Mathcounts / AMC 8 (Week 10)

Name ____________________

(1) ________ A point having whole-number coordinates is selected at random from the line $20x + y = 100$. What is the probability that the sum of the coordinates is less than 30? Express your answer as a common fraction.

(2) ________ A fast food restaurant specializes in ham sandwiches. A customer may choose to add any of the following: mayonnaise, mustard, lettuce, tomato or cheese. How many different ham sandwich combinations are possible?

(3) ________ Track practice lasts for one hour from 2:30-3:30. At a randomly selected time during track practice, Tania looks at her watch. What is the probability that the minute and hour hand on her watch form an acute angle? Express your answer as a common fraction.

(4) ________ What fraction of the eleven letters in the word "MISSISSIPPI" are I's? Express your answer as a common fraction.

(5) ________ Two numbers are chosen at random, with replacement, from the set $\{1, 2, 3, 4\}$. The two numbers are used as the numerator and denominator of a fraction. What is the probability that the fraction represents a whole number? Express your answer as a common fraction.

(6) ________ Compute: $\frac{4!+3!}{3!+2!}$. Express your answer as a decimal to the nearest hundredth.

(7) ________ There are several sets of three different numbers whose sum is 14 that can be chosen from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. What fraction of these sets contains a 4? Express your answer as a common fraction.

(8) _________ Angel wants to sell 50 identical pencils in groups of 2 or 3. In how many ways can the pencils be grouped?

(9) _________ A four-digit number is created by using each of the digits 4, 5, 8 and 9 exactly once. What is the probability that the number will be a multiple of 4? Express your answer as a common fraction.

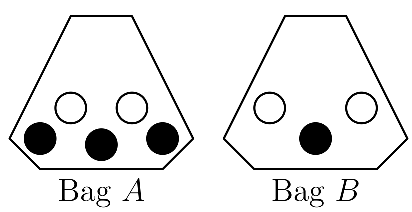
(10) _________ There are six teams in a school district competition. Each team plays each other team once. What is the total number of games played in the competition?

(11) _________ Each digit in the number 2001 is placed on a different card. In how many ways can three different cards be selected so that the product of the numbers on those cards is not zero?

(12) _________ Camy made a list of every possible distinct five-digit positive even integer that can be formed using each of the digits 1, 3, 4, 5 and 9 exactly once in each integer. What is the sum of the integers on Camy's list?

(13) _________ What is the positive difference between the probability of a fair coin landing heads up exactly 2 times out of 3 flips and the probability of a fair coin landing heads up 3 times out of 3 flips? Express your answer as a common fraction.

(14) _________ Two bags of marbles are pictured below. One marble is randomly selected from Bag A and placed into Bag B. One marble is then randomly selected from Bag B. What is the probability that the marble selected from Bag B is black? Express your answer as a common fraction.



(15) ________ The number 121 is a palindrome, because it reads the same backwards as forward. How many integer palindromes are between 100 and 500?

(16) ________ If two distinct numbers are selected at random from the first seven prime numbers, what is the probability that their sum is an even number? Express your answer as a common fraction.

(17) ________ Each day, two out of the three teams in a class are randomly selected to participate in a MATHCOUNTS trial competition. What is the probability that Team A is selected on at least two of the next three days? Express your answer as a common fraction.

(18) ________ A bag contains 7 white, 9 blue and 4 red marbles. If three marbles are pulled from the bag, what is the probability that two are blue and one is red? Express your answer as a common fraction.

(19) ________ A digital, 12-hour clock shows hours and minutes. During what fraction of the day will the clock show the digit 1 in its display? Express your answer as a common fraction.

(20) ________ The first 20 numbers of an arrangement are shown below. What would be the value of the 40^{th} number if the arrangement were continued?

- Row 1: 2, 2
- Row 2: 4, 4, 4, 4
- Row 3: 6, 6, 6, 6, 6, 6
- Row 4: 8, 8, 8, 8, 8, 8, 8, 8

(21) ________ What is the number of distinct ways of arranging the letters in the word AVERAGE?

(22) ________ What is the probability of getting an even number when a fair six-sided die is rolled? Express your answer as a common fraction.

(23) ________ P and Q are whole numbers such that $0 < P < 10$ and $0 < Q < 10$. How many common fractions $\frac{P}{Q}$ exist if $\frac{1}{2} < \frac{P}{Q} < 1$?

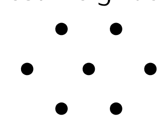
(24) ________ How many different four-digit numbers can be obtained by using any four of the digits 2, 3, 4, 4, and 4?

(25) ________ If digits may not be repeated, how many positive three-digit integers can be written using the digits 1, 2, 3 and 4?

(26) ________ Ms. Albertson is randomly selecting the order in which her 25 students will each present a report next week. Five students will present each day, Monday through Friday. What is the probability that the shortest student will present his report on Thursday? Express your answer as a common fraction.

(27) ________ What is the units digit of $1! + 3! + 5! + 7! + 9! + 11!$?

(28) ________ Each point in the hexogonal lattice is one unit from its nearest neighbor. How many circles of radius one contain at least two points of the lattice?



(29) ________ How many diagonals does a regular seven-sided polygon contain?

(30) ________ The probability that Kim has a math test today is $\frac{4}{7}$. What is the probability that Kim does not have a math test today? Express your answer as a common fraction.

Answer Sheet

Number	Answer	Problem ID
1	1/3	3455
2	32 combinations	22D4
3	1/2	C5D3
4	4/11	C203
5	1/2	DBAC
6	3.75	45B31
7	$\frac{3}{8}$	A4C3
8	9	CAAC
9	1/6	BBCC
10	15	45C3
11	0 ways	0A14
12	1199976	2241
13	$\frac{1}{4}$	0DB3
14	2/5	144C
15	40 palindromes	C041
16	5/7	B3C3
17	20/27	2522
18	12/95	53C3
19	1/2	BBAC
20	12	4D03
21	1260 ways	32C01
22	1/2	0D03
23	13	55D3
24	20	C3BC
25	24	A403
26	1/5	5203
27	7	D3C2
28	13	C3C3
29	14 diagonals	0141
30	3/7	A303

Solutions

(1) **1/3 ID: [3455]**
No solution is available at this time.

(2) **32 combinations ID: [22D4]**
No solution is available at this time.

(3) **1/2 ID: [C5D3]**
No solution is available at this time.

(4) **4/11 ID: [C203]**
There are 4 I's, and 11 letters, so the fraction is $\boxed{\frac{4}{11}}$.

(5) **1/2 ID: [DBAC]**
No solution is available at this time.

(6) **3.75 ID: [45B31]**
No solution is available at this time.

(7) $\frac{3}{8}$ **ID: [A4C3]**

We proceed by casework.

First, assume that one of the numbers chosen is a 1. The other two numbers then must sum to 13. There are three such pairs: 4+9, 5+8, or 6+7.

Now, let one of the numbers chosen be a 2. The other two numbers must sum to 12, giving three more pairs: 3+9, 4+8, and 5+7 (we can't have 6+6 because the numbers must be different).

Now consider the cases in which one of the numbers chosen is a 3. There are two pairs of numbers that sum to 11: 4+7 and 5+6.

If we try to make a case in which one of the numbers chosen is a 4, we must obtain a sum of 10 from two of 5, 6, 7, 8, and 9. This is impossible, so we've counted all of the cases. (We don't use numbers smaller than 4 to obtain a sum of 10 because we've already counted all of the cases in which 1, 2, and 3 are in the set.)

Three of these eight sets contain a 4; thus, our answer is $\boxed{\frac{3}{8}}$.

(8) **9 ID: [CAAC]**

No solution is available at this time.

(9) **1/6 ID: [BBCC]**

No solution is available at this time.

(10) **15 ID: [45C3]**

The first team plays each of the five other teams, and then it has no more games. The second team then plays the four other teams (there are only four because the first team is no longer playing), and then it is also done. There are three teams left for the third team to play, two for the fourth, and one final game between the fifth and sixth teams. Thus, $5+4+3+2+1 = \boxed{15}$ games are played.

(11) **0 ways ID: [0A14]**

There are four cards total, two of which have a zero on them. If three cards are selected, therefore, at least one of them must have a zero on it, making the product of the numbers on those three cards zero. Thus, there are $\boxed{0}$ ways.

(12) **1199976 ID: [2241]**

Since Camy's numbers are even, they all must end in 4. This leaves us free to pick the remaining digits without restriction. Note that once we fix one of the remaining digits, we have $3! = 6$ ways of forming the final number. Thus each of the digits $1, 3, 5, 9$ appears in the 10s, 100s, 1000s, and 10000s spots exactly 6 times. So the 1 contributes a total of $6 \cdot 1 \cdot (10 + 100 + 1000 + 10000) = 6 \cdot 11110$ to the sum, and similarly the 3, 5, 9 contribute $6 \cdot 33330$, $6 \cdot 55550$, $6 \cdot 99990$, respectively. Since the 4 appears in $4! = 24$ numbers, our total sum is

$$6 \cdot 11110 + 6 \cdot 33330 + 6 \cdot 55550 + 6 \cdot 99990 + 4 \cdot 24 = \boxed{1,199,976}$$

(13) $\frac{1}{4}$ **ID: [0DB3]**

The probability that a fair coin lands heads up exactly 2 times out of 3 flips is $p_1 = \binom{3}{2}(1/2)^2(1/2) = 3/8$. The probability that a fair coin lands heads up 3 times out of 3 flips is $p_2 = (1/2)^3 = 1/8$. Finally, we have $p_1 - p_2 = 2/8 = \boxed{1/4}$.

(14) **2/5 ID: [144C]**

The probability that the marble drawn is originally from bag B is $\frac{3}{4}$. Given that the marble drawn is originally from bag B, the probability that it is black is $\frac{1}{3}$. Therefore, the probability that the marble drawn from the bag is black and is originally from bag B is $\frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4}$.

The probability that the marble drawn is originally from bag A is $\frac{1}{4}$. Given that the marble drawn is originally from bag A, the probability that it is black is $\frac{3}{5}$. Therefore, the probability that the marble drawn is black and is originally from bag A is $\frac{1}{4} \cdot \frac{3}{5} = \frac{3}{20}$.

Since the marble originally came either from bag A or from bag B, the probability that it is black is the sum of the probability that it is black and originally came from bag A and the probability that it is black and originally came from bag B: $\frac{1}{4} + \frac{3}{20} = \boxed{\dfrac{2}{5}}$.

(15) **40 palindromes ID: [C041]**

The hundreds digit can be any one of 1, 2, 3 or 4. Whatever the hundreds digit is, that fixes what the units digit can be. Then, there are 10 choices for the middle (tens) digit. So, we can construct $4 \cdot 10 = \boxed{40}$ palindromes by choosing the digits.

(16) **5/7 ID: [B3C3]**

The only way for the sum to not be even is if one of the primes chosen is 2. There are six pairs where one of the primes is 2, and there are $\binom{7}{2} = 21$ total possible pairs, so the probability that the sum is NOT even is $\frac{6}{21} = \frac{2}{7}$. Therefore, the probability that the sum IS even is $1 - \frac{2}{7} = \boxed{\frac{5}{7}}$.

(17) **20/27 ID: [2522]**

We can compute this a few ways, but the numbers seem small enough that we can go ahead and just compute the probability of A being selected all three days, and the probability of A being selected exactly 2 of the three days. Team A is selected on any given day with probability $\frac{2}{3}$, because there are $\binom{3}{2} = 3$ possible pairs of teams, and 2 of them contain A. So, there is a $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$ chance of being selected all three days. Of being selected exactly twice, there is a $\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \binom{3}{2} = \frac{4}{9}$. Adding these two yields $\frac{8}{27} + \frac{4}{9} = \frac{8+12}{27} = \boxed{\frac{20}{27}}$.

(18) **12/95 ID: [53C3]**

No solution is available at this time.

(19) **1/2 ID: [BBAC]**

No solution is available at this time.

(20) **12 ID: [4D03]**

Since we are told there are 20 numbers in the first 4 Rows, we want to find the 20^{th} number starting in Row 5. Since there are 10 numbers in Row 5, and there are 12 numbers in Row 6, the 20^{th} number if we start counting in Row 5 is located at the 10^{th} spot of Row 6, which is of course a $\boxed{12}$.

(21) **1260 ways ID: [32C01]**

No solution is available at this time.

(22) **1/2 ID: [0D03]**

No solution is available at this time.

(23) **13 ID: [55D3]**

We can begin by restating the inequalities. $\frac{1}{2} < \frac{P}{Q}$ is the same as $Q < 2P$, and $\frac{P}{Q} < 1$ is the same as $P < Q$. Thus, $P < Q < 2P$. If $P = 1$, there are no possible solutions for Q. If $P = 2$, $Q = 3$ is the only possible solution. Similarly, if $P = 3$, Q could be 4 or 5; if $P = 4$, then $Q = 5$, 6, or 7; if $P = 5$, then $Q = 6$, 7, 8, or 9; if $P = 6$, $Q = 7$, 8, or 9; if $P = 7$, $Q = 8$ or 9; if $P = 8$, $Q = 9$. There are sixteen such fractions.

However, we have over-counted some fractions. $\frac{2}{3} = \frac{4}{6} = \frac{6}{9}$, and $\frac{3}{4} = \frac{6}{8}$.

Thus, there are $\boxed{13}$ possible distinct common fractions.

(24) **20 ID: [C3BC]**

No solution is available at this time.

(25) **24 ID: [A403]**

No solution is available at this time.

(26) **1/5 ID: [5203]**

No solution is available at this time.

(27) **7 ID: [D3C2]**

We observe that for all $n > 5$, $n!$ has a units digit of 0, because 5! has a factor of 5 and 2, which become a factor of 10. So, the terms in the sum, 5!, 7!, 9!, and 11! all have a 0 for the units digit. And, $1! + 3! = 1 + 6 = \boxed{7}$ is the units digit of the sum.

(28) **13 ID: [C3C3]**

No solution is available at this time.

(29) **14 diagonals ID: [0141]**

A seven-sided polygon has seven vertices. There are $\binom{7}{2} = 21$ ways to connect the pairs of these 7 points. But 7 of those pairs are pairs of consecutive vertices, so they are counted as sides. So, only $21 - 7 = \boxed{14}$ of these segments are diagonals.

(30) **3/7 ID: [A303]**

The probability that Kim does not have a math test is equal to one minus the probability she does have a math test. So, the probability of not having a math test is $1 - \frac{4}{7} = \boxed{\dfrac{3}{7}}$.