Lecture 8
Least-norm solutions of underdetermined equations

- least-norm solution of underdetermined equations
- minimum norm solutions via QR factorization
- derivation via Lagrange multipliers
- relation to regularized least-squares
- general norm minimization with equality constraints

Underdetermined linear equations

we consider

$$y = Ax$$

where $A \in \mathbf{R}^{m \times n}$ is fat ($m < n$), $i.e.$,

- there are more variables than equations
- x is *underspecified*, $i.e.$, many choices of x lead to the same y

we'll assume that A is full rank (m), so for each $y \in \mathbf{R}^m$, there is a solution

set of all solutions has form

$$\{ \, x \mid Ax = y \, \} = \{ \, x_p + z \mid z \in \mathcal{N}(A) \, \}$$

where x_p is any ('particular') solution, $i.e.$, $Ax_p = y$

- z characterizes available choices in solution
- solution has $\dim \mathcal{N}(A) = n - m$ 'degrees of freedom'
- can choose z to satisfy other specs or optimize among solutions

Least-norm solution

one particular solution is

$$x_{\rm ln} = A^T(AA^T)^{-1}y$$

(AA^T is invertible since A full rank)

in fact, $x_{\rm ln}$ is the solution of $y = Ax$ that minimizes $\|x\|$

i.e., $x_{\rm ln}$ is solution of optimization problem

$$\begin{array}{ll} \text{minimize} & \|x\| \\ \text{subject to} & Ax = y \end{array}$$

(with variable $x \in \mathbf{R}^n$)

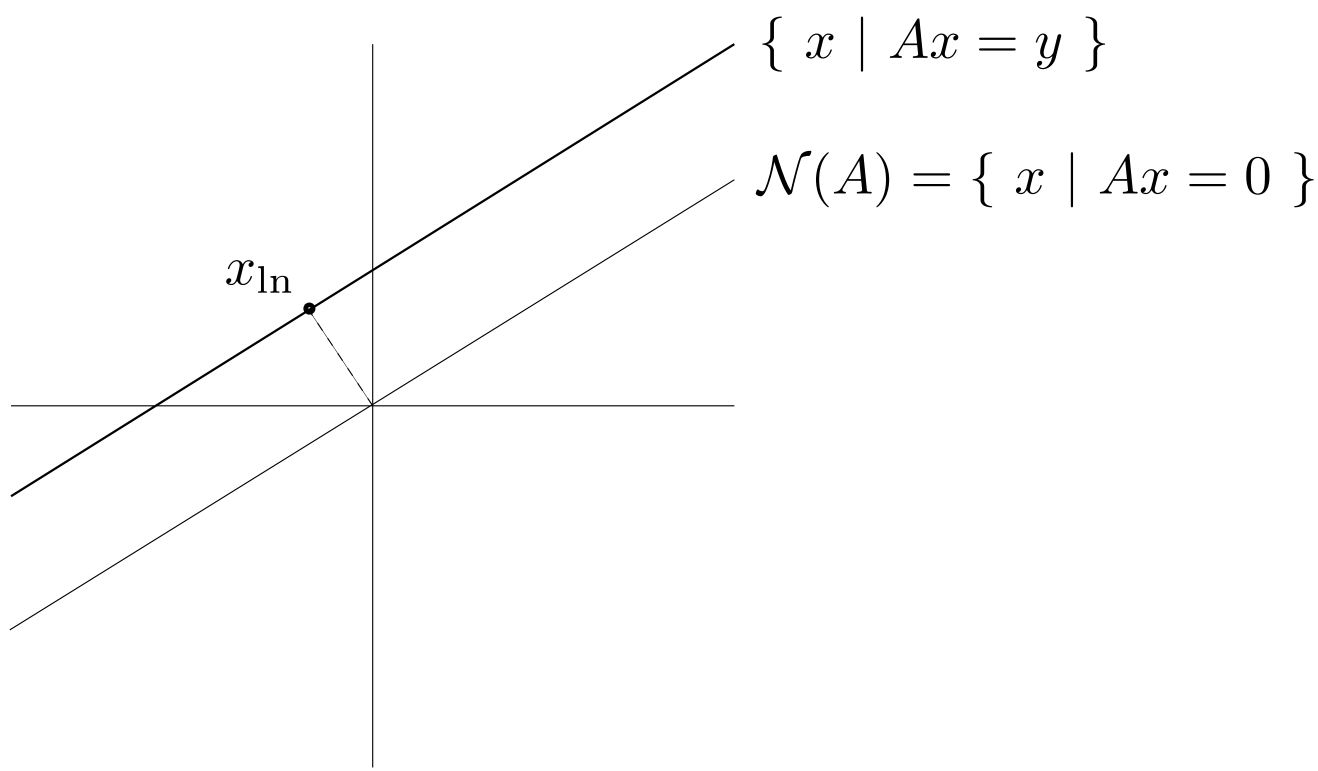
suppose $Ax = y$, so $A(x - x_{\mathrm{ln}}) = 0$ and

$$\begin{aligned} (x - x_{\mathrm{ln}})^T x_{\mathrm{ln}} &= (x - x_{\mathrm{ln}})^T A^T (AA^T)^{-1} y \\ &= (A(x - x_{\mathrm{ln}}))^T (AA^T)^{-1} y \\ &= 0 \end{aligned}$$

i.e., $(x - x_{\mathrm{ln}}) \perp x_{\mathrm{ln}}$, so

$$\|x\|^2 = \|x_{\mathrm{ln}} + x - x_{\mathrm{ln}}\|^2 = \|x_{\mathrm{ln}}\|^2 + \|x - x_{\mathrm{ln}}\|^2 \geq \|x_{\mathrm{ln}}\|^2$$

i.e., x_{ln} has smallest norm of any solution



- **orthogonality condition:** $x_{\text{ln}} \perp \mathcal{N}(A)$
- **projection interpretation:** x_{ln} is projection of 0 on solution set $\{ x \mid Ax = y \}$

- $A^\dagger = A^T(AA^T)^{-1}$ is called the *pseudo-inverse* of full rank, fat A
- $A^T(AA^T)^{-1}$ is a *right inverse* of A
- $I - A^T(AA^T)^{-1}A$ gives projection onto $\mathcal{N}(A)$

cf. analogous formulas for full rank, **skinny** matrix A:

- $A^\dagger = (A^TA)^{-1}A^T$
- $(A^TA)^{-1}A^T$ is a *left inverse* of A
- $A(A^TA)^{-1}A^T$ gives projection onto $\mathcal{R}(A)$

Least-norm solution via QR factorization

find QR factorization of A^T, *i.e.*, $A^T = QR$, with

- $Q \in \mathbf{R}^{n\times m}$, $Q^TQ = I_m$
- $R \in \mathbf{R}^{m\times m}$ upper triangular, nonsingular

then

- $x_{\rm ln} = A^T(AA^T)^{-1}y = QR^{-T}y$
- $\|x_{\rm ln}\| = \|R^{-T}y\|$

Derivation via Lagrange multipliers

- least-norm solution solves optimization problem

$$\begin{array}{ll}\text{minimize} & x^T x \\ \text{subject to} & Ax = y\end{array}$$

- introduce Lagrange multipliers: $L(x, \lambda) = x^T x + \lambda^T (Ax - y)$

- optimality conditions are

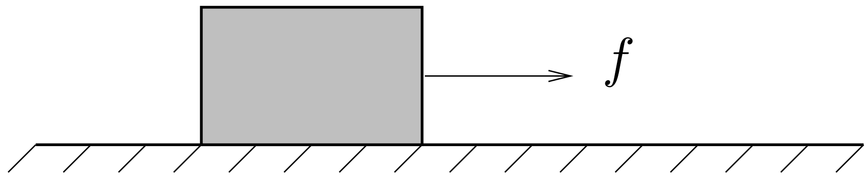
$$\nabla_x L = 2x + A^T \lambda = 0, \qquad \nabla_\lambda L = Ax - y = 0$$

- from first condition, $x = -A^T \lambda / 2$

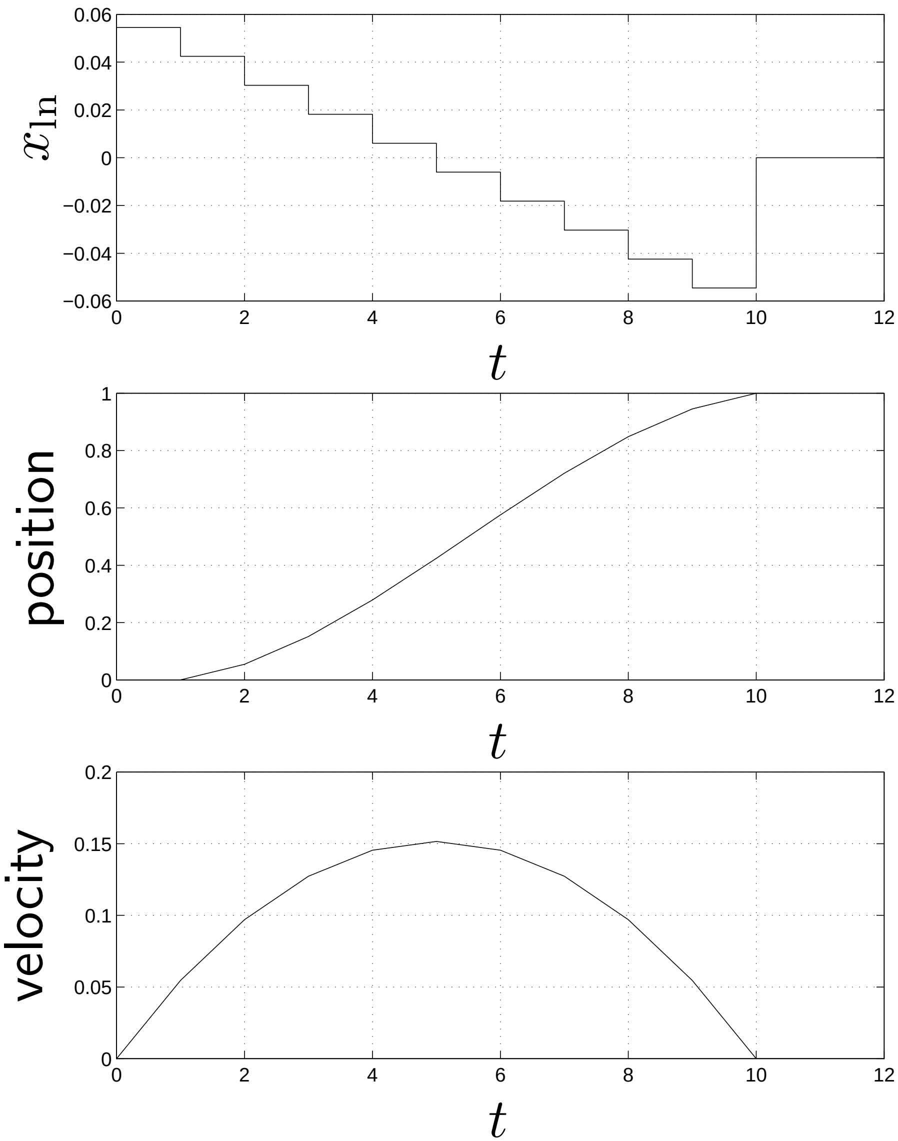
- substitute into second to get $\lambda = -2(AA^T)^{-1} y$

- hence $x = A^T (AA^T)^{-1} y$

Example: transferring mass unit distance



- unit mass at rest subject to forces x_i for $i-1 < t \leq i$, $i = 1, \ldots, 10$
- y_1 is position at $t = 10$, y_2 is velocity at $t = 10$
- $y = Ax$ where $A \in \mathbf{R}^{2\times 10}$ (A is fat)
- find least norm force that transfers mass unit distance with zero final velocity, *i.e.*, $y = (1, 0)$



$x_{\rm ln}$
t
position
t
velocity
t

Relation to regularized least-squares

- suppose $A \in \mathbf{R}^{m \times n}$ is fat, full rank
- define $J_1 = \|Ax - y\|^2$, $J_2 = \|x\|^2$
- least-norm solution minimizes J_2 with $J_1 = 0$
- minimizer of weighted-sum objective $J_1 + \mu J_2 = \|Ax - y\|^2 + \mu \|x\|^2$ is

$$x_\mu = \left(A^T A + \mu I\right)^{-1} A^T y$$

- **fact:** $x_\mu \to x_{\rm ln}$ as $\mu \to 0$, *i.e.*, regularized solution converges to least-norm solution as $\mu \to 0$
- in matrix terms: as $\mu \to 0$,

$$\left(A^T A + \mu I\right)^{-1} A^T \to A^T \left(A A^T\right)^{-1}$$

(for full rank, fat A)

General norm minimization with equality constraints

consider problem

$$\begin{array}{ll} \text{minimize} & \|Ax - b\| \\ \text{subject to} & Cx = d \end{array}$$

with variable x

- includes least-squares and least-norm problems as special cases

- equivalent to

$$\begin{array}{ll} \text{minimize} & (1/2)\|Ax - b\|^2 \\ \text{subject to} & Cx = d \end{array}$$

- Lagrangian is

$$\begin{aligned} L(x, \lambda) &= (1/2)\|Ax - b\|^2 + \lambda^T(Cx - d) \\ &= (1/2)x^T A^T Ax - b^T Ax + (1/2)b^T b + \lambda^T Cx - \lambda^T d \end{aligned}$$

- optimality conditions are

$$\nabla_x L = A^T A x - A^T b + C^T \lambda = 0, \qquad \nabla_\lambda L = Cx - d = 0$$

- write in block matrix form as

$$\begin{bmatrix} A^T A & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} A^T b \\ d \end{bmatrix}$$

- if the block matrix is invertible, we have

$$\begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} A^T A & C^T \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} A^T b \\ d \end{bmatrix}$$

if A^TA is invertible, we can derive a more explicit (and complicated) formula for x

- from first block equation we get

$$x = (A^TA)^{-1}(A^Tb - C^T\lambda)$$

- substitute into $Cx = d$ to get

$$C(A^TA)^{-1}(A^Tb - C^T\lambda) = d$$

so

$$\lambda = \left(C(A^TA)^{-1}C^T\right)^{-1}\left(C(A^TA)^{-1}A^Tb - d\right)$$

- recover x from equation above (not pretty)

$$x = (A^TA)^{-1}\left(A^Tb - C^T\left(C(A^TA)^{-1}C^T\right)^{-1}\left(C(A^TA)^{-1}A^Tb - d\right)\right)$$