Number Theory 5A3

(1) ________ How many positive integers less than 100 have an odd number of distinct factors?

(2) ________ A farmer has some pigs and some chickens. He finds that together they have 70 heads and 200 legs. How many pigs does he have?

(3) ________ If a three-digit number of the form $1D1$ is divided by D, the quotient is a two digit number of the form $2D$ remainder 5. What is the value of D?

(4) ________ 1993 is a:
(a) perfect square;
(b) prime number; or
(c) perfect number.

(5) ________ If the three-digit number $\underline{2d2}$ is divisible by 7, what is d?

(6) ________ What is the remainder when the product $1734 \times 5389 \times 80{,}607$ is divided by 10?

(7) ________ A and B are non-zero digits for which $\underline{A468B05}$ is divisible by 11. What is $A + B$?

(8) ________ Find the sum of the smallest and largest prime factors of 10101.

(9) ________ What is the smallest number divisible by integers 1 through 9?

(10) ________ Two factors, each with no digit greater than 5, have a product of 16,848. What is the largest possible 3-digit factor satisfying these conditions?

(11) ________ Determine the units digit of $17^{13} - 17$.

Answer Sheet

Number	Answer	Problem ID
1	9	AB1C
2	30	BA2A1
3	6	1C3B
4	(b) prime number	CB1C
5	5	145B
6	2	324B
7	12	3DA41
8	40	5C0C
9	2520	115B
10	324	C33B
11	0	53DA1

Solutions

(1) **9 ID: [AB1C]**
No solution is available at this time.

(2) **30 ID: [BA2A1]**
No solution is available at this time.

(3) **6 ID: [1C3B]**
No solution is available at this time.

(4) **(b) prime number ID: [CB1C]**
No solution is available at this time.

(5) **5 ID: [145B]**
Here we can use the divisibility rule for 7: drop the last digit, subtract twice its value from the number formed from the remaining digits, and check if the result is divisible by 7. (This rule isn't often used since it's not nearly as simple as the other divisibility rules, but it can still be useful!) To see how this works for the number $\underline{2d2}$, the rule says drop the last digit (2), leaving the number $\underline{2d}$; subtracting twice the last digit gives $\underline{2d} - 4$. This needs to be divisible by 7, but the only multiple of 7 between $20 - 4 = 16$ and $29 - 4 = 25$ is 21, so we must have $d = \boxed{5}$ since $25 - 4 = 21$.

(6) **2 ID: [324B]**
The remainder when a number is divided by 10 is simply the units digit of that number. So we only look for the units digit of the product. With 1734×5389, $4 \times 9 = 36$, so the result will have a units digit of 6. Then we multiply 6 by the units digit of $80,607$ and get $6 \times 7 = 42$. That means the final product will have a units digit of $\boxed{2}$.

(7) **12 ID: [3DA41]**
No solution is available at this time.

(8) **40 ID: [5C0C]**

10101 is clearly not divisible by 2 or 5. The sum of 10101's digits is 3, so it is divisible by 3, but not by 9. $10101 = 3 \cdot 3367$. $3367/7 = 481$ and $481/7 = 68\frac{5}{7}$ so $10101 = 3 \cdot 7 \cdot 481$ and 481 is not divisible by any prime number less than 11. Applying the divisibility test for 11, we have $4 - 8 + 1 = -3$, which is not divisible by 11, so 481 is not divisible by 11 either. $481/13 = 37$ and 37 is prime, so the prime factorization of 10101 is $10101 = 3 \cdot 7 \cdot 13 \cdot 37$. So, the sum of its smallest and largest prime factors is $3 + 37 = \boxed{40}$.

(9) **2520 ID: [115B]**

To find the least common multiple of 1, 2, 3, 4, 5, 6, 7, 8, and 9, we ignore 1 and prime factorize the rest to obtain $2, 3, 2^2, 5, 2 \cdot 3, 7, 2^3$, and 3^2. Taking the maximum exponent for each prime, we find that the least common multiple is $2^3 \cdot 3^2 \cdot 5 \cdot 7 = \boxed{2520}$.

(10) **324 ID: [C33B]**

No solution is available at this time.

(11) **0 ID: [53DA1]**

The units digit of $17^3 - 17$ is the same as the units digit as $7^{13} - 7$. To find the units digit of 7^{13}, we look at the first few powers of 7 modulo 10:

$$\begin{aligned} 7^0 &\equiv 1, \\ 7^1 &\equiv 7, \\ 7^2 &\equiv 7 \cdot 7 \equiv 49 \equiv 9, \\ 7^3 &\equiv 7 \cdot 9 \equiv 63 \equiv 3, \\ 7^4 &\equiv 7 \cdot 3 \equiv 21 \equiv 1 \pmod{10}. \end{aligned}$$

Since $7^4 \equiv 1 \pmod{10}$, the remainders become periodic, with period 4. Since $13 \equiv 1 \pmod{4}$, $7^{13} \equiv 7 \pmod{10}$, so the units digit of $7^{13} - 7$ is $\boxed{0}$.