1.2 The Electromagnetic Wave Equation

Rkka

August 2024

1 The electromagnetic wave equation

$$\boxed{\nabla^2\vec{E} - \epsilon\mu\frac{\partial^2}{\partial t^2}\vec{E} + (\nabla\ln\mu)\times(\nabla\times\vec{E}) + \nabla[\vec{E}\cdot(\nabla\ln\epsilon)] = 0}$$

$$\boxed{\nabla^2\vec{H} - \epsilon\mu\frac{\partial^2}{\partial t^2}\vec{H} + (\nabla\ln\epsilon)\times(\nabla\times\vec{H}) + \nabla[\vec{H}\cdot(\nabla\ln\mu)] = 0}$$

Note the following vector identities :

$$\nabla\times(f\vec{A}) = (\nabla f)\times\vec{A} + f\nabla\times\vec{A} \tag{1}$$

$$\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} \tag{2}$$

We'll consider $\rho = 0$ and $\vec{j} = 0$ case (In fact, $\rho_f = 0$ and $\vec{j}_f = 0$).

1.1 $\vec{E}$ wave equation

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\mu\frac{\partial\vec{H}}{\partial t}$$

$$\rightarrow \frac{1}{\mu}\nabla\times\vec{E} = -\frac{\partial\vec{H}}{\partial t}$$

$$\rightarrow \nabla\times(\frac{1}{\mu}\nabla\times\vec{E}) = -\frac{\partial}{\partial t}\nabla\times\vec{H}$$

(1) LHS

$$\nabla\times(\frac{1}{\mu}\nabla\times\vec{E}) = (\nabla\frac{1}{\mu})\times(\nabla\times\vec{E}) + \frac{1}{\mu}\nabla\times(\nabla\times\vec{E})$$

$$\rightarrow = (\nabla\frac{1}{\mu})\times(\nabla\times\vec{E}) + \frac{1}{\mu}\nabla(\nabla\cdot\vec{E}) - \frac{1}{\mu}\nabla^2\vec{E}$$

Here, since $\rho_f = 0$, $\nabla\cdot\vec{D} = \nabla\cdot(\epsilon\vec{E}) = (\nabla\epsilon)\cdot\vec{E} + \epsilon\nabla\cdot\vec{E} = 0$. So, we use $\nabla\cdot\vec{E} = -\frac{1}{\epsilon}(\nabla\epsilon)\cdot\vec{E} = -(\nabla\ln\epsilon)\cdot\vec{E}$

$$\rightarrow \nabla\times(\frac{1}{\mu}\times\vec{E}) = (\nabla\frac{1}{\mu})\times(\nabla\times\vec{E}) - \frac{1}{\mu}\nabla(\vec{E}\cdot(\nabla\ln\epsilon)) - \frac{1}{\mu}\nabla^2\vec{E}$$

(2) RHS

$$-\frac{\partial}{\partial t}\nabla\times\vec{H} = -\frac{\partial}{\partial t}(\frac{\partial}{\partial t}\vec{D}) = -\epsilon\frac{\partial^2}{\partial t^2}\vec{E}$$

$$\Rightarrow (1) = (2)$$

$$\frac{1}{\mu}\nabla^2\vec{E} - \epsilon\frac{\partial^2}{\partial t^2}\vec{E} - (\nabla\frac{1}{\mu})\times(\nabla\times\vec{E}) + \frac{1}{\mu}\nabla[\vec{E}\cdot(\nabla\ln\epsilon)] = 0$$

$$\boxed{\nabla^2\vec{E} - \epsilon\mu\frac{\partial^2}{\partial t^2}\vec{E} + (\nabla\ln\mu)\times(\nabla\times\vec{E}) + \nabla[\vec{E}\cdot(\nabla\ln\epsilon)] = 0}$$

where we used $-\mu(\nabla\frac{1}{\mu}) = -\mu(-\frac{1}{\mu^2})\nabla\mu = \frac{1}{\mu}\nabla\mu = \nabla(\ln\mu)$.

1.2 $\vec{H}$ wave equation

$$\nabla\times\vec{H} = \frac{\partial\vec{D}}{\partial t} = \epsilon\frac{\partial\vec{E}}{\partial t}$$

$$\rightarrow \frac{1}{\epsilon}\nabla\times\vec{H} = -\frac{\partial\vec{E}}{\partial t}$$

$$\rightarrow \nabla\times(\frac{1}{\epsilon}\nabla\times\vec{H}) = \frac{\partial}{\partial t}\nabla\times\vec{E}$$

(1) LHS

$$\nabla\times(\frac{1}{\epsilon}\nabla\times\vec{H}) = (\nabla\frac{1}{\epsilon})\times(\nabla\times\vec{H}) + \frac{1}{\epsilon}\nabla\times(\nabla\times\vec{H})$$

$$\rightarrow = (\nabla\frac{1}{\epsilon})\times(\nabla\times\vec{H}) + \frac{1}{\epsilon}\nabla(\nabla\cdot\vec{H}) - \frac{1}{\epsilon}\nabla^2\vec{H}$$

Here, $\nabla\cdot\vec{B} = \nabla\cdot(\mu\vec{H}) = (\nabla\mu)\cdot\vec{H} + \mu\nabla\cdot\vec{H} = 0$. So, we use $\nabla\cdot\vec{H} = -\frac{1}{\mu}(\nabla\mu)\cdot\vec{H} = -(\nabla\ln\mu)\cdot\vec{H}$

$$\rightarrow \nabla\times(\frac{1}{\epsilon}\nabla\times\vec{H}) = (\nabla\frac{1}{\epsilon})\times(\nabla\times\vec{H}) - \frac{1}{\epsilon}\nabla(\vec{H}\cdot(\nabla\ln\mu)) - \frac{1}{\epsilon}\nabla^2\vec{H}$$

(2) RHS

$$\frac{\partial}{\partial t}\nabla\times\vec{E} = -\frac{\partial}{\partial t}(\frac{\partial}{\partial t}\vec{B}) = -\mu\frac{\partial^2}{\partial t^2}\vec{H}$$

$$\Rightarrow (1) = (2)$$

$$\frac{1}{\epsilon}\nabla^2\vec{H} - \mu\frac{\partial^2}{\partial t^2}\vec{H} - (\nabla\frac{1}{\epsilon})\times(\nabla\times\vec{H}) + \frac{1}{\epsilon}\nabla[\vec{H}\cdot(\nabla\ln\mu)] = 0$$

$$\boxed{\nabla^2\vec{H} - \epsilon\mu\frac{\partial^2}{\partial t^2}\vec{H} + (\nabla\ln\epsilon)\times(\nabla\times\vec{H}) + \nabla[\vec{H}\cdot(\nabla\ln\mu)] = 0}$$

where we used $-\epsilon(\nabla\frac{1}{\epsilon}) = -\epsilon(-\frac{1}{\epsilon^2})\nabla\epsilon = \frac{1}{\epsilon}\nabla\epsilon = \nabla(\ln\epsilon)$.

References

[1] Born, M., Wolf, E., Bhatia, A. B. (2019). Principles of Optics: 60th Anniversary Edition. United Kingdom: Cambridge University Press.