

1.1 Electromagnetic Fields

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1 Energy law

The electric and magnetic energy in a given volume V are respectively :

$$W_e = \int_V d\tau \left(\frac{1}{2} \vec{E} \cdot \vec{D} \right) \quad (1)$$

$$W_m = \int_V d\tau \left(\frac{1}{2} \vec{H} \cdot \vec{B} \right) \quad (2)$$

The electromagnetic energy law is :

$$\boxed{\frac{dW}{dt} + \int_V d\tau (\vec{j} \cdot \vec{E}) + \int_{\partial V} \vec{S} \cdot \vec{n} da}$$

where $W = W_e + W_m$ and $\vec{S} = \vec{E} \times \vec{H}$.

(Proof)

Start from : $\nabla \times \vec{H} - \frac{\partial}{\partial t} \vec{D} = \vec{j}$. Applying $\vec{E} \cdot$ on both side,

$$\vec{E} \cdot (\nabla \times \vec{H}) - \vec{E} \cdot \frac{\partial}{\partial t} \vec{D} = \vec{E} \cdot \vec{j} \quad (3)$$

Similarly, for $\nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{B} = 0$, applying $\vec{H} \cdot$ on both side :

$$\vec{H} \cdot (\nabla \times \vec{E}) + \vec{H} \cdot \frac{\partial}{\partial t} \vec{B} = 0 \quad (4)$$

(Note) $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$. Then,

$$\begin{aligned} \vec{H} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{H}) &= -\vec{H} \cdot \frac{\partial}{\partial t} \vec{B} - \vec{E} \cdot \frac{\partial}{\partial t} \vec{D} - \vec{E} \cdot \vec{j} \\ \rightarrow \nabla \cdot (\vec{E} \times \vec{H}) + [\vec{E} \cdot \frac{\partial}{\partial t} \vec{D} + \vec{H} \cdot \frac{\partial}{\partial t} \vec{B}] + \vec{j} \cdot \vec{E} &= 0 \end{aligned}$$

In integral form,

$$\int_{\partial V} (\vec{E} \times \vec{H}) \cdot \vec{n} da + \int_V d\tau [\vec{E} \cdot \frac{\partial}{\partial t} \vec{D} + \vec{H} \cdot \frac{\partial}{\partial t} \vec{B}] + \int_V d\tau (\vec{j} \cdot \vec{E}) = 0 \quad (5)$$

This is the energy law of Electromagnetism. We can simplify when we assume the simple material equations : $\vec{j} = \sigma \vec{E}$, $\vec{D} = \epsilon \vec{E}$, $\vec{B} = \mu \vec{H}$. Then,

$$\vec{E} \cdot \frac{\partial}{\partial t} \vec{D} = \vec{E} \cdot \frac{\partial}{\partial t} \epsilon \vec{E} = \frac{1}{2} \frac{\partial}{\partial t} (\epsilon \vec{E}^2) = \frac{1}{2} \frac{\partial}{\partial t} (\vec{E} \cdot \vec{D})$$

Similarly,

$$\vec{H} \cdot \frac{\partial}{\partial t} \vec{B} = \vec{H} \cdot \frac{\partial}{\partial t} \mu \vec{H} = \frac{1}{2} \frac{\partial}{\partial t} (\mu \vec{H}^2) = \frac{1}{2} \frac{\partial}{\partial t} (\vec{H} \cdot \vec{B})$$

Then

$$\begin{aligned} \int_V d\tau [\frac{1}{2} \frac{\partial}{\partial t} (\vec{E} \cdot \vec{D}) + \frac{1}{2} \frac{\partial}{\partial t} (\vec{H} \cdot \vec{B})] + \int_V d\tau (\vec{j} \cdot \vec{E}) + \int_{\partial V} (\vec{E} \times \vec{H}) \cdot \vec{n} da &= 0 \\ \rightarrow \frac{d}{dt} \int_V d\tau [\frac{1}{2} (\vec{E} \cdot \vec{D}) + \frac{1}{2} (\vec{H} \cdot \vec{B})] + \int_V d\tau (\vec{j} \cdot \vec{E}) + \int_{\partial V} (\vec{E} \times \vec{H}) \cdot \vec{n} da &= 0 \\ \Rightarrow \boxed{\frac{dW}{dt} + \int_V d\tau (\vec{j} \cdot \vec{E}) + \int_{\partial V} \vec{S} \cdot \vec{n} da} &= 0 \\ \text{with } \vec{S} = \vec{E} \times \vec{H} \end{aligned}$$

(QED)

1.1 More physical representation

When A : total work done to charges,

$$\begin{aligned} \delta A &= \sum_k \vec{F}_k \cdot \delta \vec{x}_k = \sum_k q_k (\vec{E}_k + \vec{v}_k \times \vec{B}_k) \cdot \delta \vec{x}_k \\ &= \sum_k q_k \vec{E}_k \cdot \delta \vec{x}_k = \sum_k q_k \vec{E}_k \cdot \vec{v}_k \delta t \\ &\rightarrow \delta A = \delta t \int_V d\tau (\rho \vec{E} \cdot \vec{v}) \end{aligned}$$

But, since Maxwell's equations do not contain a velocity \vec{v} , we should remove it to restate the energy law.

Roentgen's experimental discovery (1888) : (1) Convection current (moving charges) $\vec{j}_v = \rho \vec{v}$ and (2) Conduction current (electric fields) $\vec{j}_c = \sigma \vec{E}$ give same physical (electromagnetic) effects. $\rightarrow \vec{j} = \vec{j}_c + \vec{j}_v$

So, $\delta A = \delta t \int_V d\tau \vec{j}_v \cdot \vec{E}$. Now, define a scalar Q (\equiv Joule's heat) as :

$$Q \equiv \int_V d\tau \vec{j}_c \cdot \vec{E} = \int_V d\tau \sigma \vec{E}^2$$

Then,

$$\int_V d\tau (\vec{j} \cdot \vec{E}) = \int_V d\tau (\vec{j}_c + \vec{j}_v) \cdot \vec{E} = Q + \int_V d\tau (\vec{j}_v \cdot \vec{E}) = Q + \frac{\delta A}{\delta t}$$

Hence, the energy law :

$$\boxed{\frac{dW}{dt} = -\frac{\delta A}{\delta t} - Q - \int_{\partial V} \vec{S} \cdot \vec{n} da}$$

When $\sigma = 0$ (nonconducting) and $A = 0$ (no mechanical work)

$\Rightarrow \frac{dw}{dt} + \nabla \cdot \vec{S} = 0$: note that this is a differential form.

References

- [1] Born, M., Wolf, E., Bhatia, A. B. (2019). Principles of Optics: 60th Anniversary Edition. United Kingdom: Cambridge University Press.