ASTRONOMY 9707

Computer Project

This project requires you to write a computer program to calculate the propagation of a sound wave, illustrating the cases of linear wave propagation and nonlinear steepening.

We have seen in class that finite difference approximations to the partial differential equations of hydrodynamics are not always stable. For this project, we will use the Lax method to integrate the flux conservative form of the continuity and momentum equations. For an equation of the form

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}(fv), \tag{1}$$

the Lax method is a finite-difference approximation

$$\frac{f_j^{n+1} - (f_{j+1}^n + f_{j-1}^n)/2}{\Delta t} = -\frac{f_{j+1}^n v_{j+1}^n - f_{j-1}^n v_{j-1}^n}{2\Delta x}. \tag{2}$$

Note the difference with the forward time centered space (FTCS) scheme which would be unconditionally unstable. This method is conditionally stable, i.e., it is stable if

$$\nu \equiv \frac{c\Delta t}{\Delta x} \leq 1, \tag{3}$$

where ν is known as the Courant number.

1. Derive the flux conservative form of the momentum equation for one-dimensional motions, by combining the standard form of the momentum equation with the continuity equation. The equations to solve, in flux conservative form, are

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= -\frac{\partial}{\partial x}(\rho v_x), & (4)\\ \frac{\partial(\rho v_x)}{\partial t} &= -\frac{\partial}{\partial x}(\rho v_x v_x) - c_{\rm s}^2\frac{\partial \rho}{\partial x}. & (5) \end{aligned}$$

   These equations assume that the gas is isothermal, so that the sound speed $c_{\rm s} = \sqrt{\gamma P/\rho} = \sqrt{\gamma kT/m}$ remains constant.

2. Set up a periodic finite difference grid of $N = 1000$ points. Periodicity means point $N+1$ is the same as the 1st point. Initial conditions of the density and x-velocity (recall the linearized equations) are

$$\begin{aligned} \rho(x) &= \rho_0 + \rho_1\cos(2\pi x/\lambda), & (6)\\ v_x(x) &= v_1\cos(2\pi x/\lambda), & (7) \end{aligned}$$

   where $v_1 = c_{\rm s}\rho_1/\rho_0$. Let $c_{\rm s} = 1$ in all equations. Also, let $\rho_0 = 1$. Here, x is an array of N numbers starting with zero, spaced by $\Delta x = L/N$. In this problem, let $L = 2$ and $\lambda = 1$, i.e., two wavelengths in your periodic domain.

March your solution forward in time at all points using the Lax method given above and central differencing for the pressure gradient term in the momentum equation. The wave should move to the right.

(a) Set $\rho_1 = 0.03$ (a linear wave) and integrate with time step Δt determined by using a Courant number $\nu = 0.1$. Show the profiles of v_x and $\rho - \rho_0$ at times $t = 0.3, 0.5$, and 1.0. The last time is one full wavelength crossing time.

(b) Set $\rho_1 = 0.3$ (a nonlinear wave) and integrate with time step Δt determined by using a Courant number $\nu = 0.1$. Show the same profiles as part (a) at times $t = 0.3, 0.5$, and 1.0.

(c) Investigate the effect of poorer resolution on the solution. Pick a significantly smaller value of N and show how the solution to part (a) changes. Continue to determine the time step Δt using $\nu = 0.1$. Illustrate the solution with a single plot of either v_x or $\rho - \rho_0$ comparing with one of the time outputs in part (a). Look at the numerical values of the solution carefully.

(d) Investigate the effect of Courant number. Try to solve part (a) but with $\nu = 2$. Show one figure of very early time output which illustrates the numerical instability for this value of ν.

Please provide a copy of all computer programs you write. Be sure to add in comment lines in the program. Your mark will depend on the clarity of the program. Any computing language is acceptable; the comments should clarify the logic of the program.

Write some text summarizing what you have done and the results you have obtained. Comment on the difference in the results from parts (a) and (b). Why do you think that the profile steepens in the solution to part (b)? In part (c), you investigated the effect of grid size. What did you learn? What are your thoughts about the feasibility of very high-resolution three-dimensional models? How long did your one-dimensional model take to run on the computer you were using?