divisible by 18 and whose cube root is a number between 8 and 8.1.

Answer Sheet

Number	Answer	Problem ID
1	28 %	C35D1
2	11/42	4A5B
3	270	BDC02
4	0	AAC02
5	60	542A1
6	2y	045D1
7	294	555B
8	$\frac{1}{15}$	120D1
9	512 and 1,953,125	2BC02
10	2	42B02
11	5609	4AB12
12	952	A4051
13	1197	40DA1
14	522	55A12

Solutions

(1) **28 %** **ID: [C35D1]**
For each possible hundreds digit $(1, 2, 3, \ldots 9)$, there are 100 possible three digit numbers that begin with that hundreds digit, for a total of $9 \times 100 = 900$ different three digit numbers. We now find how many of these three digit numbers contain the digit 5 at least once. All 100 of the three digit numbers with 5 in their hundreds place must have at least one 5. For each of the other eight possible hundreds digits, there are 10 three digit numbers with five in the tens place and 9 with five in the ones place but not the tens. Therefore, there are $8 \times 19 = 152$ three digit numbers with at least one five in their tens or ones places.

Adding this total to the number of three digit numbers with seven in their hundreds place, we have a total of $100 + 152 = 252$ such numbers. Therefore, the percentage of three digit numbers that contain at least one 5 is $\frac{252}{900} = \frac{28}{100} = \boxed{28\%}$.

(2) **11/42** **ID: [4A5B]**
No solution is available at this time.

(3) **270** **ID: [BDC02]**
There are many ways to solve this problem, the most obvious being to list all of the proper divisors and add them up. There is, however, a creative solution that uses the fact that the sum of the proper divisors of 18 is 21. Note that we can factor 198 into $11 \cdot 18 = 11 \cdot 2 \cdot 3 \cdot 3$. Each proper divisor will be composed of three or fewer of these factors. Those divisors that do not contain the factor 11 will be either the proper divisors of 18 or 18 itself, contributing 21 and 18, respectively, to the sum. Those divisors that do contain the factor 11 will again by the proper divisors of 18, only multiplied by 11. The sum of these divisors, therefore, is $11 \cdot 21 = 231$. Since these are all possible divisors, the sum of the proper divisors of 198 is $21 + 18 + 231 = \boxed{270}$.

(4) **0** **ID: [AAC02]**
The decimal representation of $\frac{1}{13}$ is $0.\overline{076923}$. Since the first six digits repeat, we know that after every 6th digit the pattern will restart. Since $43 \div 6 = 7r1$, the first 42 digits will be seven repetitions of the same pattern followed by the first digit of the pattern. Since the first digit is $\boxed{0}$, this is our final answer.

(5) **60** **ID: [542A1]**

No solution is available at this time.

(6) **2y** **ID: [045D1]**

We notice that if the median term in a sequence of consecutive odd integers is x, then next greatest and next least terms must be $x+2$ and $x-2$. The next greatest and next least terms after these are $x+4$ and $x-4$. We can continue this process and we find that the sequence of consecutive odd integers with n terms and median term x looks like this:

$$x-(n-1), x-(n-3), \dots, x-2, x, x+2, \dots, x+(n-3), x+(n-1).$$

We notice that since the sum of the first and the last term equals $2x$, the sum of the second and the second to last term equals $2x$, etc. and the middle term equals x, the mean of the consecutive odd integers must be equal to x. This makes x equal to y. We also have that the sum of the smallest and largest integers in this list equal to $(x-(n-1))+(x+(n-1)) = 2x$, so their sum is equal to $\boxed{2y}$.

(7) **294** **ID: [555B]**

No solution is available at this time.

(8) $\frac{1}{15}$ **ID: [120D1]**

The first ten prime numbers are $2, 3, 5, 7, 11, 13, 17, 19, 23,$ and 29. The only pairings of two of these primes whose sum adds to 24 are 5 and 19, 7 and 17, and 11 and 13. That gives us three such pairings (if the order that we select them in doesn't matter). We now find the total number of ways that we can select two different prime numbers. Since there are 10 different primes, there are $\binom{10}{2} = \frac{10\cdot 9}{2\cdot 1} = 45$ different ways to choose two primes. Therefore, the probability that the sum of the two primes that we choose equals 24 is $\dfrac{3}{45} = \boxed{\dfrac{1}{15}}$.

(9) **512 and 1,953,125** **ID: [2BC02]**

We know that 1,000,000,000 in scientific notation is equivalent to 1×10^9. Any factor of 1,000,000,000 containing a 10, however, will guarantee a zero in that number. Therefore, we must break down each of the 10's. Because $10 = 2 \times 5$, we can also write 1,000,000,000 as $1 \times (2 \times 5)^9 = 2^9 \times 5^9$. Since $2^9 = 512$ and $5^9 = 1,953,125$, we have our answer of $\boxed{512 \text{ and } 1,953,125}$.

(10) **2** **ID: [42B02]**

Since $39 = 2 \times 18 + 3$, the 39th digit past the decimal point is the same as the 3rd digit past the decimal point. To find this, we can directly divide:

$$\begin{array}{r|llll}
 & 0 & .0 & 5 & 2 \\ \hline
19 & 1 & .0 & 0 & 0 \\
 & & 9 & 5 & \downarrow \\ \cline{2-4}
 & & & 5 & 0 \\
 & & & 3 & 8 \\ \cline{4-5}
 & & & 1 & 2
\end{array}$$

Thus, the answer is $\boxed{2}$.

(11) **5609** **ID: [4AB12]**

No solution is available at this time.

(12) **952** **ID: [A4051]**

Let n be the number we are seeking. Since 7 and 8 are relatively prime then $7|n$ and $8|n$ if and only if $(7 \cdot 8)|n$. So, we must find the largest three digit multiple of $7 \cdot 8 = 56$. Since $\frac{999}{56}$ is between 17 and 18, $17 \cdot 56 = \boxed{952}$ is the largest three-digit multiple of 56.

(13) **1197** **ID: [40DA1]**

No solution is available at this time.

(14) **522** **ID: [55A12]**

We want an integer n such that $8 < \sqrt[3]{n} < 8.1$. Cubing each part of the inequality gives $8^3 < n < 8.1^3$, or $512 < n < 531.441$. We know n is a multiple of 18, so we try to find a multiple of 18 in this range (we can do this by letting $n = 18k$, and trying out different integer values of k). We find that $18 \cdot 29 = 522$ is the only multiple of 18 in this range. So $\boxed{522}$ is the answer.