1.2 The Electromagnetic Wave Equation

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August 2024

1 The electromagnetic wave equation

$$\nabla^2 \vec{E} - \epsilon\mu\frac{\partial^2}{\partial t^2}\vec{E} + (\nabla \ln \mu) \times (\nabla \times \vec{E}) + \nabla[\vec{E} \cdot (\nabla \ln \epsilon)] = 0$$

$$\nabla^2 \vec{H} - \epsilon\mu\frac{\partial^2}{\partial t^2}\vec{H} + (\nabla \ln \epsilon) \times (\nabla \times \vec{H}) + \nabla[\vec{H} \cdot (\nabla \ln \mu)] = 0$$

Note the following vector identities :

$$\nabla \times (f\vec{A}) = (\nabla f) \times \vec{A} + f\nabla \times \vec{A} \tag{1}$$

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \tag{2}$$

We'll consider $\rho = 0$ and $\vec{j} = 0$ case (In fact, $\rho_f = 0$ and $\vec{j}_f = 0$).

1.1 $\vec{E}$ wave equation

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu\frac{\partial \vec{H}}{\partial t}$$

$$\rightarrow \frac{1}{\mu}\nabla \times \vec{E} = -\frac{\partial \vec{H}}{\partial t}$$

$$\rightarrow \nabla \times (\frac{1}{\mu}\nabla \times \vec{E}) = -\frac{\partial}{\partial t}\nabla \times \vec{H}$$

(1) LHS

$$\nabla \times (\frac{1}{\mu}\nabla \times \vec{E}) = (\nabla\frac{1}{\mu}) \times (\nabla \times \vec{E}) + \frac{1}{\mu}\nabla \times (\nabla \times \vec{E})$$

$$\rightarrow = (\nabla\frac{1}{\mu}) \times (\nabla \times \vec{E}) + \frac{1}{\mu}\nabla(\nabla \cdot \vec{E}) - \frac{1}{\mu}\nabla^2 \vec{E}$$

Here, since $\rho_f = 0$, $\nabla \cdot \vec{D} = \nabla \cdot (\epsilon\vec{E}) = (\nabla\epsilon) \cdot \vec{E} + \epsilon\nabla \cdot \vec{E} = 0$. So, we use $\nabla \cdot \vec{E} = -\frac{1}{\epsilon}(\nabla\epsilon) \cdot \vec{E} = -(\nabla \ln \epsilon) \cdot \vec{E}$

$$\rightarrow \nabla \times (\frac{1}{\mu} \times \vec{E}) = (\nabla\frac{1}{\mu}) \times (\nabla \times \vec{E}) - \frac{1}{\mu}\nabla(\vec{E} \cdot (\nabla \ln \epsilon)) - \frac{1}{\mu}\nabla^2 \vec{E}$$

(2) RHS

$$-\frac{\partial}{\partial t}\nabla \times \vec{H} = -\frac{\partial}{\partial t}(\frac{\partial}{\partial t}\vec{D}) = -\epsilon\frac{\partial^2}{\partial t^2}\vec{E}$$

$$\Rightarrow (1) = (2)$$

$$\frac{1}{\mu}\nabla^2 \vec{E} - \epsilon \frac{\partial^2}{\partial t^2}\vec{E} - (\nabla \frac{1}{\mu}) \times (\nabla \times \vec{E}) + \frac{1}{\mu}\nabla[\vec{E} \cdot (\nabla \ln \epsilon)] = 0$$

$$\boxed{\nabla^2 \vec{E} - \epsilon\mu \frac{\partial^2}{\partial t^2}\vec{E} + (\nabla \ln \mu) \times (\nabla \times \vec{E}) + \nabla[\vec{E} \cdot (\nabla \ln \epsilon)] = 0}$$

where we used $-\mu(\nabla \frac{1}{\mu}) = -\mu(-\frac{1}{\mu^2})\nabla\mu = \frac{1}{\mu}\nabla\mu = \nabla(\ln \mu)$.

1.2 $\vec{H}$ wave equation

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} = \epsilon \frac{\partial \vec{E}}{\partial t}$$

$$\rightarrow \frac{1}{\epsilon}\nabla \times \vec{H} = -\frac{\partial \vec{E}}{\partial t}$$

$$\rightarrow \nabla \times (\frac{1}{\epsilon}\nabla \times \vec{H}) = \frac{\partial}{\partial t}\nabla \times \vec{E}$$

(1) LHS

$$\nabla \times (\frac{1}{\epsilon}\nabla \times \vec{H}) = (\nabla \frac{1}{\epsilon}) \times (\nabla \times \vec{H}) + \frac{1}{\epsilon}\nabla \times (\nabla \times \vec{H})$$

$$\rightarrow = (\nabla \frac{1}{\epsilon}) \times (\nabla \times \vec{H}) + \frac{1}{\epsilon}\nabla(\nabla \cdot \vec{H}) - \frac{1}{\epsilon}\nabla^2 \vec{H}$$

Here, $\nabla \cdot \vec{B} = \nabla \cdot (\mu\vec{H}) = (\nabla\mu) \cdot \vec{H} + \mu\nabla \cdot \vec{H} = 0$. So, we use
$\nabla \cdot \vec{H} = -\frac{1}{\mu}(\nabla\mu) \cdot \vec{H} = -(\nabla \ln \mu) \cdot \vec{H}$

$$\rightarrow \nabla \times (\frac{1}{\epsilon}\nabla \times \vec{H}) = (\nabla \frac{1}{\epsilon}) \times (\nabla \times \vec{H}) - \frac{1}{\epsilon}\nabla(\vec{H} \cdot (\nabla \ln \mu)) - \frac{1}{\epsilon}\nabla^2 \vec{H}$$

(2) RHS

$$\frac{\partial}{\partial t}\nabla \times \vec{E} = -\frac{\partial}{\partial t}(\frac{\partial}{\partial t}\vec{B}) = -\mu\frac{\partial^2}{\partial t^2}\vec{H}$$

$$\Rightarrow (1) = (2)$$

$$\frac{1}{\epsilon}\nabla^2 \vec{H} - \mu\frac{\partial^2}{\partial t^2}\vec{H} - (\nabla \frac{1}{\epsilon}) \times (\nabla \times \vec{H}) + \frac{1}{\epsilon}\nabla[\vec{H} \cdot (\nabla \ln \mu)] = 0$$

$$\boxed{\nabla^2 \vec{H} - \epsilon\mu\frac{\partial^2}{\partial t^2}\vec{H} + (\nabla \ln \epsilon) \times (\nabla \times \vec{H}) + \nabla[\vec{H} \cdot (\nabla \ln \mu)] = 0}$$

where we used $-\epsilon(\nabla \frac{1}{\epsilon}) = -\epsilon(-\frac{1}{\epsilon^2})\nabla\epsilon = \frac{1}{\epsilon}\nabla\epsilon = \nabla(\ln \epsilon)$.