Number Theory 3A1

Name ____________________

(1) ________ A whole number, N, is chosen so that $\frac{N}{3}$ is strictly between 7.5 and 8. What is the value of N ?

(2) ________ What fraction of the one-digit positive integers is prime? Express your answer as a common fraction.

(3) ________ What is the greatest three-digit multiple of 19?

(4) ________ What is the positive difference between the greatest and least prime factors of 2000?

(5) ________ The game of Ibish is played in rounds. In the first round, you earn 0, 10 or 11 points; in the second round, you earn 0, 10, 11 or 12 points; in the third round, you earn 0, 10, 11, 12 or 13 points, and so on, including the next greatest integer in the possible point values. What is the fewest number of rounds after which your total score can have a 9 in the units digit?

(6) ________ What is the median of all values defined by the expression $2^x - 1$, where x is a prime number between 0 and 20?

(7) ________ What is the greatest three-digit multiple of 33 that can be written using three different digits?

(8) __________ A 16-page booklet is made from a stack of four sheets of paper that is folded in half and then joined along the common fold. The 16 pages are then numbered from front to back, starting with page 1. What are the other three page numbers on the same sheet of paper as page 5?

(9) __________ Four consecutive positive integers have a product of 840. What is the largest of the four integers?

(10) __________ What is the units digit of the product of $3^{35} \times 7^{35}$?

(11) __________ What is the median of the composite integers that are greater than 20 and less than 35?

(12) __________ What is the least natural number that has exactly three factors?

(13) __________ Use the clues below to determine the value of n. The sum of the digits of n is 11, and $50 < n < 100$. When n is divided by 2, the result is a prime integer. What is the value of n?

(14) __________ What is the number of factors of the product 126×216?

(15) __________ What is the sum of the reciprocals of the natural-number factors of 6?

(16) __________ To determine whether a number N is prime, we must test for divisibility by every prime less than or equal to the square root of N. How many primes must we test to determine whether 2003 is prime?

(17) __________ A *composite number* is a number that has two or more prime factors. The number 87 can be expressed as the sum of two composite numbers in many ways. What is the minimum positive difference between two such numbers?

(18) ________ What is the least three-digit positive integer that has 2, 5 and 7 as factors?

(19) ________ How many positive integer factors of $2^2 \times 3^2 \times 5$ are multiples of 12?

(20) ________ What is the greatest three-digit number which is a multiple of 13?

(21) ________ For how many positive integers n will $\frac{60}{n}$ also be an integer?

(22) ________ What positive two-digit integer is exactly twice the sum of its digits?

(23) ________ What is the smallest positive integer n such that $2n$ is a perfect square and $3n$ is a perfect cube?

(24) ________ What is the least common multiple of 6, 8, and 10?

(25) ________ What is the sum of the prime numbers between 30 and 50?

(26) ________ What is the probability that a randomly selected positive integer less than or equal to 100 is divisible by 7? Express your answer as a common fraction.

(27) ________ What is the sum of the proper divisors of 256?

(28) ________ How many natural-number factors does N have if $N = 2^3 \cdot 3^2 \cdot 5^1$?

(29) ________ A relatively prime day is one where the month number (e.g. January = 1, February = 2, etc.) and date of the month $(1, 2, \ldots, 31)$ have no common factor other than 1. Which month has the fewest relatively prime days?

(30) ________ What is the least whole number that is divisible by 7, but leaves a remainder of 1 when divided by any integer 2 through 6?

Answer Sheet

Number	Answer	Problem ID
1	23	50D2
2	4/9	0CC2
3	988	45C1
4	3	3C4C
5	4 rounds	04C1
6	1087	A2AC
7	957	21D2
8	6, 11, 12	0113
9	7	ABA2
10	1	D0D4
11	27	D05C
12	4	1113
13	74	2103
14	60	4CA3
15	2	02A2
16	14	3213
17	3	0DC1
18	140	54C1
19	4	1103
20	988	5222
21	12	505C
22	18	534C
23	72	2BA3
24	120	C013
25	199	A2A2
26	7/50	C2A2
27	255	41B3
28	24	2413
29	June	A513
30	301	52AC

Solutions

(1) **23** **ID: [50D2]**
$7.5 < \frac{N}{3} < 8 \Rightarrow 22.5 < N < 24$. Since N is a whole number, $N = \boxed{23}$.

(2) **4/9** **ID: [0CC2]**
There are 9 one-digit positive integers. Of them, 2, 3, 5, and 7 are prime. Thus, $\boxed{\frac{4}{9}}$ of the one-digit positive integers is prime.

(3) **988** **ID: [45C1]**
Since 19 is close to 20 and $20 \cdot 50 = 1000$, we consider

$$19 \cdot 50 = 950.$$

From here we skip-count by 19s:

$$950, 969, 988, 1007, \ldots$$

The largest three-digit multiple of 19 is $\boxed{988}$.

(4) **3** **ID: [3C4C]**
The prime factorization of 2000 is $2^4 \times 5^3$ (build a factor tree if you need to see it). The greatest prime factor is 5 and the smallest prime factor is 2. The difference is $5 - 2 = \boxed{3}$.

(5) **4 rounds** **ID: [04C1]**
$0 + 0 + 1 + 2 + 3 + 4 > 9$. You can earn 0, 10, 11, 12, 13, or 14 points in the fourth round, so your total score can have a 9 in the units digit after $\boxed{4}$ rounds.

(6) **1087** **ID: [A2AC]**
No solution is available at this time.

(7) **957** **ID: [21D2]**
The largest three-digit multiple of 33 is $33 \cdot 30 = 990$, which uses only two different digits. Subtracting 33 once gives $990 - 33 = 957$, which has three different digits. Therefore, our answer is $\boxed{957}$.

(8) **6, 11, 12 ID: [0113]**
No solution is available at this time.

(9) **7 ID: [ABA2]**
We have $840 = 2^3 \cdot 3 \cdot 5 \cdot 7$. From this prime factorization, it is clear that the product of four consecutive positive integers is $840 = 2^2 \cdot 5 \cdot (2 \cdot 3) \cdot 7 = 4 \cdot 5 \cdot 6 \cdot 7$. The largest of the four integers is $\boxed{7}$.

(10) **1 ID: [D0D4]**
No solution is available at this time.

(11) **27 ID: [D05C]**
No solution is available at this time.

(12) **4 ID: [1113]**
No solution is available at this time.

(13) **74 ID: [2103]**
No solution is available at this time.

(14) **60 ID: [4CA3]**
No solution is available at this time.

(15) **2 ID: [02A2]**
The natural-number factors of 6 are 1, 6, 2, 3. The sum of their reciprocals is
$1/1 + 1/6 + 1/2 + 1/3 = 6/6 + 1/6 + 3/6 + 2/6 = 12/6 = \boxed{2}$.

(16) **14 ID: [3213]**
We must test every prime less than or equal to $\sqrt{2003} < 45$. There are $\boxed{14}$ such primes: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, and 43.

(17) **3** **ID: [0DC1]**

The minimum difference between two numbers whose sum is 87 is achieved when the numbers are as close as possible to $87 \div 2 = 43.5$. These numbers are 43 and 44, but 43 is prime, so we consider the next pair, 42 and 45, both of which are composite. Thus, the minimum positive difference is $45 - 42 = \boxed{3}$.

(18) **140** **ID: [54C1]**

Since 2, 5, 7 are pairwise relatively prime (meaning that no two of them share a prime factor), we must find the least three-digit positive integer that is divisible by $2 \cdot 5 \cdot 7 = 70$. That integer is $70 \cdot 2 = \boxed{140}$.

(19) **4** **ID: [1103]**

No solution is available at this time.

(20) **988** **ID: [5222]**

Since 1001 is $7 \cdot 11 \cdot 13$, we know that 1001 is a multiple of 13. The greatest 3-digit multiple of 13 is therefore

$$1001 - 13 = \boxed{988}.$$

(21) **12** **ID: [505C]**

No solution is available at this time.

(22) **18** **ID: [534C]**

Let the tens digit of the two-digit integer be a and let its units digit be b. The equation

$$10a + b = 2(a + b)$$

is given. Distributing on the right-hand side and subtracting $2a + b$ from both sides gives $8a = b$. Since $8a > 9$ for any digit $a > 1$, we have $a = 1$, $b = 8$, and $10a + b = \boxed{18}$.

(23) **72** **ID: [2BA3]**

If $2n$ is a perfect square, then n must be divisible by 2. Now if $3n$ is a perfect cube and n is divisible by 2, then n must be divisible by $3^2 = 9$ and by $2^3 = 8$. Therefore, the smallest positive integer n such that $2n$ is a perfect square and $3n$ is a perfect cube is $9 \times 8 = \boxed{72}$.

(24) **120 ID: [C013]**

$6 = 2 \cdot 3$, $8 = 2^3$, and $10 = 2 \cdot 5$, so the least common multiple of 6, 8, and 10 is $2^3 \cdot 3 \cdot 5 = \boxed{120}$.

(25) **199 ID: [A2A2]**

The prime numbers between 30 and 50 are 31, 37, 41, 43, and 47. Their sum is $\boxed{199}$.

(26) **7/50 ID: [C2A2]**

No solution is available at this time.

(27) **255 ID: [41B3]**

Since $256 = 2^8$ the divisors of 256 are the powers of 2 up to 2^8. So the sum of the proper factors of 256 is $2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 = \boxed{255}$.

(28) **24 ID: [2413]**

Any positive integer divisor of N must take the form $2^a \cdot 3^b \cdot 5^c$ where $0 \le a \le 3$, $0 \le b \le 2$ and $0 \le c \le 1$. In other words, there are 4 choices for a, 3 choices for b and 2 choices for c. So there are $4 \cdot 3 \cdot 2 = \boxed{24}$ natural-number factors of N.

(29) **June ID: [A513]**

Since exactly 1 in every n consecutive dates is divisible by n, the month with the fewest relatively prime days is the month with the greatest number of distinct small prime divisors. This reasoning gives us June ($6 = 2 \cdot 3$) and December ($12 = 2^2 \cdot 3$). December, however, has one more relatively prime day, namely December 31, than does June, which has only 30 days. Therefore, $\boxed{\text{June}}$ has the fewest relatively prime days.

(30) **301 ID: [52AC]**

If n leaves a remainder of 1 when divided by 2, 3, 4, 5, and 6, then $n-1$ is divisible by all of those integers. In other words, $n-1$ is a multiple of the least common multiple of 2, 3, 4, 5, and 6. Prime factorizing 2, 3, 4, 5, and 6, we find that their least common multiple is $2^2 \cdot 3 \cdot 5 = 60$. Thus the possible values for an integer n which is one more than a multiple of 2, 3, 4, 5, and 6 are 61, 121, 181, 241, 301 and so on. Checking them one at a time, we find that the least of these integers which is divisible by 7 is $\boxed{301}$.