Holography

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1 Introduction

1.1 Definition

In this article, we will follow a definition first proposed by Dr. D. Gabor, who pioneered this concept. A hologram is a complete record of the amplitudes and phases of wave fields.

1.2 Motivation

Gabor conceived the idea of holography to address a problem in electron microscopy related to resolution limits. The resolution limitation of electron microscopy was mainly due to the use of the electron objective lens. At this point, Dr. Gabor proposed a new microscope principle where no electron objective lens is used. This new microscope principle is holography.

1.3 Recording of a hologram

A beam is illuminated on objects. This beam is called a *primary wavefront*. Through interactions between the primary wavefront and the objects, a *secondary wavefront* is generated, which is called the *object field*.

When the object is smaller than the area illuminated on the object plane, interference between the primary and secondary wavefronts occurs. When the phase of the primary wavefront coincides with the phase of the secondary wavefront, constructive interference occurs. These superposed waves are recorded on a 2D photographic plate. The recorded pattern is a *hologram*.

The intensity of the hologram includes phase information from the interference, successfully recording the phase of the wave. Additionally, the intensity of the light wave, which corresponds to the square of the amplitude, is also recorded. This allows the hologram to capture both the amplitude and phase information of the wave fields, providing a complete representation of the wave's characteristics.

1.4 Reconstructing original object waves

A beam identical to the one used in the recording process is illuminated on the hologram. Through the interference between this illumination beam and the hologram, wave fields very similar to the original object waves are created.

A striking feature of holograms, recognized by Dr. Gabor, is that they constitute records of three-dimensional as well as plane objects. This feature has become a major reason for the widespread interest and study of holography. The ability to reconstruct three-dimensional images from a two-dimensional hologram allows for applications in various fields, including microscopy, data storage, and art.

2 Diffraction theory

2.1 Green's Identities

2.1.1 First identities

$$\boxed{\int_V (\psi\nabla^2\phi + \nabla\psi\cdot\nabla\phi)d\tau = \oint_{\partial V}\psi\nabla\phi dS}$$

use divergence theorem with $F = \psi\nabla\phi$:

$$\int_V \nabla\cdot(\psi\nabla\phi)d\tau = \oint_{\partial V}\psi\nabla\phi\cdot d\vec{S}$$

(QED)

2.1.2 Second identities

$$\boxed{\int_V (\psi\nabla^2\phi - \phi\nabla^2\psi)d\tau = \oint_{\partial V}(\psi\nabla\phi - \phi\nabla\psi)\cdot d\vec{S}}$$

first, use $F = \psi\epsilon\nabla\phi - \phi\epsilon\nabla\psi$ then :

$$\int_V \nabla\cdot\vec{F}d\tau = \int_V \psi\nabla\cdot(\epsilon\nabla\phi) - \phi\nabla\cdot(\epsilon\nabla\psi)d\tau$$

$$= \oint_{\partial V}\vec{F}\cdot d\vec{S} = \oint \epsilon(\psi\frac{\partial\phi}{\partial\vec{n}} - \phi\frac{\partial\psi}{\partial\vec{n}})dS$$

set $\epsilon = 1$ then :

$$\int_V (\psi\nabla^2\phi - \phi\nabla^2\psi)d\tau = \oint_{\partial V}(\psi\nabla\phi - \phi\nabla\psi)\cdot d\vec{S}$$

(QED)

2.2 Kirchhoff Diffraction Formula

Let P_1 be a source point and P_0 be an observer point. Consider the Helmholtz equation : $(\nabla^2 + k^2)U = 0$ in free space. Choose a Green function as $G(P_1) = \frac{e^{ikr_{01}}}{r_{01}}$: which is an outgoing spherical wave. Then, with the second Green's identity, we have

$$\int_{V'} d\tau'(U\nabla^2 G - G\nabla^2 U) = \oint_{\partial V'} d\vec{a}'\cdot(U\nabla G - G\nabla U) =$$

$$= \oint_{\partial V'} da'(U\frac{\partial G}{\partial n} - G\frac{\partial U}{\partial n})$$

Since the integration volume V' contains source point P_0, we cannot use the Green's function directly. We need to remove the P_0. So, take small sphere S_ϵ containing P_0. Then $S' = S + S_\epsilon$.

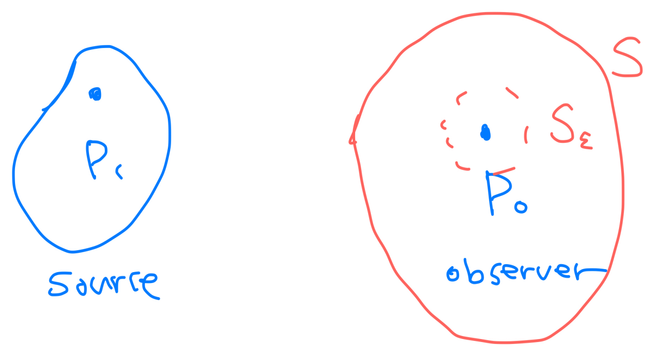

Figure 1: Small sphere surrounding P_0

(1) $\iiint_{V'} d\tau'(U\nabla^2 G - G\nabla^2 U) = \iiint_{V'} d\tau'(-k^2 UG + k^2 UG) = 0.$

(2) $\iint_{S'} da'(U\frac{\partial G}{\partial n} - G\frac{\partial U}{\partial n}) = \iiint_{V'} d\tau'(U\nabla^2 G - G\nabla^2 U) = 0$ with (1).

(3) From $G(P_1) = \frac{e^{ikr_{01}}}{r_{01}}$

$\rightarrow \frac{\partial U}{\partial n} = cos(\vec{n}, \vec{r_{01}})\frac{ikr_{01}e^{ikr_{01}} - e^{ikr_{01}}}{r_{01}^2} = cos(\vec{n}, \vec{r_{01}})[ik - \frac{1}{r_{01}}]\frac{e^{ikr_{01}}}{r_{01}}$

(4) If P_1 is on S_ϵ, $cos(\vec{n}, \vec{r_{01}}) = -1$

$\rightarrow G(P_1) = \frac{e^{ik\epsilon}}{\epsilon}$ and $\frac{\partial G(P_1)}{\partial n} = (\frac{1}{\epsilon} - ik)\frac{e^{ik\epsilon}}{\epsilon}$

(5) Hence, for S_ϵ :

$\lim_{\epsilon\to 0}\iint_{S_\epsilon}[U\frac{\partial G}{\partial n} - G\frac{\partial U}{\partial n}]da = \lim_{\epsilon\to 0} 4\pi\epsilon^2 \cdot [U(P_0)(\frac{1}{\epsilon} - ik)\frac{e^{ik\epsilon}}{\epsilon} - \frac{e^{ik\epsilon}}{\epsilon}\frac{\partial U(P_0)}{\partial n}]$

$= ... = 4\pi U(P_0)$

(6) $\therefore \iint_{S'}(U\frac{\partial G}{\partial n} - G\frac{\partial U}{\partial n})da = \iiint_{V'} d\tau'(U\nabla^2 G - G\nabla^2 U) = 0$

Since $S' = S + S_\epsilon$, $\rightarrow \iint_{S_\epsilon} = -\iint_S$

$\Rightarrow U(P_0) = -\frac{1}{4\pi}\iint_S(U\frac{\partial G}{\partial n} - G\frac{\partial U}{\partial n})da = \frac{1}{4\pi}\iint_S[\frac{e^{ikr_{01}}}{r_{01}}\frac{\partial U}{\partial n} - U\frac{\partial}{\partial n}(\frac{e^{ikr_{01}}}{r_{01}})]da$

(7) From $S = S_1 + S_2$, it seems reasonable that $\iint_{S_1} \rightarrow 0$. In fact, when the Sommerfeld radiation condition $\lim_{R\to\infty} R(\frac{\partial U}{\partial n} - ikU) = 0$ is satisfied, the S_1 integral vanishes. Then, remaining is S_2 integral and here most of contribution comes from an aperture so Σ : $U(P_0) = \frac{1}{4\pi}\iint_\Sigma(G\frac{\partial U}{\partial n} - U\frac{\partial G}{\partial n})da$

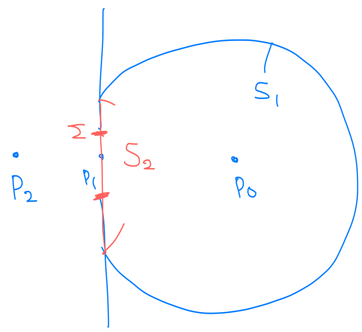

Figure 2: Kirchhoff's diffraction theory

(8) When a distance between source and observer is much larger than a wavelength, then $k >> \frac{1}{r_{01}} \rightarrow \frac{\partial G}{\partial n} \approx ik\frac{e^{ikr_{01}}}{r_{01}}cos(\vec{n}, \vec{r_{01}})$

$\Rightarrow U(P_0) \approx \frac{1}{4\pi} \iint_\Sigma \frac{e^{ikr_{01}}}{r_{01}}[\frac{\partial U}{\partial n} - ikUcos(\vec{n}, \vec{r_{01}})]da$

(9) Now, suppose $U|\Sigma$: an outgoing spherical wave from a point P_2 on a left side of the aperture : $U(P_1) = A\frac{e^{ikr_{21}}}{r_{21}}$ and then $\frac{\partial U}{\partial n} \approx A\frac{e^{ikr_{21}}}{r_{21}} \cdot ikcos(\vec{n}, \vec{r_{01}})$

(10) Hence $U(P_0) = \frac{1}{4\pi} \iint_\Sigma \frac{e^{ikr_{01}}}{r_{01}} \cdot A\frac{e^{ikr_{21}}}{r_{21}}[ikcos(\vec{n}, \vec{r_{21}}) - ikcos(\vec{n}, \vec{r_{01}})]da$

$= -\frac{ik}{4\pi} \iint_\Sigma A\frac{e^{ik(r_{01}+r_{21})}}{r_{01}r_{21}}[cos(\vec{n}, \vec{r_{01}}) - cos(\vec{n}, \vec{r_{01}})]da$

$= \frac{A}{i\lambda} \iint_\Sigma \frac{e^{ik(r_{01}+r_{21})}}{r_{01}r_{21}} \frac{[cos(\vec{n},\vec{r_{01}}) - cos(\vec{n},\vec{r_{01}})]}{2} da$

$$\boxed{U(P_0) = \frac{A}{i\lambda} \iint_\Sigma \frac{e^{ik(r_{01}+r_{21})}}{r_{01}r_{21}} \frac{[cos(\vec{n},\vec{r_{01}}) - cos(\vec{n},\vec{r_{01}})]}{2} da}$$

This is the *Fresnel Kirchhoff's diffraction formula.* There are self-contradictions in this formula, but the result is same. We usually use the following formula :

$$\boxed{U(P_0) = \frac{1}{i\lambda} \iint_\Sigma daU(P_1)\frac{e^{ikr_{01}}}{r_{01}}cos\theta}$$

Note that the $i\lambda$ comes from $\frac{\partial G}{\partial n}$.

2.3 Fresnel Diffraction and Fraunhofer Diffraction

2.3.1 Fresnel Diffraction

Approximation :

$r_{01} = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \approx z + z \cdot \frac{(x-x')^2+(y-y')^2}{2z^2} =$

$z + \frac{(x-x')^2+(y-y')^2}{2z}$

$\Rightarrow U(x,y,z) \approx \frac{1}{i\lambda} \iint dx'dy'U(x',y')\frac{1}{z}e^{i\frac{2\pi}{\lambda}[z+\frac{(x-x')^2+(y-y')^2}{2x}]} =$

$\frac{1}{i\lambda z}e^{i\frac{2\pi}{\lambda}z}e^{i\frac{2\pi}{\lambda}(x^2+y^2)} \iint dx'dy'[U(x',y')e^{i\frac{2\pi}{\lambda}\cdot(\frac{x'^2+y'^2}{2z})}]e^{-i\frac{2\pi}{\lambda z}(xx'+yy')}$

Hence, the *Fresnel diffraction* becomes :

$$\boxed{U(x,y,z) = \frac{1}{i\lambda z}e^{i\frac{2\pi}{\lambda}z}e^{i\frac{2\pi}{\lambda}(x^2+y^2)} \iint dx'dy'[U(x',y')e^{i\frac{2\pi}{\lambda}\cdot(\frac{x'^2+y'^2}{2z})}]e^{-i\frac{2\pi}{\lambda z}(xx'+yy')}}$$

We can see this as a Fourier transform :

$$\boxed{U(x,y,z) = \frac{1}{i\lambda z}e^{i\frac{2\pi}{\lambda}z}e^{i\frac{2\pi}{\lambda}(x^2+y^2)}\mathcal{F}[U(x',y')e^{i\frac{2\pi}{\lambda}\cdot(\frac{x'^2+y'^2}{2z})}]|_{u=\frac{x}{\lambda z},\nu=\frac{y}{\lambda z}}}$$

References

[1] GABOR, D. A New Microscopic Principle. Nature 161, 777–778 (1948). https://doi.org/10.1038/161777a0

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