Sampling and Aliasing

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July 30, 2024

Abstract

This article is about sampling and aliasing problems that I have encountered while studying numerical simulation of wave optics with computer to implement computer generated holography (CGH).

1 The Nyquist-Shannon sampling theorem

1.1 Statement of the theorem

If a function $x(t)$ contains no frequencied higher than B hertz, then it can be completely determined from its ordinates at a sequence of points spaced lass than $1/(2B)$ seconds apart.

We can rewrite the above theorem with the following expressions : $f_s \geq 2B$. In other words, when we denote a sampling interval as Δx and a period of an original signal as λ, the theorem is : $\Delta x < (\lambda/2)$. Intuitively, it says that we need to sample the original signal sufficiently densely in order to represent the signal without loss of information.

1.2 Motivation

The sampling theorem is required to avoid a type of distortion called *aliasing*.

2 Sampling theory

2.1 Dirac comb

The *Dirac comb* function is defined as :

$$\boxed{comb_T(t) \equiv \sum_{n=-\infty}^{\infty} \delta(t - nT)}$$

where $\delta(t)$ is the Dirac delta function. In the above definition the comb function has a period T. This function is important in sampling theory because every periodically sampled function $f_s(x)$ can be expressed as : $f_s(x) = f(x)comb_T(x)$ where T is a sampling period.
To understand a useful fourier transform property of the comb function, we need the following lemma :

$$\boxed{\textbf{Lemma. } \delta(u) = \int_{-\infty}^{\infty} dx \cdot e^{-i2\pi ux}}$$

Proof. $\mathcal{F}^{-1}[\delta(u)] = \int_{-\infty}^{\infty} du \cdot \delta(u) e^{i2\pi ux} = 1$

$$\rightarrow \mathcal{F}[\mathcal{F}^{-1}[\delta(u)]] = \delta(u) = \int_{-\infty}^{\infty} dx \cdot 1 \cdot e^{-i2\pi ux} = \int_{-\infty}^{\infty} dx \cdot e^{-i2\pi ux}$$

Hence, $\delta(u) = \int_{-\infty}^{\infty} dx \cdot e^{-i2\pi ux}$

(QED)

Now, we can prove the following useful theorem :

$$\boxed{\textbf{Theorem. } \mathcal{F}[comb_T(t)] = \frac{1}{T} comb_{\frac{1}{T}}(u)}$$

Proof. First, consider a Fourier series of $comb_T(t)$:

$$comb_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi \frac{n}{T} t} \text{ with } c_n = \frac{1}{T} \int_{-T/2}^{T/2} dt \cdot comb_T(t) e^{-i2\pi \frac{n}{T} t}$$

$$\rightarrow comb_T(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{i2\pi \frac{n}{T} t}$$

Then, with a Fourier transform :

$$\mathcal{F}[comb_T(t)] = \int_{-\infty}^{\infty} dt \cdot comb_T(t) e^{-i2\pi ut} = \int_{-\infty}^{\infty} dt \cdot \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{i2\pi \frac{n}{T} t} e^{-i2\pi ut}$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dt \cdot e^{-i2\pi (u - \frac{n}{T}) t} = \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta(u - \frac{n}{T})$$

$$= \frac{1}{T} comb_{\frac{1}{T}}(u)$$

(QED)

2.2 Discrete Fourier Transform

With the Dirac comb function, we can now understand the *Discrete Fourier Transform*, or the *Fast Fourier Transform*. They're practically same, so I'll call them as the FFT.

$F_s(u) = \mathcal{F}[f_s(x)] = \mathcal{F}[f(x) \cdot comb_T(x)] = \mathcal{F}[f(x)] * \mathcal{F}[comb_T(x)]$

$= \frac{1}{T} F(u) * comb_{\frac{1}{T}}(u) = \frac{1}{T} \sum_{n=-\infty}^{\infty} F(u - \frac{n}{T})$

$$\boxed{F_s(u) = \frac{1}{T} \sum_{n=-\infty}^{\infty} F(u - \frac{n}{T})}$$

So, the Fourier transform of a sampled function : $\mathcal{F}[f_s(x)] = F_s(u)$ becomes a periodic function with a period $\frac{1}{T}$. At this point an aliasing can occur.

To avoid such aliasing, we need $Pdu = \frac{1}{dx}$. Similarily, $Mdx = \frac{1}{du}$. Hence, $P = M$ and $du = \frac{1}{Mdx}$. This naturally leads to the following Discrete Fourier Transform(DFT) :

$F(u_p) = \sum_{m=-M/2}^{M/2-1} f(x_m) e^{-i2\pi u_p x_m}$

$\rightarrow F(p \cdot du) = \sum_{m=-M/2}^{M/2-1} f(m \cdot dx) e^{-i2\pi \frac{p}{Mdx} \cdot mdx} = \sum_{m=-M/2}^{M/2-1} f(m \cdot dx) e^{-i2\pi \frac{pm}{M}}$

Or,

$$\boxed{\textbf{1D DFT :}\ \ F[p] = \sum_{m=-M/2}^{M/2-1} f[m] e^{-i2\pi \frac{pm}{M}}}$$

$$\boxed{\textbf{2D DFT :}\ \ F[p,q] = \sum_{m=-M/2}^{M/2-1} \sum_{n=-N/2}^{N/2-1} f[m,n] e^{-i2\pi (\frac{pm}{M} + \frac{qn}{N})}}$$

Note that in the IFT(Inverse Fourier Transform) we have coefficient $\frac{1}{M}$ for 1D and $\frac{1}{MN}$ for 2D case :

$$\boxed{\textbf{1D IFT :}\ \ f[m] = \frac{1}{M} \sum_{p=-M/2}^{M/2-1} F[p] e^{i2\pi \frac{pm}{M}}}$$

$$\boxed{\textbf{2D IFT :}\ \ f[m,n] = \frac{1}{MN} \sum_{p=-M/2}^{M/2-1} \sum_{q=-N/2}^{N/2-1} F[p,q] e^{i2\pi (\frac{pm}{M} + \frac{qn}{N})}}$$

3 Aliasing

3.1 Definition

In signal processing, aliasing is the overlapping of frequency components[1]. For example, suppose we sample a high frequency(f_1) signal $x_1(t)$ with a sampling frequency f_s lower then the Nyquist frequency($f_s < f_{Nyquist} = 2f_1$). And denote the resulting sampled signal $x_1[n]$. The problem is, there is a lower frequency(f_2) signal $x_2(t)$ such that when we sample $x_2(t)$ with the same sampling frequency f_s, the sampled signal $x_2[n]$ becomes identical to $x_1[n]$: $x_1[n] = x_2[n]$.

In other words, when we sample a signal sparsely, then high-frequency information is lost and thus appears as a lower-frequency signal.

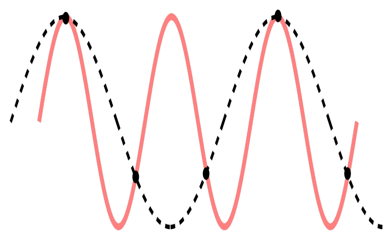

Figure 1: Aliasing. Red line represents original signal and dotted line represents sampled signal. You can see the original signal is sampled as a lower frequency signal.

3.2 Examples

3.2.1 Plane Wave

With a given sampling resolution dx and a wavelength λ, the maximum diffraction angle - hence maximum oblique angle - is determined as :

$$\boxed{\theta_{max,x} = sin^{-1}(\frac{\lambda}{2 \cdot dx})}$$

When $\theta \leq \theta_{max,x}$, it guarantees no aliasing. y-direction is same.

3.2.2 Spherical Wave

$$\boxed{|x| \leq \frac{z_0}{\sqrt{(\frac{2 \cdot dx}{\lambda})^2 - 1}}}$$

y-direction is same.

3.3 Mathematical description

References

[1] Nyquist–Shannon sampling theorem. In Wikipedia, The Free Encyclopedia.

[2] Matsushima K. (2020) *Introduction to Computer Holography*, Springer.