13.1 Scalar Scattering Theory

rkka

August 2024

1 The First Born Approximation and Associated Diffraction Theorem

1.1 The First Born Approximation

From [13.1 The Basic Integral Equation], we have

$$U^{(s)}(\vec{r},w) = \int_V d\tau' F(\vec{r},w)U(\vec{r}',w)\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}$$

The *First Born Approximation* is to replace $U(\vec{r}',w)$ in the integrand to $U^{(i)}(\vec{r}',w) = e^{ik\vec{s}_0\cdot\vec{r}'}$:

$$\boxed{U^{(s)}(\vec{r},w) = \int_V d\tau' F(\vec{r},w)e^{ik\vec{s}_0\cdot\vec{r}'}\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}}$$

1.2 Angular Decomposition with the Weyl's Expansion

Now we do angular decomposition using the *Weyl's expansion for a spherical wave* :

$$\frac{e^{ik|\vec{r}-\vec{r}|}}{|\vec{r}-\vec{r}'|} = \frac{ik}{2\pi}\iint_{-\infty}^{+\infty} ds_x ds_y \frac{1}{s_z} e^{ik[s_x(x-x')+s_y(y-y')+s_z|z-z'|]}$$

Then

$$U_1^{(s)}(\vec{r},w) = \int_V d\tau' F(\vec{r}',w)e^{ik\vec{s}_0\cdot\vec{r}'}\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}$$

$$= \int_V d\tau' F(\vec{r}',w)e^{ik\vec{s}_0\cdot\vec{r}'}\frac{ik}{2\pi}\iint_{-\infty}^{+\infty} ds_x ds_y \frac{1}{s_z} e^{ik[s_x(x-x')+s_y(y-y')+s_z|z-z'|]}$$

$$\iint ds_x ds_y \frac{ik}{2\pi s_z}[\int_V d\tau' F(\vec{r}',w)e^{ik[(s_x-s_{0x})x'+(s_y-s_{0y})y'+(\pm s_z-s_{0z})z']}] \\ \times e^{ik(s_x x+s_y y\pm s_z z)}$$

$$= \iint ds_x ds_y a^{(\pm)}(s_x,s_y;s_{0x},s_{0y})e^{ik(s_x x+s_y y\pm s_z z)}$$

where we defined $a^{(\pm)}(s_x,s_y;s_{0x},s_{0y}) \equiv \frac{ik}{2\pi s_z}\int_V d\tau' F(\vec{r}',w)e^{ik[(s_x-s_{0x})x'+(s_y-s_{0y})y'+(\pm s_z-s_{0z})z']}$:

$$\boxed{a^{(\pm)}(s_x,s_y;s_{0x},s_{0y}) = \frac{ik}{2\pi s_z}\hat{F}[k(s_x-s_{0x}),k(s_y-s_{0y}),k(\pm s_z-s_{0z})]}$$

1.3 Associated Diffraction Theorem

Note that we assume the First Born Approximation. The idea is to relate $U_1^{(s)}$ with $\hat{F}$. To achive this, we apply an inverse Fourier transform to $U_1^{(s)}$ for x, y:

$$U_1^{(s)}(\vec{r}, w) = \iint d(\tfrac{ks_x}{2\pi})d(\tfrac{ks_y}{2\pi})\hat{U}_1^{(s)}(ks_x, ks_y, z^{(\pm)}; \vec{s}_0)e^{i2\pi\cdot\frac{k}{2\pi}(s_x x + s_y y)}$$

Compare this with $U_1^{(s)}(\vec{r}, w) = \iint ds_x ds_y a^{(\pm)}(s_x, s_y; s_{0x}, s_{0y})e^{ik(s_x x + s_y y \pm s_z z)}$. For each s_x and s_y we get :

$$(\tfrac{k}{2\pi})^2\hat{U}_1^{(s)}(ks_x, ks_y, z^{(\pm)}; \vec{s}_0) = a^{(\pm)}(s_x, s_y; s_{0x}, s_{0y}) \equiv \tfrac{ks_z}{2\pi i}\hat{U}_1^{(s)}(ks_x, ks_y, z^{(\pm)}; \vec{s}_0)$$

So,

$$\boxed{\hat{F}[k(s_x - s_{0x}), k(s_y - s_{0y}), k(\pm s_z - s_{0z})] = \tfrac{ks_z}{2\pi i}\hat{U}_1^{(s)}(ks_x, ks_y, z^{(\pm)}; \vec{s}_0)}$$

Note that $\hat{U}_1^{(s)}$ is 2D information while $\hat{F}$ is 3D. By measuring 2D U we can reconstruct 3D F, the scattering potential.