Design and Implementation of a PID controller for a Self-Balancing Vertical Stick System.

Abstract:

The goal of this project is to design and build a controller for a self-balancing vertical stick system (inverted pendulum). The system comprises of a stick positioned on a pivot and outfitted with an angular position sensor. The goal is to create a control system that can keep the stick upright by altering the motor torque depending on sensor data. The controller will be created with a combination of state space control & PID control. Simulation models will be developed to assess the controller's performance and test its efficacy in producing steady and accurate self-balancing. This project's successful completion will contribute to the knowledge and development of control systems for self-balancing devices, which have potential applications in disciplines such as robotics and stabilization systems.

Literature review

System identification is the first step when designing any controller. Parameters of the system depend on the dynamics of the components in it. According to our problem statement, stick, moving robot and environment where the system is placed are the factors we need to look out for.

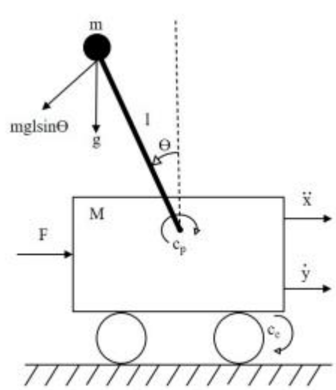

Figure.1.1 Free body diagram of inverted pendulum

The above figure explains some of the characteristics of the system. Along with these, speed of the cart, angular velocity of the stick is considered as state space variables.

To start with, the nonlinear model of an Inverted Pendulum on a Cart system is presented in this article. When a force is given to the system based on Newton's second law of motion, the cart suffers a retarding force owing to friction between the wheels and the ground. The pendulum revolves around a pivot axis, and its acceleration is moderated by gravity and frictional force.[4]

The author derived the following cart and pendulum system dynamics equations incorporated with DC motor dynamics using Newton-Euler equations.[4]

$$(m+M)r\ddot{\psi} = \frac{2K}{Rr}V - \frac{2K^2}{Rr}\dot{\psi} - \frac{2Iw}{r}\ddot{\psi} - c_c r\dot{\psi} + ml\dot{\theta}^2 sin(\theta) - mlcos(\theta)\ddot{\theta}$$

$$ml^2\ddot{\theta} = -c_p\dot{\theta} - mglsin(\theta) - mlr\ddot{\psi}cos(\theta)$$

Equation 1.1

Using these equations, where V is the input voltage, and the outputs are angular position of the wheel ψ and the angle of pendulum θ. The position of cart can be obtained by taking a product of radius of wheel and the angle of the wheel.[4]

The state space model of the system can be evaluated using Taylor's expansion theorem. [4]

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -a_1 & a_2 & -a_3 \\ 0 & 0 & 0 & 1 \\ 0 & -a_4 & a_5 & -a_6 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ b_1 \\ 0 \\ b_2 \end{bmatrix} \quad C = \begin{bmatrix} c_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

Equation 1.2

where,

$$a_1 = \frac{\frac{2K^2+c_c Rr^2}{rR}}{Mr+\frac{2I_w}{r}}, \quad a_2 = \frac{mg}{Mr+\frac{2I_w}{r}}, \quad a_3 = \frac{\frac{c_p}{l}}{Mr+\frac{2I_w}{r}},$$

$$a_4 = \frac{\frac{2K^2+c_c Rr^2}{Rl}}{Mr+\frac{2I_w}{r}}, \quad a_5 = \frac{\frac{g}{l}\left((m+M)r+\frac{2I_w}{r}\right)}{Mr+\frac{2I_w}{r}}, \quad a_6 = \frac{\frac{c_p}{ml^2}\left((m+M)r+\frac{2I_w}{r}\right)}{Mr+\frac{2I_w}{r}},$$

$$b_1 = \frac{\frac{2K}{Rr}}{Mr+\frac{2I_w}{r}}, \quad b_2 = \frac{\frac{2K}{Rl}}{Mr+\frac{2I_w}{r}}, \quad c_1 = r$$

Equation 1.3

The following transfer functions can be computed using these state space equations.

$$\frac{y(s)}{V(s)} = \frac{c_1[b_1 s^2 - (a_3 b_2 - a_6 b_1)s + a_2 b_2 - a_5 b_1]}{s[s^3 + (a_1 + a_6)s^2 + (a_1 a_6 - a_3 a_4 - a_5)s - a_1 a_5 + a_2 a_4]}$$

$$\frac{\theta(s)}{V(s)} = \frac{-b_2 s + a_1 b_2 - a_4 b_1}{s^3 + (a_1 + a_6)s^2 + (a_1 a_6 - a_3 a_4 - a_5)s - a_1 a_5 + a_2 a_4}$$

Equation 1.4

The author calculated and estimated some parameters which we can apply to the given transfer function.

Parameter	Value
M	$1.21\ kg$
m	$0.020\ kg$
g	$9.81\ \frac{m}{s^2}$
r	$0.0375\ m$
l	$0.115\ m$
I_w	$2.8395\ \times 10^{-5}\ kg\, m^2$
I_p	$26.45\ \times 10^{-5}\ kg\, m^2$
K	0.0586
R	$2.1429\ \Omega$
c_c	10
c_p	0.0005

Substituting these values in the Equation 1.3 and Equation 1.4 we get the final transfer function as,

$$\frac{y(s)}{V(s)} = \frac{1.168s^2 + 2.209s - 99.67}{s^4 + 11.74s^3 - 68.1s^2 - 838s}$$

$$\frac{\theta(s)}{V(s)} = \frac{582.1s}{s^3 + 11.74s^2 - 68.1s - 838}$$

Equation 1.5

Using a similar approach, we are going to derive a transfer function which can be applied and simulated in MATLAB along with PID controller.

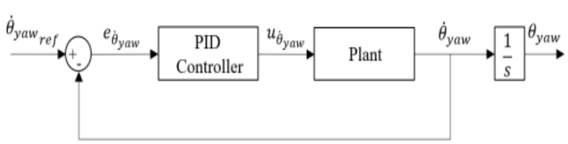

Figure.1.2 Closed loop system [2]

Other methods:

The second method I reviewed was FSF control method. FSF control, also known as Pole Placement, is a mechanism used in state feedback control theory to arrange a plant's closed-loop poles at pre-determined places on the s-plane. Pole placement is beneficial because it influences the eigenvalues of the system, which affect the features of the system response. The FSF algorithm is an automated approach for locating a suitable state-feedback controller.[1]

Similar to the previous approach, this method also uses state space representation. But due to a different strategy in identifying the system, state space equations are different.

$$\begin{bmatrix}\dot{\theta}\\ \dot{\alpha}\\ \ddot{\theta}\\ \ddot{\alpha}\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 39.32 & -14.52 & 0\\ 0 & 81.78 & -13.98 & 0\end{bmatrix}\begin{bmatrix}\theta\\ \alpha\\ \dot{\theta}\\ \dot{\alpha}\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 25.54\\ 24.59\end{bmatrix} V_m$$

$$Y = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}\theta\\ \alpha\end{bmatrix} + \begin{bmatrix}0\\ 0\end{bmatrix} V_m$$

Equation 2.1[1]

By using this state space equation, we can derive the following transfer functions.

$$\frac{\theta_s}{V_m} = \frac{25.54\, s^2 - 4.537e{-}014\, s - 1122}{s^4 + 14.52\, s^3 - 81.78\, s^2 - 638\, s}$$

$$\frac{\theta_s}{V_m} = \frac{25.54\, s^2 - 4.537e{-}014\, s - 1122}{s^4 + 14.52\, s^3 - 81.78\, s^2 - 638\, s}$$

Equation 2.2

Overall block diagram of the controller is,

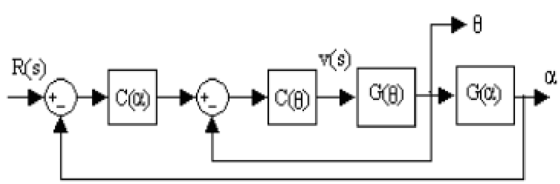

Figure.2.2 Block diagram of controller

Based on the dominant pole of the system and design requirements settling time and overshoot is calculated using root locus method. After finding damping ratio and natural frequency we can calculate the desired pole [1].

Thus, we can design, test and simulate a controller using root locus.

The same system of state space equation can be used to design a FSF controller based on Ackerman's formula which is represented as,

$$K = [0 \ \cdots \ 0 \ 1] M_c^{-1} \emptyset_d(A)$$

$$M_c = \begin{bmatrix} B & AB & \cdots & A^{(n-1)}B \end{bmatrix}$$

Equation 2.3

Mc indicates the controllability matrix and $\emptyset_d(A)$ is the characteristic of closed loop pole.

By using the Ackerman formula, gain matrix K can be calculated.

$K = (-8.9144 \ \ 26.4116 \ \ -2.9755 \ \ 3.3539)$

The system can be stabilized by using the obtained K and control signal u = -Kx. A step response is then plotted to find out the stability of the system. [1]

Methodology:

For the design of our controller, we are going to consider a single input output system. To implement the controller for inverted pendulum system, we need to have a transfer function on which the controller is to be implemented. To find it, we must take every force acting on the system into consideration.

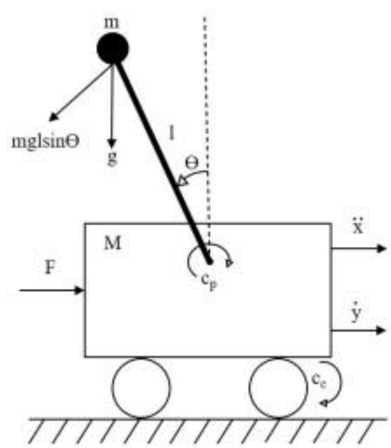

Figure.3.1

Here, a free body diagram of simple cart pendulum system is shown. Gravitational force due to mass and length of the pendulum, force along the normal, mass of the cart, rotational force of pendulum and wheel of the cart are the forces taken into consideration. We will also assume the settling time of the system should be below 2s and amplitude of deflection of the pendulum must be within 0.1.

Using Newton-Euler equations for cart and pendulum system dynamics, combined with DC motor dynamics the following equation can be derived for the necessary forces,

$$(m+M)r\ddot{\psi} = \frac{2K}{Rr}V - \frac{2K^2}{Rr}\dot{\psi} - \frac{2Iw}{r}\ddot{\psi} - c_c r\dot{\psi} + ml\dot{\theta}^2 sin(\theta) - mlcos(\theta)\ddot{\theta}$$

$$ml^2\ddot{\theta} = -c_p\dot{\theta} - mglsin(\theta) - mlr\ddot{\psi}cos(\theta)$$

Equation 3.1

Here V is the input voltage given to the motor of the wheel, ψ is the angular position of the wheel and θ is the angle of pendulum.

Taylor's expansion theorem can be used to make the state space model of system.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -a_1 & a_2 & -a_3 \\ 0 & 0 & 0 & 1 \\ 0 & -a_4 & a_5 & -a_6 \end{bmatrix} \; B = \begin{bmatrix} 0 \\ b_1 \\ 0 \\ b_2 \end{bmatrix} C = \begin{bmatrix} c_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \; D = \begin{bmatrix} 0 \end{bmatrix}$$

Equation 3.2

where,

$$a_1 = \frac{\frac{2K^2+c_cRr^2}{rR}}{Mr+\frac{2I_w}{r}}, \quad a_2 = \frac{mg}{Mr+\frac{2I_w}{r}}, \quad a_3 = \frac{\frac{c_p}{l}}{Mr+\frac{2I_w}{r}},$$
$$a_4 = \frac{\frac{2K^2+c_cRr^2}{Rl}}{Mr+\frac{2I_w}{r}}, \quad a_5 = \frac{\frac{g}{l}((m+M)r+\frac{2I_w}{r})}{Mr+\frac{2I_w}{r}}, \quad a_6 = \frac{\frac{c_p}{ml^2}((m+M)r+\frac{2I_w}{r})}{Mr+\frac{2I_w}{r}},$$
$$b_1 = \frac{\frac{2K}{Rr}}{Mr+\frac{2I_w}{r}}, \quad b_2 = \frac{\frac{2K}{Rl}}{Mr+\frac{2I_w}{r}}, \quad c_1 = r$$

By simplifying the state space equation and substituting the values, the following transfer functions can be computed.

$$\frac{y(s)}{V(s)} = \frac{c_1[b_1s^2 - (a_3b_2 - a_6b_1)s + a_2b_2 - a_5b_1]}{s[s^3 + (a_1+a_6)s^2 + (a_1a_6 - a_3a_4 - a_5)s - a_1a_5 + a_2a_4]}$$

$$\frac{\theta(s)}{V(s)} = \frac{-b_2s + a_1b_2 - a_4b_1}{s^3 + (a_1+a_6)s^2 + (a_1a_6 - a_3a_4 - a_5)s - a_1a_5 + a_2a_4}$$

Equation.3.3

By simplifying this equation, we get out final continuous transfer function as,

$$\frac{\Phi(s)}{U(s)} = \frac{\frac{ml}{q}s}{s^3 + \frac{b(I+ml^2)}{q}s^2 - \frac{(M+m)mgl}{q}s - \frac{bmgl}{q}} \qquad [\frac{rad}{N}]$$

Equation.3.4

For our consideration, we are assuming the following values,

mass of the cart (M) = 0.8
mass of the pendulum (m) = 0.2
coefficient of friction for cart (b) = 0.1
mass moment of inertia of the pendulum (I) = 0.006
gravitational constant (g) = 9.8
length to pendulum center of mass (l) = 0.3

and q is,

$$q = (M+m)(I+ml^2) - (ml)^2$$

Hence q = 0.0132

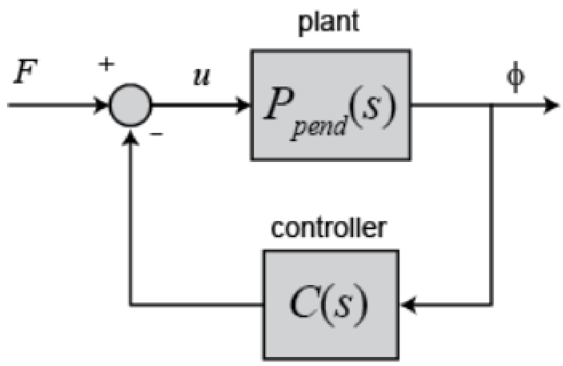

Figure 3.2

This is the overall block diagram of the system where a controller (PID) is applied to the plant model.

$$u(k) = K_P\theta_1(k) + K_I \sum\nolimits_{i=0}^{k} \theta_1(i)\Delta T + K_D\frac{\theta_1(k) - \theta_1(k-1)}{\Delta T}$$

Equation.3.5

The plant transfer function using the above equation and assumed values becomes,

```
pendulum_tf =

                   2.497e-05 s
  ---------------------------------------------------
  8.49e-06 s^3 + 9.988e-07 s^2 - 0.000245 s - 2.45e-05
```

Equation.3.6

Now to convert this into a discrete time domain we will use c2d() function in MATLAB and assuming sampling frequency as 20Hz.

```
pendulum_tf_disc =

  0.003691 z^2 - 7.258e-06 z - 0.003684
  -------------------------------------
    z^3 - 3.067 z^2 + 3.06 z - 0.9941

Sample time: 0.05 seconds
Discrete-time transfer function.
```

Equation.3.7

As the transfer function is set up, we will implement the PID controller for the pendulum to stay in upright position by controlling the cart. We will keep tuning the PID controller until the output requirements are not met.

Results:

As we go into the tuning procedure of the PID controller we need to test various combinations of proportional, integral and derivative gains that will keep the system stable and in the defined margin.

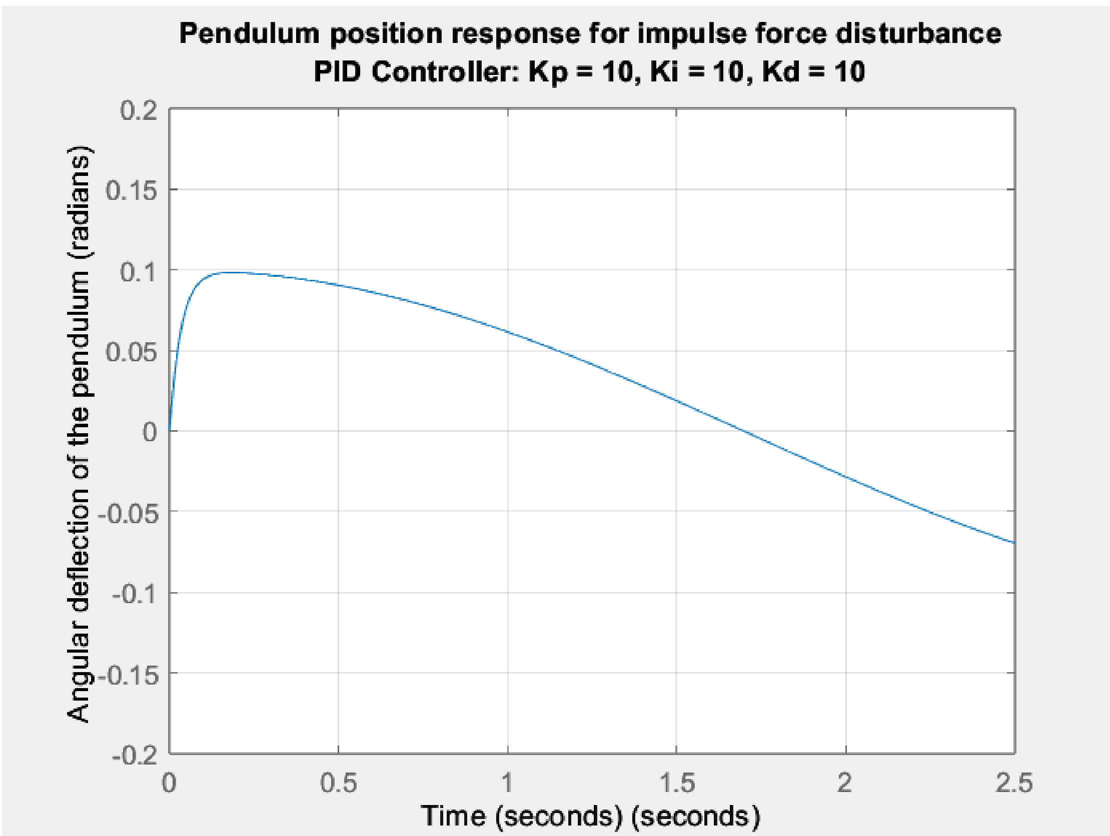

Figure 4.1

Above is the figure generated when we keep the Kp = 10, Ki = 10, Kd = 10. We can clearly see that the system is not reaching to a point where steady state error is zero. **Hence, we will reject this controller.**

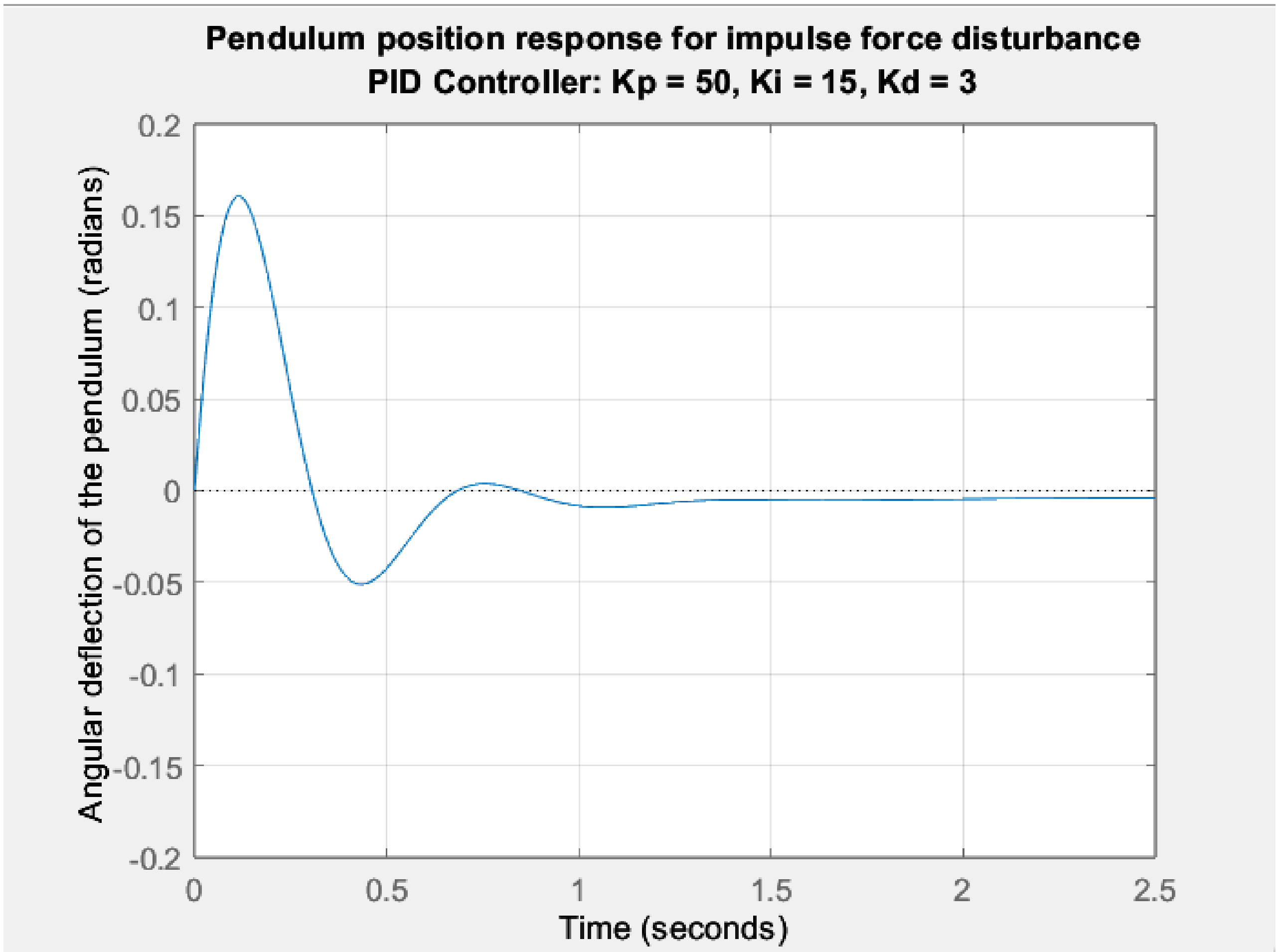

Figure 4.2

Using Kp = 50, Ki = 15, Kd = 3 we do get the steady state error near to zero but the settling time criteria(1s) and angular deflection criteria (0.1 rad) are not satisfied.

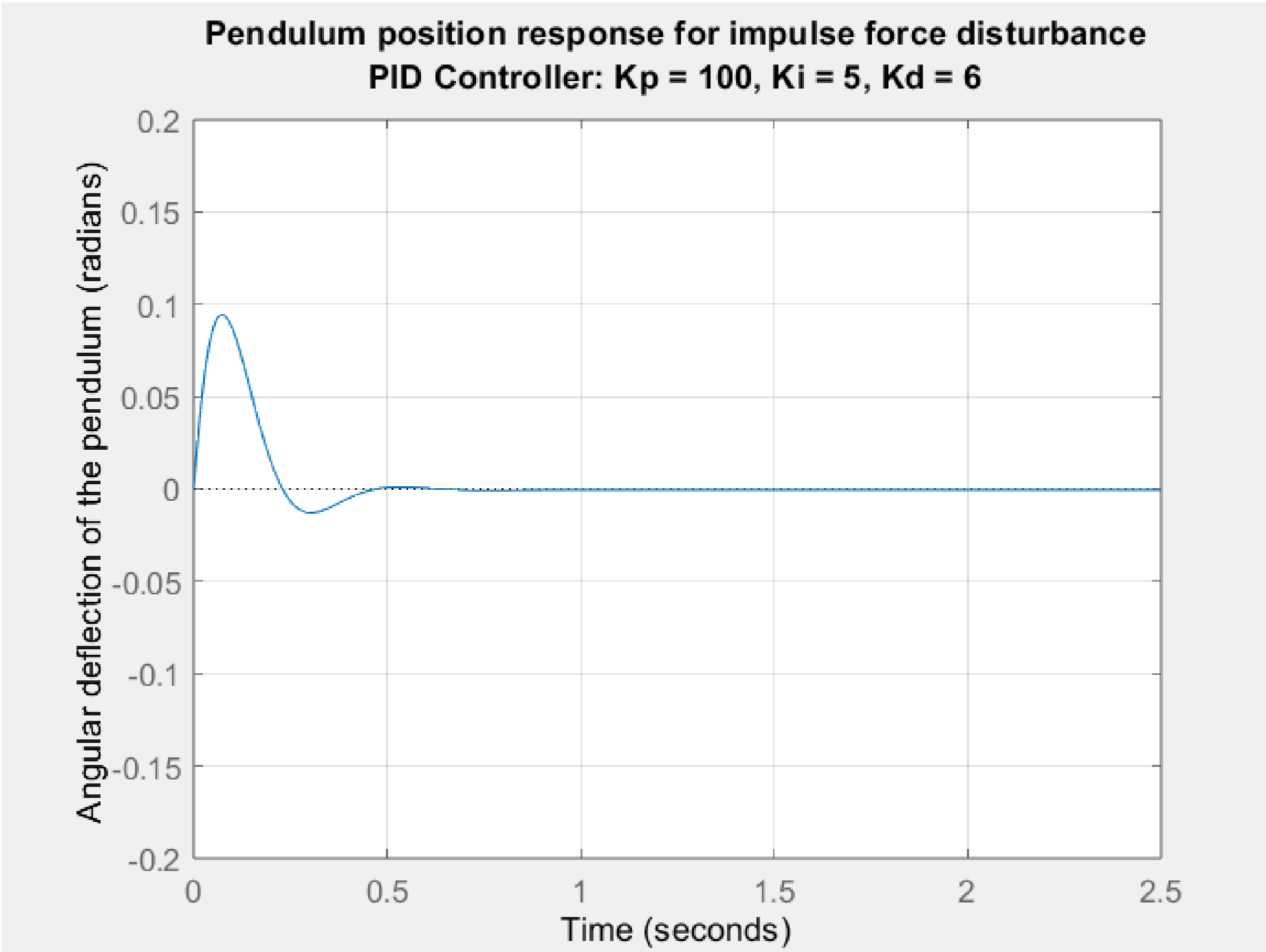

Figure 4.3

After various iterations of the gains of PID controller we found the right controller which will satisfy both the conditions (settling time < 1 & Angular deflection < 0.1).

Hence, we will select this as the final controller for our inverted pendulum system.

Conclusion:

The PID controller has proved to be an excellent control approach for simulation of stabilizing an inverted pendulum system. The position and stability of the pendulum is maintained in the presence of external disturbances by designing and implementing a PID controller.

The controller can achieve desirable performance characteristics such as quick response, low overshoot, and resilience to disturbances by properly adjusting the PID gains (proportional, integral, and derivative). The proportional component allows the controller to respond proportionately to the error, the integral term aids in the elimination of steady-state errors, and the derivative term offers damping and increases stability.

This is the story for **pendulum position control. But what about cart position control?**

$$\frac{X(s)}{F(s)} = \frac{P_{cart}(s)}{1 + P_{pend}(s)C(s)}$$

Equation 5.1

This is the transfer function of output cart position with respect to input force applied to the cart.

$$\frac{X(s)}{U(s)} = \frac{\frac{(I+ml^2)s^2 - gml}{q}}{s^4 + \frac{b(I+ml^2)}{q}s^3 - \frac{(M+m)mgl}{q}s^2 - \frac{bmgl}{q}s} \qquad \left[\frac{m}{N}\right]$$

$$q = [(M+m)(I+ml^2) - (ml)^2]$$

Equation 5.2

Like the pendulum position we found, the continuous time transfer function of the cart position and converted that to discrete time domain.

```
  0.002269 z^3 - 0.002416 z^2 - 0.002395 z + 0.002262
  ---------------------------------------------------
     z^4 - 4.069 z^3 + 6.129 z^2 - 4.051 z + 0.991

Sample time: 0.05 seconds
Discrete-time transfer function.
```

Equation 5.3

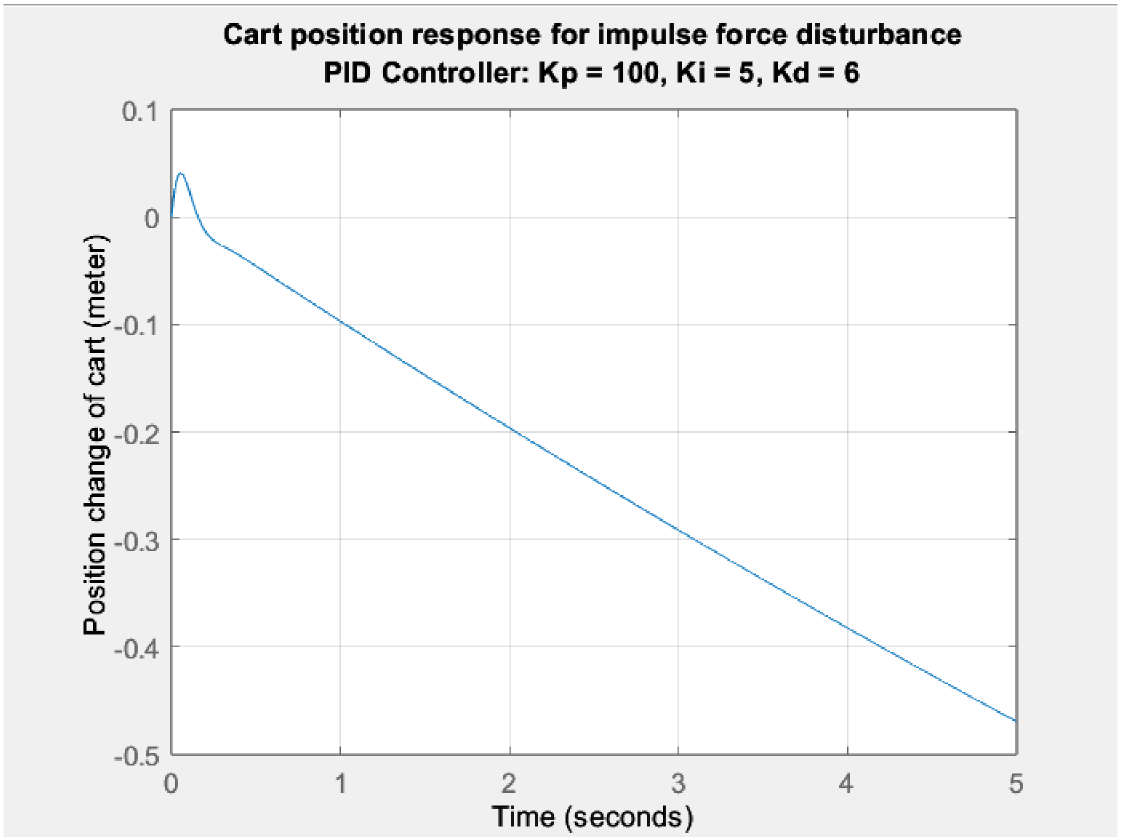

Figure 5.1

We can see that using the same controller used for pendulum angle control, cannot provide steady state action of cart. The cart moves in the negative direction with constant velocity.

Even if the pendulum angle is controlled by PID, we cannot maintain the position of the cart. Hence, this type of system is not feasible to implement on actual physical system in real world, but provides a very good understanding of the PID controller.

It is important to remember that PID controllers have several limitations. The controller's performance is heavily dependent on the accuracy of the system model and the PID gains used. Furthermore, the PID controller may not be appropriate for highly nonlinear or time-varying systems that require more complex control approaches.

In general, the PID controller is a basic and commonly used control technique for inverted pendulum systems. Its ease of installation, simplicity, and efficacy make it a powerful tool in a variety of applications, including simple robots, stabilization systems, and dynamic system control.

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