School of Mathematical Sciences | Lancaster University

# Why are we all using biased estimates of autocorrelation? and Why are most time series simulation studies poor?

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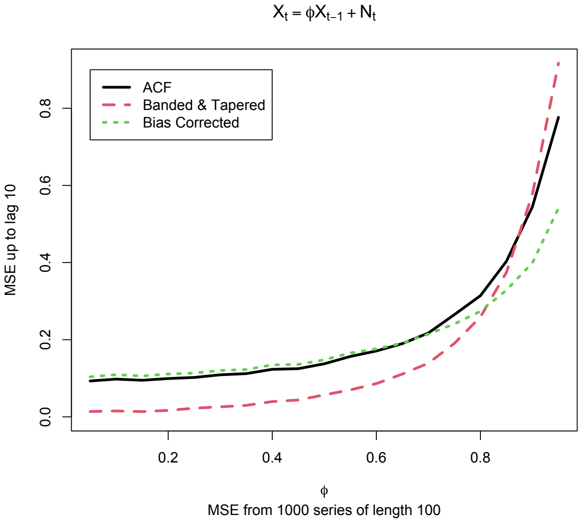
Joint work with Colin Gallagher, Xiyan Tan, and Xinyi Li (Clemson)
Clemson Nov 2025

# Aim



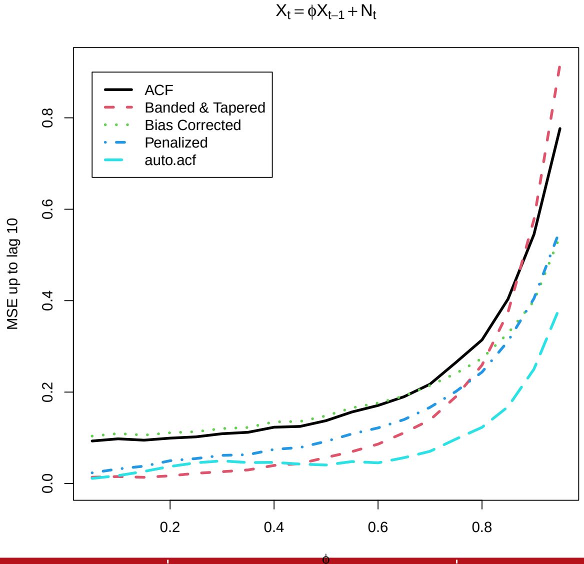

$X_t = \phi X_{t-1} + N_t$
ACF
Banded & Tapered
Bias Corrected
MSE up to lag 10
0.0
0.2
0.4
0.6
0.8
0.2
0.4
0.6
0.8
$\phi$
MSE from 1000 series of length 100

# Aim

- Provide better estimates of the (p)acf by adaptive shrinkage
- Give more intuition on causality
- Motivate better time series simulation studies.

# Results

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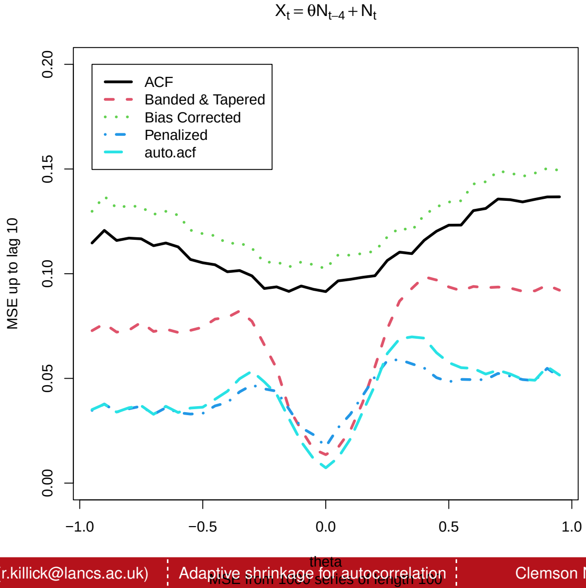
# Results


$X_t = \theta N_{t-4} + N_t$
ACF
Banded & Tapered
Bias Corrected
Penalized
auto.acf
MSE up to lag 10
0.00
0.05
0.10
0.15
0.20
−1.0
−0.5
0.0
0.5
1.0
theta
MSE from 1000 series of length 100

# Method



For $y_t = x_t - \bar{x}$, sample ACF at lag $h$, $\rho(h)$ minimizes:

$$Q_h(\rho_h) = \sum_{t=h+1}^{n+h} (y_t - \rho_h y_{t-h})^2 ,$$

To shrink toward zero (reduce variance) add a penalty:

$$Q_h(\rho_h) + \lambda_h \left( \sum_{t=1}^{n} y_t^2 \right) (\rho_h)^2$$

To move toward a bias corrected estimator, $\hat{\rho}_b$:

$$Q_h(\rho_h) + \lambda_h \left( \sum_{t=1}^{n} y_t^2 \right) (\rho_h - \hat{\rho}_b)^2$$

Let the data tell us which one to use and how much to shrink.

We minimize

$$Q_h(\rho_h) + \lambda_h \left( \sum_{t=1}^{n} y_t^2 \right) (\rho_h - \rho_\tau(h))^2$$

The solution is

$$\tilde{\rho}(h) = w_h \rho_\tau(h) + (1 - w_h)\hat{\rho}(h),$$

where $w_h = \lambda_h/(1+\lambda_h)$, and $\hat{\rho}(h)$ is the ordinary ACF at lag $h$.\

Note: General framework includes, ordinary ACF, shrinkage toward zero, bias correction, Banded & Tapered, ...

# Implementation



We use plug-in estimators of the tuning parameters.

Shrink toward zero if $\hat{\rho}_b(h)$ is small, move toward bias correction if larger.

$$\rho_\tau(h) = \begin{cases} 0 & |\hat{\rho}_b(h)| < \ell_h \\ \hat{\rho}_b(h) & |\hat{\rho}_b(h)| > \ell_h \end{cases}$$

Select $\lambda_h$ ($w$) to move more aggressively near boundaries, and to smoothly transition between making smaller and increasing magnitude:

$$\lambda_h = \begin{cases} \frac{h(\ell_h - r)}{r^2} & r < \ell_h \\ \frac{10\log_{10}(n)h(r-\ell_h)(1-\ell_h)}{(1-r)^2} & r > \ell_h, \end{cases}$$

where $r = |\hat{\rho}_b(h)|$.

# Asymptotics



$\ell_h$ controls the asymptotic distribution of $\tilde{\rho}(h)$ for $\rho(h) = 0$:

1. If $\sqrt{n}\ell_h \to 0$ as $n \to \infty$, $\hat{\rho}$ and $\tilde{\rho}(h)$ have same limiting distribution.
2. If $\sqrt{n}\ell_h \to C$ as $n \to \infty$, $\sqrt{n}\tilde{\rho}(h)$ has a non-normal limit with smaller variance than $\sqrt{n}\hat{\rho}(h)$.
3. If $\ell_h \to 0$ and $\sqrt{n}\ell_h \to \infty$ as $n \to \infty$, $\sqrt{n}\tilde{\rho}(h) \to 0$ in probability.

$\sqrt{n}(\tilde{\rho}(h) - \rho(h))$ has same limit as $\sqrt{n}(\hat{\rho}(h) - \rho(h))$, when$\rho(h) \neq 0$, \
We use

$$\ell_h = c_h\sqrt{\log(n)/n},$$

where $c_h$ depends on the estimated ACF.

# PACF

Similar development for the PACF at lag $h$, $\pi(h)$ gives:

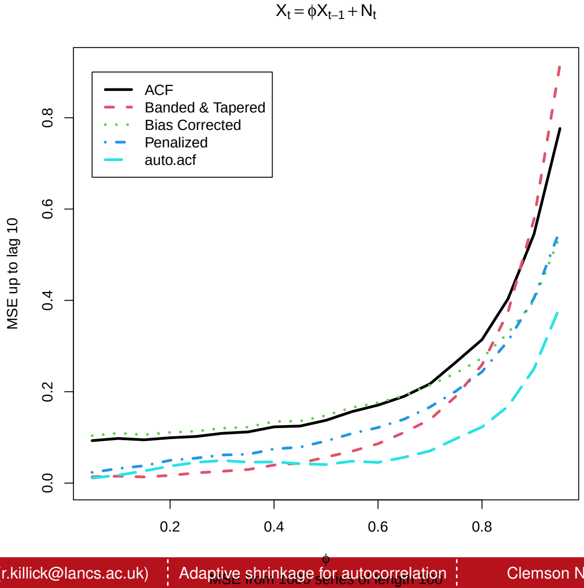
$$\tilde{\pi}(h) = w_h \pi_\tau(h) + (1 - w_h)\hat{\pi}(h),$$

where $w_h = \lambda_h/(1 + \lambda_h)$, and $\hat{\pi}(h)$ is the ordinary PACF at lag $h$.\ Use $\ell_h$, $\hat{\pi}_b(h)$, and $\lambda_h$ as above (but using PACF).

$$\pi_\tau(h) = \left\{ \begin{array}{ll} 0 & |\hat{\pi}_b(h)| < \ell_h \\ \hat{\pi}_b(h) & |\hat{\pi}_b(h)| > \ell_h. \end{array} \right\}$$

Asymptotic results again controlled by $\ell_h$ (we use our plug-in tuning parameters below).

# Results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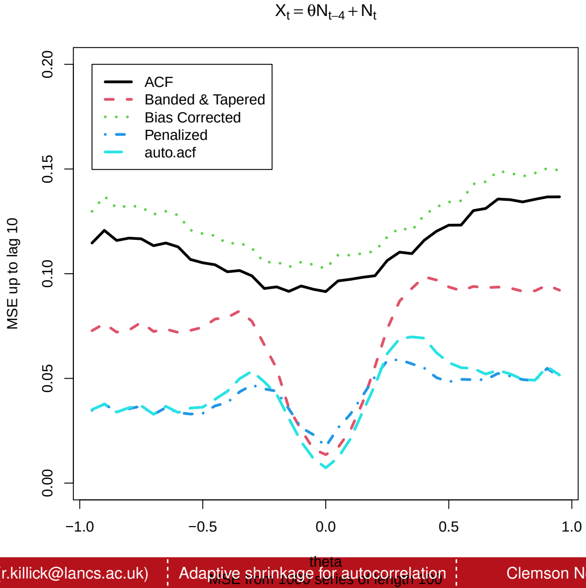
# Results

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ACF
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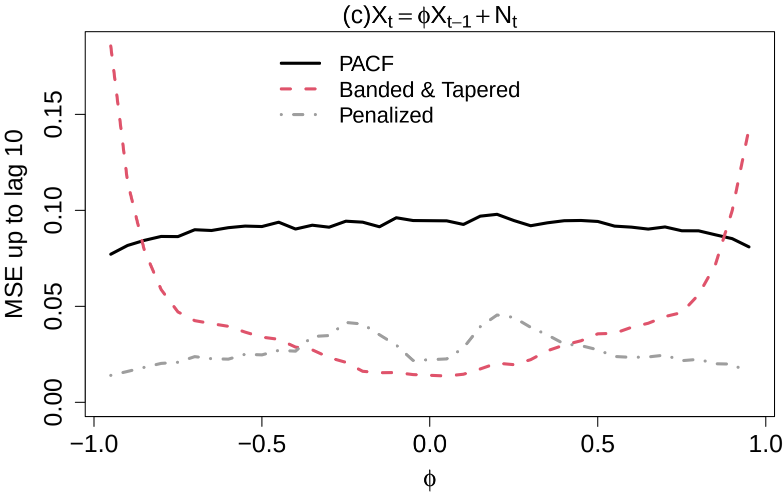
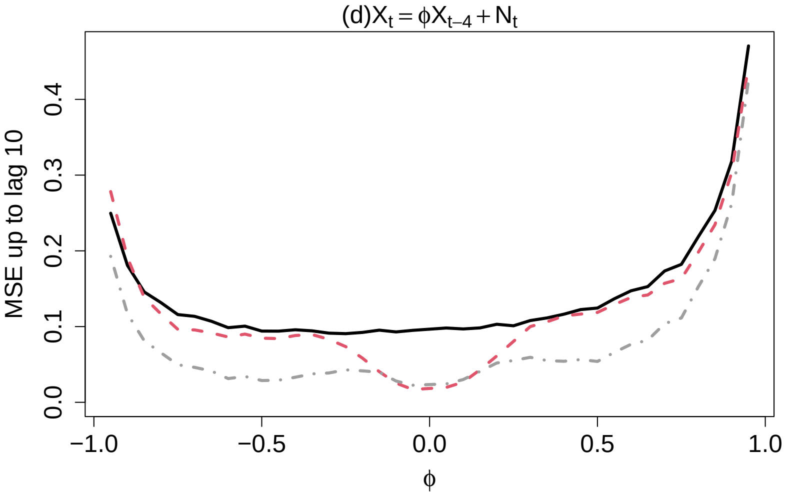
# Results




(c)$X_t = \phi X_{t-1} + N_t$
PACF
Banded & Tapered
Penalized
MSE up to lag 10
0.00
0.05
0.10
0.15
−1.0
−0.5
0.0
0.5
1.0
$\phi$

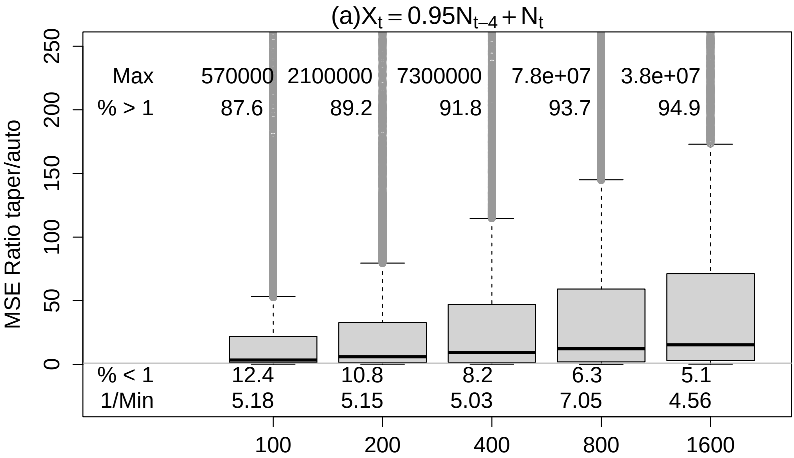
# Results



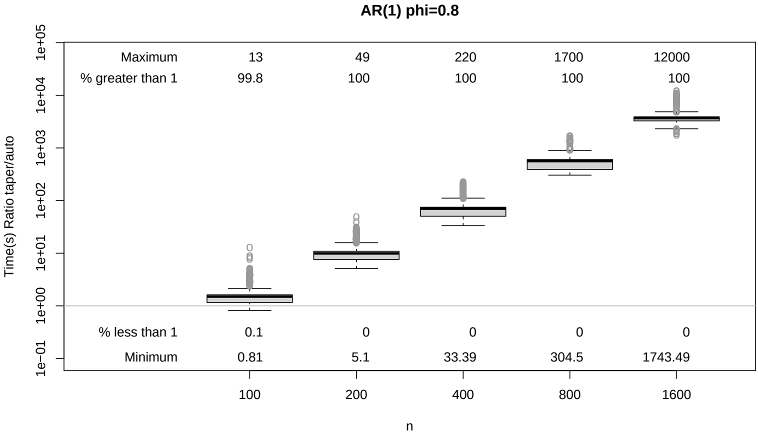


# Grow N




$(a) X_t = 0.95N_{t-4} + N_t$
MSE Ratio taper/auto
0
50
100
150
200
250
Max
570000
2100000
7300000
7.8e+07
3.8e+07
% > 1
87.6
89.2
91.8
93.7
94.9
% < 1
12.4
10.8
8.2
6.3
5.1
1/Min
5.18
5.15
5.03
7.05
4.56
100
200
400
800
1600

# Grow N


AR(1) phi=0.8
Time(s) Ratio taper/auto
1e+05
1e+04
1e+03
1e+02
1e+01
1e+00
1e−01
Maximum 13 49 220 1700 12000
% greater than 1 99.8 100 100 100 100
% less than 1 0.1 0 0 0 0
Minimum 0.81 5.1 33.39 304.5 1743.49
100
200
400
800
1600
n

# Sims

- Want to simulate across the causal space
- For a wide set of AR, MA and ARMA processes

HOW?

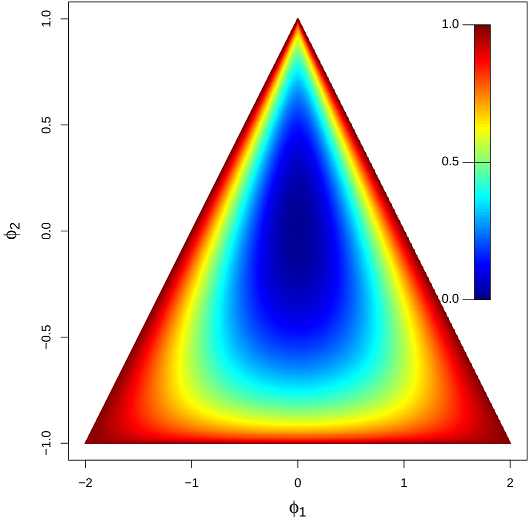
# Sims

- Want to simulate across the causal space
- For a wide set of AR, MA and ARMA processes

HOW?

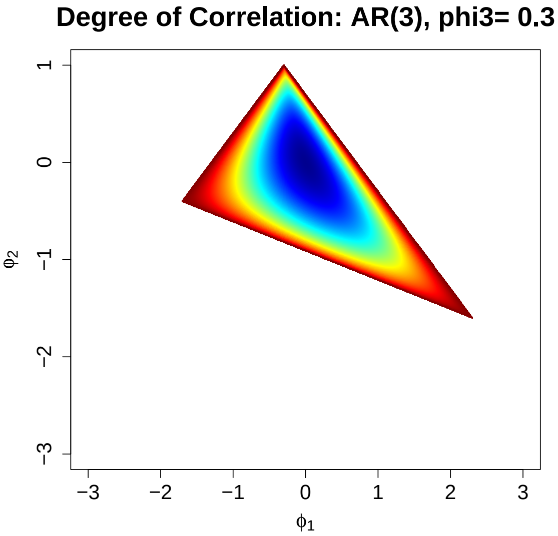
- JTSA published 44 papers in 2024, 39 had an AR simulation
- Only 2 papers selected coefficients randomly (in AR(1))
- Of the remaining 37, 25 used a single AR parameter setting!
- Only 12/39 considered orders larger than 1!

# Solution AR2



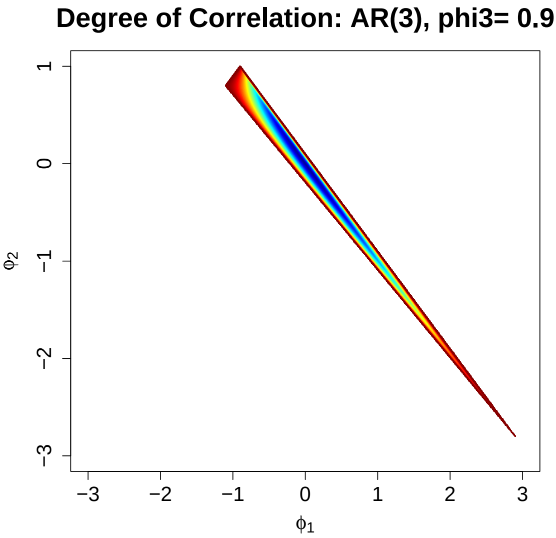
# Solution AR3





# Solution AR3





# Problem - Causality



AR(p):

$$Y_t = \sum_{j=1}^{p} \phi_j Y_{t-j} + \epsilon_t \quad t = 0, \pm 1, \pm 2, \dots,$$

Causal:

$$Y_t = \sum_{j=0}^{\infty} c_j \epsilon_{t-j} \quad \sum_{j=0}^{\infty} |c_j| < \infty;$$

How to check causality?

# Problem - Causality



AR(p):

$$Y_t = \sum_{j=1}^{p} \phi_j Y_{t-j} + \epsilon_t \quad t = 0, \pm 1, \pm 2, \dots,$$

Causal:

$$Y_t = \sum_{j=0}^{\infty} c_j \epsilon_{t-j} \quad \sum_{j=0}^{\infty} |c_j| < \infty;$$

How to check causality?

Roots of the polynomial:

$$\Phi(x) = 1 - \phi_{p,1} x - \phi_{p,2} x^2 - \cdots \phi_{p,p} x^p$$

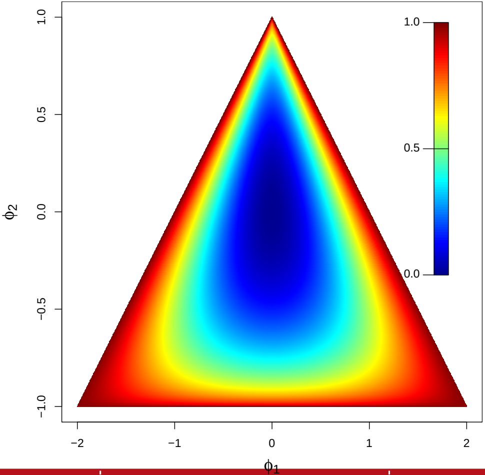
Are the roots outside the unit circle?

# Known solution spaces



AR1: $\phi_{1,1} \in (-1, 1)$

AR2: What about higher orders?



# A different way



PACF at lag $h$: $\alpha(h)$.

$$|\alpha(h)| < 1 \quad \text{for} \quad 1 \leq h \leq p \quad \text{and} \quad \alpha(h) = 0 \quad \text{for} \quad h > p.$$

Ramsey (1974) noted a 1:1 mapping. Any point in $(-1, 1)^p$ corresponds to a unique point in the causal parameter space:

$$(\phi_{p,1}, \ldots, \phi_{p,p}) = g\,(\alpha(1), \ldots, \alpha(p))$$

$$(\alpha(1), \ldots, \alpha(p)) = g^{-1}\left(\phi_{p,1}, \ldots, \phi_{p,p}\right).$$

# Durbin-Levinson

The map $g$ can be described via the Durbin-Levinson algorithm, which provides

$$\phi_{k,k} = \alpha(k)$$
$$\phi_{k,j} = \phi_{k-1,j} - \phi_{k,k}\phi_{k-1,k-j} \qquad k > 1, 1 < j < k-1.$$

# Equivalent approaches



Let $R_p$ denote the causal parameter space for an order $p$ AR. Any of the following conditions can be used to assess if $\phi \in R_p$.

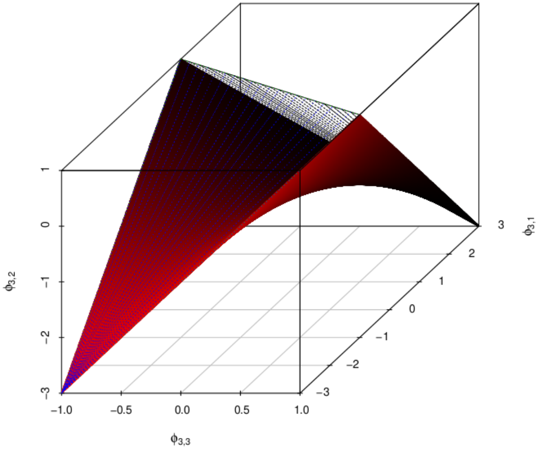
1. The roots of $\Phi(x) = 1 - \phi_{p,1}x - \phi_{p,2}x^2 - \cdots \phi_{p,p}x^p$ are outside the unit circle.
2. $|\phi_{h,h}| < 1$ for $1 \leq h \leq p$, and for each $k = 2, \ldots, p$ and $j = 1, \ldots, k$, $\phi_{k,j} = \phi_{k-1,j} - \phi_{k,k}\phi_{k-1,k-j}$.
3. $|\phi_{p,p}| < 1$, $\phi_{2i,2i} > -1$ for $i < p/2$, and for each $k = 2, \ldots, p$ and $j = 1, \ldots, k$,

   $$\sum_{j=1}^{k} \phi_{k,j} < 1 \quad \text{and} \quad \sum_{j=1}^{k} (-1)^{j+1}\phi_{k,j} > -1,$$

   with $\phi_{k,j} = \phi_{k-1,j} - \phi_{k,k}\phi_{k-1,k-j}$.
4. For $p > 1$, $(\phi_{p-1,1}, \ldots, \phi_{p-1,p-1}) \in R_{p-1}$, $|\phi_{p,p}| < 1$ and $\phi_{p,j} = \phi_{p-1,j} - \phi_{p,p}\phi_{p-1,p-j}$ for $j = 1, \ldots, p-1$.

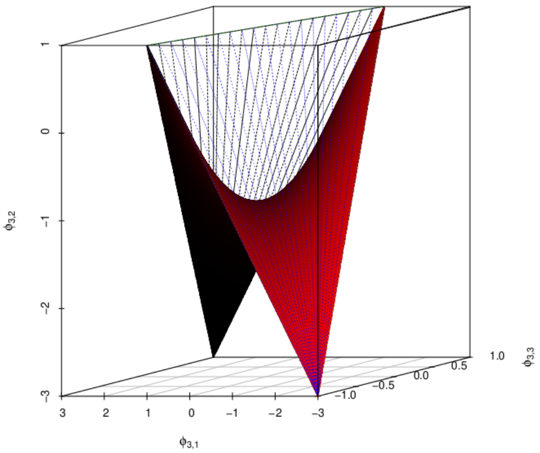
# Causal Region AR3





# Causal Region AR3





# Causal Region AR4

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Dropbox Link

# So what?



Need to summarise: $R_n(i,j) = corr(Y_i, Y_j)$, for $1 \leq i,j \leq n$

Measure of total correlation:

$$\rho(\{Y_t\}) = 1 - \det(R_n) = 1 - \prod_{k=1}^{p}(1-\alpha(k)^2)^{p+1-k}.$$

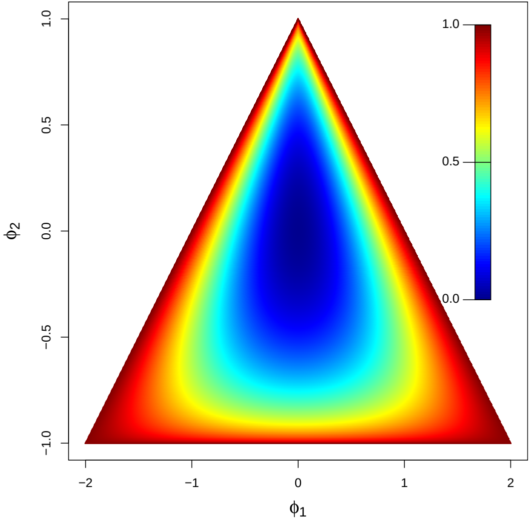
where $\det$ denotes the determinant.

$\rho(\{Y_t\}) \to 1$, high correlation

$\rho(\{Y_t\}) \to 0$, low correlation

# AR2 Total Correlation





# Simulation Studies



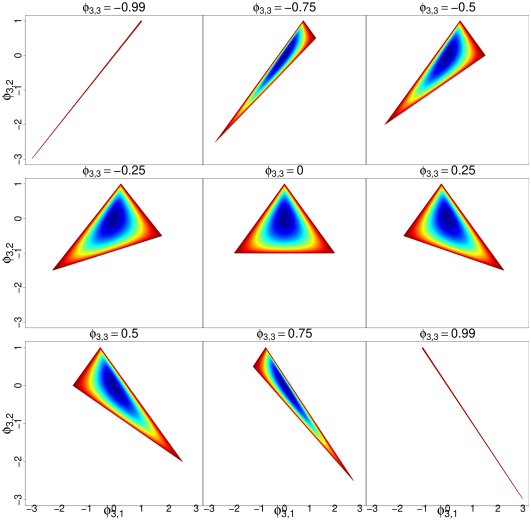
1. Generate $\{X_i\}_{i\in(1,\dots,p)}$ independently having, $0 \le \ell, u \le 1$ Uniform density
$$f(x) = \frac{1}{(u-\ell)}\mathbb{I}_{\ell<x<u}.$$
2. Generate $\{R_i\}$ independent of $\{X_i\}$, $P(R=1) = P(R=-1) = 0.5$.
3. Let each $\alpha(i) = R_i X_i$ and solve for $\{\phi_{p,j}\}$.

Larger values of $(\ell, u)$ for one or more $i$ correspond to more correlation.

# AR(3)

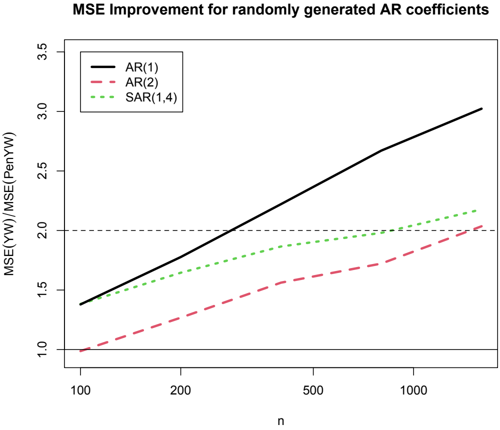


$\ell = 0.8$, $u = 0.9$, 10,000 simulations, $\rho(\{Y_t\}) > 0.9$, in 90% of simulations.

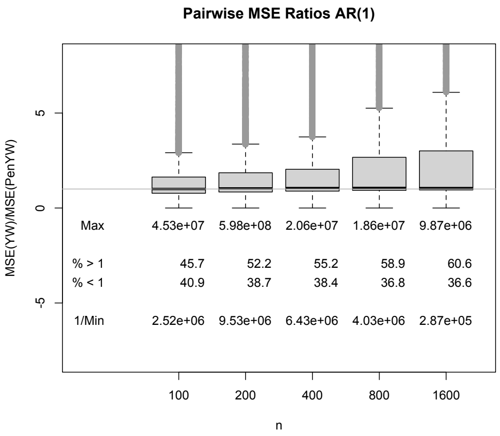


MSE for YW AR($p$) fits based on $\hat{\pi}$ vs $\tilde{\pi}$, 10,000 replications

Parameters randomly generated over parameter space.



Pairwise MSE for YW AR(1) fits based on $\hat{\pi}$ vs $\tilde{\pi}$.



# Summary



- Provided a new approach for (P)ACF estimation adaptively shrinking towards zero and target at each lag
- Improves over sample ACF, often beats Banded and Tapered too
- Computationally fast
- Adapts to a wide set of correlation structures

- Showed how PACF and AR coefficients are linked
- Used this link to describe causality regions in higher dimensions
- Visualized up to AR(4)
- Utilized the new description for simulation study generation

Available in `PenalizedCor` on Github.