1.3 Scalar Waves

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1 Scalar waves

From [1.2 The Wave Equations], consider

$$\nabla^2\vec{E} - \epsilon\mu\frac{\partial^2\vec{E}}{\partial t^2} + (\nabla \ln \mu) \times (\nabla \times \vec{E}) + \nabla \cdot [\vec{E} \cdot (\nabla \ln \epsilon)]$$

In homogeneous medium with no charges and currents($\rho = 0, \vec{j} = 0$), we have $\nabla^2\vec{E} - \frac{1}{v^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0$. Since each vector components of the field $\vec{E}$ are decoupled, we have the scalar wave equation :

$$\boxed{\nabla^2 V - \frac{1}{v^2}\frac{\partial^2 V}{\partial t^2} = 0} \tag{1}$$

where V is one of the vector components. A solution $V(\vec{r}, t)$ is called a *scalar wave.*

1.1 Plane wave

Intuition : At a fixed time t_0, V is constant over a plane satisfying $\vec{r} \cdot \vec{s} = const.$ So, a solution is of the form $V = V(\vec{r} \cdot \vec{s}, t)$. Then solve the scalar equation (1).

Denote $\vec{r} \cdot \vec{s} \equiv \zeta$ then $\frac{\partial}{\partial x} = \frac{\partial \zeta}{\partial x}\frac{\partial}{\partial \zeta} = s_x \frac{\partial}{\partial \zeta}$, etc. $\rightarrow \nabla^2 V = \frac{\nabla^2 V}{\nabla \zeta^2}$ hence

$$\frac{\partial^2 V}{\partial \zeta^2} - \frac{1}{v^2}\frac{\partial^2 V}{\partial t^2} = 0$$

We can solve this by introducing new variables $p \equiv \zeta - vt$ and $q \equiv \zeta + vt$ then $\frac{\partial^2 V}{\partial p \partial q} = 0$. A general solution for this equation is :

$$\boxed{V = V_1(p) + V_2(q) = V_1(\vec{r} \cdot \vec{s} - vt) + V_2(\vec{r} \cdot \vec{s} + vt)}$$

where V_1 and V_2 are arbitrary functions.

1.2 Spherical wave

Intuition : At a fixed time t_0, V is constant over a spherical surface with radius r. So, a solution is of the form $V = V(r,t)$ where $r = \sqrt{x^2+y^2+z^2}$. Now solve the scalar wave equation (1), noting that $\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} = \frac{x}{r}\frac{\partial}{\partial r}$ and $\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}(\frac{x}{r}\frac{\partial}{\partial r}) = \frac{1}{r}\frac{\partial}{\partial r} - \frac{x^2}{r^3}\frac{\partial}{\partial r} + \frac{x^2}{r^2}\frac{\partial^2}{\partial r^2}$, etc. hence $\nabla^2 V = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rV)$:

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rV) - \frac{1}{v^2}\frac{\partial^2 V}{\partial t^2} = 0$$

$$\rightarrow \frac{\partial^2}{\partial r^2}(rV) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}(rV) = 0$$

A general solution for this equation is :

$$\boxed{V(r,t) = \frac{V_1(r-vt)}{r} + \frac{V_2(r+vt)}{r}}$$

1.3 Harmonic wave (=monochromatic wave)

Note) For a fixed space point $\vec{r}_0$, $V(\vec{r_0},t) = F(t)$: a function of time. $F(t)$ being periodic is of particular interest (ex: colour). So, we consider the following form : $V(\vec{r}_0,t) = F(t) = a\cos{(wt+\delta)}$ with $a > 0$.
In general, we can recover propagation by replacing : $wt \rightarrow wt - g(\vec{r})$. Then,

$$\boxed{V(\vec{r},t) = a(\vec{r})\cos{[wt - g(\vec{r}) + \delta]}}$$

with $a(\vec{r}) > 0$ being real. Surfaces satisfying $g(\vec{r}) = const.$ are called the *cophasal surfaces*. For reverse direction propagation, replace $wt \rightarrow wt + g(\vec{r})$. Note that a general solution is given by superposition of both :

$$V(\vec{r},t) = a_1(\vec{r})\cos{[wt - g(\vec{r}) + \delta]} + a_2(\vec{r})\cos{[wt + g(\vec{r}) + \delta]}$$

1.3.1 Harmonic plane wave

Replace $wt \rightarrow wt - w\frac{\vec{r}\cdot\vec{s}}{v} = w(t - \frac{\vec{r}\cdot\vec{s}}{v})$ then

$$V(\vec{r},t) = a(\vec{r})\cos{[w(t - \tfrac{\vec{r}\cdot\vec{s}}{v}) + \delta]}$$

Introduce $k \equiv w/v$ and the *wavevector* $\vec{k} \equiv k\vec{s}$ then

$$V(\vec{r},t) = a(\vec{r})\cos{(wt - \vec{k}\cdot\vec{r} + \delta)}$$

1.3.2 Harmonic spherical wave

Replace $wt \rightarrow wt - w\frac{r}{v}$ then

$$V(r,t) = a(r)\frac{\cos{[w(t-\frac{r}{v})+\delta]}}{r} = a(r)\frac{\cos{[wt-kr+\delta]}}{r}$$

1.3.3 Exponential notation

We can write

$$V(\vec{r},t) = Re\{U(\vec{r})e^{-iwt}\}$$

where $U(\vec{r})$ is called the *complex amplitude*. Unless we apply nonlinear operations(ex: squaring for calculating the energy), we can drop the $Re\{\cdot\}$ notation and just use $V(\vec{r},t) = U(\vec{r})e^{-iwt}$. If the nonlinear operations are needed, one must take real part of the field and then can apply the nonlinear operations safely.

References

[1] Born, M., Wolf, E., Bhatia, A. B. (2019). Principles of Optics: 60th Anniversary Edition. United Kingdom: Cambridge University Press.