# 1.1 Electromagnetic Fields

Rkka

August 2024

## 1 Energy law

The electric and magnetic energy in a given volume $V$ are respectively :

$$W_e = \int_V d\tau(\frac{1}{2}\vec{E}\cdot\vec{D}) \tag{1}$$

$$W_m = \int_V d\tau(\frac{1}{2}\vec{H}\cdot\vec{B}) \tag{2}$$

The electromagnetic energy law is :

$$\boxed{\frac{dW}{dt} + \int_V d\tau(\vec{j}\cdot\vec{E}) + \int_{\partial V}\vec{S}\cdot\vec{n}da}$$

where $W = W_e + W_m$ and $\vec{S} = \vec{E}\times\vec{H}$.

**(Proof)**

Start from : $\nabla\times\vec{H} - \frac{\partial}{\partial t}D = \vec{j}$. Applying $\vec{E}\cdot$ on both side,

$$\vec{E}\cdot(\nabla\times\vec{H}) - \vec{E}\cdot\frac{\partial}{\partial t}\vec{D} = \vec{E}\cdot\vec{j} \tag{3}$$

Similarily, for $\nabla\times E + \frac{\partial}{\partial t}B = 0$, applying $\vec{H}\cdot$ on both side :

$$\vec{H}\cdot(\nabla\times\vec{E}) + \vec{H}\cdot\frac{\partial}{\partial t}\vec{B} = 0 \tag{4}$$

(Note) $\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$ . Then,

$$\vec{H}\cdot(\nabla\times\vec{E}) - \vec{E}\cdot(\nabla\times\vec{H}) = -\vec{H}\cdot\tfrac{\partial}{\partial t}\vec{B} - \vec{E}\cdot\tfrac{\partial}{\partial t}\vec{D} - \vec{E}\cdot\vec{j}$$

$$\rightarrow \nabla\cdot(\vec{E}\times\vec{H}) + [\vec{E}\cdot\tfrac{\partial}{\partial t}\vec{D} + \vec{H}\cdot\tfrac{\partial}{\partial t}\vec{B}] + \vec{j}\cdot\vec{E} = 0$$

In integral form,

$$\int_{\partial V}(\vec{E}\times\vec{H})\cdot\vec{n}da + \int_V d\tau[\vec{E}\cdot\frac{\partial}{\partial t}D + \vec{H}\cdot\frac{\partial}{\partial t}\vec{B}] + \int_V d\tau(\vec{j}\cdot\vec{E}) = 0 \tag{5}$$

This is the energy law of Electromagnetism. We can simplify when we assume the simple material equations : $\vec{j} = \sigma\vec{E}$, $\vec{D} = \epsilon\vec{E}$, $\vec{B} = \mu\vec{H}$. Then,

$$\vec{E} \cdot \frac{\partial}{\partial t}\vec{D} = \vec{E} \cdot \frac{\partial}{\partial t}\epsilon\vec{E} = \frac{1}{2}\frac{\partial}{\partial t}(\epsilon\vec{E}^2) = \frac{1}{2}\frac{\partial}{\partial t}(\vec{E} \cdot \vec{D})$$

Similarily,

$$\vec{H} \cdot \frac{\partial}{\partial t}\vec{B} = \vec{H} \cdot \frac{\partial}{\partial t}\mu\vec{H} = \frac{1}{2}\frac{\partial}{\partial t}(\mu\vec{H}^2) = \frac{1}{2}\frac{\partial}{\partial t}(\vec{H} \cdot \vec{B})$$

Then

$$\int_V d\tau[\tfrac{1}{2}\tfrac{\partial}{\partial t}(\vec{E} \cdot \vec{D}) + \tfrac{1}{2}\tfrac{\partial}{\partial t}(\vec{H} \cdot \vec{B})] + \int_V d\tau(\vec{j} \cdot \vec{E}) + \int_{\partial V}(\vec{E} \times \vec{H}) \cdot \vec{n}da = 0$$

$$\rightarrow \frac{d}{dt}\int_V d\tau[\tfrac{1}{2}(\vec{E} \cdot \vec{D}) + \tfrac{1}{2}(\vec{H} \cdot \vec{D})] + \int_V d\tau(\vec{j} \cdot \vec{E}) + \int_{\partial V}(\vec{E} \times \vec{H}) \cdot \vec{n}da = 0$$

$$\Rightarrow \boxed{\frac{dW}{dt} + \int_V d\tau(\vec{j} \cdot \vec{E}) + \int_{\partial V}\vec{S} \cdot \vec{n}da = 0}$$

with $\vec{S} = \vec{E} \times \vec{H}$

(QED)

## 1.1 More physical representation

When $A$ : total work done to charges,

$$\delta A = \sum_k \vec{F}_k \cdot \delta\vec{x}_k = \sum q_k(\vec{E}_k + \vec{v}_k \times \vec{B}_k) \cdot \delta\vec{x}_k$$

$$= \sum_k q_k\vec{E}_k \cdot \delta\vec{x}_k = \sum_k q_k\vec{E}_k \cdot \vec{v}_k\delta t$$

$$\rightarrow \delta A = \delta t\int_V d\tau(\rho\vec{E} \cdot \vec{v})$$

But, since Maxwell's equations do not contain a velocity $\vec{v}$, we should remove it to restate the energy law.

**Roentgen's experimental discovery (1888) :** (1) Convection current(moving charges) $\vec{j}_v = \rho\vec{v}$ and (2) Conduction current(electric fields) $\vec{j}_c = \sigma\vec{E}$ give same physical(electromagnetic) effects. $\rightarrow \vec{j} = \vec{j}_c + \vec{j}_v$

So, $\delta A = \delta t\int_V d\tau\vec{j}_v \cdot \vec{E}$. Now, define a scalar $Q$ ($\equiv$ *Joule's heat*) as :

$$Q \equiv \int_V d\tau\vec{j}_c \cdot \vec{E} = \int_V d\tau\sigma\vec{E}^2$$

Then,

$$\int_V d\tau(\vec{j} \cdot \vec{E}) = \int_V d\tau(\vec{j}_c + \vec{j}_v) \cdot \vec{E} = Q + \int_V d\tau(\vec{j}_v \cdot \vec{E}) = Q + \frac{\delta A}{\delta t}$$

Hence, the energy law :

$$\boxed{\frac{dW}{dt} = -\frac{\delta A}{\delta t} - Q - \int_{\partial V}\vec{S} \cdot \vec{n}da}$$

When $\sigma = 0$ (nonconducting) and $A = 0$ (no mechanical work)
$\Rightarrow \frac{dw}{dt} + \nabla \cdot \vec{S} = 0$ : note that this is a differential form.