13.1 Scalar Scattering Theory

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August 2024

1 The Basic Integral Equation

From [1.2 EM Fields] :

$$\boxed{\nabla^2\vec{E} - \epsilon\mu\frac{\partial^2\vec{E}}{\partial t^2} + (\nabla\ln\mu)\times(\nabla\times\vec{E}) + \nabla[\vec{E}\cdot(\nabla\ln\epsilon)] = 0}$$

Specialize the above equation to a nonmagnetic, monochromatic case :
$\nabla\ln\mu = 0$ and $\epsilon\mu = \frac{1}{v^2} = \frac{1}{c^2}\cdot\frac{c^2}{v^2} = \frac{n^2}{c^2}$:

$$\nabla^2\vec{E}(\vec{r},w) + \frac{n^2(\vec{r},w)}{c^2}w^2\vec{E}(\vec{r},w) + \nabla[\vec{E}(\vec{r},w)\cdot(\nabla\ln\epsilon(\vec{r},w))] = 0$$

Denote $k \equiv \frac{w}{c}$:

$$\nabla^2\vec{E}(\vec{r},w) + k^2n^2(\vec{r},w)\vec{E}(\vec{r},w) + \nabla[\vec{E}(\vec{r},w)\cdot(\nabla\ln\epsilon(\vec{r},w))] = 0$$

If the dielectric constant varies so slowly such that $\nabla\ln\epsilon(\vec{r},w) = 0$ then

$$\boxed{\nabla^2\vec{E}(\vec{r},w) + k^2n^2(\vec{r},w)\vec{E}(\vec{r},w) = 0}$$

A nonmagnetic, time-harmonic, monochromatic, $\epsilon(\vec{r},w)$ slowly varying wave equation. Note that each Cartesian components of $\vec{E}$ are decoupled now. So, just let $U(\vec{r},w)$ one of the components :

$$\nabla^2U(\vec{r},w) + k^2n^2(\vec{r},w)U(\vec{r},w) = 0$$

To solve the differential equation with homogeneous/inhomogeneous part, manipulate the equation :

$$\nabla^2U(\vec{r},w) + k^2U(\vec{r},w) = -k^2[n^2(\vec{r},w) - 1]U(\vec{r},w)$$

Define the *scattering potential of the medium* $F(\vec{r},w) = \frac{1}{4\pi}k^2[n^2(\vec{r},w) - 1]$:

$$\boxed{\nabla^2U(\vec{r},w) + k^2U(\vec{r},w) = -4\pi F(\vec{r},w)U(\vec{r},w)}$$

This is a differential form of the scalar scattering equation. We'll rewrite this into a integral form.

Express $U(\vec{r}, w) = U^{(i)}(\vec{r}, w) + U^{(s)}(\vec{r}, w)$ where $U^{(i)}$ is an incident field and $U^{(s)}$ is a scattered field.
The incident field is usually a plane wave so $(\nabla^2 + k^2)U^{(i)}(\vec{r}, w) = 0$:

$$(\nabla^2 + k^2)U^{(s)}(\vec{r}, w) = -4\pi F(\vec{r}, w)U(\vec{r}, w) \tag{1}$$

Denote $G(\vec{r} - \vec{r}\,')$: a Green's function of the Helmholtz operator $(\nabla^2 + k^2)$:

$$(\nabla^2 + k^2)G(\vec{r} - \vec{r}\,') = -4\pi\delta(\vec{r} - \vec{r}\,') \tag{2}$$

(1) $\times\, G(\vec{r} - \vec{r}\,', w)$ $-$ (2) $\times\, U^{(s)}(\vec{r}, w)$ then

$$G\nabla^2 U^{(s)} - U^{(s)}\nabla^2 G = -4\pi GFU + 4\pi U^{(s)}\delta(\vec{r} - \vec{r}\,')$$

Take volume integral over V_R (larger than the medium volume V and has a radius R) :

(1) LHS

With the Green's second identity,

$$\int_{V_R} d\tau' [G\nabla^2 U^{(s)} - U^{(s)}\nabla^2 G] = \int_{S_R} (G\frac{\partial U^{(s)}}{\partial \vec{n}} - U^{(s)}\frac{\partial G}{\partial \vec{n}}) \cdot d\vec{S}$$

(2) RHS

$$\int_{V_R} d\tau' [-4\pi GFU + 4\pi U^{(s)}\delta(\vec{r} - \vec{r}\,')]$$

$$= -4\pi \int_{V_R} d\tau' G(\vec{r} - \vec{r}\,')F(\vec{r}\,')U(\vec{r}\,') + 4\pi \int_{V_R} d\tau' U^{(s)}(\vec{r}\,')\delta(\vec{r} - \vec{r}\,')$$

$$= 4\pi U^{(s)}(\vec{r}) - 4\pi \int_V d\tau' G(\vec{r} - \vec{r}\,')F(\vec{r}\,')U(\vec{r}\,')$$

Then, (1)=(2)

$$U^{(s)}(\vec{r}) = \int_V d\tau' G(\vec{r} - \vec{r}\,')F(\vec{r}\,')U(\vec{r}\,') + \int_{S_R} (G\frac{\partial U^{(s)}}{\partial \vec{n}} - U^{(s)}\frac{\partial G}{\partial \vec{n}}) \cdot d\vec{S}$$

Now, we'll show the surface integral vanishes when $R \to \infty$. To do that, choose the Green function :

$$G(\vec{r} - \vec{r}\,', w) = \frac{e^{ik|\vec{r} - \vec{r}\,'|}}{|\vec{r} - \vec{r}\,'|}$$

the *outgoing free-space Green's function.* Note that $G \to 0$ as $R \to \infty$ in this case. We can also assume when $R \to \infty$, $U^{(s)}$ behaves as an outgoing spherical wave. Then, $U^{(s)} \to 0$ as $R \to \infty$. With these choice and assumption, we get :

$$\boxed{U^{(s)}(\vec{r}, w) = \int_V d\tau' G(\vec{r} - \vec{r}\,', w)F(\vec{r}\,', w)U(\vec{r}\,', w)}$$

Or,

$$U^{(s)}(\vec{r}, w) = \int_V d\tau' F(\vec{r}\,', w)U(\vec{r}\,', w)\frac{e^{ik|\vec{r} - \vec{r}\,'|}}{|\vec{r} - \vec{r}\,'|}$$

Using a plane incident wave $U^{(i)}(\vec{r}, w) = e^{ik\vec{s_0}\cdot\vec{r}}$, we have

$$\boxed{U(\vec{r}, w) = e^{ik\vec{s_0}\cdot\vec{r}} + \int_V d\tau' F(\vec{r}\,', w) U(\vec{r}\,', w) \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}}$$

the *integral equation of potential scattering.*

References

[1] Born, M., Wolf, E., Bhatia, A. B. (2019). Principles of Optics: 60th Anniversary Edition. United Kingdom: Cambridge University Press.