

Lecture 1 Overview

- course mechanics
- outline & topics
- what is a linear dynamical system?
- why study linear systems?
- some examples

Course mechanics

- all class info, lectures, homeworks, announcements on class web page:

www.stanford.edu/class/ee263

course requirements:

- weekly homework
- takehome midterm exam (date TBD)
- takehome final exam (date TBD)

Prerequisites

- exposure to linear algebra (*e.g.*, Math 104)
- exposure to Laplace transform, differential equations

not needed, but might increase appreciation:

- control systems
- circuits & systems
- dynamics

Major topics & outline

- linear algebra & applications
- autonomous linear dynamical systems
- linear dynamical systems with inputs & outputs
- basic quadratic control & estimation

Linear dynamical system

continuous-time linear dynamical system (CT LDS) has the form

$$\frac{dx}{dt} = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t) + D(t)u(t)$$

where:

- $t \in \mathbf{R}$ denotes *time*
- $x(t) \in \mathbf{R}^n$ is the *state* (vector)
- $u(t) \in \mathbf{R}^m$ is the *input* or *control*
- $y(t) \in \mathbf{R}^p$ is the *output*

- $A(t) \in \mathbf{R}^{n \times n}$ is the *dynamics matrix*
- $B(t) \in \mathbf{R}^{n \times m}$ is the *input matrix*
- $C(t) \in \mathbf{R}^{p \times n}$ is the *output or sensor matrix*
- $D(t) \in \mathbf{R}^{p \times m}$ is the *feedthrough matrix*

for lighter appearance, equations are often written

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

- CT LDS is a first order vector *differential equation*
- also called *state equations*, or ‘ m -input, n -state, p -output’ LDS

Some LDS terminology

- most linear systems encountered are *time-invariant*: A, B, C, D are constant, *i.e.*, don't depend on t
- when there is no input u (hence, no B or D) system is called *autonomous*
- very often there is no feedthrough, *i.e.*, $D = 0$
- when $u(t)$ and $y(t)$ are scalar, system is called *single-input, single-output* (SISO); when input & output signal dimensions are more than one, MIMO

Discrete-time linear dynamical system

discrete-time linear dynamical system (DT LDS) has the form

$$x(t+1) = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t) + D(t)u(t)$$

where

- $t \in \mathbf{Z} = \{0, \pm 1, \pm 2, \dots\}$
- (vector) signals x, u, y are *sequences*

DT LDS is a first-order vector *recursion*

Why study linear systems?

applications arise in **many** areas, *e.g.*

- automatic control systems
- signal processing
- communications
- economics, finance
- circuit analysis, simulation, design
- mechanical and civil engineering
- aeronautics
- navigation, guidance

Usefulness of LDS

- depends on availability of **computing power**, which is large & increasing exponentially
- used for
 - analysis & design
 - implementation, embedded in real-time systems
- like DSP, was a specialized topic & technology 30 years ago

Origins and history

- parts of LDS theory can be traced to 19th century
- builds on classical circuits & systems (1920s on) (transfer functions . . .) but with more emphasis on linear algebra
- first engineering application: aerospace, 1960s
- transitioned from specialized topic to ubiquitous in 1980s (just like digital signal processing, information theory, . . .)

Nonlinear dynamical systems

many dynamical systems are **nonlinear** (a fascinating topic) so why study **linear** systems?

- most techniques for nonlinear systems are based on linear methods
- methods for linear systems often work unreasonably well, in practice, for nonlinear systems
- if you don't understand linear dynamical systems you certainly can't understand nonlinear dynamical systems

Examples (ideas only, no details)

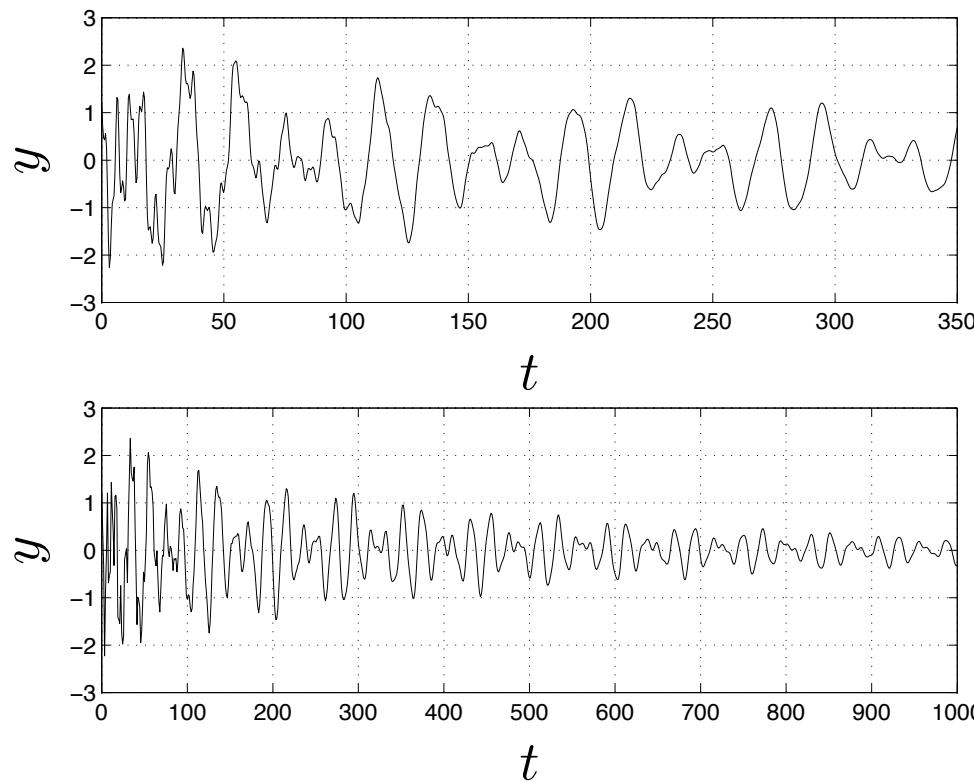
- let's consider a specific system

$$\dot{x} = Ax, \quad y = Cx$$

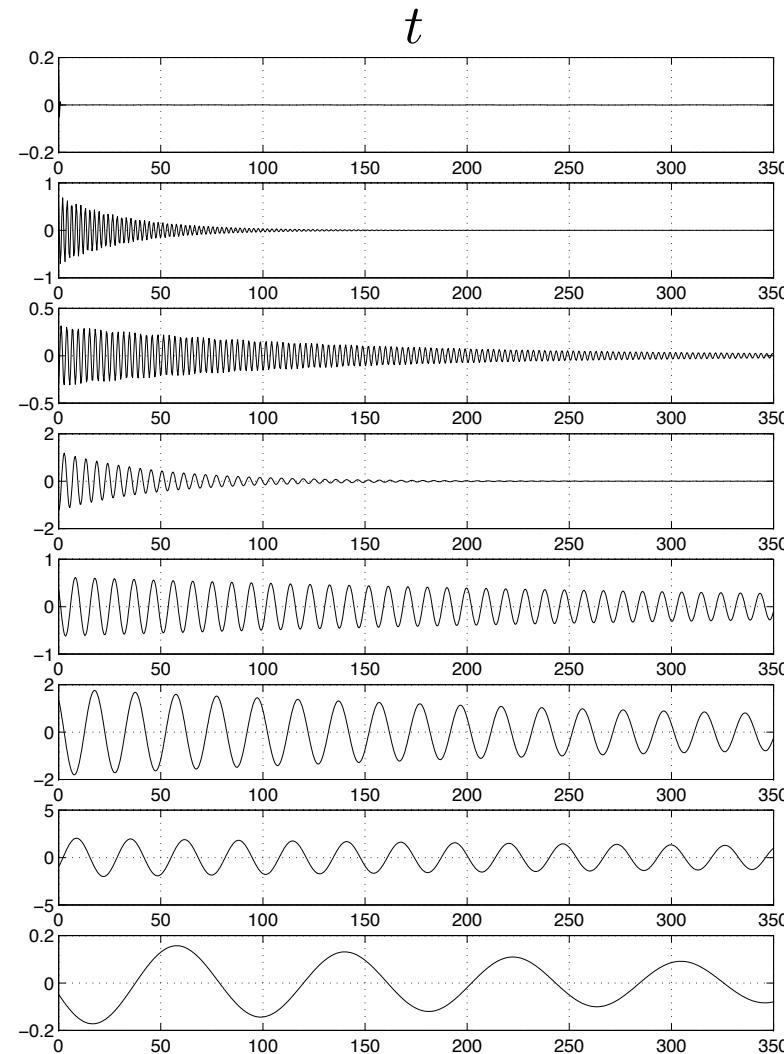
with $x(t) \in \mathbf{R}^{16}$, $y(t) \in \mathbf{R}$ (a '16-state single-output system')

- model of a lightly damped mechanical system, but it doesn't matter

typical output:



- output waveform is very complicated; looks almost random and unpredictable
- we'll see that such a solution can be decomposed into much simpler (modal) components



(idea probably familiar from ‘poles’)

Input design

add two inputs, two outputs to system:

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad x(0) = 0$$

where $B \in \mathbf{R}^{16 \times 2}$, $C \in \mathbf{R}^{2 \times 16}$ (same A as before)

problem: find appropriate $u : \mathbf{R}_+ \rightarrow \mathbf{R}^2$ so that $y(t) \rightarrow y_{\text{des}} = (1, -2)$

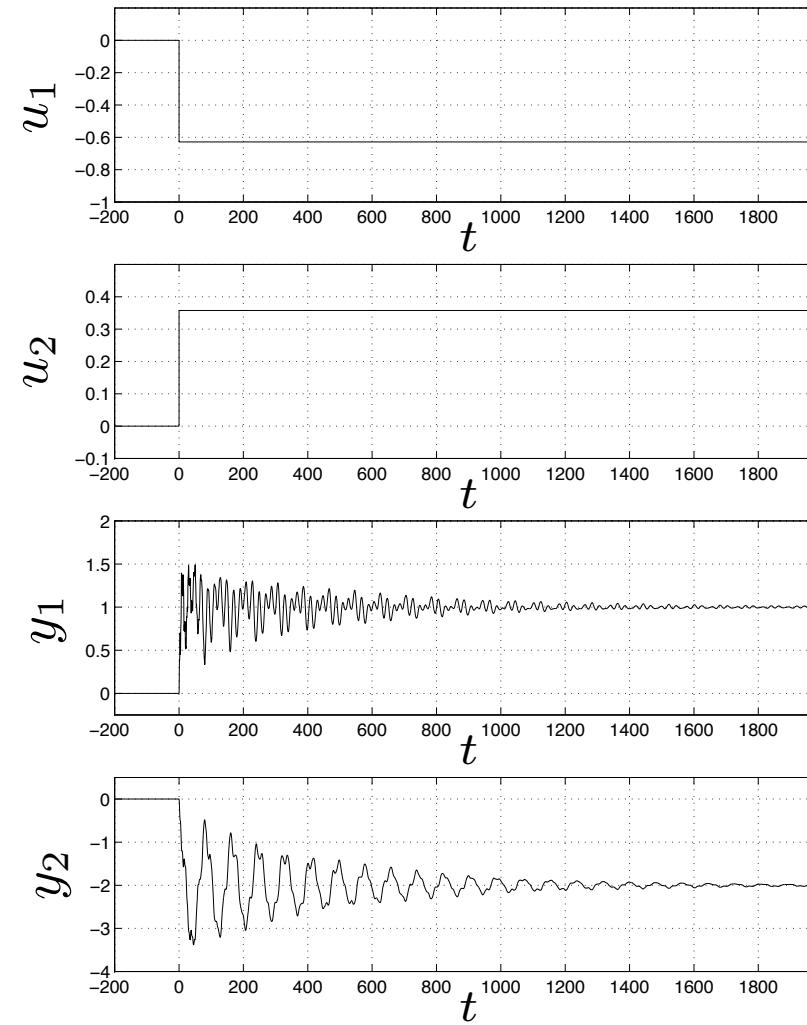
simple approach: consider static conditions (u , x , y constant):

$$\dot{x} = 0 = Ax + Bu_{\text{static}}, \quad y = y_{\text{des}} = Cx$$

solve for u to get:

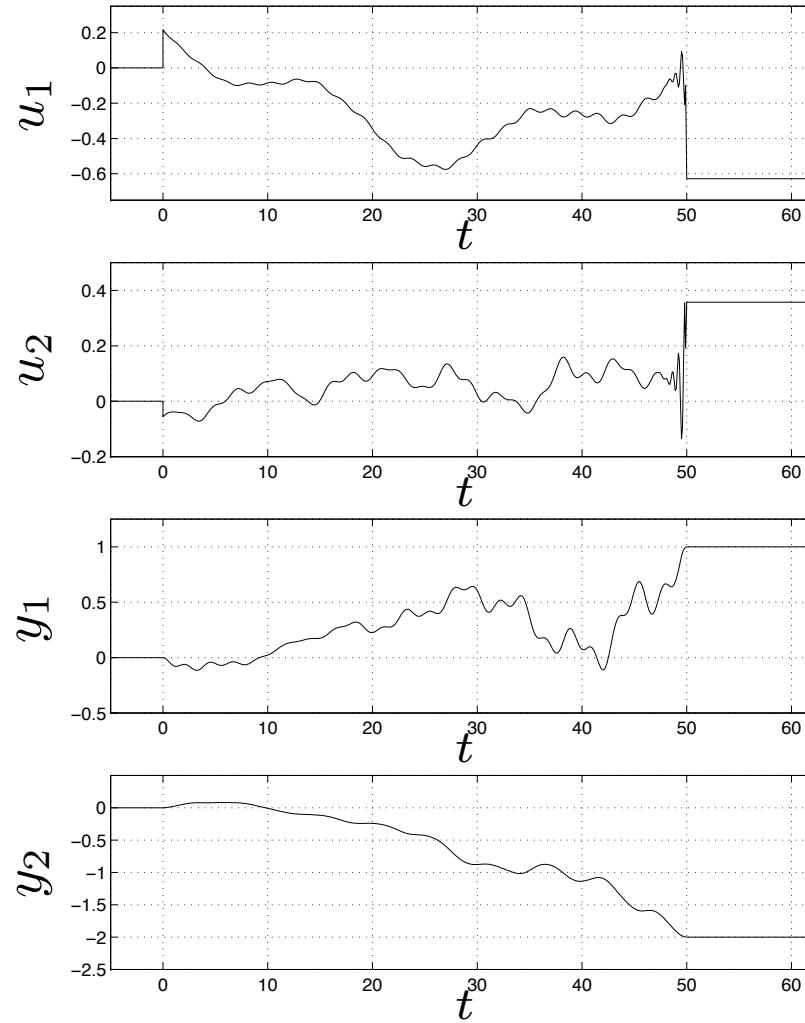
$$u_{\text{static}} = (-CA^{-1}B)^{-1}y_{\text{des}} = \begin{bmatrix} -0.63 \\ 0.36 \end{bmatrix}$$

let's apply $u = u_{\text{static}}$ and just wait for things to settle:



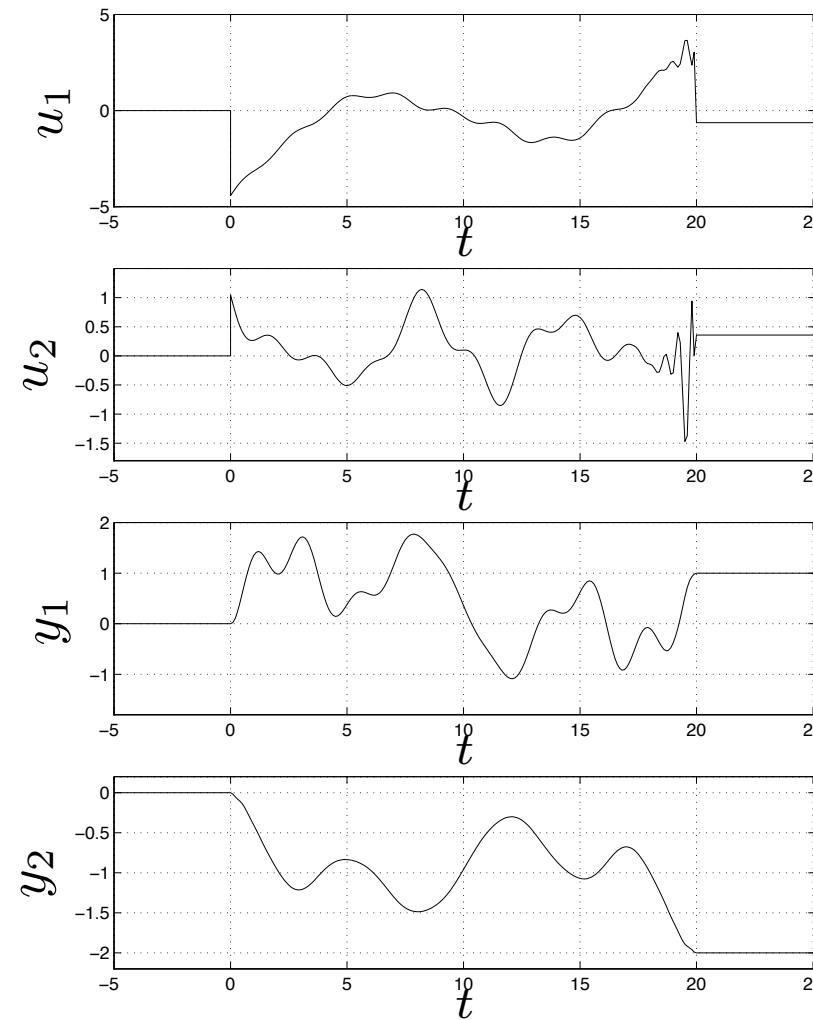
... takes about 1500 sec for $y(t)$ to converge to y_{des}

using very clever input waveforms (EE263) we can do much better, e.g.



. . . here y converges exactly in 50 sec

in fact by using larger inputs we do still better, e.g.

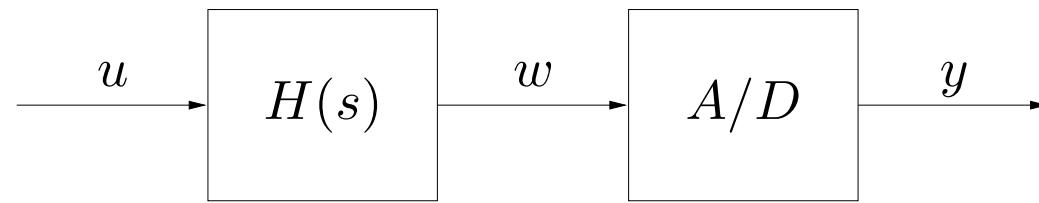


. . . here we have (exact) convergence in 20 sec

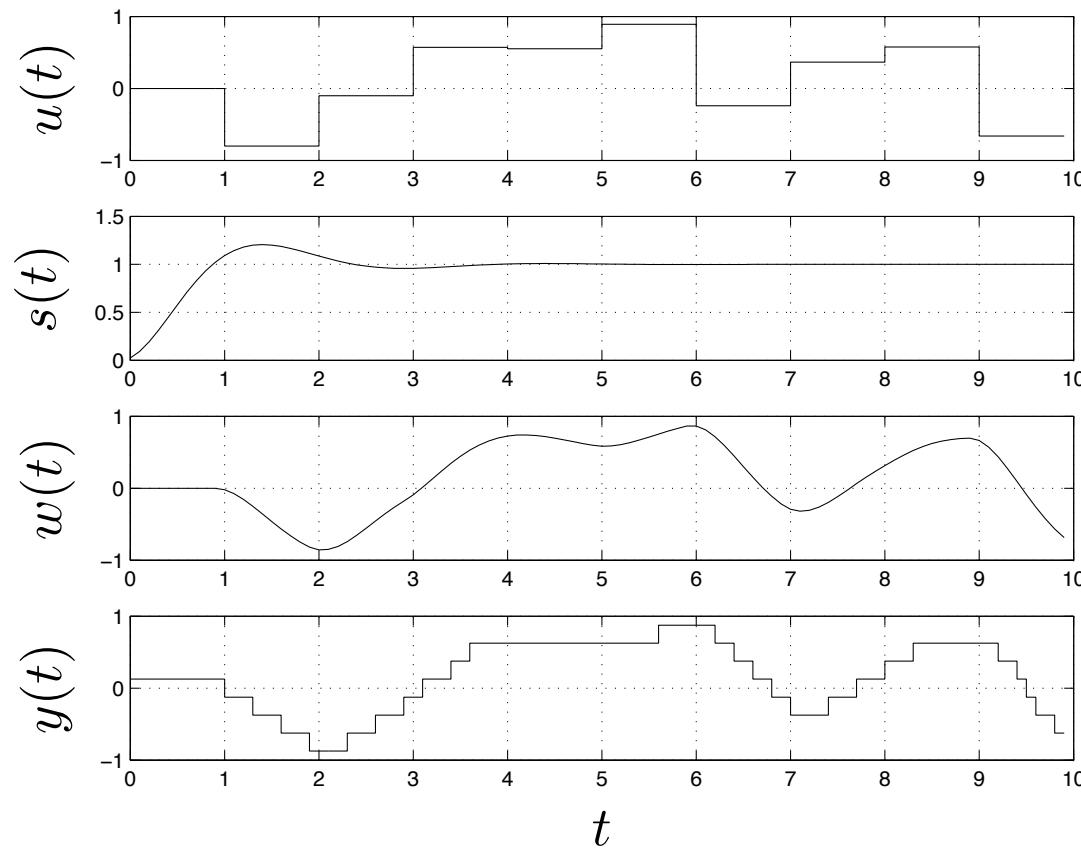
in this course we'll study

- how to synthesize or design such inputs
- the tradeoff between size of u and convergence time

Estimation / filtering



- signal u is piecewise constant (period 1 sec)
- filtered by 2nd-order system $H(s)$, step response $s(t)$
- A/D runs at 10Hz, with 3-bit quantizer

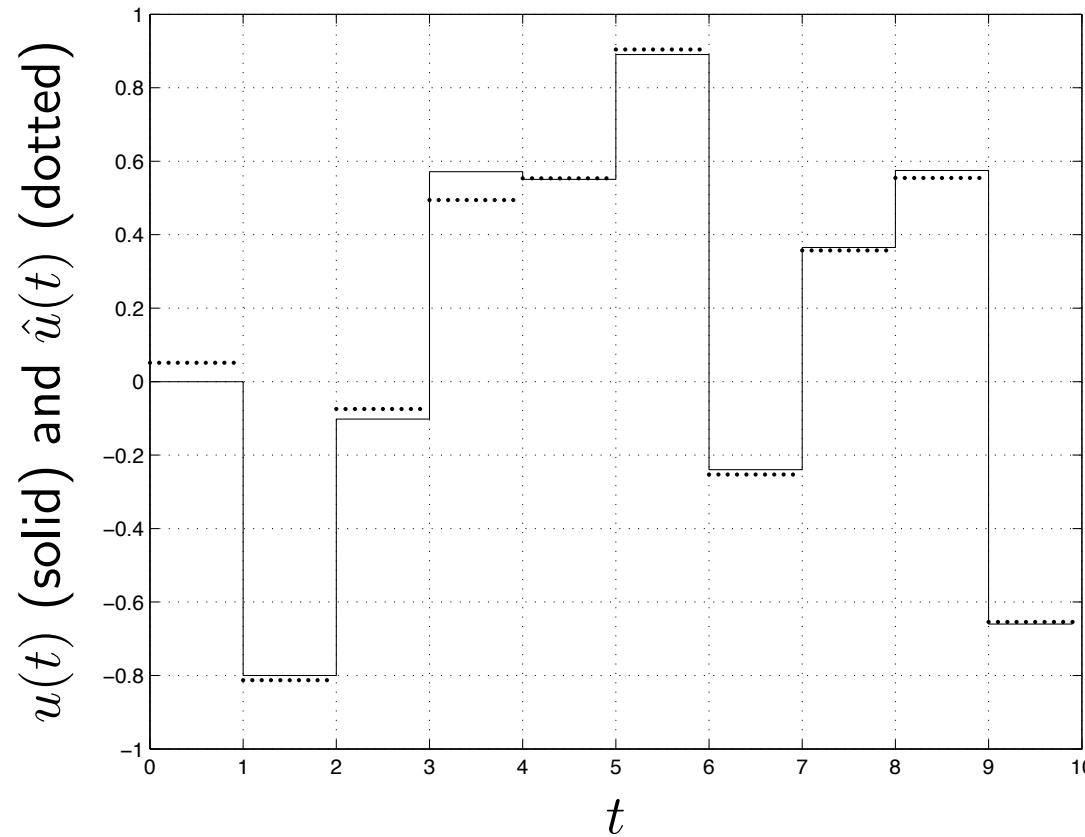


problem: estimate original signal u , given quantized, filtered signal y

simple approach:

- ignore quantization
 - design equalizer $G(s)$ for $H(s)$ (*i.e.*, $GH \approx 1$)
 - approximate u as $G(s)y$
- . . . yields terrible results

formulate as *estimation problem* (EE263) . . .



RMS error 0.03, well **below** quantization error (!)

Lecture 2

Linear functions and examples

- linear equations and functions
- engineering examples
- interpretations

Linear equations

consider system of linear equations

$$\begin{aligned}y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\&\vdots \\y_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n\end{aligned}$$

can be written in matrix form as $y = Ax$, where

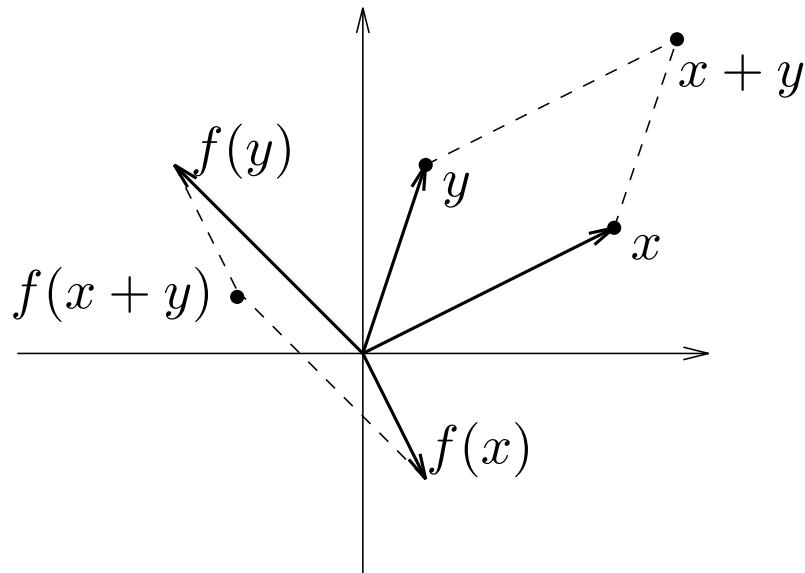
$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Linear functions

a function $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is *linear* if

- $f(x + y) = f(x) + f(y), \forall x, y \in \mathbf{R}^n$
- $f(\alpha x) = \alpha f(x), \forall x \in \mathbf{R}^n \forall \alpha \in \mathbf{R}$

i.e., *superposition* holds



Matrix multiplication function

- consider function $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ given by $f(x) = Ax$, where $A \in \mathbf{R}^{m \times n}$
- matrix multiplication function f is linear
- **converse** is true: **any** linear function $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ can be written as $f(x) = Ax$ for some $A \in \mathbf{R}^{m \times n}$
- representation via matrix multiplication is unique: for any linear function f there is only one matrix A for which $f(x) = Ax$ for all x
- $y = Ax$ is a concrete representation of a generic linear function

Interpretations of $y = Ax$

- y is measurement or observation; x is unknown to be determined
- x is ‘input’ or ‘action’; y is ‘output’ or ‘result’
- $y = Ax$ defines a function or transformation that maps $x \in \mathbf{R}^n$ into $y \in \mathbf{R}^m$

Interpretation of a_{ij}

$$y_i = \sum_{j=1}^n a_{ij}x_j$$

a_{ij} is *gain factor* from j th input (x_j) to i th output (y_i)

thus, *e.g.*,

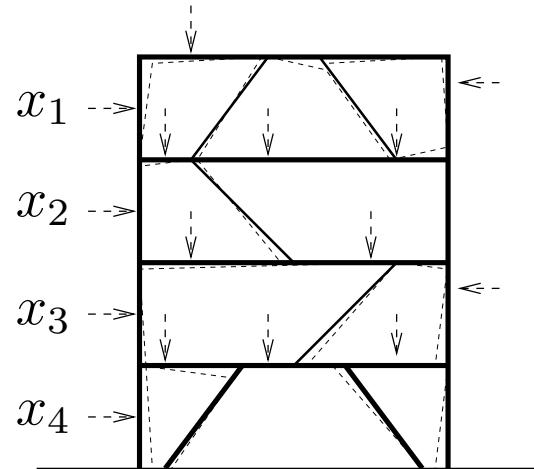
- i th *row* of A concerns i th *output*
- j th *column* of A concerns j th *input*
- $a_{27} = 0$ means 2nd output (y_2) doesn't depend on 7th input (x_7)
- $|a_{31}| \gg |a_{3j}|$ for $j \neq 1$ means y_3 depends mainly on x_1

- $|a_{52}| \gg |a_{i2}|$ for $i \neq 5$ means x_2 affects mainly y_5
- A is lower triangular, i.e., $a_{ij} = 0$ for $i < j$, means y_i only depends on x_1, \dots, x_i
- A is diagonal, i.e., $a_{ij} = 0$ for $i \neq j$, means i th output depends only on i th input

more generally, **sparsity pattern** of A , i.e., list of zero/nonzero entries of A , shows which x_j affect which y_i

Linear elastic structure

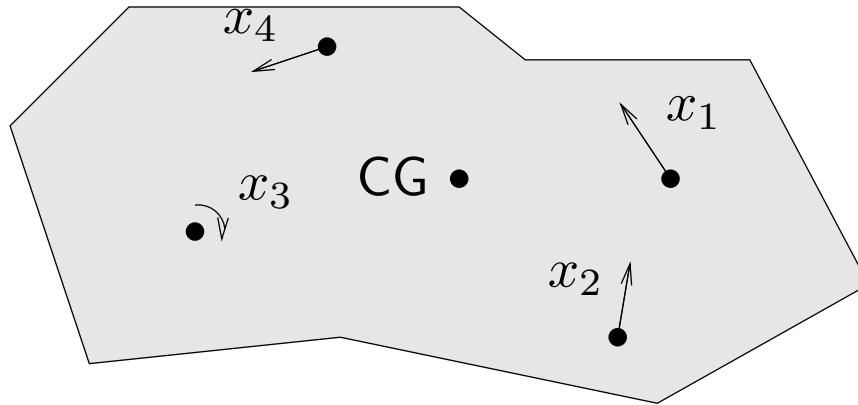
- x_j is external force applied at some node, in some fixed direction
- y_i is (small) deflection of some node, in some fixed direction



(provided x, y are small) we have $y \approx Ax$

- A is called the *compliance matrix*
- a_{ij} gives deflection i per unit force at j (in m/N)

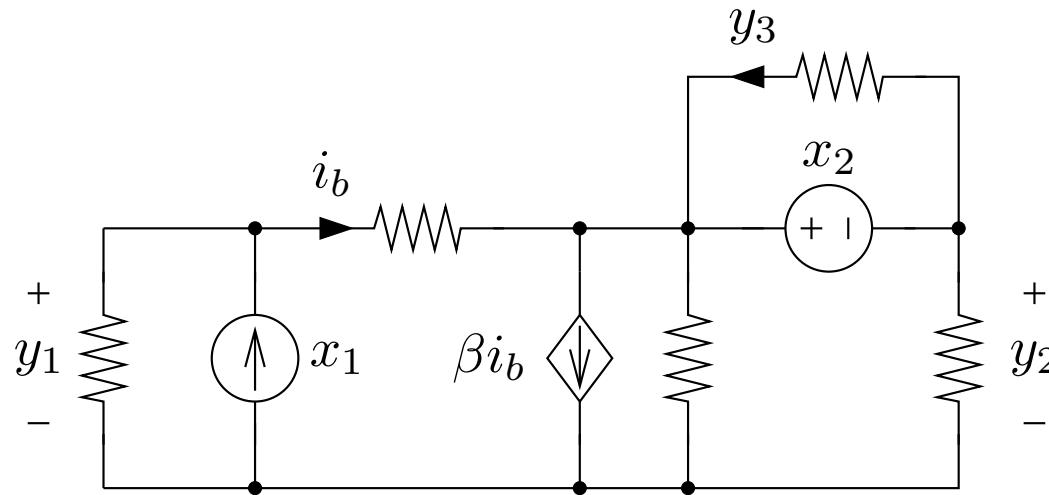
Total force/torque on rigid body



- x_j is external force/torque applied at some point/direction/axis
- $y \in \mathbf{R}^6$ is resulting total force & torque on body
(y_1, y_2, y_3 are x-, y-, z- components of total force,
 y_4, y_5, y_6 are x-, y-, z- components of total torque)
- we have $y = Ax$
- A depends on geometry
(of applied forces and torques with respect to center of gravity CG)
- j th column gives resulting force & torque for unit force/torque j

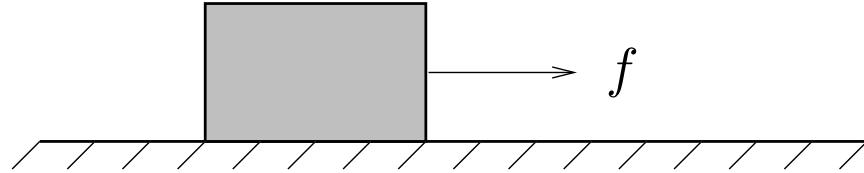
Linear static circuit

interconnection of resistors, linear dependent (controlled) sources, and independent sources



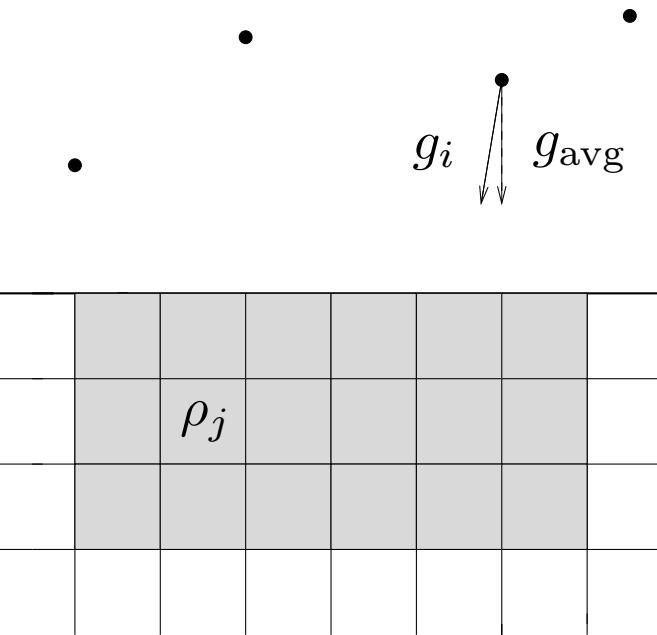
- x_j is value of independent source j
- y_i is some circuit variable (voltage, current)
- we have $y = Ax$
- if x_j are currents and y_i are voltages, A is called the *impedance* or *resistance* matrix

Final position/velocity of mass due to applied forces



- unit mass, zero position/velocity at $t = 0$, subject to force $f(t)$ for $0 \leq t \leq n$
- $f(t) = x_j$ for $j - 1 \leq t < j$, $j = 1, \dots, n$
(x is the sequence of applied forces, constant in each interval)
- y_1, y_2 are final position and velocity (i.e., at $t = n$)
- we have $y = Ax$
- a_{1j} gives influence of applied force during $j - 1 \leq t < j$ on final position
- a_{2j} gives influence of applied force during $j - 1 \leq t < j$ on final velocity

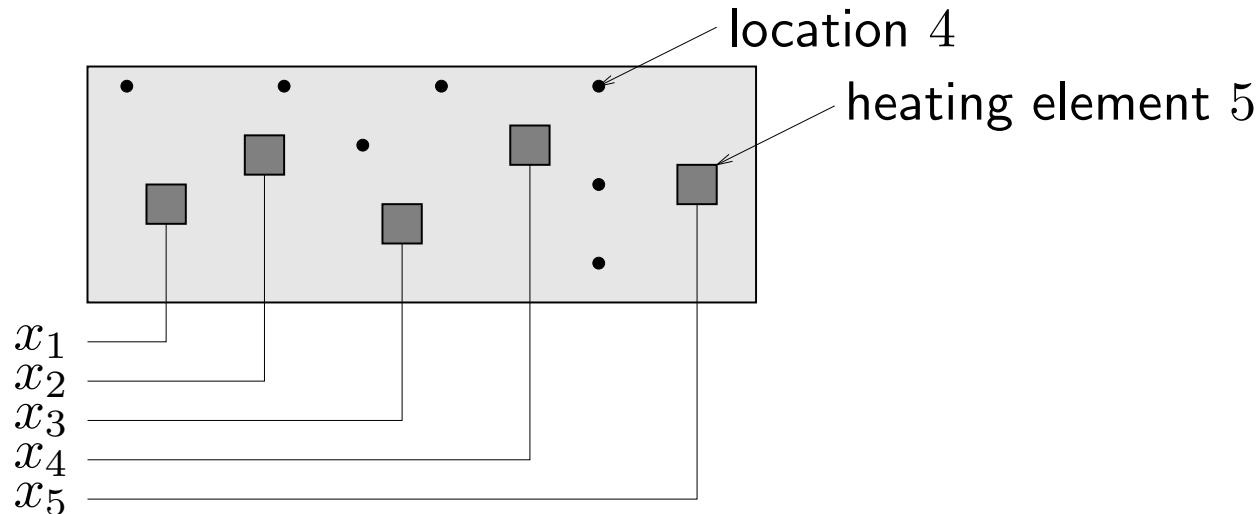
Gravimeter prospecting



- $x_j = \rho_j - \rho_{\text{avg}}$ is (excess) mass density of earth in voxel j ;
- y_i is measured *gravity anomaly* at location i , i.e., some component (typically vertical) of $g_i - g_{\text{avg}}$
- $y = Ax$

- A comes from physics and geometry
- j th column of A shows sensor readings caused by unit density anomaly at voxel j
- i th row of A shows sensitivity pattern of sensor i

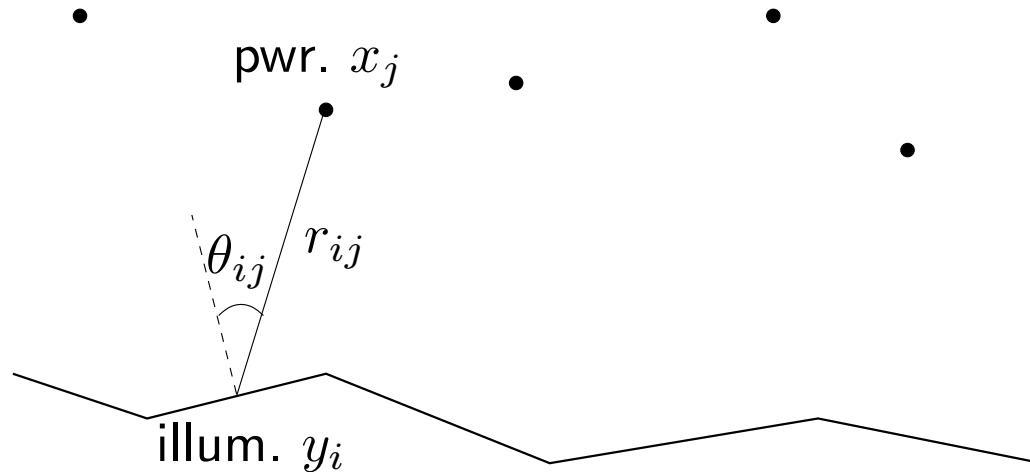
Thermal system



- x_j is power of j th heating element or heat source
- y_i is change in steady-state temperature at location i
- thermal transport via conduction
- $y = Ax$

- a_{ij} gives influence of heater j at location i (in $^{\circ}\text{C}/\text{W}$)
- j th column of A gives pattern of steady-state temperature rise due to 1W at heater j
- i th row shows how heaters affect location i

Illumination with multiple lamps



- n lamps illuminating m (small, flat) patches, no shadows
- x_j is power of j th lamp; y_i is illumination level of patch i
- $y = Ax$, where $a_{ij} = r_{ij}^{-2} \max\{\cos \theta_{ij}, 0\}$
($\cos \theta_{ij} < 0$ means patch i is shaded from lamp j)
- j th column of A shows illumination pattern from lamp j

Signal and interference power in wireless system

- n transmitter/receiver pairs
- transmitter j transmits to receiver j (and, inadvertently, to the other receivers)
- p_j is power of j th transmitter
- s_i is received signal power of i th receiver
- z_i is received interference power of i th receiver
- G_{ij} is path gain from transmitter j to receiver i
- we have $s = Ap$, $z = Bp$, where

$$a_{ij} = \begin{cases} G_{ii} & i = j \\ 0 & i \neq j \end{cases} \quad b_{ij} = \begin{cases} 0 & i = j \\ G_{ij} & i \neq j \end{cases}$$

- A is diagonal; B has zero diagonal (ideally, A is ‘large’, B is ‘small’)

Cost of production

production *inputs* (materials, parts, labor, . . .) are combined to make a number of *products*

- x_j is price per unit of production input j
- a_{ij} is units of production input j required to manufacture one unit of product i
- y_i is production cost per unit of product i
- we have $y = Ax$
- i th row of A is *bill of materials* for unit of product i

production inputs needed

- q_i is quantity of product i to be produced
- r_j is total quantity of production input j needed
- we have $r = A^T q$

total production cost is

$$r^T x = (A^T q)^T x = q^T A x$$

Network traffic and flows

- n flows with rates f_1, \dots, f_n pass from their source nodes to their destination nodes over fixed routes in a network
- t_i , traffic on link i , is sum of rates of flows passing through it
- flow routes given by *flow-link incidence matrix*

$$A_{ij} = \begin{cases} 1 & \text{flow } j \text{ goes over link } i \\ 0 & \text{otherwise} \end{cases}$$

- traffic and flow rates related by $t = Af$

link delays and flow latency

- let d_1, \dots, d_m be link delays, and l_1, \dots, l_n be latency (total travel time) of flows
- $l = A^T d$
- $f^T l = f^T A^T d = (Af)^T d = t^T d$, total # of packets in network

Linearization

- if $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is differentiable at $x_0 \in \mathbf{R}^n$, then

$$x \text{ near } x_0 \implies f(x) \text{ very near } f(x_0) + Df(x_0)(x - x_0)$$

where

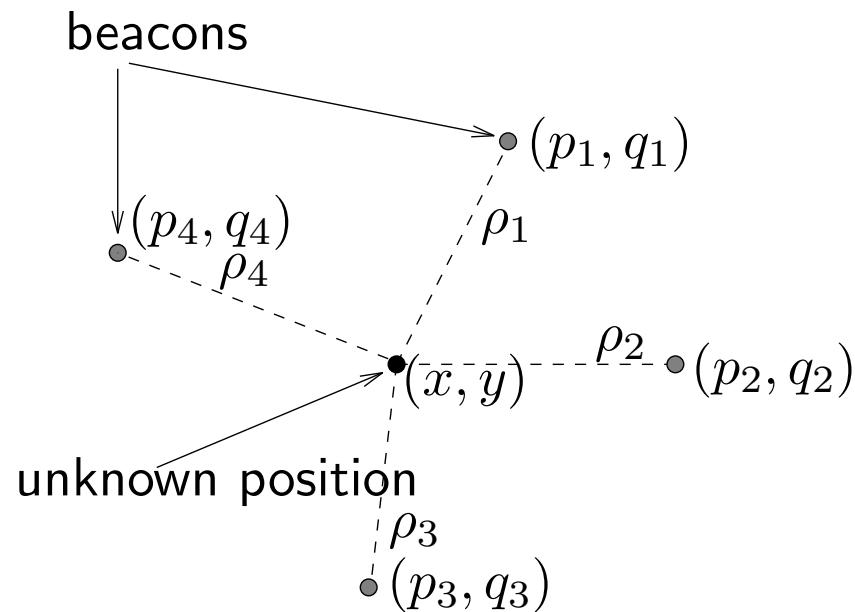
$$Df(x_0)_{ij} = \frac{\partial f_i}{\partial x_j} \Big|_{x_0}$$

is derivative (Jacobian) matrix

- with $y = f(x)$, $y_0 = f(x_0)$, define *input deviation* $\delta x := x - x_0$, *output deviation* $\delta y := y - y_0$
- then we have $\delta y \approx Df(x_0)\delta x$
- when deviations are small, they are (approximately) related by a linear function

Navigation by range measurement

- (x, y) unknown coordinates in plane
- (p_i, q_i) known coordinates of beacons for $i = 1, 2, 3, 4$
- ρ_i measured (known) distance or range from beacon i



- $\rho \in \mathbf{R}^4$ is a nonlinear function of $(x, y) \in \mathbf{R}^2$:

$$\rho_i(x, y) = \sqrt{(x - p_i)^2 + (y - q_i)^2}$$

- linearize around (x_0, y_0) : $\delta\rho \approx A \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}$, where

$$a_{i1} = \frac{(x_0 - p_i)}{\sqrt{(x_0 - p_i)^2 + (y_0 - q_i)^2}}, \quad a_{i2} = \frac{(y_0 - q_i)}{\sqrt{(x_0 - p_i)^2 + (y_0 - q_i)^2}}$$

- i th row of A shows (approximate) change in i th range measurement for (small) shift in (x, y) from (x_0, y_0)
- first column of A shows sensitivity of range measurements to (small) change in x from x_0
- obvious application: (x_0, y_0) is last navigation fix; (x, y) is current position, a short time later

Broad categories of applications

linear model or function $y = Ax$

some broad categories of applications:

- estimation or inversion
- control or design
- mapping or transformation

(this list is not exclusive; can have combinations . . .)

Estimation or inversion

$$y = Ax$$

x has n components

- y_i is i th measurement or sensor reading (which we know)
- x_j is j th parameter to be estimated or determined
- a_{ij} is sensitivity of i th sensor to j th parameter

sample problems:

- find x , given y
what if there are infinitely many x's?
- find all x 's that result in y (i.e., all x 's consistent with measurements)
- if there is no x such that $y = Ax$, find x s.t. $y \approx Ax$ (i.e., if the sensor readings are inconsistent, find x which is almost consistent)

the third one is the most common of these problems
sensors aren't perfect

Control or design

$$y = Ax$$

- x is vector of design parameters or inputs (which we can choose)
- y is vector of results, or outcomes
- A describes how input choices affect results

sample problems:

- find x so that $y = y_{\text{des}}$
 - even though it has the same equation as the estimation problem we approach the two problems differently, use different tools
 - errors/noise enters differently into the problem
 - really different problems
- find all x 's that result in $y = y_{\text{des}}$ (*i.e.*, find all designs that meet specifications)
- among x 's that satisfy $y = y_{\text{des}}$, find a small one (*i.e.*, find a small or efficient x that meets specifications)

Mapping or transformation

- x is mapped or transformed to y by linear function $y = Ax$

sample problems:

- determine if there is an x that maps to a given y
- (if possible) find *an* x that maps to y
- find *all* x 's that map to a given y
- if there is only one x that maps to y , find it (*i.e.*, decode or undo the mapping)

Matrix multiplication as mixture of columns

write $A \in \mathbf{R}^{m \times n}$ in terms of its columns:

has n columns (each column has m rows)

$$A = [\begin{array}{cccc} a_1 & a_2 & \cdots & a_n \end{array}]$$

where $a_j \in \mathbf{R}^m$

then $y = Ax$ can be written as

$$y = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

— think of x as acting on a 's instead of A acting on x

(x_j 's are scalars, a_j 's are m -vectors)

- y is a (linear) combination or mixture of the columns of A
- coefficients of x give coefficients of mixture

an important example: $x = e_j$, the j th *unit vector*

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

then $Ae_j = a_j$, the j th column of A
(e_j corresponds to a pure mixture, giving only column j)

Matrix multiplication as inner product with rows

write A in terms of its rows:

$$A = \begin{bmatrix} \tilde{a}_1^T \\ \tilde{a}_2^T \\ \vdots \\ \tilde{a}_m^T \end{bmatrix}$$

— A written in terms of its rows

where $\tilde{a}_i \in \mathbf{R}^n$

— i th row has n columns

then $y = Ax$ can be written as

$$y = \begin{bmatrix} \tilde{a}_1^T x \\ \tilde{a}_2^T x \\ \vdots \\ \tilde{a}_m^T x \end{bmatrix}$$

thus $y_i = \langle \tilde{a}_i, x \rangle$, i.e., y_i is inner product of i th row of A with x

- right way of thinking when it comes to estimation problems
- each of the rows of A is a sensor (for estimation problems)

geometric interpretation:

$y_i = \tilde{a}_i^T x = \alpha$ is a hyperplane in \mathbb{R}^n (normal to \tilde{a}_i)

— the ith sensor tells us that x has to satisfy this equation

— each a_i generates a set of hyperplanes

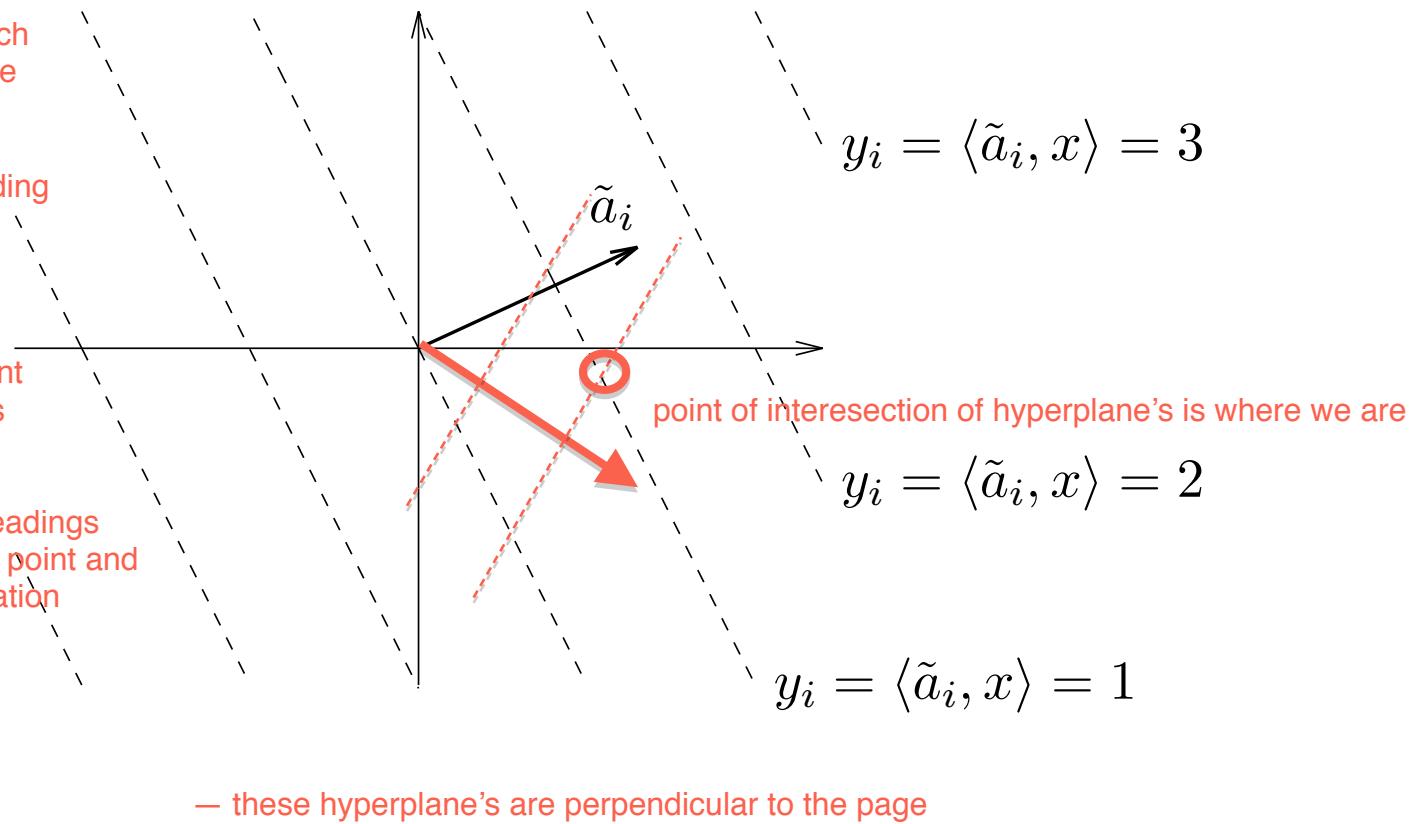
$$y_i = \langle \tilde{a}_i, x \rangle = 0$$

— this equation tells us which one of these hyperplanes we are on

— we know what x is by finding the place where all of the hyperplanes intersect

— we need to have enough sensors to find a unique point where all of the hyperplanes intersect

— if we have inconsistent readings then we don't have a unique point and where the problem of estimation comes in



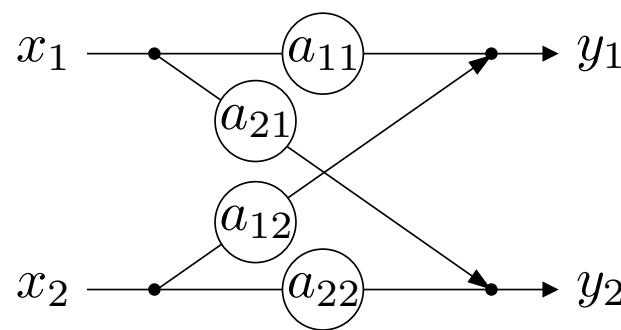
Block diagram representation

$y = Ax$ can be represented by a *signal flow graph* or *block diagram*

e.g. for $m = n = 2$, we represent

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

as



—used to be more useful when we tried to implement matrix multiplication as a circuit

- a_{ij} is the gain along the path from j th input to i th output
- (by not drawing paths with zero gain) shows sparsity structure of A (e.g., diagonal, block upper triangular, arrow . . .)

example: block upper triangular, *i.e.*,

—block means that each of the A's is itself a matrix

—So A_{11} could a 2×2 matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where $A_{11} \in \mathbf{R}^{m_1 \times n_1}$, $A_{12} \in \mathbf{R}^{m_1 \times n_2}$, $A_{21} \in \mathbf{R}^{m_2 \times n_1}$, $A_{22} \in \mathbf{R}^{m_2 \times n_2}$

partition x and y conformably as

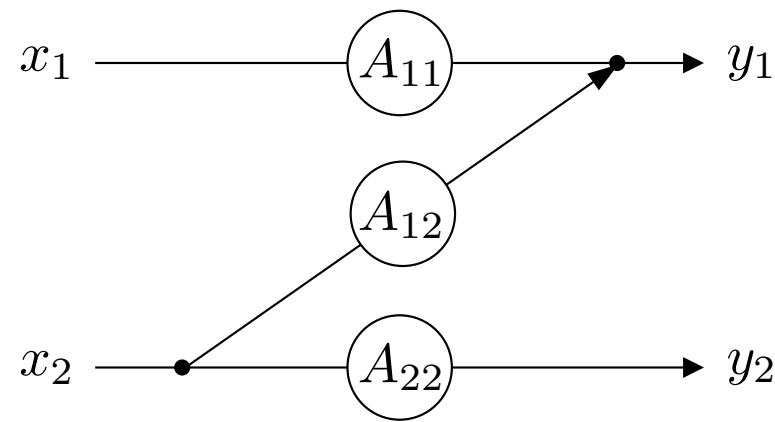
$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$(x_1 \in \mathbf{R}^{n_1}, x_2 \in \mathbf{R}^{n_2}, y_1 \in \mathbf{R}^{m_1}, y_2 \in \mathbf{R}^{m_2})$ so

$$y_1 = A_{11}x_1 + A_{12}x_2, \quad y_2 = A_{22}x_2,$$

i.e., y_2 doesn't depend on x_1

block diagram:



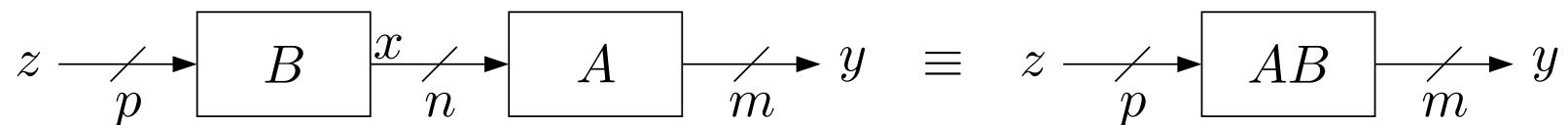
. . . no path from x_1 to y_2 , so y_2 doesn't depend on x_1

Matrix multiplication as composition

for $A \in \mathbf{R}^{m \times n}$ and $B \in \mathbf{R}^{n \times p}$, $C = AB \in \mathbf{R}^{m \times p}$ where

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

composition interpretation: $y = Cz$ represents composition of $y = Ax$ and $x = Bz$



(note that B is on left in block diagram)

Column and row interpretations

can write product $C = AB$ as

$$C = [\ c_1 \cdots c_p \] = AB = [\ Ab_1 \cdots Ab_p \]$$

i.e., i th column of C is A acting on i th column of B

similarly we can write

$$C = \begin{bmatrix} \tilde{c}_1^T \\ \vdots \\ \tilde{c}_m^T \end{bmatrix} = AB = \begin{bmatrix} \tilde{a}_1^T B \\ \vdots \\ \tilde{a}_m^T B \end{bmatrix}$$

i.e., i th row of C is i th row of A acting (on left) on B

Inner product interpretation

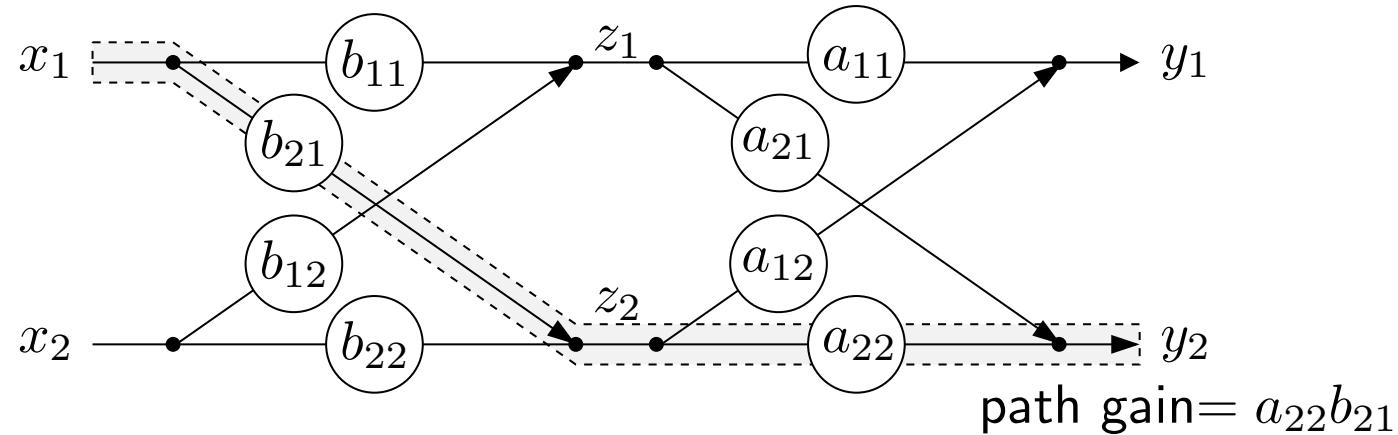
inner product interpretation:

$$c_{ij} = \tilde{a}_i^T b_j = \langle \tilde{a}_i, b_j \rangle$$

i.e., entries of C are inner products of rows of A and columns of B

- $c_{ij} = 0$ means i th row of A is orthogonal to j th column of B
- **Gram matrix** of vectors f_1, \dots, f_n defined as $G_{ij} = f_i^T f_j$
(gives inner product of each vector with the others)
- $G = [f_1 \ \cdots \ f_n]^T [f_1 \ \cdots \ f_n]$

Matrix multiplication interpretation via paths



- $a_{ik}b_{kj}$ is gain of path from input j to output i via k
- c_{ij} is sum of gains over *all* paths from input j to output i

Lecture 3

Linear algebra review

- vector space, subspaces
- independence, basis, dimension
- range, nullspace, rank
- change of coordinates
- norm, angle, inner product

Vector spaces

a *vector space* or *linear space* (over the reals) consists of

- a set \mathcal{V}
- a vector sum $+ : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$
- a scalar multiplication : $\mathbf{R} \times \mathcal{V} \rightarrow \mathcal{V}$
- a distinguished element $0 \in \mathcal{V}$

which satisfy a list of properties

- $x + y = y + x, \quad \forall x, y \in \mathcal{V} \quad (+ \text{ is commutative})$
- $(x + y) + z = x + (y + z), \quad \forall x, y, z \in \mathcal{V} \quad (+ \text{ is associative})$
- $0 + x = x, \quad \forall x \in \mathcal{V} \quad (0 \text{ is additive identity})$
- $\forall x \in \mathcal{V} \quad \exists(-x) \in \mathcal{V} \text{ s.t. } x + (-x) = 0 \quad (\text{existence of additive inverse})$
- $(\alpha\beta)x = \alpha(\beta x), \quad \forall \alpha, \beta \in \mathbf{R} \quad \forall x \in \mathcal{V} \quad (\text{scalar mult. is associative})$
- $\alpha(x + y) = \alpha x + \alpha y, \quad \forall \alpha \in \mathbf{R} \quad \forall x, y \in \mathcal{V} \quad (\text{right distributive rule})$
this is equality in Right
- $(\alpha + \beta)x = \alpha x + \beta x, \quad \forall \alpha, \beta \in \mathbf{R} \quad \forall x \in \mathcal{V} \quad (\text{left distributive rule})$
this is a plus in R
this plus is a plus of vectors
- $1x = x, \quad \forall x \in \mathcal{V}$

Examples

- $\mathcal{V}_1 = \mathbf{R}^n$, with standard (componentwise) vector addition and scalar multiplication
- $\mathcal{V}_2 = \{0\}$ (where $0 \in \mathbf{R}^n$)
- $\mathcal{V}_3 = \text{span}(v_1, v_2, \dots, v_k)$ where

$$\text{span}(v_1, v_2, \dots, v_k) = \{\alpha_1 v_1 + \dots + \alpha_k v_k \mid \alpha_i \in \mathbf{R}\}$$

span is all possible linear combinations

and $v_1, \dots, v_k \in \mathbf{R}^n$

Subspaces

- a *subspace* of a vector space is a *subset* of a vector space which is itself a vector space
- roughly speaking, a subspace is closed under vector addition and scalar multiplication
 - subspace must go through the origin so that the scalar multiplication by 0 exists
- examples $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ above are subspaces of \mathbf{R}^n

Vector spaces of functions

- $\mathcal{V}_4 = \{x : \mathbf{R}_+ \rightarrow \mathbf{R}^n \mid x \text{ is differentiable}\}$, where vector sum is sum of functions:
$$(x + z)(t) = x(t) + z(t)$$
plus in vectors of V4 plus in vectors in Rⁿ
this is a vector in V4 (x+z)

and scalar multiplication is defined by

$$(\alpha x)(t) = \alpha x(t)$$

(a *point* in \mathcal{V}_4 is a *trajectory* in \mathbf{R}^n)

- $\mathcal{V}_5 = \{x \in \mathcal{V}_4 \mid \dot{x} = Ax\}$
(*points* in \mathcal{V}_5 are *trajectories* of the linear system $\dot{x} = Ax$)
- \mathcal{V}_5 is a subspace of \mathcal{V}_4
— not subset but superspace

Independent set of vectors

a set of vectors $\{v_1, v_2, \dots, v_k\}$ is *independent* if

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0 \implies \alpha_1 = \alpha_2 = \cdots = 0$$

independence is an attribute of a set of vectors, not of vectors

some equivalent conditions:

- coefficients of $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$ are uniquely determined, i.e.,

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_k v_k$$

implies $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \dots, \alpha_k = \beta_k$

- no vector v_i can be expressed as a linear combination of the other vectors $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_k$
 - this comes directly from the definition of independence

Basis and dimension

basis is an attribute of a set of vectors

set of vectors $\{v_1, v_2, \dots, v_k\}$ is a *basis* for a vector space \mathcal{V} if

- v_1, v_2, \dots, v_k span \mathcal{V} , i.e., $\mathcal{V} = \text{span}(v_1, v_2, \dots, v_k)$
- $\{v_1, v_2, \dots, v_k\}$ is independent

equivalent: every $v \in \mathcal{V}$ can be uniquely expressed as

$$v = \alpha_1 v_1 + \cdots + \alpha_k v_k$$

fact: for a given vector space \mathcal{V} , the number of vectors in any basis is the same

a vector space can have multiple basis, but they each have to have the same number of elements (same dimension)

number of vectors in any basis is called the *dimension* of \mathcal{V} , denoted $\dim \mathcal{V}$
(we assign $\dim \{0\} = 0$, and $\dim \mathcal{V} = \infty$ if there is no basis)

Nullspace of a matrix

the *nullspace* of $A \in \mathbf{R}^{m \times n}$ is defined as

$$\mathcal{N}(A) = \{ x \in \mathbf{R}^n \mid Ax = 0 \} \text{ this is a subspace of } \mathbf{R}^n$$

— columns of A being linearly independent, makes the nullspace of A = 0

- $\mathcal{N}(A)$ is set of vectors mapped to zero by $y = Ax$
- $\mathcal{N}(A)$ is set of vectors orthogonal to all rows of A

$\mathcal{N}(A)$ gives *ambiguity* in x given $y = Ax$:

- if $y = Ax$ and $z \in \mathcal{N}(A)$, then $y = A(x + z)$ because $A^*z = 0$
- conversely, if $y = Ax$ and $y = A\tilde{x}$, then $\tilde{x} = x + z$ for some $z \in \mathcal{N}(A)$

Zero nullspace

A is called *one-to-one* if 0 is the only element of its nullspace:

$$\mathcal{N}(A) = \{0\} \iff$$

- x can always be uniquely determined from $y = Ax$
(*i.e.*, the linear transformation $y = Ax$ doesn't 'lose' information)
- mapping from x to Ax is one-to-one: different x 's map to different y 's
- columns of A are independent (hence, a basis for their span)
Ax=0
— the only way to make
this work is to have each
of the coefficients = 0
. A is independent
- A has a *left inverse*, *i.e.*, there is a matrix $B \in \mathbf{R}^{n \times m}$ s.t. $BA = I$
I is dimension nx
- $\det(A^T A) \neq 0$
for the last two bullet points look at the lecture video for explanation

(we'll establish these later)

Interpretations of nullspace

listen to the video for this slide too

suppose $z \in \mathcal{N}(A)$

$y = Ax$ represents **measurement** of x

- z is undetectable from sensors — get zero sensor readings
- x and $x + z$ are indistinguishable from sensors: $Ax = A(x + z)$

$\mathcal{N}(A)$ characterizes *ambiguity* in x from measurement $y = Ax$

$y = Ax$ represents **output** resulting from input x

- z is an input with no result
- x and $x + z$ have same result

$\mathcal{N}(A)$ characterizes *freedom of input choice* for given result

Range of a matrix

the *range* of $A \in \mathbf{R}^{m \times n}$ is defined as

$$\mathcal{R}(A) = \{Ax \mid x \in \mathbf{R}^n\} \subseteq \mathbf{R}^m \text{ (output space)}$$

$\mathcal{R}(A)$ can be interpreted as

- the set of vectors that can be ‘hit’ by linear mapping $y = Ax$
- the span of columns of A
- the set of vectors y for which $Ax = y$ has a solution

the set of vectors which can be generated using A

Onto matrices

A is called *onto* if $\mathcal{R}(A) = \mathbf{R}^m \iff$

- $Ax = y$ can be solved in x for any y
- columns of A span \mathbf{R}^m
- A has a *right inverse*, i.e., there is a matrix $B \in \mathbf{R}^{n \times m}$ s.t. $AB = I$
- rows of A are independent
- $\mathcal{N}(A^T) = \{0\}$
- $\det(AA^T) \neq 0$

(some of these are not obvious; we'll establish them later)

Interpretations of range

suppose $v \in \mathcal{R}(A)$, $w \notin \mathcal{R}(A)$

$y = Ax$ represents **measurement** of x

- $y = v$ is a *possible* or *consistent* sensor signal Y=Ax works and we can generate v
- $y = w$ is *impossible* or *inconsistent*; sensors have failed or model is wrong Y=Ax doesnt work anymore and we cannot generate w

$y = Ax$ represents **output** resulting from input x

- v is a possible result or output listen to the example on this slide
- w cannot be a result or output

$\mathcal{R}(A)$ characterizes the *possible results* or *achievable outputs*

Inverse

$A \in \mathbf{R}^{n \times n}$ is *invertible* or *nonsingular* if $\det A \neq 0$
square matrix

equivalent conditions:

- columns of A are a basis for \mathbf{R}^n
- rows of A are a basis for \mathbf{R}^n
- $y = Ax$ has a unique solution x for every $y \in \mathbf{R}^n$
- A has a (left and right) inverse denoted $A^{-1} \in \mathbf{R}^{n \times n}$, with
 $AA^{-1} = A^{-1}A = I$
BA=I
AB=I
- $\mathcal{N}(A) = \{0\}$
- $\mathcal{R}(A) = \mathbf{R}^n$
- $\det A^T A = \det AA^T \neq 0$
det(A^T)*det(A) =/ 0
— because $\det(A)$ is not 0 (shown above)

Interpretations of inverse

suppose $A \in \mathbf{R}^{n \times n}$ has inverse $B = A^{-1}$

- mapping associated with B undoes mapping associated with A (applied either before or after!)
- $x = By$ is a perfect (pre- or post-) *equalizer* for the *channel* $y = Ax$
- $x = By$ is unique solution of $Ax = y$

Dual basis interpretation

watch this slide

- let a_i be columns of A , and \tilde{b}_i^T be rows of $B = A^{-1}$

- from $y = x_1a_1 + \dots + x_na_n$ and $x_i = \tilde{b}_i^Ty$, we get

$$\begin{aligned} b_i^T a_j &= 0 \quad i \neq j \\ &= 1 \quad i=j \end{aligned}$$

$$y = \sum_{i=1}^n (\tilde{b}_i^T y) a_i$$

thus, inner product with *rows of inverse matrix* gives the coefficients in the *expansion of a vector in the columns of the matrix*

- $\{\tilde{b}_1, \dots, \tilde{b}_n\}$ and $\{a_1, \dots, a_n\}$ are called *dual bases*

Rank of a matrix

we define the *rank* of $A \in \mathbf{R}^{m \times n}$ as

$$\mathbf{rank}(A) = \dim \mathcal{R}(A)$$

(nontrivial) facts:

- $\mathbf{rank}(A) = \mathbf{rank}(A^T)$
- $\mathbf{rank}(A)$ is maximum number of independent columns (or rows) of A
hence $\mathbf{rank}(A) \leq \min(m, n)$ rank cant be bigger than height or width of matrix
- $\mathbf{rank}(A) + \dim \mathcal{N}(A) = n$

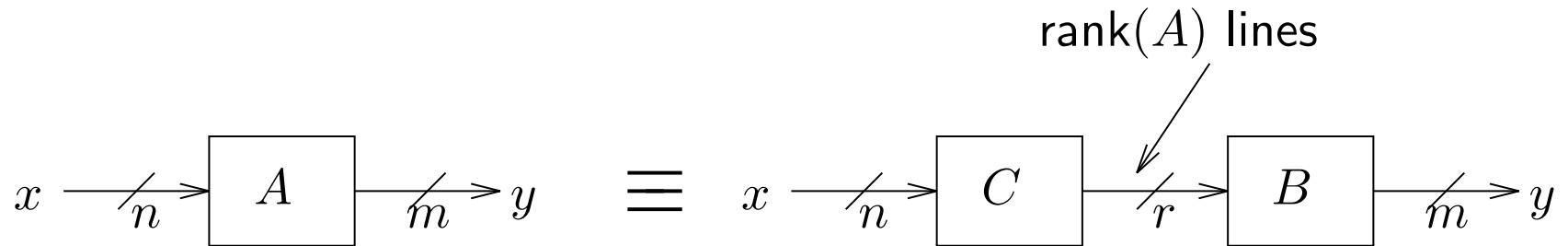
Conservation of dimension

interpretation of $\text{rank}(A) + \dim \mathcal{N}(A) = n$:

- $\text{rank}(A)$ is dimension of set ‘hit’ by the mapping $y = Ax$
- $\dim \mathcal{N}(A)$ is dimension of set of x ‘crushed’ to zero by $y = Ax$
- ‘conservation of dimension’: each dimension of input is either crushed to zero or ends up in output
 - Ex:
If you have ten knobs affecting an output. If you have rank 8, then your output will be only be 8 dimensional (as opposed to the expected 10).
This means that the dimension of the null space is 2. So all the knobs work, but if you were to stop using the 2 knobs in the null space, you would be
- roughly speaking:
 - n is number of degrees of freedom in input x
 - $\dim \mathcal{N}(A)$ is number of degrees of freedom lost in the mapping from x to $y = Ax$
 - $\text{rank}(A)$ is number of degrees of freedom in output y

'Coding' interpretation of rank

- rank of product: $\text{rank}(BC) \leq \min\{\text{rank}(B), \text{rank}(C)\}$
- hence if $A = BC$ with $B \in \mathbf{R}^{m \times r}$, $C \in \mathbf{R}^{r \times n}$, then $\text{rank}(A) \leq r$
as well as $\text{rank}(A) \leq m$
and $\text{rank}(A) \leq n$
- conversely: if $\text{rank}(A) = r$ then $A \in \mathbf{R}^{m \times n}$ can be factored as $A = BC$ with $B \in \mathbf{R}^{m \times r}$, $C \in \mathbf{R}^{r \times n}$:



- $\text{rank}(A) = r$ is minimum size of vector needed to faithfully reconstruct y from x

Application: fast matrix-vector multiplication

[view this slide in video](#)

- need to compute matrix-vector product $y = Ax$, $A \in \mathbf{R}^{m \times n}$
- A has known factorization $A = BC$, $B \in \mathbf{R}^{m \times r}$
- computing $y = Ax$ directly: mn operations
- computing $y = Ax$ as $y = B(Cx)$ (compute $z = Cx$ first, then $y = Bz$): $rn + mr = (m + n)r$ operations
- savings can be considerable if $r \ll \min\{m, n\}$

Full rank matrices

watch this slide

for $A \in \mathbf{R}^{m \times n}$ we always have $\text{rank}(A) \leq \min(m, n)$

we say A is *full rank* if $\text{rank}(A) = \min(m, n)$

- for **square** matrices, full rank means nonsingular
- for **skinny** matrices ($m \geq n$), full rank means columns are independent
- for **fat** matrices ($m \leq n$), full rank means rows are independent

Change of coordinates

watch all of these slides

'standard' basis vectors in \mathbf{R}^n : (e_1, e_2, \dots, e_n) where

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

(1 in i th component)

obviously we have

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

x_i are called the coordinates of x (in the standard basis)

if (t_1, t_2, \dots, t_n) is another basis for \mathbf{R}^n , we have

$$x = \tilde{x}_1 t_1 + \tilde{x}_2 t_2 + \cdots + \tilde{x}_n t_n \stackrel{=Tx}{\sim}$$

where \tilde{x}_i are the coordinates of x in the basis (t_1, t_2, \dots, t_n)

define $T = [\begin{array}{cccc} t_1 & t_2 & \cdots & t_n \end{array}]$ so $x = T\tilde{x}$, hence

$$\tilde{x} = T^{-1}x$$

(T is invertible since t_i are a basis)

T^{-1} transforms (standard basis) coordinates of x into t_i -coordinates

inner product i th row of T^{-1} with x extracts t_i -coordinate of x

consider linear transformation $y = Ax$, $A \in \mathbf{R}^{n \times n}$

express y and x in terms of t_1, t_2, \dots, t_n :

$$x = T\tilde{x}, \quad y = T\tilde{y}$$

so

$$\tilde{y} = (T^{-1}AT)\tilde{x}$$

- $A \rightarrow T^{-1}AT$ is called *similarity transformation*
- similarity transformation by T expresses linear transformation $y = Ax$ in coordinates t_1, t_2, \dots, t_n

(Euclidean) norm

for $x \in \mathbf{R}^n$ we define the (Euclidean) norm as

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{x^T x}$$

$\|x\|$ measures length of vector (from origin)

important properties:

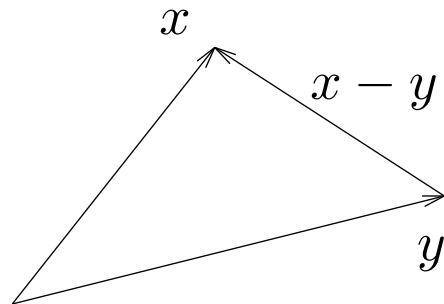
- $\|\alpha x\| = |\alpha| \|x\|$ (homogeneity)
- $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)
- $\|x\| \geq 0$ (nonnegativity)
- $\|x\| = 0 \iff x = 0$ (definiteness)

RMS value and (Euclidean) distance

root-mean-square (RMS) value of vector $x \in \mathbf{R}^n$:

$$\mathbf{rms}(x) = \left(\frac{1}{n} \sum_{i=1}^n x_i^2 \right)^{1/2} = \frac{\|x\|}{\sqrt{n}}$$

norm defines distance between vectors: $\mathbf{dist}(x, y) = \|x - y\|$



Inner product

$$\langle x, y \rangle := x_1y_1 + x_2y_2 + \cdots + x_ny_n = x^T y$$

important properties:

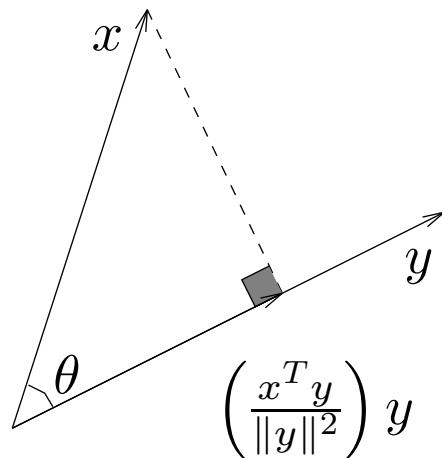
- $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- $\langle x, y \rangle = \langle y, x \rangle$
- $\langle x, x \rangle \geq 0$
- $\langle x, x \rangle = 0 \iff x = 0$

$f(y) = \langle x, y \rangle$ is linear function : $\mathbf{R}^n \rightarrow \mathbf{R}$, with linear map defined by row vector x^T

Cauchy-Schwarz inequality and angle between vectors

- for any $x, y \in \mathbf{R}^n$, $|x^T y| \leq \|x\| \|y\|$
- (unsigned) angle between vectors in \mathbf{R}^n defined as

$$\theta = \angle(x, y) = \cos^{-1} \frac{x^T y}{\|x\| \|y\|}$$



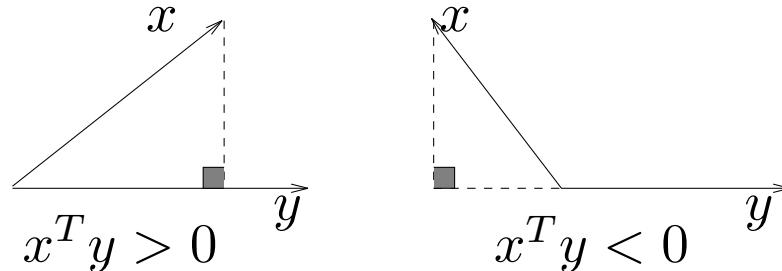
thus $x^T y = \|x\| \|y\| \cos \theta$

special cases:

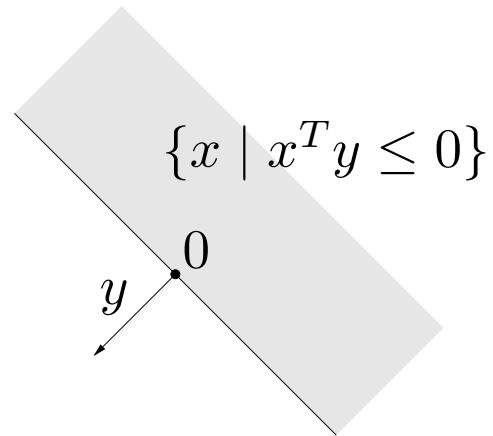
- x and y are *aligned*: $\theta = 0$; $x^T y = \|x\| \|y\|$;
(if $x \neq 0$) $y = \alpha x$ for some $\alpha \geq 0$
- x and y are *opposed*: $\theta = \pi$; $x^T y = -\|x\| \|y\|$
(if $x \neq 0$) $y = -\alpha x$ for some $\alpha \geq 0$
- x and y are *orthogonal*: $\theta = \pi/2$ or $-\pi/2$; $x^T y = 0$
denoted $x \perp y$

interpretation of $x^T y > 0$ and $x^T y < 0$:

- $x^T y > 0$ means $\angle(x, y)$ is acute
- $x^T y < 0$ means $\angle(x, y)$ is obtuse



$\{x \mid x^T y \leq 0\}$ defines a *halfspace* with outward normal vector y , and boundary passing through 0



Lecture 4

Orthonormal sets of vectors and QR factorization

- orthonormal set of vectors
- Gram-Schmidt procedure, QR factorization
- orthogonal decomposition induced by a matrix

Orthonormal set of vectors

set of vectors $\{u_1, \dots, u_k\} \subset \mathbf{R}^n$ is

- *normalized* if $\|u_i\| = 1$, $i = 1, \dots, k$
(u_i are called *unit vectors* or *direction vectors*)
- *orthogonal* if $u_i \perp u_j$ for $i \neq j$
- *orthonormal* if both

slang: we say ‘ u_1, \dots, u_k are orthonormal vectors’ but orthonormality (like independence) is a property of a *set* of vectors, not vectors individually

in terms of $U = [u_1 \ \cdots \ u_k]$, orthonormal means

$$U^T U = I_k$$

- an orthonormal set of vectors is independent
(multiply $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_k u_k = 0$ by u_i^T)
- hence $\{u_1, \dots, u_k\}$ is an orthonormal basis for

$$\text{span}(u_1, \dots, u_k) = \mathcal{R}(U)$$

- **warning:** if $k < n$ then $UU^T \neq I$ (since its rank is at most k)
(more on this matrix later . . .)

Geometric properties

suppose columns of $U = [u_1 \ \cdots \ u_k]$ are orthonormal

if $w = Uz$, then $\|w\| = \|z\|$

- multiplication by U does not change norm
- mapping $w = Uz$ is *isometric*: it preserves distances
- simple derivation using matrices:

$$\|w\|^2 = \|Uz\|^2 = (Uz)^T(Uz) = z^T U^T U z = z^T z = \|z\|^2$$

- *inner products* are also preserved: $\langle Uz, U\tilde{z} \rangle = \langle z, \tilde{z} \rangle$

- if $w = Uz$ and $\tilde{w} = U\tilde{z}$ then

$$\langle w, \tilde{w} \rangle = \langle Uz, U\tilde{z} \rangle = (Uz)^T(U\tilde{z}) = z^T U^T U \tilde{z} = \langle z, \tilde{z} \rangle$$

- norms and inner products preserved, so *angles* are preserved:
 $\angle(Uz, U\tilde{z}) = \angle(z, \tilde{z})$
- thus, multiplication by U preserves inner products, angles, and distances

Orthonormal basis for \mathbf{R}^n

- suppose u_1, \dots, u_n is an orthonormal *basis* for \mathbf{R}^n
- then $U = [u_1 \cdots u_n]$ is called **orthogonal**: it is square and satisfies $U^T U = I$
(you'd think such matrices would be called *orthonormal*, not *orthogonal*)
- it follows that $U^{-1} = U^T$, and hence also $UU^T = I$, i.e.,

$$\sum_{i=1}^n u_i u_i^T = I$$

Expansion in orthonormal basis

suppose U is orthogonal, so $x = UU^T x$, i.e.,

$$x = \sum_{i=1}^n (u_i^T x) u_i$$

- $u_i^T x$ is called the *component* of x in the direction u_i
- $a = U^T x$ *resolves* x into the vector of its u_i components
- $x = Ua$ *reconstitutes* x from its u_i components
- $x = Ua = \sum_{i=1}^n a_i u_i$ is called the (u_i) -*expansion* of x

the identity $I = UU^T = \sum_{i=1}^n u_i u_i^T$ is sometimes written (in physics) as

$$I = \sum_{i=1}^n |u_i\rangle\langle u_i|$$

since

$$x = \sum_{i=1}^n |u_i\rangle\langle u_i|x\rangle$$

(but we won't use this notation)

Geometric interpretation

if U is orthogonal, then transformation $w = Uz$

- preserves *norm* of vectors, i.e., $\|Uz\| = \|z\|$
- preserves *angles* between vectors, i.e., $\angle(Uz, U\tilde{z}) = \angle(z, \tilde{z})$

examples:

- rotations (about some axis)
- reflections (through some plane)

Example: rotation by θ in \mathbf{R}^2 is given by

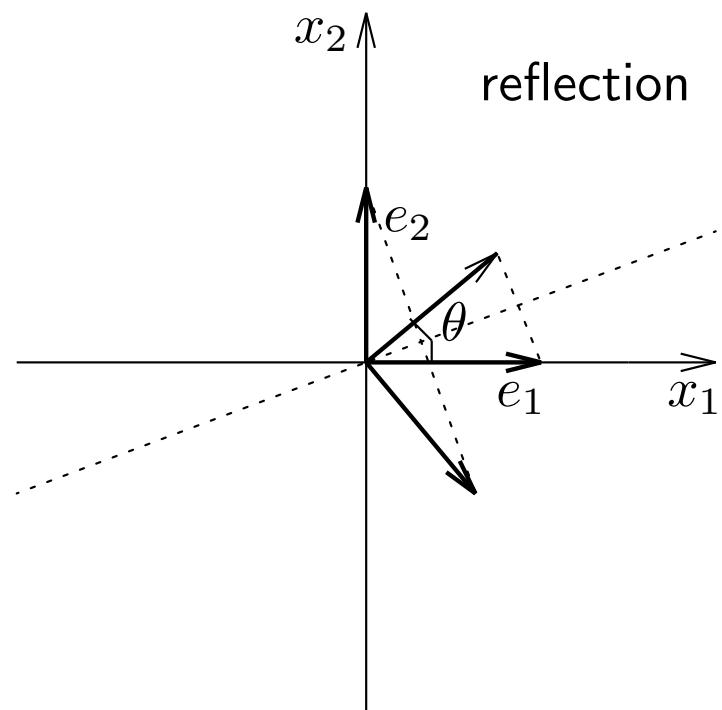
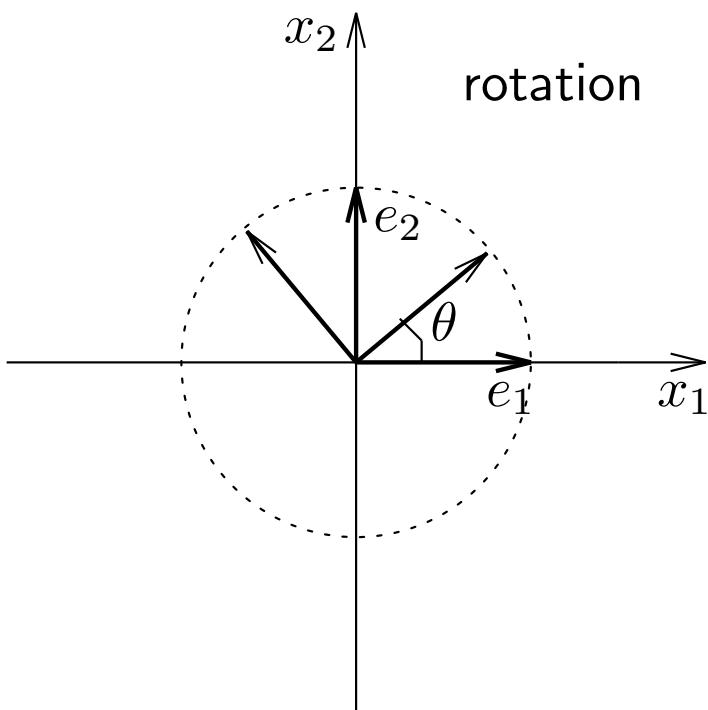
$$y = U_\theta x, \quad U_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

since $e_1 \rightarrow (\cos \theta, \sin \theta)$, $e_2 \rightarrow (-\sin \theta, \cos \theta)$

reflection across line $x_2 = x_1 \tan(\theta/2)$ is given by

$$y = R_\theta x, \quad R_\theta = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}$$

since $e_1 \rightarrow (\cos \theta, \sin \theta)$, $e_2 \rightarrow (\sin \theta, -\cos \theta)$



can check that U_θ and R_θ are orthogonal

Gram-Schmidt procedure

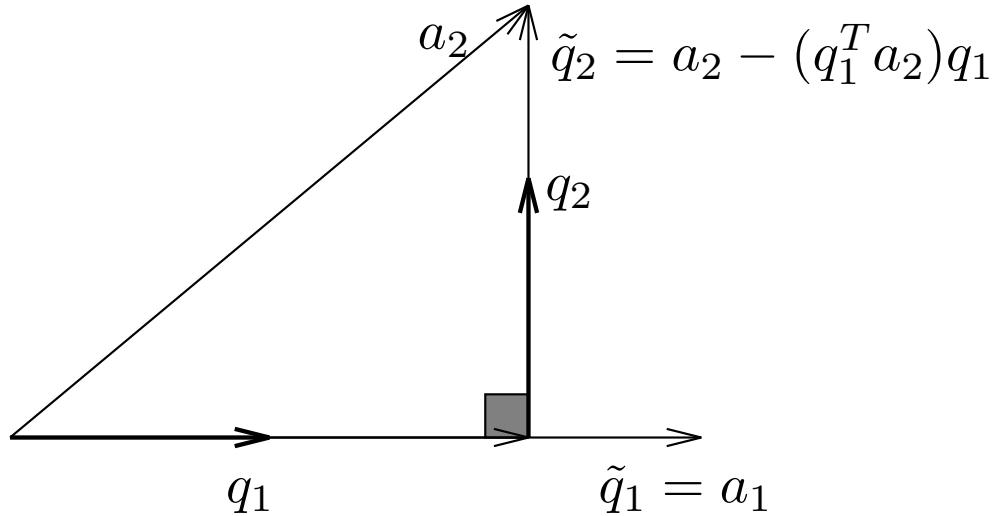
- given independent vectors $a_1, \dots, a_k \in \mathbf{R}^n$, G-S procedure finds orthonormal vectors q_1, \dots, q_k s.t.

$$\text{span}(a_1, \dots, a_r) = \text{span}(q_1, \dots, q_r) \quad \text{for } r \leq k$$

- thus, q_1, \dots, q_r is an orthonormal basis for $\text{span}(a_1, \dots, a_r)$
- rough idea of method: first *orthogonalize* each vector w.r.t. previous ones; then *normalize* result to have norm one

Gram-Schmidt procedure

- step 1a. $\tilde{q}_1 := a_1$
- step 1b. $q_1 := \tilde{q}_1 / \|\tilde{q}_1\|$ (normalize)
- step 2a. $\tilde{q}_2 := a_2 - (q_1^T a_2) q_1$ (remove q_1 component from a_2)
- step 2b. $q_2 := \tilde{q}_2 / \|\tilde{q}_2\|$ (normalize)
- step 3a. $\tilde{q}_3 := a_3 - (q_1^T a_3) q_1 - (q_2^T a_3) q_2$ (remove q_1, q_2 components)
- step 3b. $q_3 := \tilde{q}_3 / \|\tilde{q}_3\|$ (normalize)
- etc.



for $i = 1, 2, \dots, k$ we have

$$\begin{aligned}
 a_i &= (q_1^T a_i) q_1 + (q_2^T a_i) q_2 + \cdots + (q_{i-1}^T a_i) q_{i-1} + \|\tilde{q}_i\| q_i \\
 &= r_{1i} q_1 + r_{2i} q_2 + \cdots + r_{ii} q_i
 \end{aligned}$$

(note that the r_{ij} 's come right out of the G-S procedure, and $r_{ii} \neq 0$)

QR decomposition

written in matrix form: $A = QR$, where $A \in \mathbf{R}^{n \times k}$, $Q \in \mathbf{R}^{n \times k}$, $R \in \mathbf{R}^{k \times k}$:

$$\underbrace{\begin{bmatrix} a_1 & a_2 & \cdots & a_k \end{bmatrix}}_A = \underbrace{\begin{bmatrix} q_1 & q_2 & \cdots & q_k \end{bmatrix}}_Q \underbrace{\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1k} \\ 0 & r_{22} & \cdots & r_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{kk} \end{bmatrix}}_R$$

- $Q^T Q = I_k$, and R is upper triangular & invertible
- called **QR decomposition** (or factorization) of A
- usually computed using a variation on Gram-Schmidt procedure which is less sensitive to numerical (rounding) errors
- columns of Q are orthonormal basis for $\mathcal{R}(A)$

General Gram-Schmidt procedure

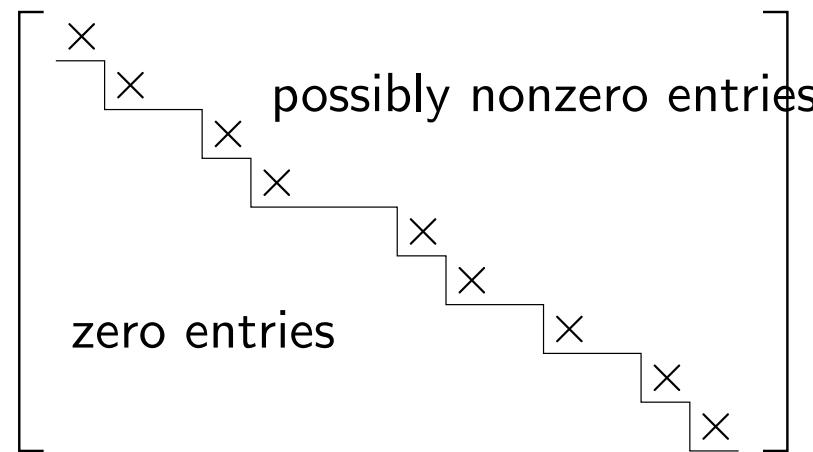
- in basic G-S we assume $a_1, \dots, a_k \in \mathbf{R}^n$ are independent
- if a_1, \dots, a_k are dependent, we find $\tilde{q}_j = 0$ for some j , which means a_j is linearly dependent on a_1, \dots, a_{j-1}
- modified algorithm: when we encounter $\tilde{q}_j = 0$, skip to next vector a_{j+1} and continue:

```
r = 0;  
for i = 1, ..., k  
{  
     $\tilde{a} = a_i - \sum_{j=1}^r q_j q_j^T a_i;$   
    if  $\tilde{a} \neq 0$  {  $r = r + 1$ ;  $q_r = \tilde{a}/\|\tilde{a}\|$ ; }  
}
```

on exit,

- q_1, \dots, q_r is an orthonormal basis for $\mathcal{R}(A)$ (hence $r = \text{Rank}(A)$)
- each a_i is linear combination of previously generated q_j 's

in matrix notation we have $A = QR$ with $Q^T Q = I_r$ and $R \in \mathbf{R}^{r \times k}$ in *upper staircase form*:



‘corner’ entries (shown as \times) are nonzero

can permute columns with \times to front of matrix:

$$A = Q[\tilde{R} \ S]P$$

where:

- $Q^T Q = I_r$
- $\tilde{R} \in \mathbf{R}^{r \times r}$ is upper triangular and invertible
- $P \in \mathbf{R}^{k \times k}$ is a permutation matrix
(which moves forward the columns of a which generated a new q)

Applications

- directly yields orthonormal basis for $\mathcal{R}(A)$
- yields factorization $A = BC$ with $B \in \mathbf{R}^{n \times r}$, $C \in \mathbf{R}^{r \times k}$, $r = \text{Rank}(A)$
- to check if $b \in \text{span}(a_1, \dots, a_k)$: apply Gram-Schmidt to $[a_1 \ \cdots \ a_k \ b]$
- staircase pattern in R shows which columns of A are dependent on previous ones

works incrementally: one G-S procedure yields QR factorizations of $[a_1 \ \cdots \ a_p]$ for $p = 1, \dots, k$:

$$[a_1 \ \cdots \ a_p] = [q_1 \ \cdots \ q_s] R_p$$

where $s = \text{Rank}([a_1 \ \cdots \ a_p])$ and R_p is leading $s \times p$ submatrix of R

‘Full’ QR factorization

with $A = Q_1 R_1$ the QR factorization as above, write

$$A = [Q_1 \quad Q_2] \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$$

where $[Q_1 \ Q_2]$ is orthogonal, i.e., columns of $Q_2 \in \mathbf{R}^{n \times (n-r)}$ are orthonormal, orthogonal to Q_1

to find Q_2 :

- find any matrix \tilde{A} s.t. $[A \ \tilde{A}]$ has rank n (e.g., $\tilde{A} = I$)
- apply general Gram-Schmidt to $[A \ \tilde{A}]$
- Q_1 are orthonormal vectors obtained from columns of A
- Q_2 are orthonormal vectors obtained from extra columns (\tilde{A})

i.e., any set of orthonormal vectors can be extended to an orthonormal basis for \mathbf{R}^n

$\mathcal{R}(Q_1)$ and $\mathcal{R}(Q_2)$ are called *complementary subspaces* since

- they are orthogonal (i.e., every vector in the first subspace is orthogonal to every vector in the second subspace)
- their sum is \mathbf{R}^n (i.e., every vector in \mathbf{R}^n can be expressed as a sum of two vectors, one from each subspace)

this is written

- $\mathcal{R}(Q_1) \stackrel{\perp}{+} \mathcal{R}(Q_2) = \mathbf{R}^n$
- $\mathcal{R}(Q_2) = \mathcal{R}(Q_1)^\perp$ (and $\mathcal{R}(Q_1) = \mathcal{R}(Q_2)^\perp$)
(each subspace is the *orthogonal complement* of the other)

we know $\mathcal{R}(Q_1) = \mathcal{R}(A)$; but what is its orthogonal complement $\mathcal{R}(Q_2)$?

Orthogonal decomposition induced by A

from $A^T = \begin{bmatrix} R_1^T & 0 \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix}$ we see that

$$A^T z = 0 \iff Q_1^T z = 0 \iff z \in \mathcal{R}(Q_2)$$

so $\mathcal{R}(Q_2) = \mathcal{N}(A^T)$

(in fact the columns of Q_2 are an orthonormal basis for $\mathcal{N}(A^T)$)

we conclude: $\mathcal{R}(A)$ and $\mathcal{N}(A^T)$ are *complementary subspaces*:

- $\mathcal{R}(A)^\perp + \mathcal{N}(A^T) = \mathbf{R}^n$ (recall $A \in \mathbf{R}^{n \times k}$)
- $\mathcal{R}(A)^\perp = \mathcal{N}(A^T)$ (and $\mathcal{N}(A^T)^\perp = \mathcal{R}(A)$)
- called *orthogonal decomposition* (of \mathbf{R}^n) induced by $A \in \mathbf{R}^{n \times k}$

- every $y \in \mathbf{R}^n$ can be written uniquely as $y = z + w$, with $z \in \mathcal{R}(A)$, $w \in \mathcal{N}(A^T)$ (we'll soon see what the vector z is . . .)
- can now prove most of the assertions from the linear algebra review lecture
- switching $A \in \mathbf{R}^{n \times k}$ to $A^T \in \mathbf{R}^{k \times n}$ gives decomposition of \mathbf{R}^k :

$$\mathcal{N}(A) \overset{\perp}{+} \mathcal{R}(A^T) = \mathbf{R}^k$$

Lecture 5

Least-squares

- least-squares (approximate) solution of overdetermined equations
- projection and orthogonality principle
- least-squares estimation
- BLUE property

Overdetermined linear equations

consider $y = Ax$ where $A \in \mathbf{R}^{m \times n}$ is (strictly) skinny, i.e., $m > n$

- called *overdetermined* set of linear equations
(more equations than unknowns)
- for most y , cannot solve for x

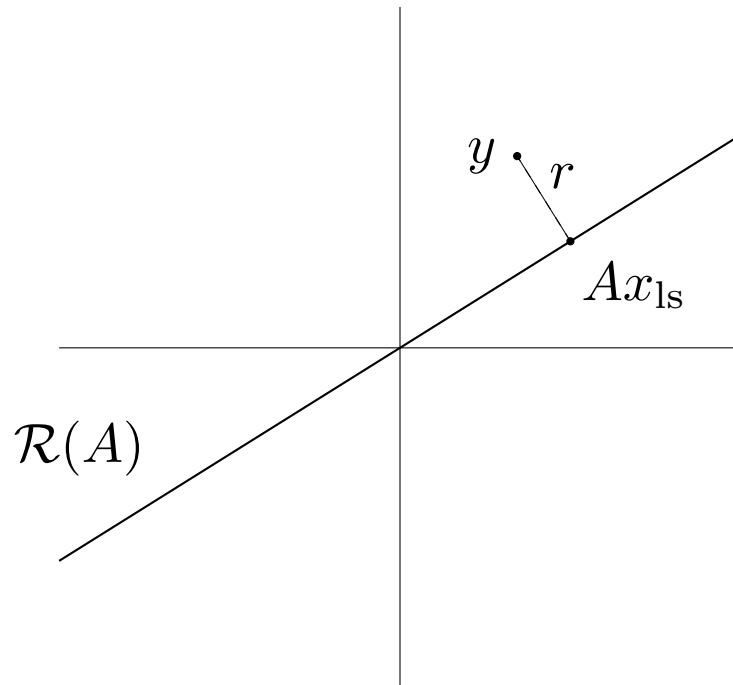
one approach to *approximately* solve $y = Ax$:

- define *residual* or error $r = Ax - y$
- find $x = x_{\text{ls}}$ that minimizes $\|r\|$

x_{ls} called *least-squares* (approximate) solution of $y = Ax$

Geometric interpretation

Ax_{ls} is point in $\mathcal{R}(A)$ closest to y (Ax_{ls} is *projection* of y onto $\mathcal{R}(A)$)



Least-squares (approximate) solution

- assume A is full rank, skinny
- to find x_{ls} , we'll minimize norm of residual squared,

$$\|r\|^2 = x^T A^T A x - 2y^T A x + y^T y$$

- set gradient w.r.t. x to zero:

$$\nabla_x \|r\|^2 = 2A^T A x - 2A^T y = 0$$

- yields the *normal equations*: $A^T A x = A^T y$
- assumptions imply $A^T A$ invertible, so we have

$$x_{\text{ls}} = (A^T A)^{-1} A^T y$$

. . . a very famous formula

- x_{ls} is linear function of y
- $x_{\text{ls}} = A^{-1}y$ if A is square
- x_{ls} solves $y = Ax_{\text{ls}}$ if $y \in \mathcal{R}(A)$
- $A^\dagger = (A^T A)^{-1} A^T$ is called the *pseudo-inverse* of A
- A^\dagger is a *left inverse* of (full rank, skinny) A :

$$A^\dagger A = (A^T A)^{-1} A^T A = I$$

Projection on $\mathcal{R}(A)$

Ax_{ls} is (by definition) the point in $\mathcal{R}(A)$ that is closest to y , i.e., it is the *projection* of y onto $\mathcal{R}(A)$

$$Ax_{\text{ls}} = \mathcal{P}_{\mathcal{R}(A)}(y)$$

- the projection function $\mathcal{P}_{\mathcal{R}(A)}$ is linear, and given by

$$\mathcal{P}_{\mathcal{R}(A)}(y) = Ax_{\text{ls}} = A(A^T A)^{-1} A^T y$$

- $A(A^T A)^{-1} A^T$ is called the *projection matrix* (associated with $\mathcal{R}(A)$)

Orthogonality principle

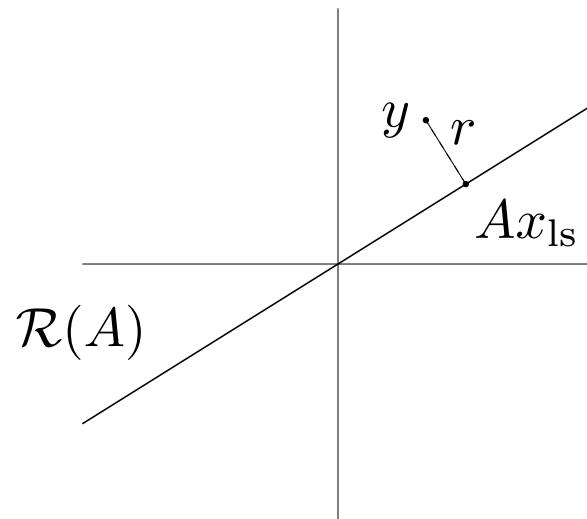
optimal residual

$$r = Ax_{\text{ls}} - y = (A(A^T A)^{-1} A^T - I)y$$

is orthogonal to $\mathcal{R}(A)$:

$$\langle r, Az \rangle = y^T (A(A^T A)^{-1} A^T - I)^T A z = 0$$

for all $z \in \mathbb{R}^n$



Completion of squares

since $r = Ax_{ls} - y \perp A(x - x_{ls})$ for any x , we have

$$\begin{aligned}\|Ax - y\|^2 &= \|(Ax_{ls} - y) + A(x - x_{ls})\|^2 \\ &= \|Ax_{ls} - y\|^2 + \|A(x - x_{ls})\|^2\end{aligned}$$

this shows that for $x \neq x_{ls}$, $\|Ax - y\| > \|Ax_{ls} - y\|$

Least-squares via QR factorization

- $A \in \mathbf{R}^{m \times n}$ skinny, full rank
- factor as $A = QR$ with $Q^T Q = I_n$, $R \in \mathbf{R}^{n \times n}$ upper triangular, invertible
- pseudo-inverse is

$$(A^T A)^{-1} A^T = (R^T Q^T Q R)^{-1} R^T Q^T = R^{-1} Q^T$$

$$\text{so } x_{\text{ls}} = R^{-1} Q^T y$$

- projection on $\mathcal{R}(A)$ given by matrix

$$A(A^T A)^{-1} A^T = A R^{-1} Q^T = Q Q^T$$

Least-squares via full QR factorization

- full QR factorization:

$$A = [Q_1 \ Q_2] \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$$

with $[Q_1 \ Q_2] \in \mathbf{R}^{m \times m}$ orthogonal, $R_1 \in \mathbf{R}^{n \times n}$ upper triangular, invertible

- multiplication by orthogonal matrix doesn't change norm, so

$$\begin{aligned}\|Ax - y\|^2 &= \left\| [Q_1 \ Q_2] \begin{bmatrix} R_1 \\ 0 \end{bmatrix} x - y \right\|^2 \\ &= \left\| [Q_1 \ Q_2]^T [Q_1 \ Q_2] \begin{bmatrix} R_1 \\ 0 \end{bmatrix} x - [Q_1 \ Q_2]^T y \right\|^2\end{aligned}$$

$$\begin{aligned}
&= \left\| \begin{bmatrix} R_1 x - Q_1^T y \\ -Q_2^T y \end{bmatrix} \right\|^2 \\
&= \|R_1 x - Q_1^T y\|^2 + \|Q_2^T y\|^2
\end{aligned}$$

- this is evidently minimized by choice $x_{\text{ls}} = R_1^{-1} Q_1^T y$ (which makes first term zero)
- residual with optimal x is

$$Ax_{\text{ls}} - y = -Q_2 Q_2^T y$$

- $Q_1 Q_1^T$ gives projection onto $\mathcal{R}(A)$
- $Q_2 Q_2^T$ gives projection onto $\mathcal{R}(A)^\perp$

Least-squares estimation

many applications in inversion, estimation, and reconstruction problems have form

$$y = Ax + v$$

- x is what we want to estimate or reconstruct
- y is our sensor measurement(s)
- v is an unknown *noise* or *measurement error* (assumed small)
- i th row of A characterizes i th sensor

least-squares estimation: choose as estimate \hat{x} that minimizes

$$\|A\hat{x} - y\|$$

i.e., deviation between

- what we actually observed (y), and
- what we would observe if $x = \hat{x}$, and there were no noise ($v = 0$)

least-squares estimate is just $\hat{x} = (A^T A)^{-1} A^T y$

BLUE property

linear measurement with noise:

$$y = Ax + v$$

with A full rank, skinny

consider a *linear estimator* of form $\hat{x} = By$

- called *unbiased* if $\hat{x} = x$ whenever $v = 0$
(*i.e.*, no estimation error when there is no noise)

same as $BA = I$, *i.e.*, B is left inverse of A

- estimation error of unbiased linear estimator is

$$x - \hat{x} = x - B(Ax + v) = -Bv$$

obviously, then, we'd like B 'small' (and $BA = I$)

- **fact:** $A^\dagger = (A^T A)^{-1} A^T$ is the *smallest* left inverse of A , in the following sense:

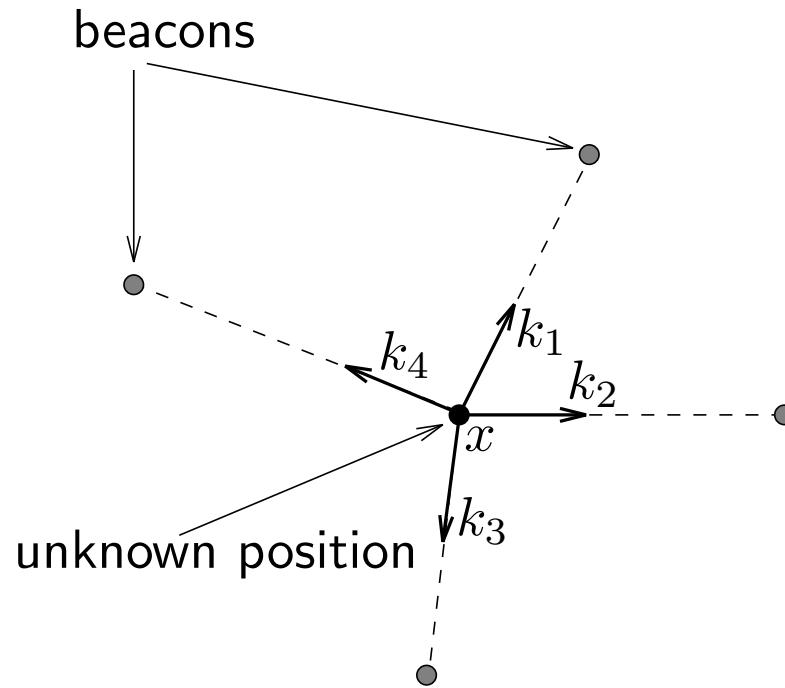
for any B with $BA = I$, we have

$$\sum_{i,j} B_{ij}^2 \geq \sum_{i,j} A_{ij}^{\dagger 2}$$

i.e., least-squares provides the *best linear unbiased estimator* (BLUE)

Navigation from range measurements

navigation using range measurements from *distant* beacons



beacons far from unknown position $x \in \mathbf{R}^2$, so linearization around $x = 0$ (say) nearly exact

ranges $y \in \mathbf{R}^4$ measured, with measurement noise v :

$$y = - \begin{bmatrix} k_1^T \\ k_2^T \\ k_3^T \\ k_4^T \end{bmatrix} x + v$$

where k_i is unit vector from 0 to beacon i

measurement errors are independent, Gaussian, with standard deviation 2
(details not important)

problem: estimate $x \in \mathbf{R}^2$, given $y \in \mathbf{R}^4$

(roughly speaking, a 2:1 measurement redundancy ratio)

actual position is $x = (5.59, 10.58)$;
measurement is $y = (-11.95, -2.84, -9.81, 2.81)$

Just enough measurements method

y_1 and y_2 suffice to find x (when $v = 0$)

compute estimate \hat{x} by inverting top (2×2) half of A :

$$\hat{x} = B_{je}y = \begin{bmatrix} 0 & -1.0 & 0 & 0 \\ -1.12 & 0.5 & 0 & 0 \end{bmatrix} y = \begin{bmatrix} 2.84 \\ 11.9 \end{bmatrix}$$

(norm of error: 3.07)

Least-squares method

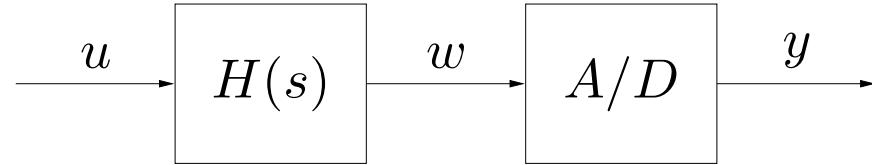
compute estimate \hat{x} by least-squares:

$$\hat{x} = A^\dagger y = \begin{bmatrix} -0.23 & -0.48 & 0.04 & 0.44 \\ -0.47 & -0.02 & -0.51 & -0.18 \end{bmatrix} y = \begin{bmatrix} 4.95 \\ 10.26 \end{bmatrix}$$

(norm of error: 0.72)

- B_{je} and A^\dagger are both left inverses of A
- larger entries in B lead to larger estimation error

Example from overview lecture



- signal u is piecewise constant, period 1 sec, $0 \leq t \leq 10$:

$$u(t) = x_j, \quad j - 1 \leq t < j, \quad j = 1, \dots, 10$$

- filtered by system with impulse response $h(t)$:

$$w(t) = \int_0^t h(t - \tau)u(\tau) d\tau$$

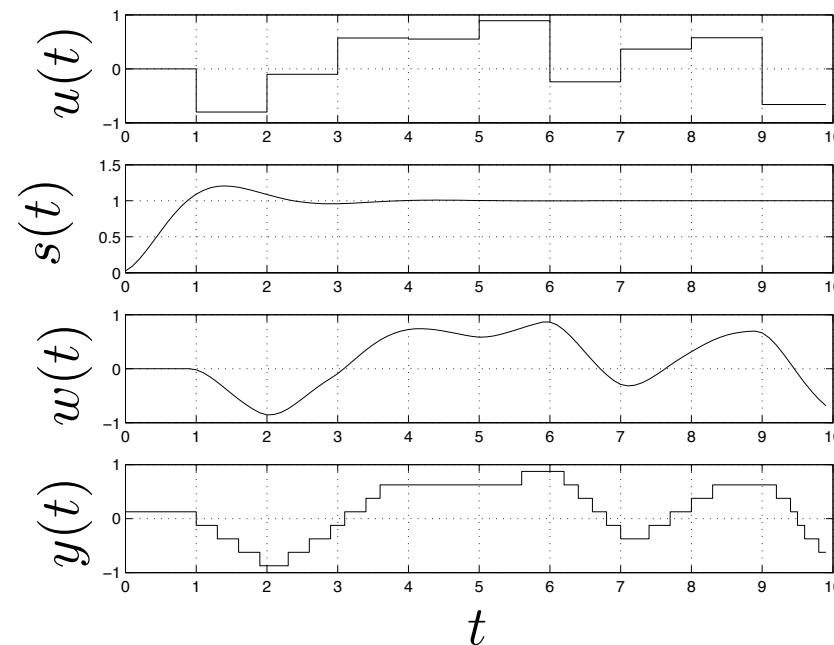
- sample at 10Hz: $\tilde{y}_i = w(0.1i)$, $i = 1, \dots, 100$

- 3-bit quantization: $y_i = Q(\tilde{y}_i)$, $i = 1, \dots, 100$, where Q is 3-bit quantizer characteristic

$$Q(a) = (1/4) (\text{round}(4a + 1/2) - 1/2)$$

- **problem:** estimate $x \in \mathbf{R}^{10}$ given $y \in \mathbf{R}^{100}$

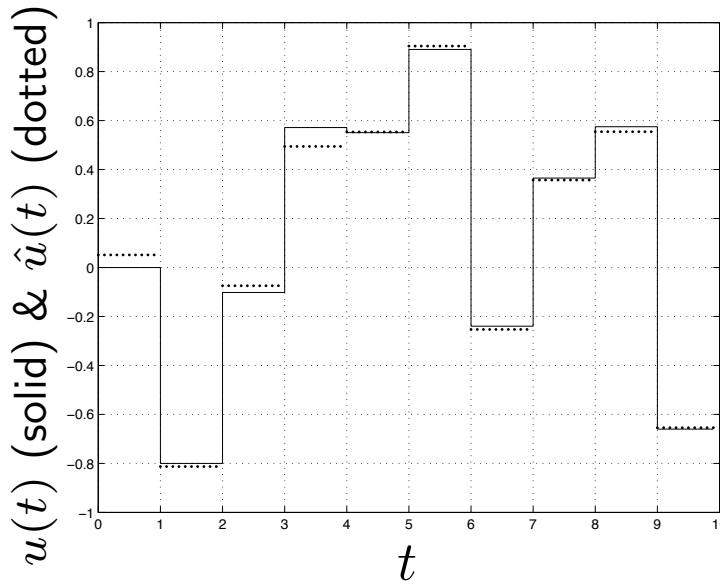
example:



we have $y = Ax + v$, where

- $A \in \mathbf{R}^{100 \times 10}$ is given by $A_{ij} = \int_{j-1}^j h(0.1i - \tau) d\tau$
- $v \in \mathbf{R}^{100}$ is *quantization error*: $v_i = Q(\tilde{y}_i) - \tilde{y}_i$ (so $|v_i| \leq 0.125$)

least-squares estimate: $x_{\text{ls}} = (A^T A)^{-1} A^T y$

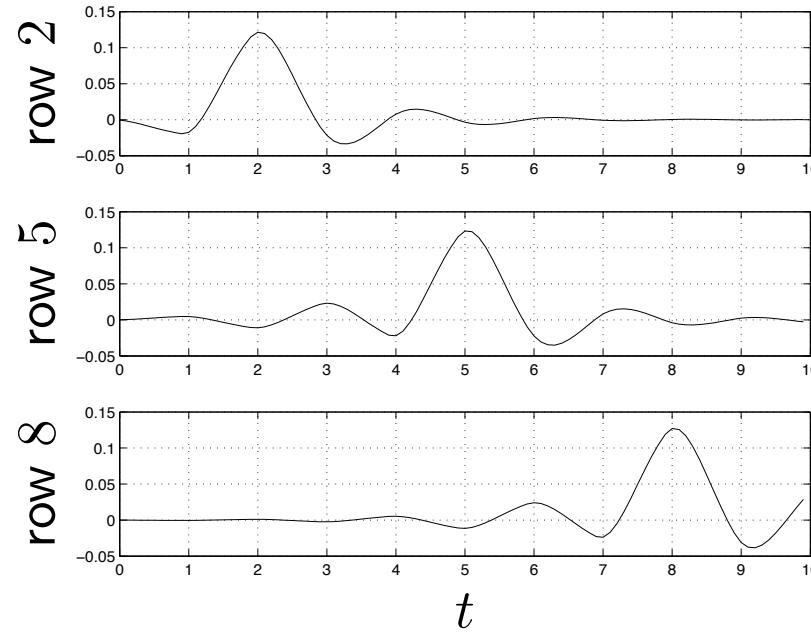


RMS error is $\frac{\|x - x_{ls}\|}{\sqrt{10}} = 0.03$

better than if we had no filtering! (RMS error 0.07)

more on this later . . .

some rows of $B_{ls} = (A^T A)^{-1} A^T$:



- rows show how sampled measurements of y are used to form estimate of x_i for $i = 2, 5, 8$
- to estimate x_5 , which is the original input signal for $4 \leq t < 5$, we mostly use $y(t)$ for $3 \leq t \leq 7$

Lecture 6

Least-squares applications

- least-squares data fitting
- growing sets of regressors
- system identification
- growing sets of measurements and recursive least-squares

Least-squares data fitting

we are given:

- functions $f_1, \dots, f_n : S \rightarrow \mathbf{R}$, called *regressors* or *basis functions*
- *data or measurements* (s_i, g_i) , $i = 1, \dots, m$, where $s_i \in S$ and (usually) $m \gg n$

problem: find coefficients $x_1, \dots, x_n \in \mathbf{R}$ so that

$$x_1 f_1(s_i) + \cdots + x_n f_n(s_i) \approx g_i, \quad i = 1, \dots, m$$

i.e., find linear combination of functions that fits data

least-squares fit: choose x to minimize total square fitting error:

$$\sum_{i=1}^m (x_1 f_1(s_i) + \cdots + x_n f_n(s_i) - g_i)^2$$

- using matrix notation, total square fitting error is $\|Ax - g\|^2$, where $A_{ij} = f_j(s_i)$
- hence, least-squares fit is given by

$$x = (A^T A)^{-1} A^T g$$

(assuming A is skinny, full rank)

- corresponding function is

$$f_{\text{lsfit}}(s) = x_1 f_1(s) + \cdots + x_n f_n(s)$$

- applications:
 - interpolation, extrapolation, smoothing of data
 - developing simple, approximate model of data

Least-squares polynomial fitting

problem: fit polynomial of degree $< n$,

$$p(t) = a_0 + a_1 t + \cdots + a_{n-1} t^{n-1},$$

to data (t_i, y_i) , $i = 1, \dots, m$

- basis functions are $f_j(t) = t^{j-1}$, $j = 1, \dots, n$
- matrix A has form $A_{ij} = t_i^{j-1}$

$$A = \begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots & & & & \vdots \\ 1 & t_m & t_m^2 & \cdots & t_m^{n-1} \end{bmatrix}$$

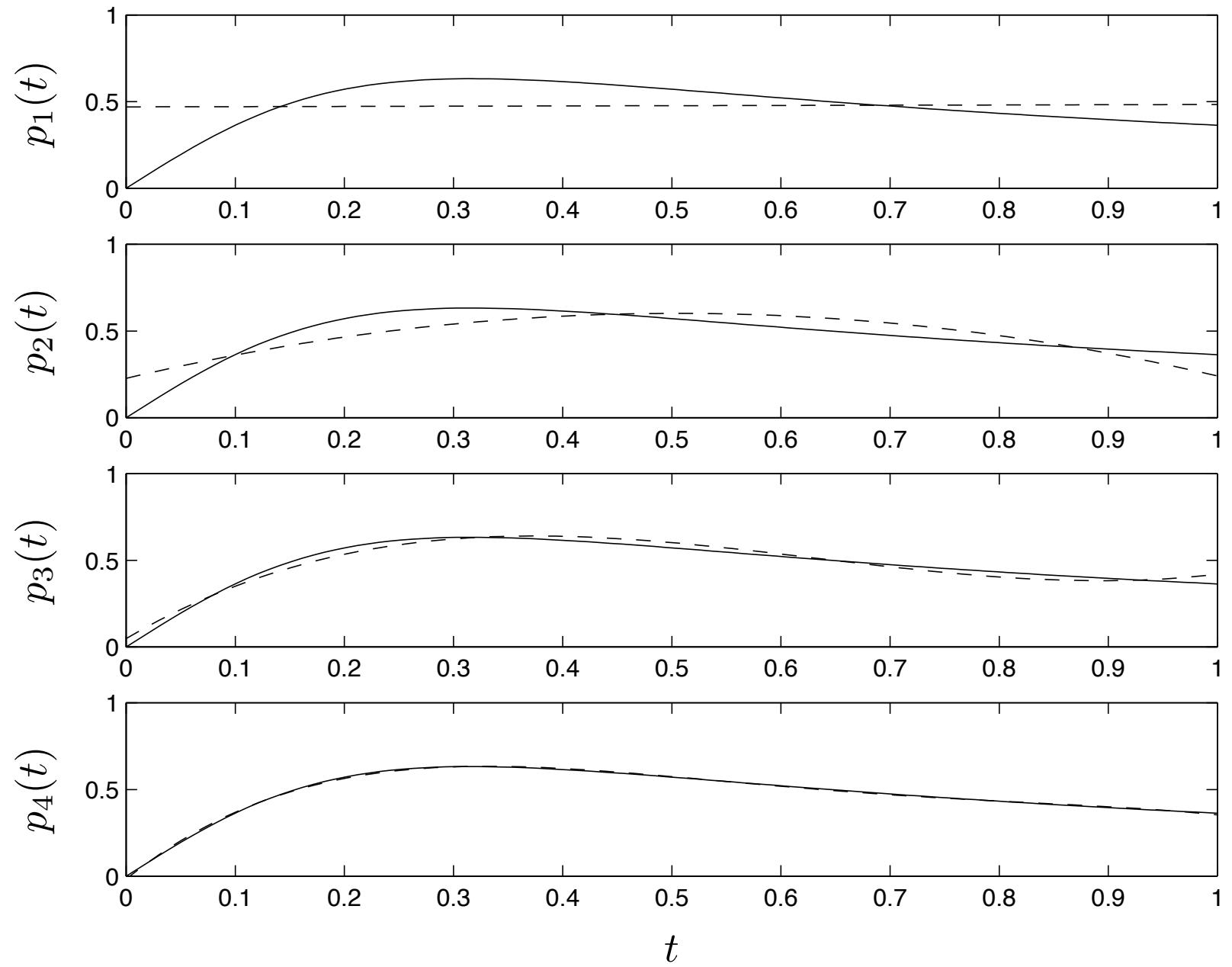
(called a *Vandermonde matrix*)

assuming $t_k \neq t_l$ for $k \neq l$ and $m \geq n$, A is full rank:

- suppose $Aa = 0$
- corresponding polynomial $p(t) = a_0 + \cdots + a_{n-1}t^{n-1}$ vanishes at m points t_1, \dots, t_m
- by fundamental theorem of algebra p can have no more than $n - 1$ zeros, so p is identically zero, and $a = 0$
- columns of A are independent, *i.e.*, A full rank

Example

- fit $g(t) = 4t/(1 + 10t^2)$ with polynomial
- $m = 100$ points between $t = 0$ & $t = 1$
- least-squares fit for degrees 1, 2, 3, 4 have RMS errors .135, .076, .025, .005, respectively



Growing sets of regressors

consider *family* of least-squares problems

$$\text{minimize} \quad \left\| \sum_{i=1}^p x_i a_i - y \right\|$$

for $p = 1, \dots, n$

(a_1, \dots, a_p are called *regressors*)

- approximate y by linear combination of a_1, \dots, a_p
- project y onto $\text{span}\{a_1, \dots, a_p\}$
- regress y on a_1, \dots, a_p
- as p increases, get better fit, so optimal residual decreases

solution for each $p \leq n$ is given by

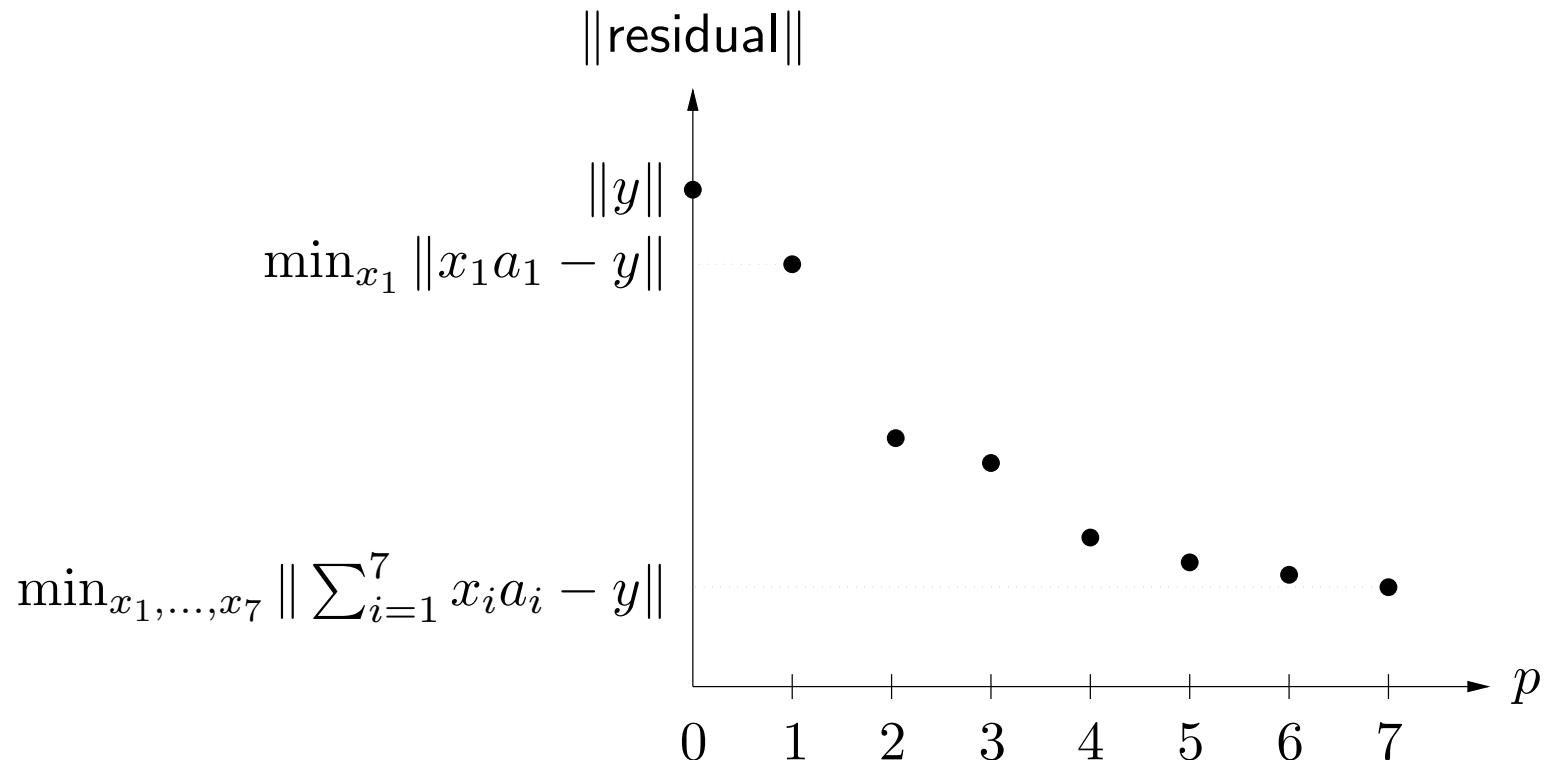
$$x_{\text{ls}}^{(p)} = (A_p^T A_p)^{-1} A_p^T y = R_p^{-1} Q_p^T y$$

where

- $A_p = [a_1 \cdots a_p] \in \mathbf{R}^{m \times p}$ is the first p columns of A
- $A_p = Q_p R_p$ is the QR factorization of A_p
- $R_p \in \mathbf{R}^{p \times p}$ is the leading $p \times p$ submatrix of R
- $Q_p = [q_1 \cdots q_p]$ is the first p columns of Q

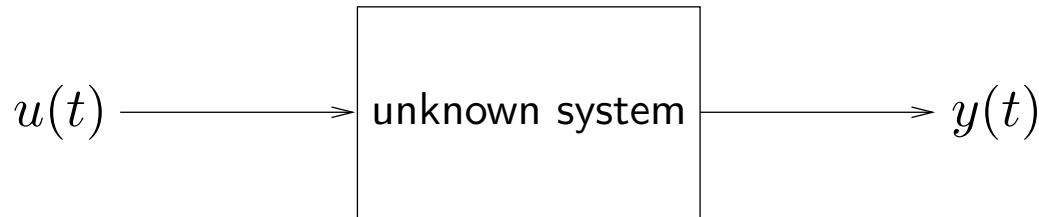
Norm of optimal residual versus p

plot of optimal residual versus p shows how well y can be matched by linear combination of a_1, \dots, a_p , as function of p



Least-squares system identification

we measure input $u(t)$ and output $y(t)$ for $t = 0, \dots, N$ of unknown system



system identification problem: find reasonable model for system based on measured I/O data u, y

example with scalar u, y (vector u, y readily handled): fit I/O data with moving-average (MA) model with n delays

$$\hat{y}(t) = h_0 u(t) + h_1 u(t-1) + \cdots + h_n u(t-n)$$

where $h_0, \dots, h_n \in \mathbf{R}$

we can write model or predicted output as

$$\begin{bmatrix} \hat{y}(n) \\ \hat{y}(n+1) \\ \vdots \\ \hat{y}(N) \end{bmatrix} = \begin{bmatrix} u(n) & u(n-1) & \cdots & u(0) \\ u(n+1) & u(n) & \cdots & u(1) \\ \vdots & \vdots & & \vdots \\ u(N) & u(N-1) & \cdots & u(N-n) \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_n \end{bmatrix}$$

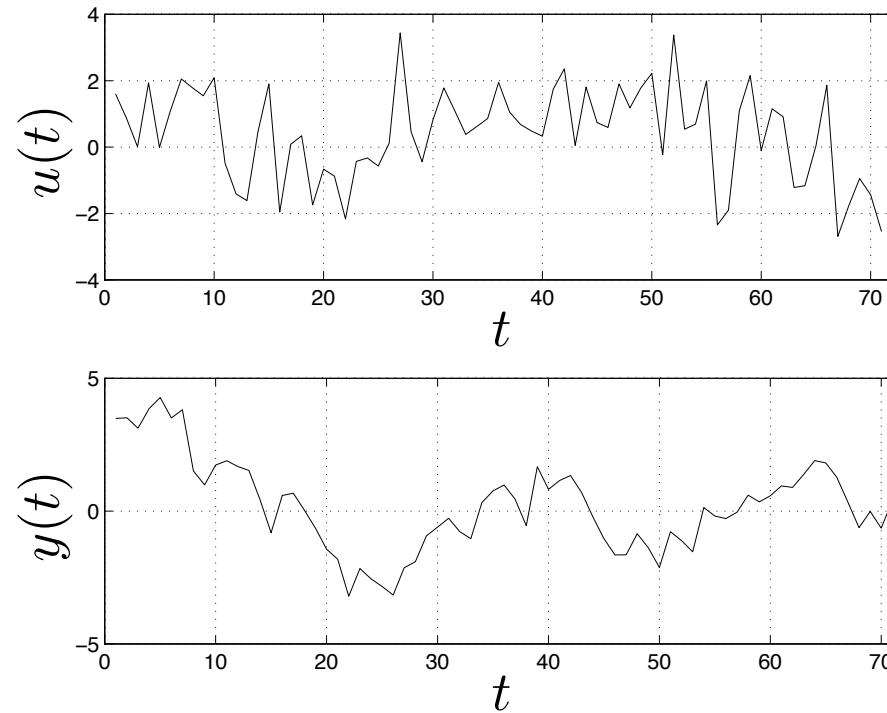
model prediction error is

$$e = (y(n) - \hat{y}(n), \dots, y(N) - \hat{y}(N))$$

least-squares identification: choose model (*i.e.*, h) that minimizes norm of model prediction error $\|e\|$

. . . a least-squares problem (with variables h)

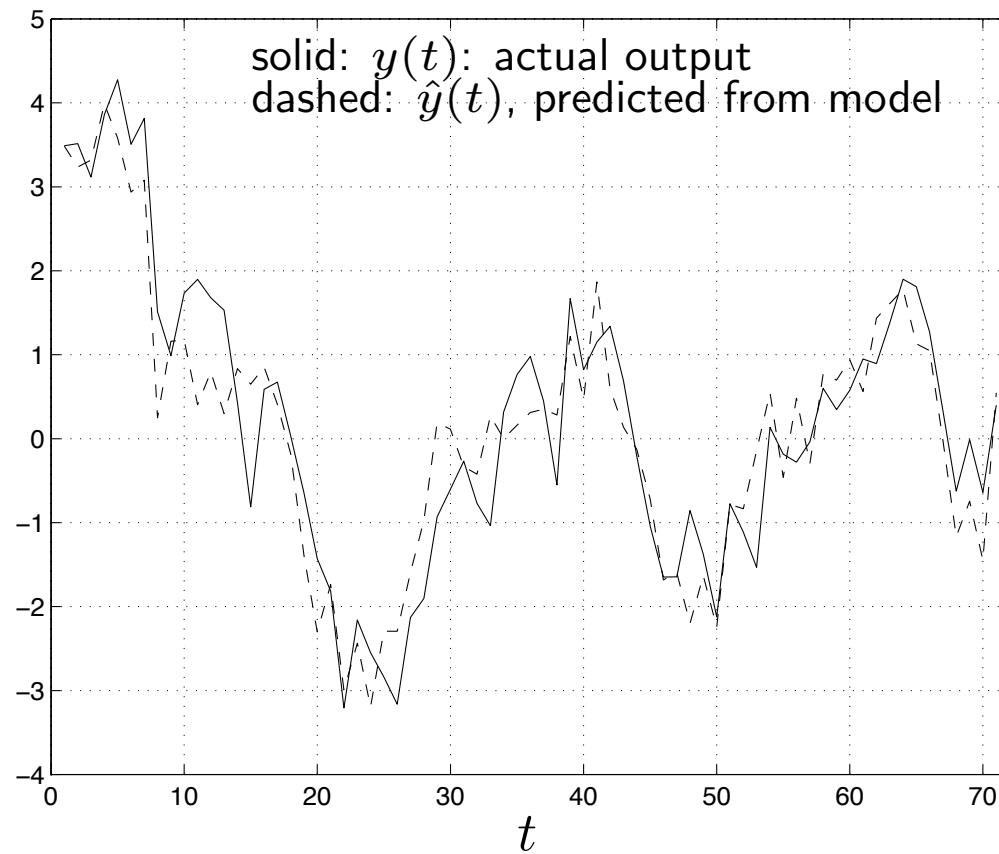
Example



for $n = 7$ we obtain MA model with

$$(h_0, \dots, h_7) = (.024, .282, .418, .354, .243, .487, .208, .441)$$

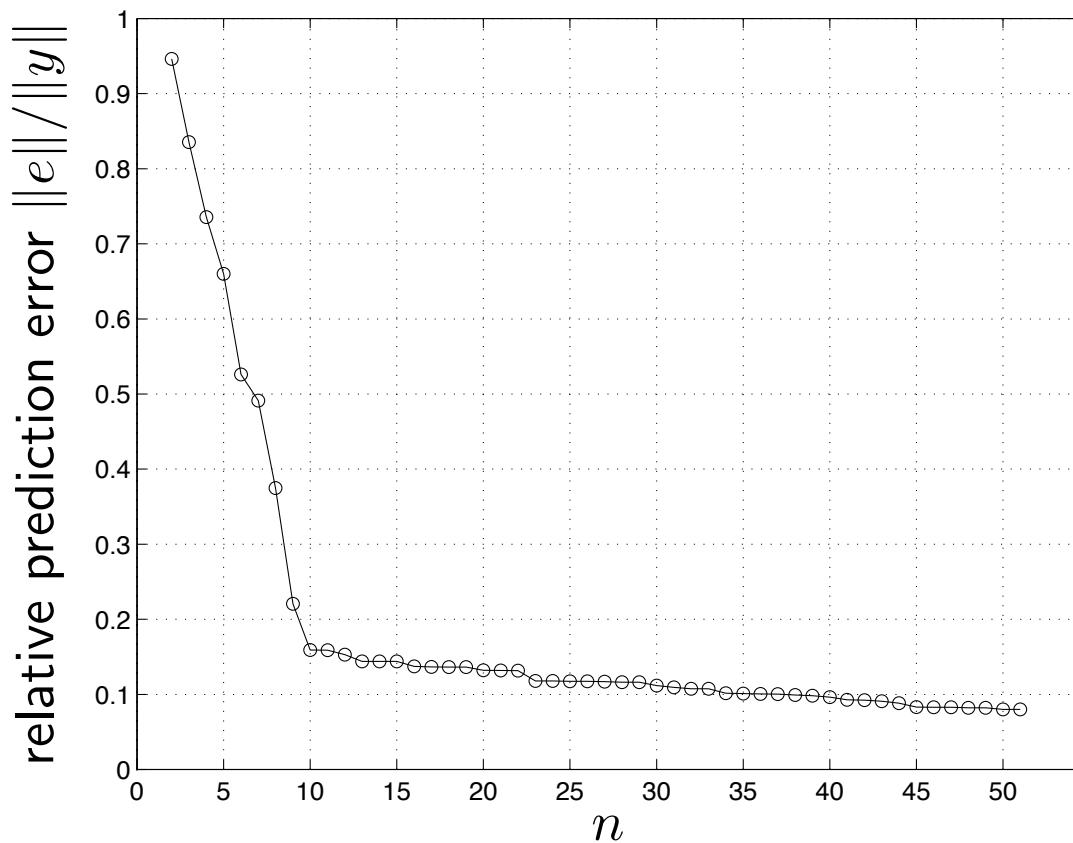
with relative prediction error $\|e\|/\|y\| = 0.37$



Model order selection

question: how large should n be?

- obviously the larger n , the smaller the prediction error *on the data used to form the model*
- suggests using largest possible model order for smallest prediction error

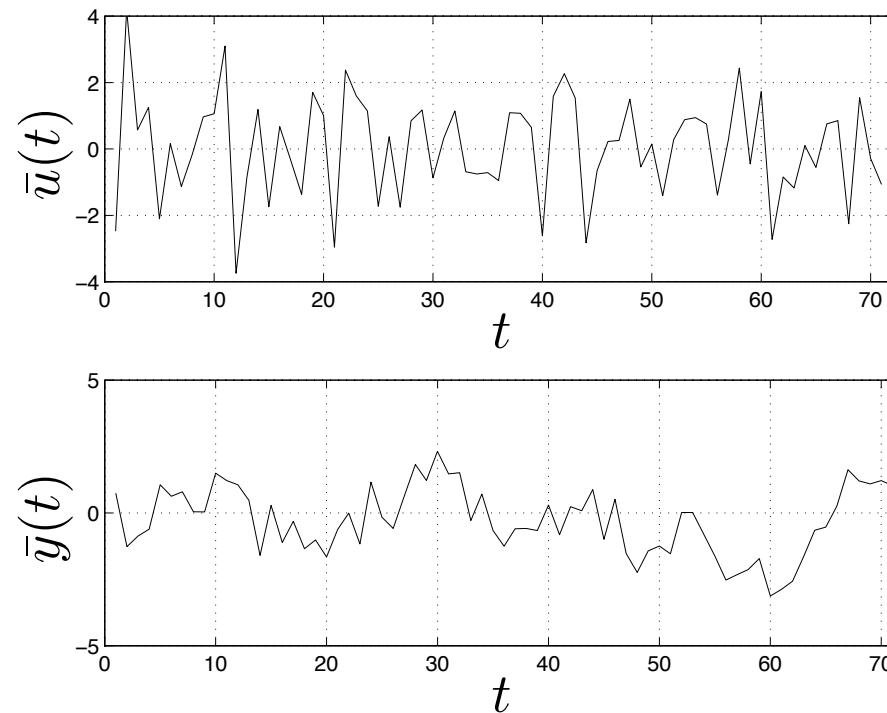


difficulty: for n too large the *predictive ability* of the model on other I/O data (from the same system) becomes worse

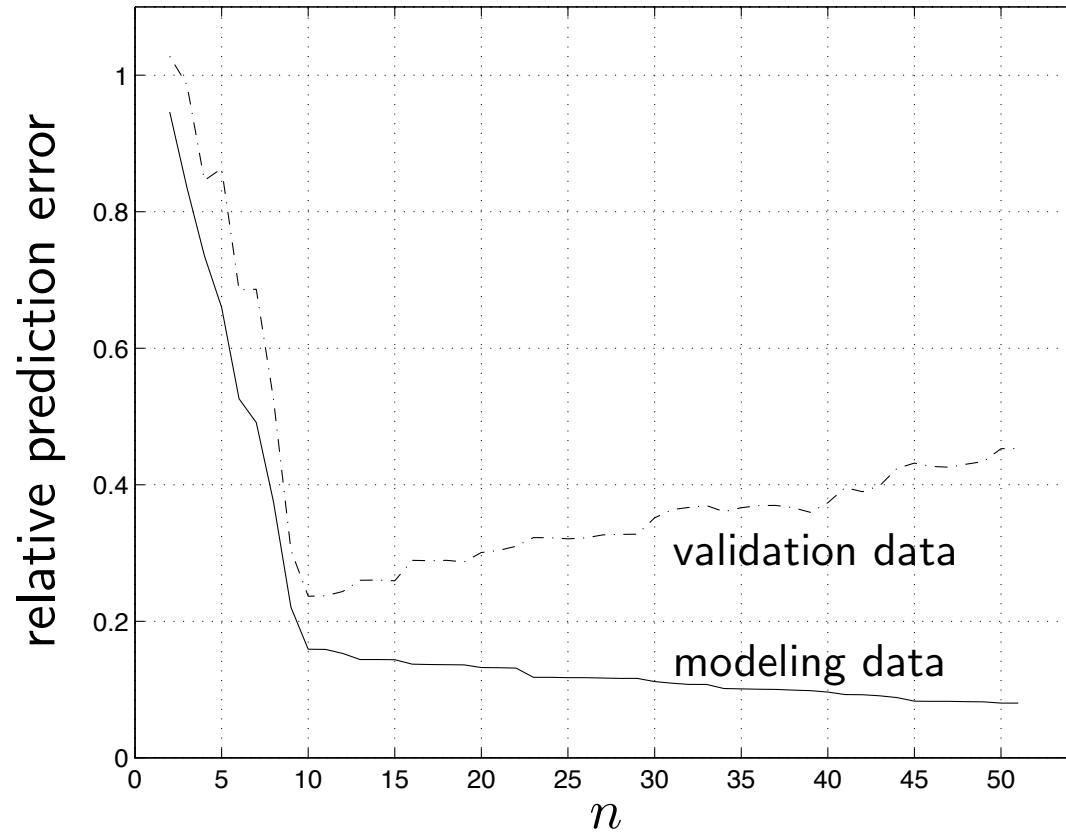
Out of sample validation

evaluate model predictive performance on another I/O data set *not used to develop model*

model validation data set:

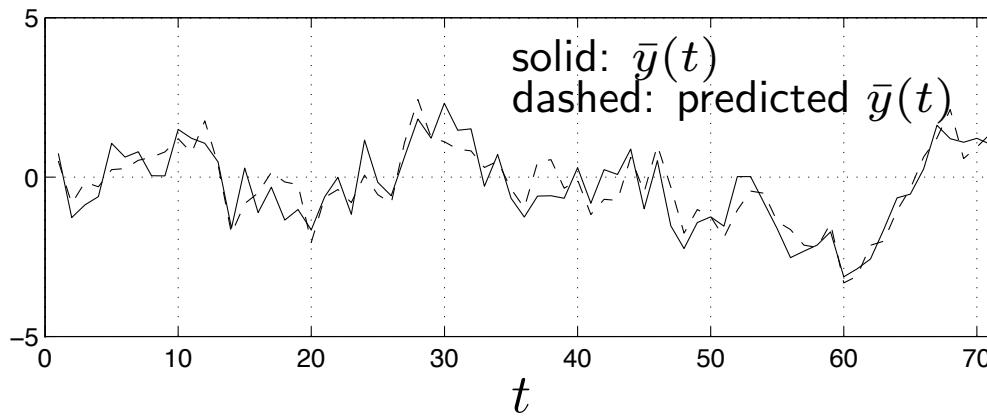
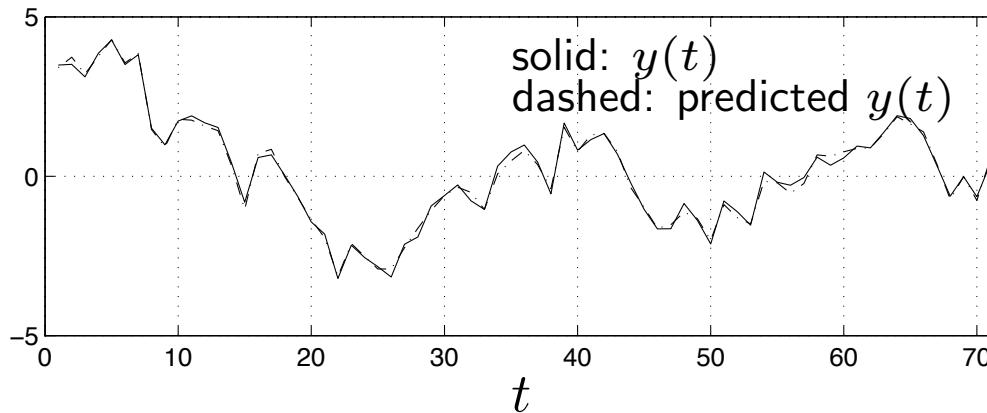


now check prediction error of models (developed using *modeling data*) on *validation data*:



plot suggests $n = 10$ is a good choice

for $n = 50$ the actual and predicted outputs on system identification and model validation data are:



loss of predictive ability when n too large is called *model overfit* or *overmodeling*

Growing sets of measurements

least-squares problem in ‘row’ form:

$$\text{minimize} \quad \|Ax - y\|^2 = \sum_{i=1}^m (\tilde{a}_i^T x - y_i)^2$$

where \tilde{a}_i^T are the rows of A ($\tilde{a}_i \in \mathbf{R}^n$)

- $x \in \mathbf{R}^n$ is some vector to be estimated
- each pair \tilde{a}_i, y_i corresponds to one measurement
- solution is

$$x_{\text{ls}} = \left(\sum_{i=1}^m \tilde{a}_i \tilde{a}_i^T \right)^{-1} \sum_{i=1}^m y_i \tilde{a}_i$$

- suppose that \tilde{a}_i and y_i become available sequentially, *i.e.*, m increases with time

Recursive least-squares

we can compute $x_{\text{ls}}(m) = \left(\sum_{i=1}^m \tilde{a}_i \tilde{a}_i^T \right)^{-1} \sum_{i=1}^m y_i \tilde{a}_i$ recursively

- initialize $P(0) = 0 \in \mathbf{R}^{n \times n}$, $q(0) = 0 \in \mathbf{R}^n$
- for $m = 0, 1, \dots,$

$$P(m+1) = P(m) + \tilde{a}_{m+1} \tilde{a}_{m+1}^T \quad q(m+1) = q(m) + y_{m+1} \tilde{a}_{m+1}$$

- if $P(m)$ is invertible, we have $x_{\text{ls}}(m) = P(m)^{-1} q(m)$
- $P(m)$ is invertible $\iff \tilde{a}_1, \dots, \tilde{a}_m$ span \mathbf{R}^n
(so, once $P(m)$ becomes invertible, it stays invertible)

Fast update for recursive least-squares

we can calculate

$$P(m+1)^{-1} = (P(m) + \tilde{a}_{m+1}\tilde{a}_{m+1}^T)^{-1}$$

efficiently from $P(m)^{-1}$ using the *rank one update formula*

$$(P + \tilde{a}\tilde{a}^T)^{-1} = P^{-1} - \frac{1}{1 + \tilde{a}^T P^{-1} \tilde{a}} (P^{-1} \tilde{a})(P^{-1} \tilde{a})^T$$

valid when $P = P^T$, and P and $P + \tilde{a}\tilde{a}^T$ are both invertible

- gives an $O(n^2)$ method for computing $P(m+1)^{-1}$ from $P(m)^{-1}$
- standard methods for computing $P(m+1)^{-1}$ from $P(m+1)$ are $O(n^3)$

Verification of rank one update formula

$$\begin{aligned} & (P + \tilde{a}\tilde{a}^T) \left(P^{-1} - \frac{1}{1 + \tilde{a}^T P^{-1} \tilde{a}} (P^{-1} \tilde{a})(P^{-1} \tilde{a})^T \right) \\ = & I + \tilde{a}\tilde{a}^T P^{-1} - \frac{1}{1 + \tilde{a}^T P^{-1} \tilde{a}} P (P^{-1} \tilde{a})(P^{-1} \tilde{a})^T \\ & - \frac{1}{1 + \tilde{a}^T P^{-1} \tilde{a}} \tilde{a}\tilde{a}^T (P^{-1} \tilde{a})(P^{-1} \tilde{a})^T \\ = & I + \tilde{a}\tilde{a}^T P^{-1} - \frac{1}{1 + \tilde{a}^T P^{-1} \tilde{a}} \tilde{a}\tilde{a}^T P^{-1} - \frac{\tilde{a}^T P^{-1} \tilde{a}}{1 + \tilde{a}^T P^{-1} \tilde{a}} \tilde{a}\tilde{a}^T P^{-1} \\ = & I \end{aligned}$$

Lecture 7

Regularized least-squares and Gauss-Newton method

- multi-objective least-squares
- regularized least-squares
- nonlinear least-squares
- Gauss-Newton method

Multi-objective least-squares

in many problems we have two (or more) objectives

- we want $J_1 = \|Ax - y\|^2$ small
- and also $J_2 = \|Fx - g\|^2$ small

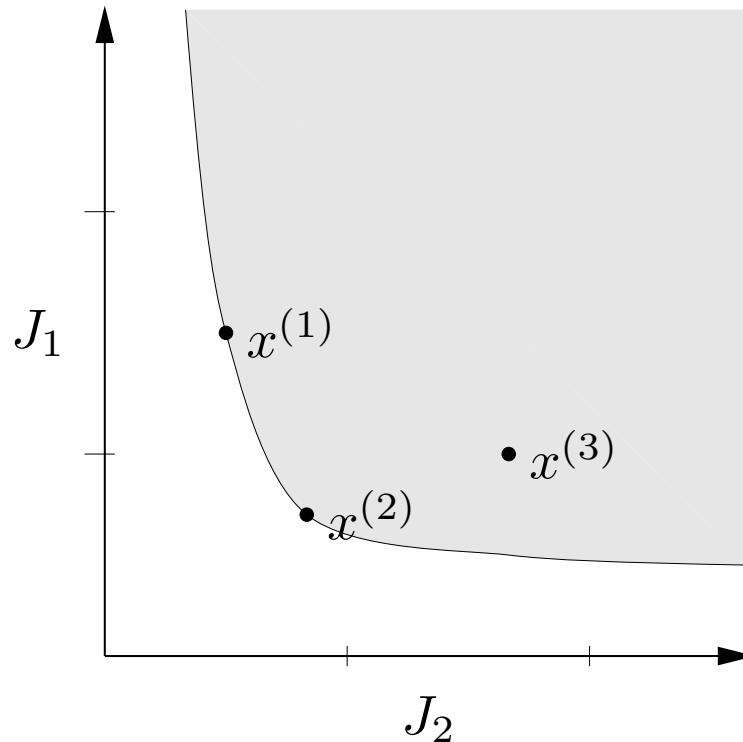
($x \in \mathbf{R}^n$ is the variable)

- usually the objectives are *competing*
- we can make one smaller, at the expense of making the other larger

common example: $F = I$, $g = 0$; we want $\|Ax - y\|$ small, with small x

Plot of achievable objective pairs

plot (J_2, J_1) for every x :



note that $x \in \mathbf{R}^n$, but this plot is in \mathbf{R}^2 ; point labeled $x^{(1)}$ is really $(J_2(x^{(1)}), J_1(x^{(1)}))$

- shaded area shows (J_2, J_1) achieved by some $x \in \mathbf{R}^n$
- clear area shows (J_2, J_1) not achieved by any $x \in \mathbf{R}^n$
- boundary of region is called *optimal trade-off curve*
- corresponding x are called *Pareto optimal*
(for the two objectives $\|Ax - y\|^2$, $\|Fx - g\|^2$)

three example choices of x : $x^{(1)}$, $x^{(2)}$, $x^{(3)}$

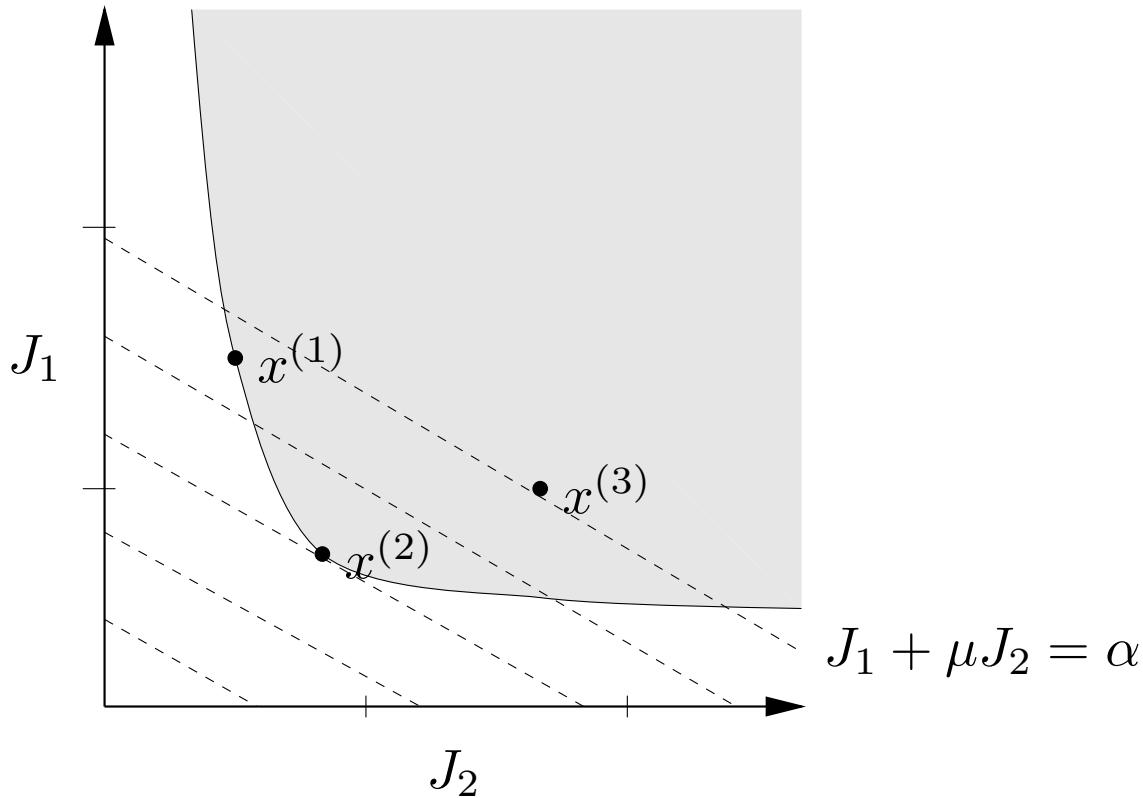
- $x^{(3)}$ is worse than $x^{(2)}$ on both counts (J_2 and J_1)
- $x^{(1)}$ is better than $x^{(2)}$ in J_2 , but worse in J_1

Weighted-sum objective

- to find Pareto optimal points, *i.e.*, x 's on optimal trade-off curve, we minimize *weighted-sum objective*

$$J_1 + \mu J_2 = \|Ax - y\|^2 + \mu\|Fx - g\|^2$$

- parameter $\mu \geq 0$ gives relative weight between J_1 and J_2
- points where weighted sum is constant, $J_1 + \mu J_2 = \alpha$, correspond to line with slope $-\mu$ on (J_2, J_1) plot



- $x^{(2)}$ minimizes weighted-sum objective for μ shown
- by varying μ from 0 to $+\infty$, can sweep out entire *optimal tradeoff curve*

Minimizing weighted-sum objective

can express weighted-sum objective as ordinary least-squares objective:

$$\begin{aligned}\|Ax - y\|^2 + \mu\|Fx - g\|^2 &= \left\| \begin{bmatrix} A \\ \sqrt{\mu}F \end{bmatrix} x - \begin{bmatrix} y \\ \sqrt{\mu}g \end{bmatrix} \right\|^2 \\ &= \|\tilde{A}x - \tilde{y}\|^2\end{aligned}$$

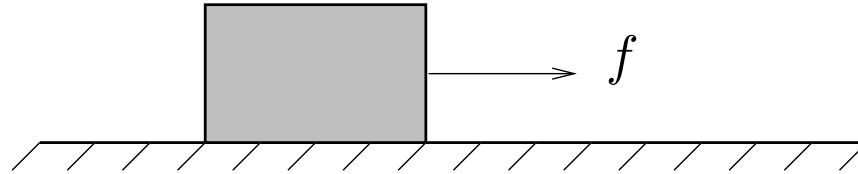
where

$$\tilde{A} = \begin{bmatrix} A \\ \sqrt{\mu}F \end{bmatrix}, \quad \tilde{y} = \begin{bmatrix} y \\ \sqrt{\mu}g \end{bmatrix}$$

hence solution is (assuming \tilde{A} full rank)

$$\begin{aligned}x &= (\tilde{A}^T \tilde{A})^{-1} \tilde{A}^T \tilde{y} \\ &= (A^T A + \mu F^T F)^{-1} (A^T y + \mu F^T g)\end{aligned}$$

Example



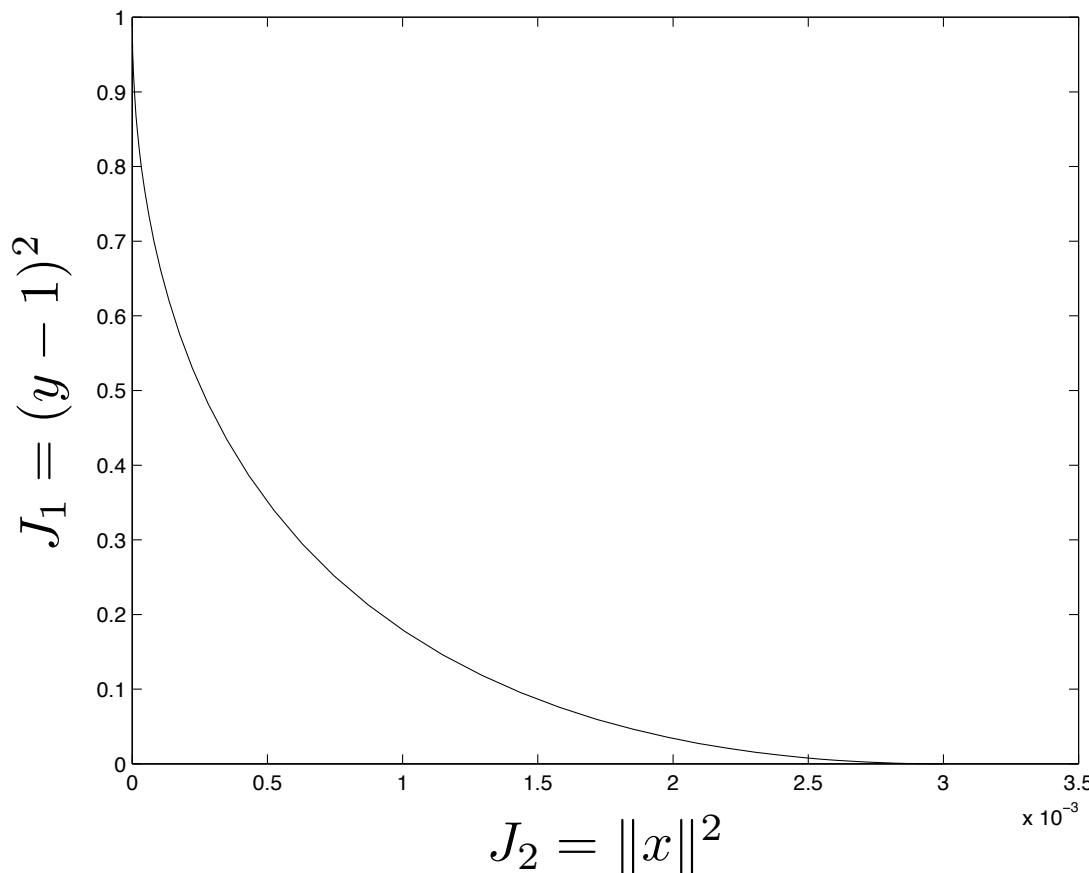
- unit mass at rest subject to forces x_i for $i - 1 < t \leq i$, $i = 1, \dots, 10$
- $y \in \mathbf{R}$ is position at $t = 10$; $y = a^T x$ where $a \in \mathbf{R}^{10}$
- $J_1 = (y - 1)^2$ (final position error squared)
- $J_2 = \|x\|^2$ (sum of squares of forces)

weighted-sum objective: $(a^T x - 1)^2 + \mu \|x\|^2$

optimal x :

$$x = (aa^T + \mu I)^{-1} a$$

optimal trade-off curve:



- upper left corner of optimal trade-off curve corresponds to $x = 0$
- bottom right corresponds to input that yields $y = 1$, i.e., $J_1 = 0$

Regularized least-squares

when $F = I$, $g = 0$ the objectives are

$$J_1 = \|Ax - y\|^2, \quad J_2 = \|x\|^2$$

minimizer of weighted-sum objective,

$$x = (A^T A + \mu I)^{-1} A^T y,$$

is called *regularized* least-squares (approximate) solution of $Ax \approx y$

- also called *Tychonov regularization*
- for $\mu > 0$, works for *any* A (no restrictions on shape, rank . . .)

estimation/inversion application:

- $Ax - y$ is sensor residual
- prior information: x small
- or, model only accurate for x small
- regularized solution trades off sensor fit, size of x

Nonlinear least-squares

nonlinear least-squares (NLLS) problem: find $x \in \mathbf{R}^n$ that minimizes

$$\|r(x)\|^2 = \sum_{i=1}^m r_i(x)^2,$$

where $r : \mathbf{R}^n \rightarrow \mathbf{R}^m$

- $r(x)$ is a vector of ‘residuals’
- reduces to (linear) least-squares if $r(x) = Ax - y$

Position estimation from ranges

estimate position $x \in \mathbf{R}^2$ from approximate distances to beacons at locations $b_1, \dots, b_m \in \mathbf{R}^2$ *without* linearizing

- we measure $\rho_i = \|x - b_i\| + v_i$
(v_i is range error, unknown but assumed small)
- NLLS estimate: choose \hat{x} to minimize

$$\sum_{i=1}^m r_i(x)^2 = \sum_{i=1}^m (\rho_i - \|x - b_i\|)^2$$

Gauss-Newton method for NLLS

NLLS: find $x \in \mathbf{R}^n$ that minimizes $\|r(x)\|^2 = \sum_{i=1}^m r_i(x)^2$, where
 $r : \mathbf{R}^n \rightarrow \mathbf{R}^m$

- in general, very hard to solve exactly
- many good heuristics to compute *locally optimal* solution

Gauss-Newton method:

given starting guess for x

repeat

 linearize r near current guess

 new guess is linear LS solution, using linearized r

until convergence

Gauss-Newton method (more detail):

- linearize r near current iterate $x^{(k)}$:

$$r(x) \approx r(x^{(k)}) + Dr(x^{(k)})(x - x^{(k)})$$

where Dr is the Jacobian: $(Dr)_{ij} = \partial r_i / \partial x_j$

- write linearized approximation as

$$r(x^{(k)}) + Dr(x^{(k)})(x - x^{(k)}) = A^{(k)}x - b^{(k)}$$

$$A^{(k)} = Dr(x^{(k)}), \quad b^{(k)} = Dr(x^{(k)})x^{(k)} - r(x^{(k)})$$

- at k th iteration, we approximate NLLS problem by linear LS problem:

$$\|r(x)\|^2 \approx \|A^{(k)}x - b^{(k)}\|^2$$

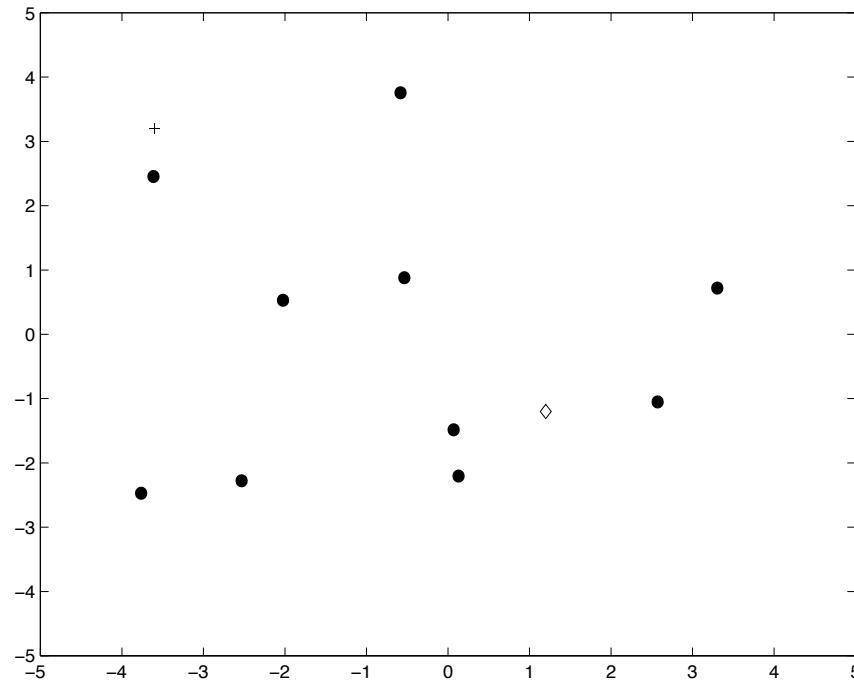
- next iterate solves this linearized LS problem:

$$x^{(k+1)} = \left(A^{(k)T} A^{(k)} \right)^{-1} A^{(k)T} b^{(k)}$$

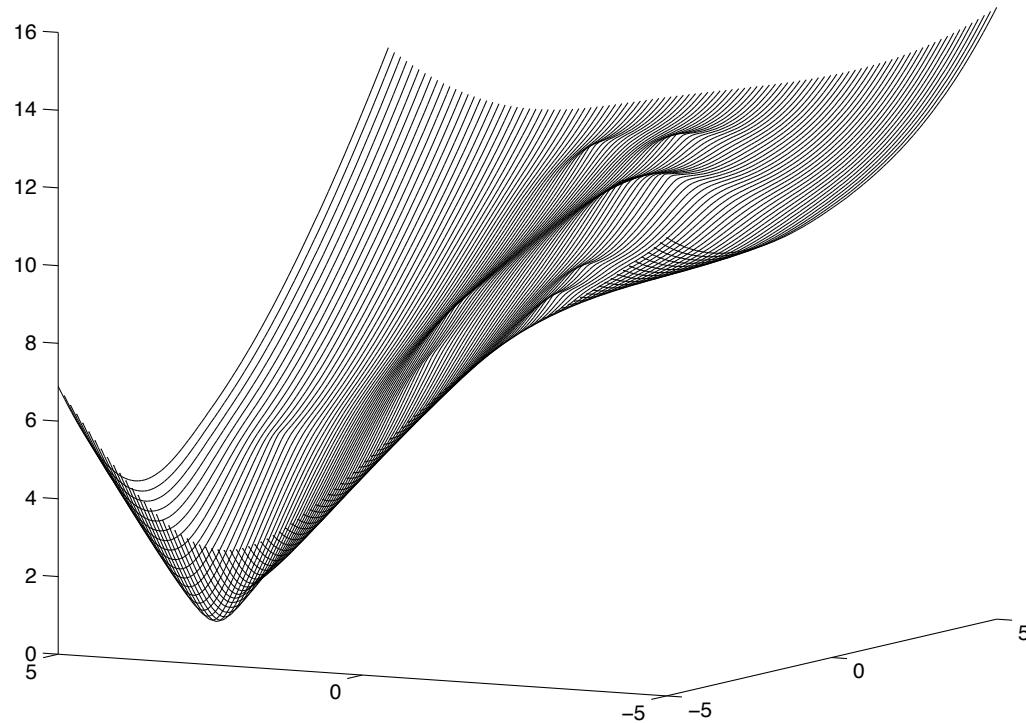
- repeat until convergence (which *isn't* guaranteed)

Gauss-Newton example

- 10 beacons
- + true position $(-3.6, 3.2)$; \diamond initial guess $(1.2, -1.2)$
- range estimates accurate to ± 0.5

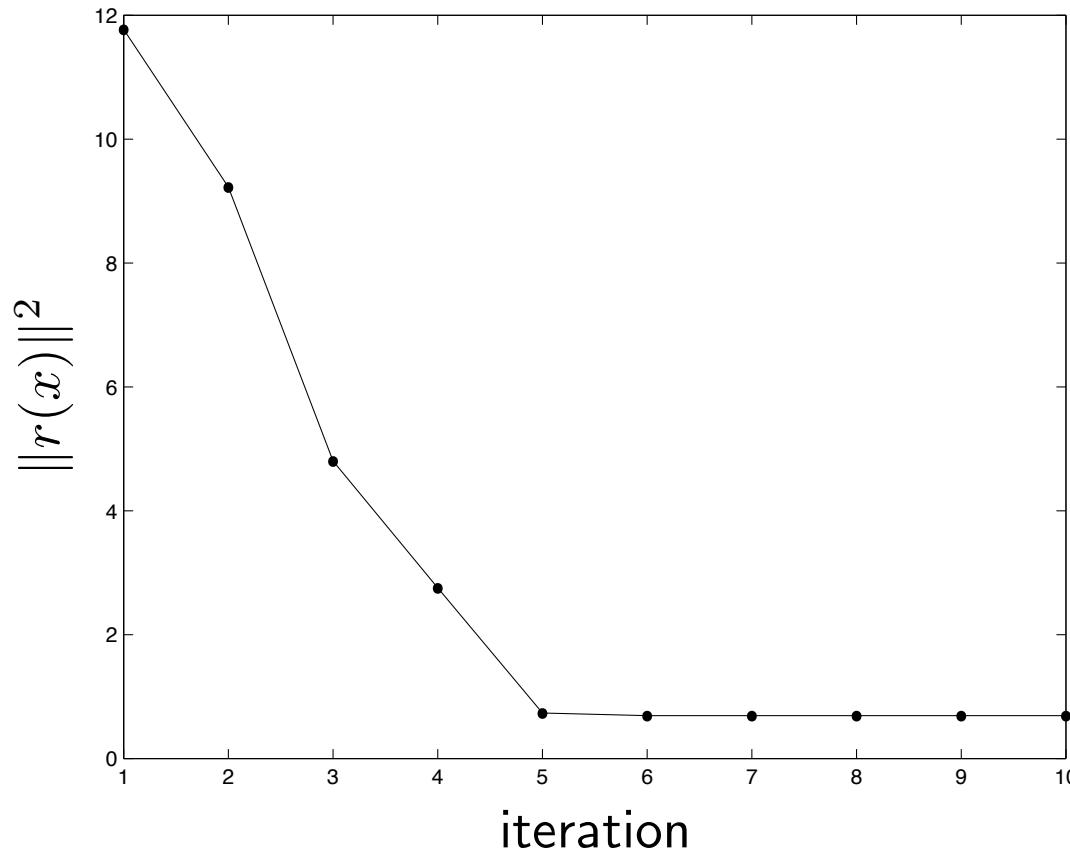


NLLS objective $\|r(x)\|^2$ versus x :



- for a linear LS problem, objective would be nice quadratic ‘bowl’
- bumps in objective due to strong nonlinearity of r

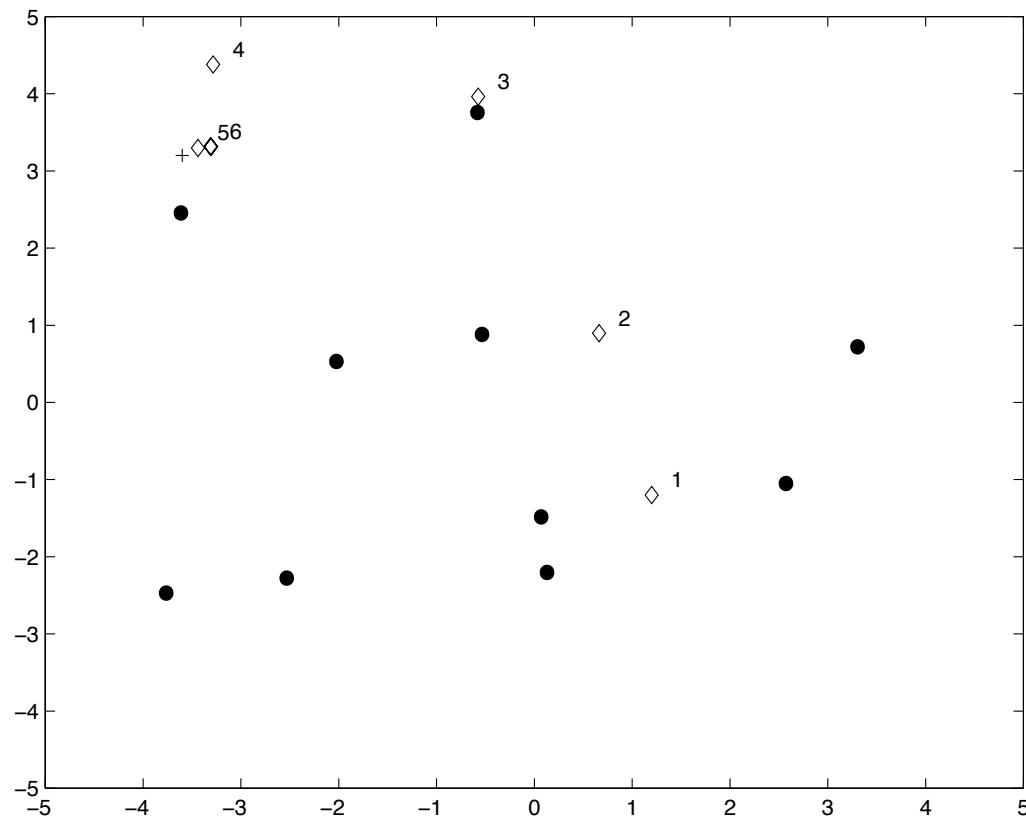
objective of Gauss-Newton iterates:



- $x^{(k)}$ converges to (in this case, global) minimum of $\|r(x)\|^2$
- convergence takes only five or so steps

- final estimate is $\hat{x} = (-3.3, 3.3)$
- estimation error is $\|\hat{x} - x\| = 0.31$
(substantially smaller than range accuracy!)

convergence of Gauss-Newton iterates:



useful variation on Gauss-Newton: add regularization term

$$\|A^{(k)}x - b^{(k)}\|^2 + \mu\|x - x^{(k)}\|^2$$

so that next iterate is not too far from previous one (hence, linearized model still pretty accurate)

Lecture 8

Least-norm solutions of underdetermined equations

- least-norm solution of underdetermined equations
- minimum norm solutions via QR factorization
- derivation via Lagrange multipliers
- relation to regularized least-squares
- general norm minimization with equality constraints

Underdetermined linear equations

we consider

$$y = Ax$$

where $A \in \mathbf{R}^{m \times n}$ is fat ($m < n$), i.e.,

- there are more variables than equations
- x is *underspecified*, i.e., many choices of x lead to the same y

we'll assume that A is full rank (m), so for each $y \in \mathbf{R}^m$, there is a solution set of all solutions has form

$$\{ x \mid Ax = y \} = \{ x_p + z \mid z \in \mathcal{N}(A) \}$$

where x_p is any ('particular') solution, i.e., $Ax_p = y$

- z characterizes available choices in solution
- solution has $\dim \mathcal{N}(A) = n - m$ ‘degrees of freedom’
- can choose z to satisfy other specs or optimize among solutions

Least-norm solution

one particular solution is

$$x_{\text{ln}} = A^T(AA^T)^{-1}y$$

(AA^T is invertible since A full rank)

in fact, x_{ln} is the solution of $y = Ax$ that minimizes $\|x\|$

i.e., x_{ln} is solution of optimization problem

$$\begin{array}{ll}\text{minimize} & \|x\| \\ \text{subject to} & Ax = y\end{array}$$

(with variable $x \in \mathbf{R}^n$)

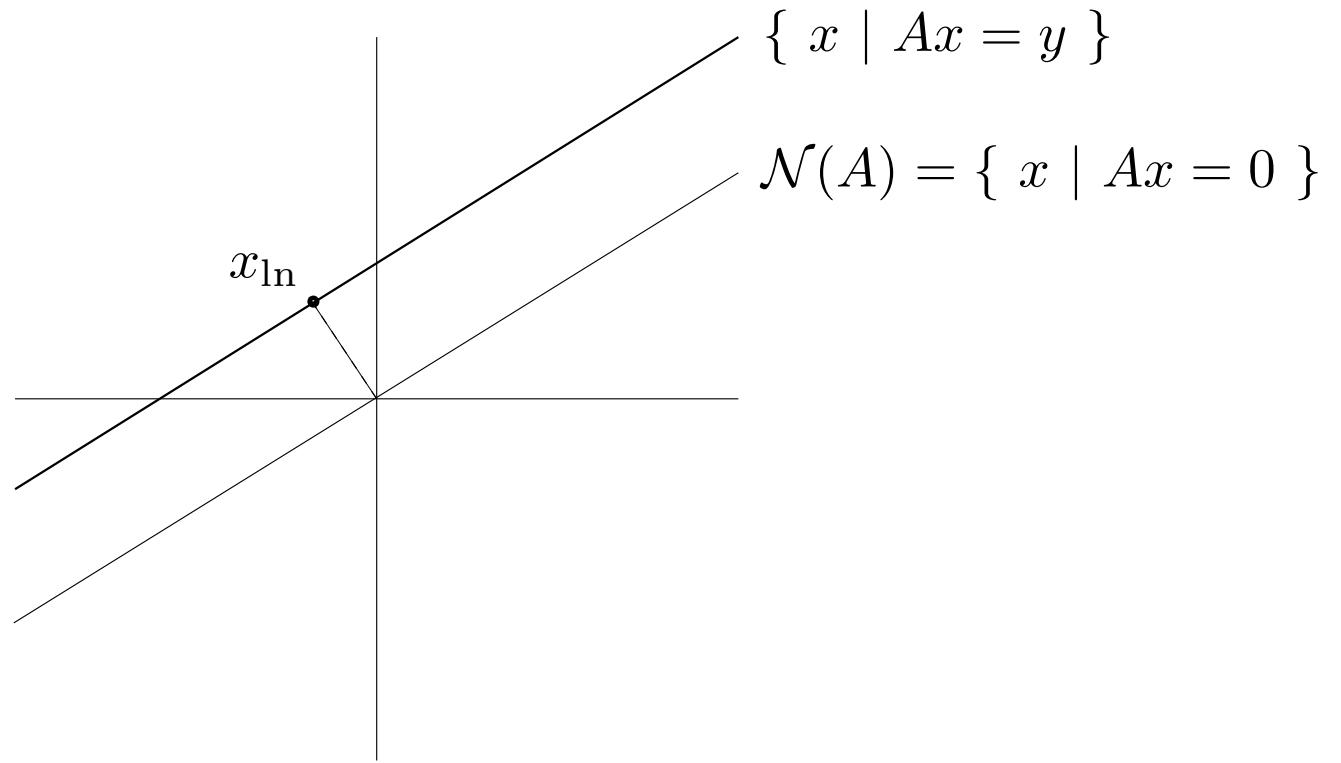
suppose $Ax = y$, so $A(x - x_{\text{ln}}) = 0$ and

$$\begin{aligned}(x - x_{\text{ln}})^T x_{\text{ln}} &= (x - x_{\text{ln}})^T A^T (A A^T)^{-1} y \\&= (A(x - x_{\text{ln}}))^T (A A^T)^{-1} y \\&= 0\end{aligned}$$

i.e., $(x - x_{\text{ln}}) \perp x_{\text{ln}}$, so

$$\|x\|^2 = \|x_{\text{ln}} + x - x_{\text{ln}}\|^2 = \|x_{\text{ln}}\|^2 + \|x - x_{\text{ln}}\|^2 \geq \|x_{\text{ln}}\|^2$$

i.e., x_{ln} has smallest norm of any solution



- **orthogonality condition:** $x_{ln} \perp \mathcal{N}(A)$
- **projection interpretation:** x_{ln} is projection of 0 on solution set $\{ x \mid Ax = y \}$

- $A^\dagger = A^T(AA^T)^{-1}$ is called the *pseudo-inverse* of full rank, fat A
- $A^T(AA^T)^{-1}$ is a *right inverse* of A
- $I - A^T(AA^T)^{-1}A$ gives projection onto $\mathcal{N}(A)$

cf. analogous formulas for full rank, **skinny** matrix A :

- $A^\dagger = (A^TA)^{-1}A^T$
- $(A^TA)^{-1}A^T$ is a *left inverse* of A
- $A(A^TA)^{-1}A^T$ gives projection onto $\mathcal{R}(A)$

Least-norm solution via QR factorization

find QR factorization of A^T , i.e., $A^T = QR$, with

- $Q \in \mathbf{R}^{n \times m}$, $Q^T Q = I_m$
- $R \in \mathbf{R}^{m \times m}$ upper triangular, nonsingular

then

- $x_{\text{ln}} = A^T (A A^T)^{-1} y = Q R^{-T} y$
- $\|x_{\text{ln}}\| = \|R^{-T} y\|$

Derivation via Lagrange multipliers

- least-norm solution solves optimization problem

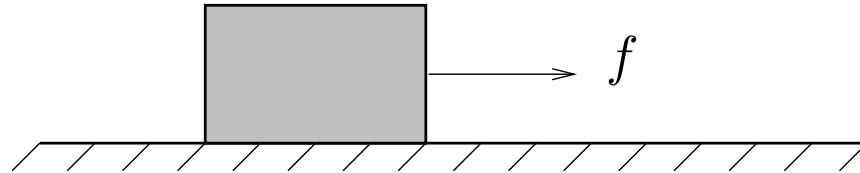
$$\begin{aligned} & \text{minimize} && x^T x \\ & \text{subject to} && Ax = y \end{aligned}$$

- introduce Lagrange multipliers: $L(x, \lambda) = x^T x + \lambda^T (Ax - y)$
- optimality conditions are

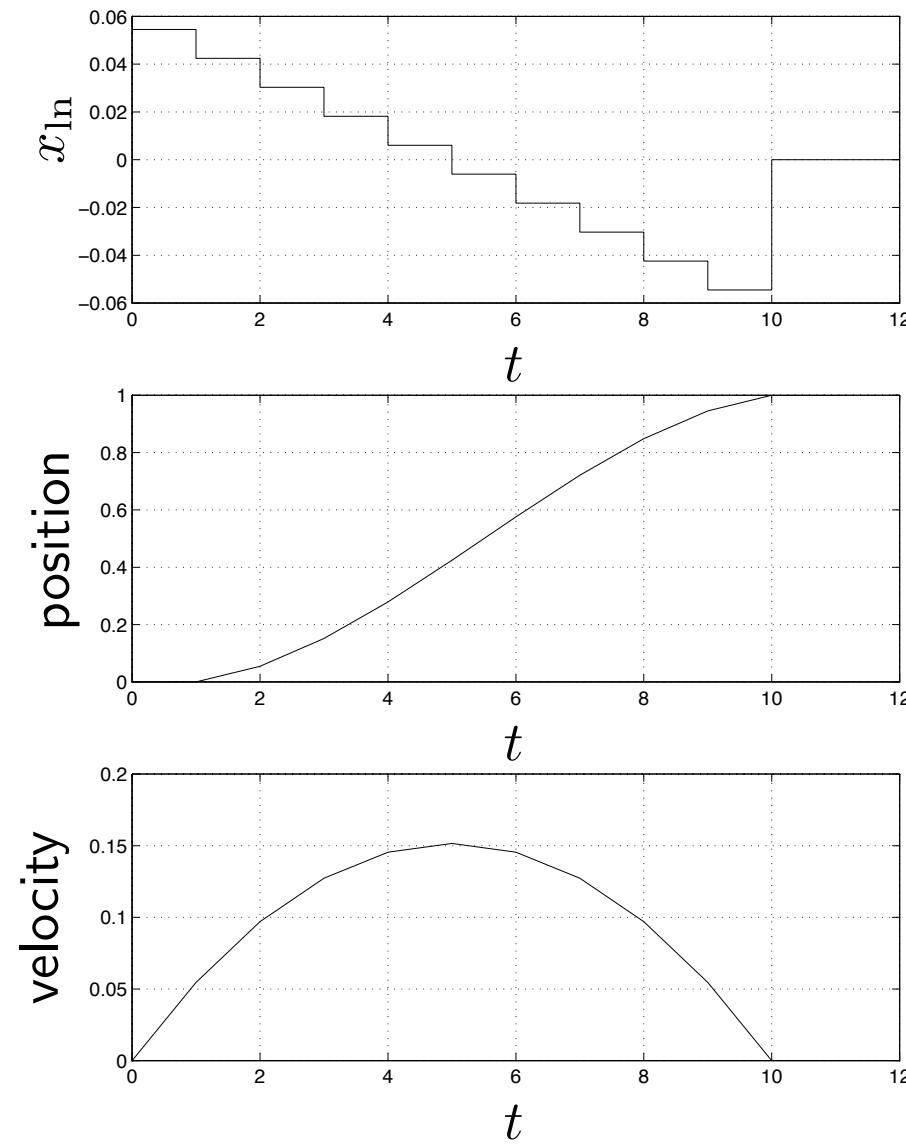
$$\nabla_x L = 2x + A^T \lambda = 0, \quad \nabla_\lambda L = Ax - y = 0$$

- from first condition, $x = -A^T \lambda / 2$
- substitute into second to get $\lambda = -2(AA^T)^{-1}y$
- hence $x = A^T(AA^T)^{-1}y$

Example: transferring mass unit distance



- unit mass at rest subject to forces x_i for $i - 1 < t \leq i$, $i = 1, \dots, 10$
- y_1 is position at $t = 10$, y_2 is velocity at $t = 10$
- $y = Ax$ where $A \in \mathbf{R}^{2 \times 10}$ (A is fat)
- find least norm force that transfers mass unit distance with zero final velocity, *i.e.*, $y = (1, 0)$



Relation to regularized least-squares

- suppose $A \in \mathbf{R}^{m \times n}$ is fat, full rank
- define $J_1 = \|Ax - y\|^2$, $J_2 = \|x\|^2$
- least-norm solution minimizes J_2 with $J_1 = 0$
- minimizer of weighted-sum objective $J_1 + \mu J_2 = \|Ax - y\|^2 + \mu\|x\|^2$ is

$$x_\mu = (A^T A + \mu I)^{-1} A^T y$$

- **fact:** $x_\mu \rightarrow x_{\text{ln}}$ as $\mu \rightarrow 0$, i.e., regularized solution converges to least-norm solution as $\mu \rightarrow 0$
- in matrix terms: as $\mu \rightarrow 0$,

$$(A^T A + \mu I)^{-1} A^T \rightarrow A^T (A A^T)^{-1}$$

(for full rank, fat A)

General norm minimization with equality constraints

consider problem

$$\begin{aligned} & \text{minimize} && \|Ax - b\| \\ & \text{subject to} && Cx = d \end{aligned}$$

with variable x

- includes least-squares and least-norm problems as special cases
- equivalent to

$$\begin{aligned} & \text{minimize} && (1/2)\|Ax - b\|^2 \\ & \text{subject to} && Cx = d \end{aligned}$$

- Lagrangian is

$$\begin{aligned} L(x, \lambda) &= (1/2)\|Ax - b\|^2 + \lambda^T(Cx - d) \\ &= (1/2)x^T A^T A x - b^T A x + (1/2)b^T b + \lambda^T C x - \lambda^T d \end{aligned}$$

- optimality conditions are

$$\nabla_x L = A^T A x - A^T b + C^T \lambda = 0, \quad \nabla_\lambda L = C x - d = 0$$

- write in block matrix form as

$$\begin{bmatrix} A^T A & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} A^T b \\ d \end{bmatrix}$$

- if the block matrix is invertible, we have

$$\begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} A^T A & C^T \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} A^T b \\ d \end{bmatrix}$$

if $A^T A$ is invertible, we can derive a more explicit (and complicated) formula for x

- from first block equation we get

$$x = (A^T A)^{-1} (A^T b - C^T \lambda)$$

- substitute into $Cx = d$ to get

$$C(A^T A)^{-1} (A^T b - C^T \lambda) = d$$

so

$$\lambda = (C(A^T A)^{-1} C^T)^{-1} (C(A^T A)^{-1} A^T b - d)$$

- recover x from equation above (not pretty)

$$x = (A^T A)^{-1} \left(A^T b - C^T (C(A^T A)^{-1} C^T)^{-1} (C(A^T A)^{-1} A^T b - d) \right)$$

Lecture 9

Autonomous linear dynamical systems

- autonomous linear dynamical systems
- examples
- higher order systems
- linearization near equilibrium point
- linearization along trajectory

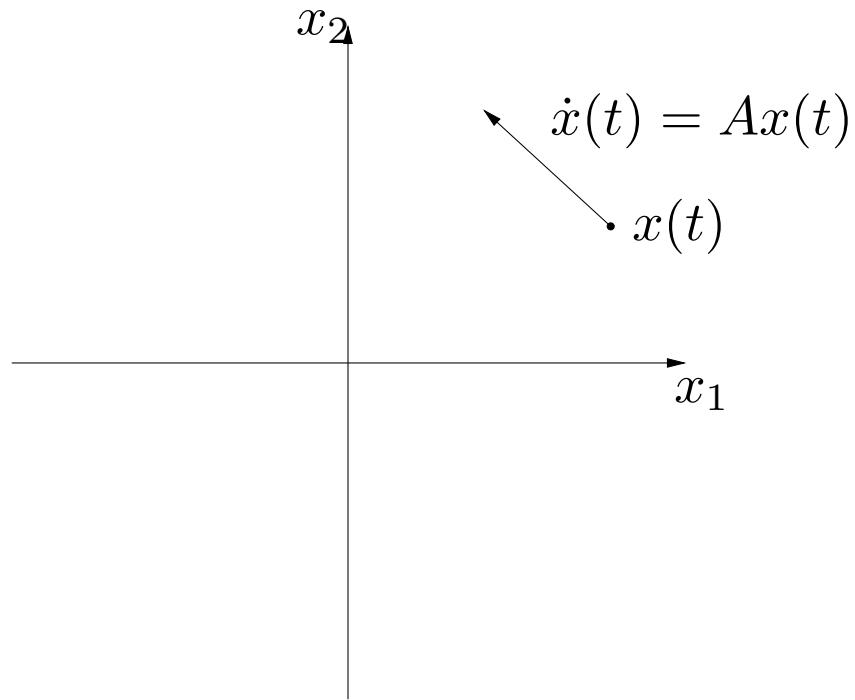
Autonomous linear dynamical systems

continuous-time autonomous LDS has form

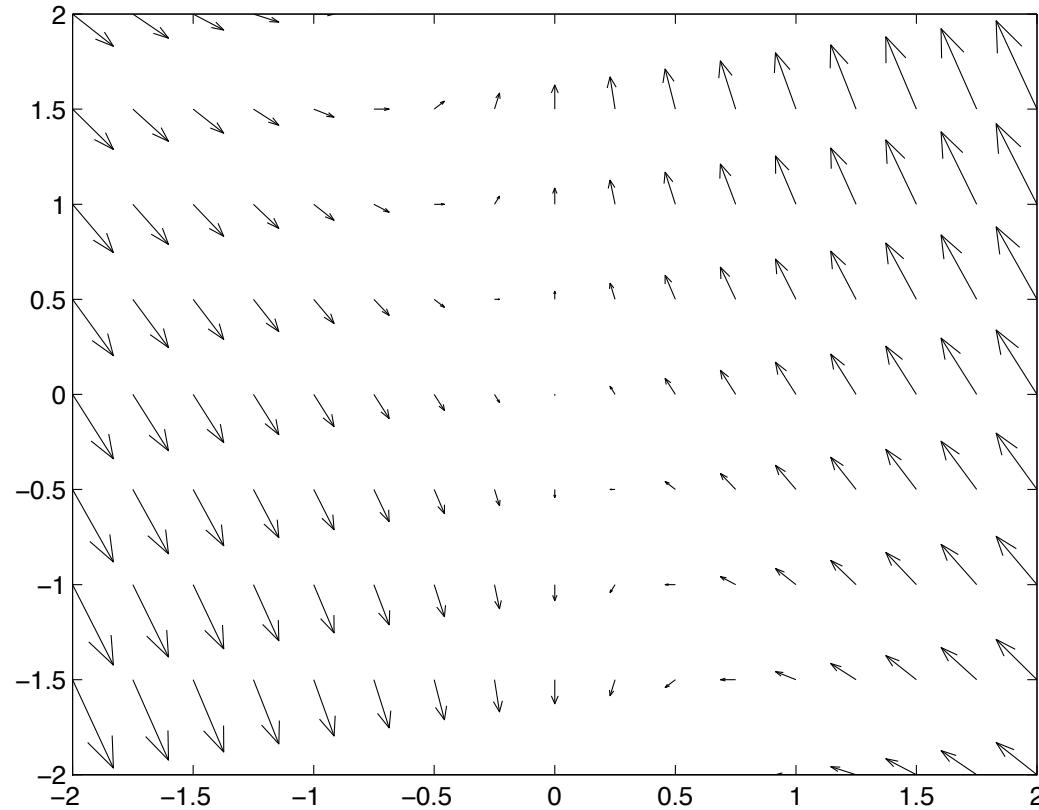
$$\dot{x} = Ax$$

- $x(t) \in \mathbf{R}^n$ is called the state
- n is the *state dimension* or (informally) the *number of states*
- A is the *dynamics matrix*
(system is *time-invariant* if A doesn't depend on t)

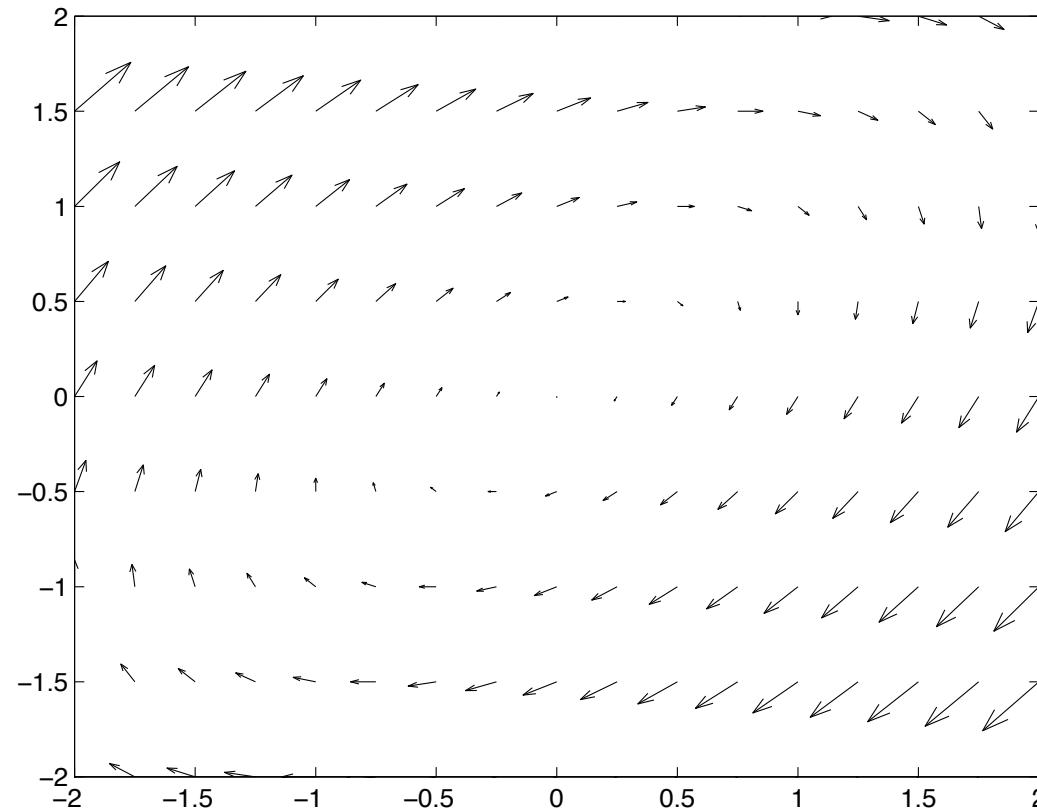
picture (*phase plane*):



example 1: $\dot{x} = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} x$

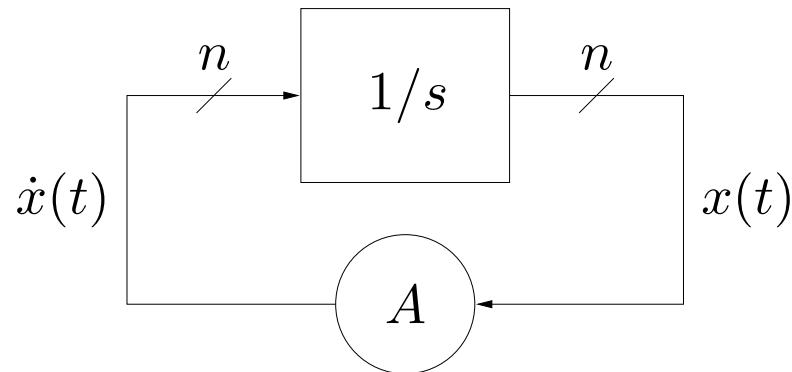


example 2: $\dot{x} = \begin{bmatrix} -0.5 & 1 \\ -1 & 0.5 \end{bmatrix} x$



Block diagram

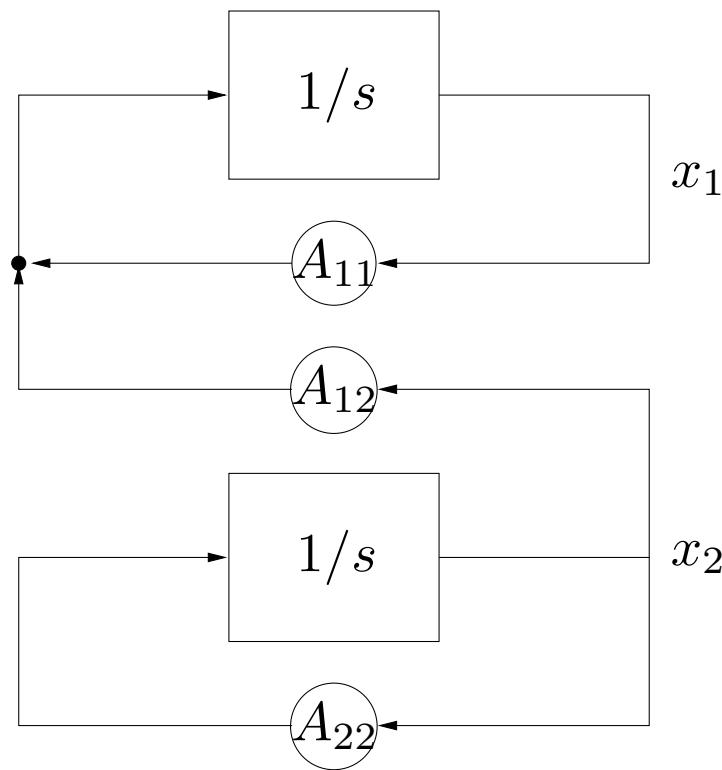
block diagram representation of $\dot{x} = Ax$:



- $1/s$ block represents n parallel scalar integrators
- coupling comes from dynamics matrix A

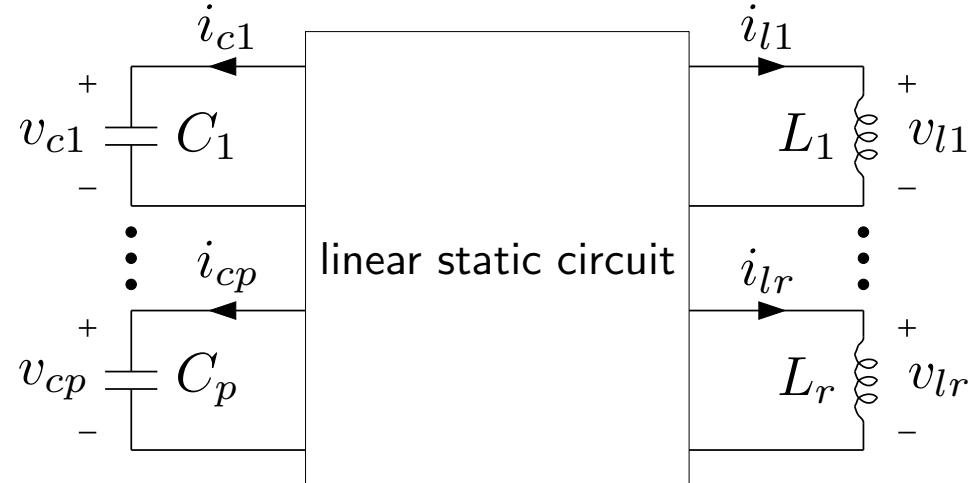
useful when A has structure, e.g., block upper triangular:

$$\dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} x$$



here x_1 doesn't affect x_2 at all

Linear circuit



circuit equations are

$$C \frac{dv_c}{dt} = i_c, \quad L \frac{di_l}{dt} = v_l, \quad \begin{bmatrix} i_c \\ v_l \end{bmatrix} = F \begin{bmatrix} v_c \\ i_l \end{bmatrix}$$

$$C = \text{diag}(C_1, \dots, C_p), \quad L = \text{diag}(L_1, \dots, L_r)$$

with state $x = \begin{bmatrix} v_c \\ i_l \end{bmatrix}$, we have

$$\dot{x} = \begin{bmatrix} C^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix} Fx$$

Chemical reactions

- reaction involving n chemicals; x_i is concentration of chemical i
- linear model of reaction kinetics

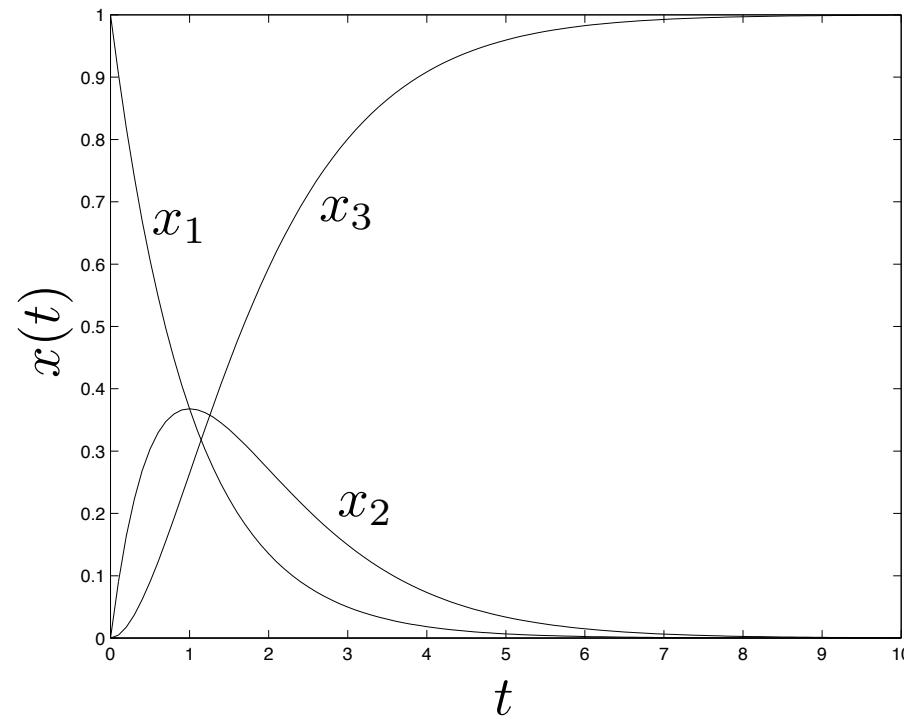
$$\frac{dx_i}{dt} = a_{i1}x_1 + \cdots + a_{in}x_n$$

- good model for some reactions; A is usually sparse

Example: series reaction $A \xrightarrow{k_1} B \xrightarrow{k_2} C$ with linear dynamics

$$\dot{x} = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{bmatrix} x$$

plot for $k_1 = k_2 = 1$, initial $x(0) = (1, 0, 0)$



Finite-state discrete-time Markov chain

$z(t) \in \{1, \dots, n\}$ is a random sequence with

$$\mathbf{Prob}(z(t+1) = i \mid z(t) = j) = P_{ij}$$

where $P \in \mathbf{R}^{n \times n}$ is the matrix of *transition probabilities*

can represent probability distribution of $z(t)$ as n -vector

$$p(t) = \begin{bmatrix} \mathbf{Prob}(z(t) = 1) \\ \vdots \\ \mathbf{Prob}(z(t) = n) \end{bmatrix}$$

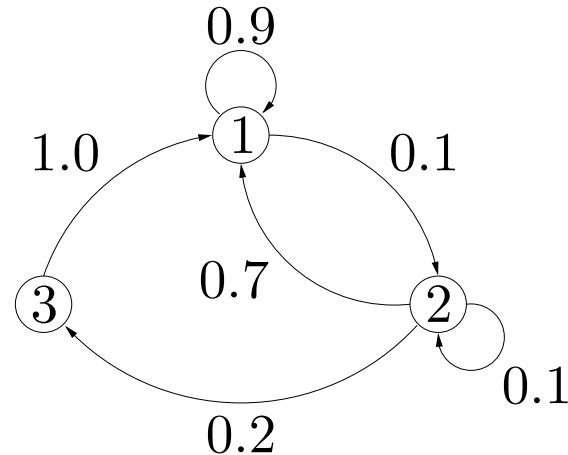
(so, e.g., $\mathbf{Prob}(z(t) = 1, 2, \text{ or } 3) = [1 \ 1 \ 1 \ 0 \cdots 0]p(t)$)

then we have $p(t+1) = Pp(t)$

P is often sparse; Markov chain is depicted graphically

- nodes are states
- edges show transition probabilities

example:



- state 1 is ‘system OK’
- state 2 is ‘system down’
- state 3 is ‘system being repaired’

$$p(t+1) = \begin{bmatrix} 0.9 & 0.7 & 1.0 \\ 0.1 & 0.1 & 0 \\ 0 & 0.2 & 0 \end{bmatrix} p(t)$$

Numerical integration of continuous system

compute approximate solution of $\dot{x} = Ax, x(0) = x_0$

suppose h is small time step (x doesn't change much in h seconds)

simple ('forward Euler') approximation:

$$x(t + h) \approx x(t) + h\dot{x}(t) = (I + hA)x(t)$$

by carrying out this recursion (discrete-time LDS), starting at $x(0) = x_0$, we get approximation

$$x(kh) \approx (I + hA)^k x(0)$$

(forward Euler is never used in practice)

Higher order linear dynamical systems

$$x^{(k)} = A_{k-1}x^{(k-1)} + \cdots + A_1x^{(1)} + A_0x, \quad x(t) \in \mathbf{R}^n$$

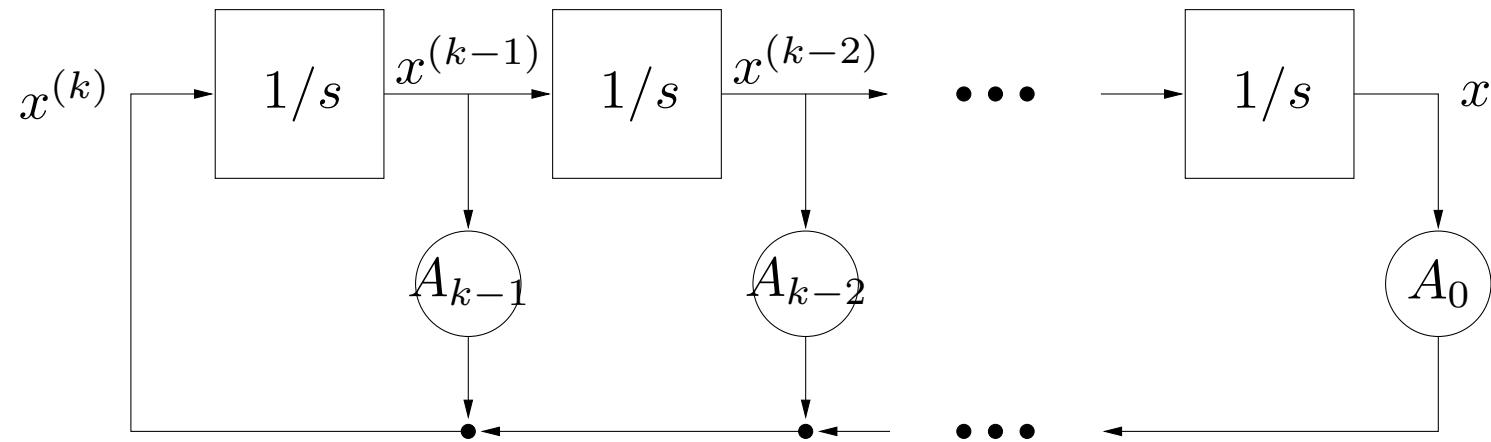
where $x^{(m)}$ denotes m th derivative

define new variable $z = \begin{bmatrix} x \\ x^{(1)} \\ \vdots \\ x^{(k-1)} \end{bmatrix} \in \mathbf{R}^{nk}$, so

$$\dot{z} = \begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(k)} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & I \\ A_0 & A_1 & A_2 & \cdots & A_{k-1} \end{bmatrix} z$$

a (first order) LDS (with bigger state)

block diagram:



Mechanical systems

mechanical system with k degrees of freedom undergoing small motions:

$$M\ddot{q} + D\dot{q} + Kq = 0$$

- $q(t) \in \mathbf{R}^k$ is the vector of generalized displacements
- M is the *mass matrix*
- K is the *stiffness matrix*
- D is the *damping matrix*

with state $x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$ we have

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} x$$

Linearization near equilibrium point

nonlinear, time-invariant differential equation (DE):

$$\dot{x} = f(x)$$

where $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$

suppose x_e is an *equilibrium point*, i.e., $f(x_e) = 0$
(so $x(t) = x_e$ satisfies DE)

now suppose $x(t)$ is near x_e , so

$$\dot{x}(t) = f(x(t)) \approx f(x_e) + Df(x_e)(x(t) - x_e)$$

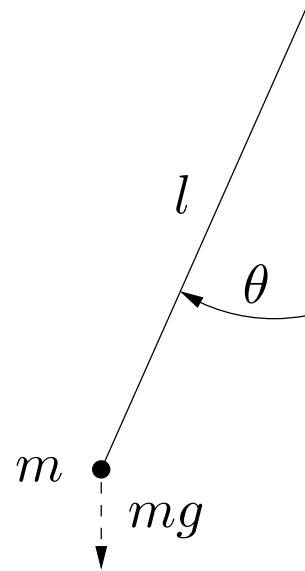
with $\delta x(t) = x(t) - x_e$, rewrite as

$$\dot{\delta x}(t) \approx Df(x_e)\delta x(t)$$

replacing \approx with $=$ yields *linearized approximation* of DE near x_e

we *hope* solution of $\dot{\delta x} = Df(x_e)\delta x$ is a good approximation of $x - x_e$
(more later)

example: pendulum



2nd order nonlinear DE $ml^2\ddot{\theta} = -lmg \sin \theta$

rewrite as first order DE with state $x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$:

$$\dot{x} = \begin{bmatrix} x_2 \\ -(g/l) \sin x_1 \end{bmatrix}$$

equilibrium point (pendulum down): $x = 0$

linearized system near $x_e = 0$:

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 \\ -g/l & 0 \end{bmatrix} \delta x$$

Does linearization ‘work’?

the linearized system usually, but not always, gives a good idea of the system behavior near x_e

example 1: $\dot{x} = -x^3$ near $x_e = 0$

for $x(0) > 0$ solutions have form $x(t) = (x(0)^{-2} + 2t)^{-1/2}$

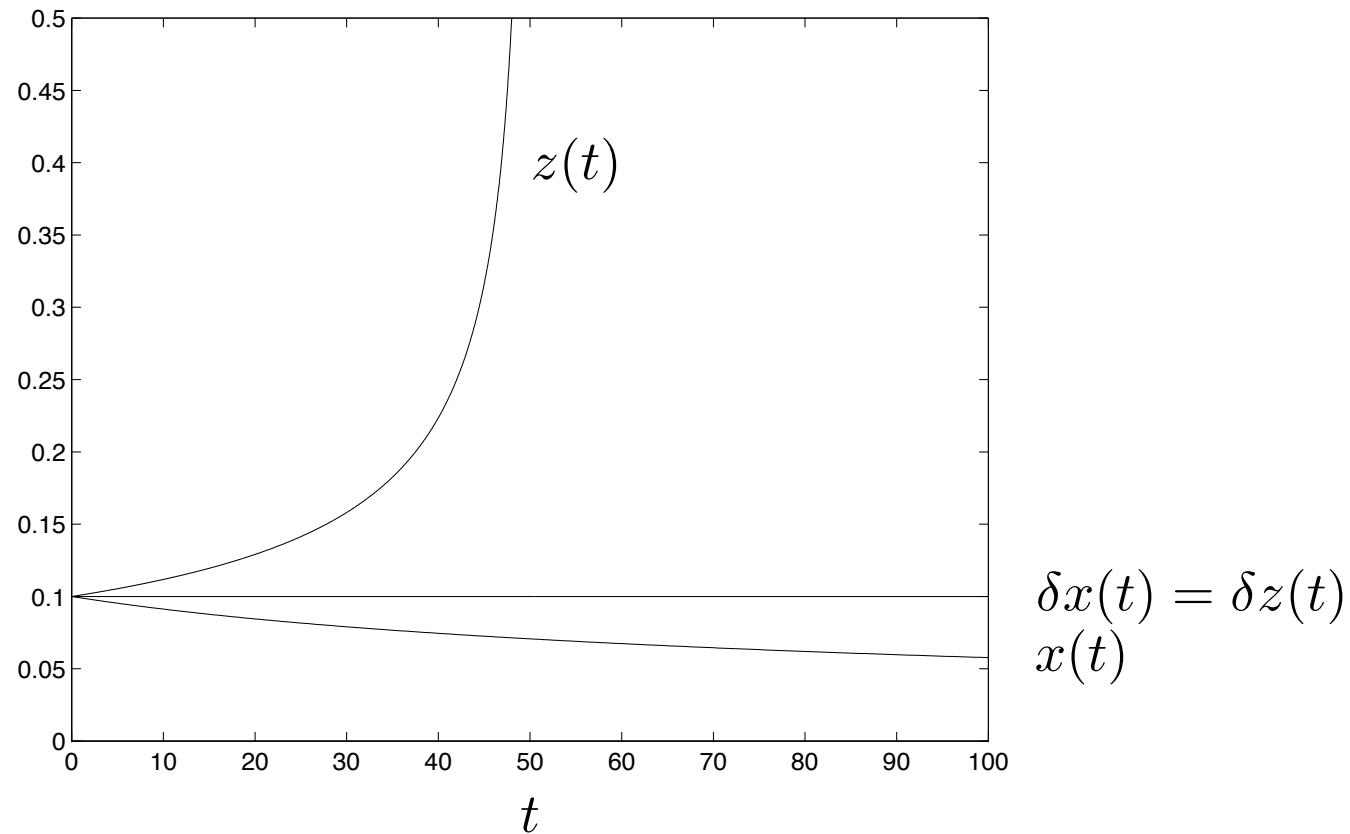
linearized system is $\dot{\delta x} = 0$; solutions are constant

example 2: $\dot{z} = z^3$ near $z_e = 0$

for $z(0) > 0$ solutions have form $z(t) = (z(0)^{-2} - 2t)^{-1/2}$

(finite escape time at $t = z(0)^{-2}/2$)

linearized system is $\dot{\delta z} = 0$; solutions are constant



- systems with very different behavior have same linearized system
- linearized systems do not predict qualitative behavior of either system

Linearization along trajectory

- suppose $x_{\text{traj}} : \mathbf{R}_+ \rightarrow \mathbf{R}^n$ satisfies $\dot{x}_{\text{traj}}(t) = f(x_{\text{traj}}(t), t)$
- suppose $x(t)$ is another trajectory, i.e., $\dot{x}(t) = f(x(t), t)$, and is near $x_{\text{traj}}(t)$
- then

$$\frac{d}{dt}(x - x_{\text{traj}}) = f(x, t) - f(x_{\text{traj}}, t) \approx D_x f(x_{\text{traj}}, t)(x - x_{\text{traj}})$$

- (time-varying) LDS

$$\dot{\delta x} = D_x f(x_{\text{traj}}, t)\delta x$$

is called *linearized* or *variational system* along trajectory x_{traj}

example: linearized oscillator

suppose $x_{\text{traj}}(t)$ is T -periodic solution of nonlinear DE:

$$\dot{x}_{\text{traj}}(t) = f(x_{\text{traj}}(t)), \quad x_{\text{traj}}(t + T) = x_{\text{traj}}(t)$$

linearized system is

$$\dot{\delta x} = A(t)\delta x$$

where $A(t) = Df(x_{\text{traj}}(t))$

$A(t)$ is T -periodic, so linearized system is called *T -periodic linear system*.

used to study:

- startup dynamics of clock and oscillator circuits
- effects of power supply and other disturbances on clock behavior

Lecture 10

Solution via Laplace transform and matrix exponential

- Laplace transform
- solving $\dot{x} = Ax$ via Laplace transform
- state transition matrix
- matrix exponential
- qualitative behavior and stability

Laplace transform of matrix valued function

suppose $z : \mathbf{R}_+ \rightarrow \mathbf{R}^{p \times q}$

Laplace transform: $Z = \mathcal{L}(z)$, where $Z : D \subseteq \mathbf{C} \rightarrow \mathbf{C}^{p \times q}$ is defined by

$$Z(s) = \int_0^\infty e^{-st} z(t) dt$$

- integral of matrix is done term-by-term
- convention: upper case denotes Laplace transform
- D is the *domain or region of convergence* of Z
- D includes at least $\{s \mid \Re s > a\}$, where a satisfies $|z_{ij}(t)| \leq \alpha e^{at}$ for $t \geq 0$, $i = 1, \dots, p$, $j = 1, \dots, q$

Derivative property

$$\mathcal{L}(\dot{z}) = sZ(s) - z(0)$$

to derive, integrate by parts:

$$\begin{aligned}\mathcal{L}(\dot{z})(s) &= \int_0^\infty e^{-st} \dot{z}(t) dt \\ &= e^{-st} z(t) \Big|_{t=0}^{t \rightarrow \infty} + s \int_0^\infty e^{-st} z(t) dt \\ &= sZ(s) - z(0)\end{aligned}$$

Laplace transform solution of $\dot{x} = Ax$

consider continuous-time time-invariant (TI) LDS

$$\dot{x} = Ax$$

for $t \geq 0$, where $x(t) \in \mathbf{R}^n$

- take Laplace transform: $sX(s) - x(0) = AX(s)$
- rewrite as $(sI - A)X(s) = x(0)$
- hence $X(s) = (sI - A)^{-1}x(0)$
- take inverse transform

$$x(t) = \mathcal{L}^{-1} \left((sI - A)^{-1} \right) x(0)$$

Resolvent and state transition matrix

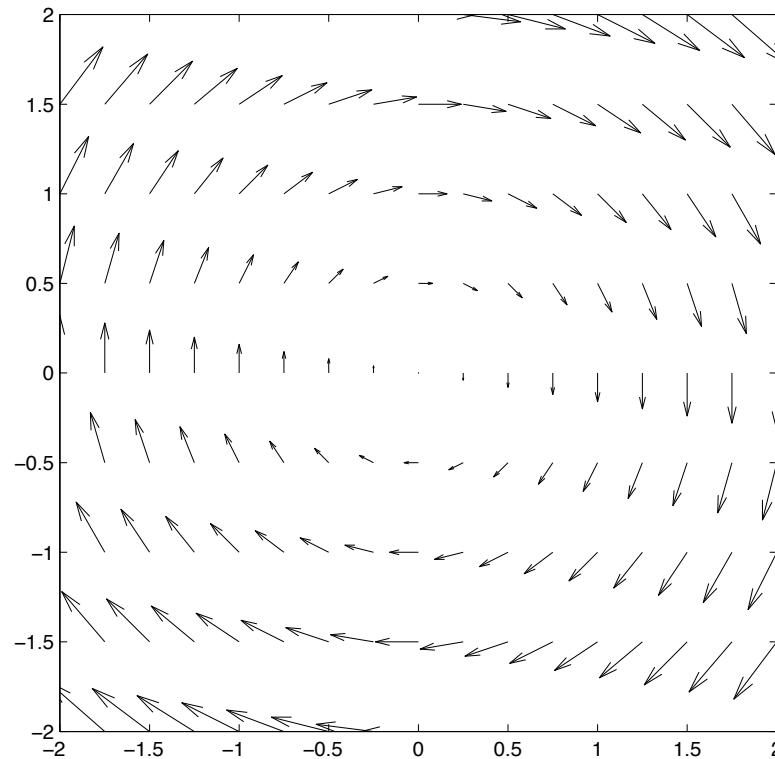
- $(sI - A)^{-1}$ is called the *resolvent* of A
- resolvent defined for $s \in \mathbf{C}$ except eigenvalues of A , i.e., s such that $\det(sI - A) = 0$
- $\Phi(t) = \mathcal{L}^{-1}((sI - A)^{-1})$ is called the *state-transition matrix*; it maps the initial state to the state at time t :

$$x(t) = \Phi(t)x(0)$$

(in particular, state $x(t)$ is a linear function of initial state $x(0)$)

Example 1: Harmonic oscillator

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$$



$sI - A = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}$, so resolvent is

$$(sI - A)^{-1} = \begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix}$$

(eigenvalues are $\pm i$)

state transition matrix is

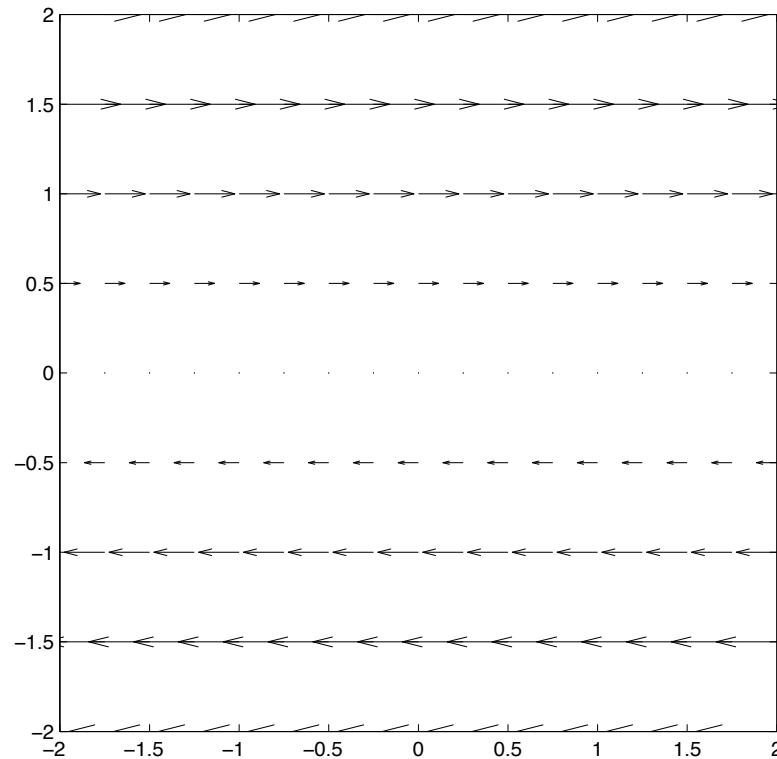
$$\Phi(t) = \mathcal{L}^{-1} \left(\begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix} \right) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

a rotation matrix ($-t$ radians)

so we have $x(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x(0)$

Example 2: Double integrator

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x$$



$$sI - A = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}, \text{ so resolvent is}$$

$$(sI - A)^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix}$$

(eigenvalues are 0, 0)

state transition matrix is

$$\Phi(t) = \mathcal{L}^{-1} \left(\begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix} \right) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$\text{so we have } x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0)$$

Characteristic polynomial

$\mathcal{X}(s) = \det(sI - A)$ is called the *characteristic polynomial* of A

- $\mathcal{X}(s)$ is a polynomial of degree n , with leading (*i.e.*, s^n) coefficient one
- roots of \mathcal{X} are the eigenvalues of A
- \mathcal{X} has real coefficients, so eigenvalues are either real or occur in conjugate pairs
- there are n eigenvalues (if we count multiplicity as roots of \mathcal{X})

Eigenvalues of A and poles of resolvent

i, j entry of resolvent can be expressed via Cramer's rule as

$$(-1)^{i+j} \frac{\det \Delta_{ij}}{\det(sI - A)}$$

where Δ_{ij} is $sI - A$ with j th row and i th column deleted

- $\det \Delta_{ij}$ is a polynomial of degree less than n , so i, j entry of resolvent has form $f_{ij}(s)/\mathcal{X}(s)$ where f_{ij} is polynomial with degree less than n
- poles of entries of resolvent must be eigenvalues of A
- but not all eigenvalues of A show up as poles of each entry
(when there are cancellations between $\det \Delta_{ij}$ and $\mathcal{X}(s)$)

Matrix exponential

$$(I - C)^{-1} = I + C + C^2 + C^3 + \dots \text{ (if series converges)}$$

- series expansion of resolvent:

$$(sI - A)^{-1} = (1/s)(I - A/s)^{-1} = \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \dots$$

(valid for $|s|$ large enough) so

$$\Phi(t) = \mathcal{L}^{-1} \left((sI - A)^{-1} \right) = I + tA + \frac{(tA)^2}{2!} + \dots$$

- looks like ordinary power series

$$e^{at} = 1 + ta + \frac{(ta)^2}{2!} + \dots$$

with square matrices instead of scalars . . .

- define **matrix exponential** as

$$e^M = I + M + \frac{M^2}{2!} + \dots$$

for $M \in \mathbf{R}^{n \times n}$ (which in fact converges for all M)

- with this definition, state-transition matrix is

$$\Phi(t) = \mathcal{L}^{-1}((sI - A)^{-1}) = e^{tA}$$

Matrix exponential solution of autonomous LDS

solution of $\dot{x} = Ax$, with $A \in \mathbf{R}^{n \times n}$ and constant, is

$$x(t) = e^{tA}x(0)$$

generalizes scalar case: solution of $\dot{x} = ax$, with $a \in \mathbf{R}$ and constant, is

$$x(t) = e^{ta}x(0)$$

- matrix exponential is *meant* to look like scalar exponential
- some things you'd guess hold for the matrix exponential (by analogy with the scalar exponential) do in fact hold
- but **many things you'd guess are wrong**

example: you might guess that $e^{A+B} = e^A e^B$, but it's false (in general)

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$e^A = \begin{bmatrix} 0.54 & 0.84 \\ -0.84 & 0.54 \end{bmatrix}, \quad e^B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$e^{A+B} = \begin{bmatrix} 0.16 & 1.40 \\ -0.70 & 0.16 \end{bmatrix} \neq e^A e^B = \begin{bmatrix} 0.54 & 1.38 \\ -0.84 & -0.30 \end{bmatrix}$$

however, we do have $e^{A+B} = e^A e^B$ if $AB = BA$, i.e., A and B commute

thus for $t, s \in \mathbf{R}$, $e^{(tA+sA)} = e^{tA} e^{sA}$

with $s = -t$ we get

$$e^{tA} e^{-tA} = e^{tA-tA} = e^0 = I$$

so e^{tA} is nonsingular, with inverse

$$(e^{tA})^{-1} = e^{-tA}$$

example: let's find e^A , where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

we already found

$$e^{tA} = \mathcal{L}^{-1}(sI - A)^{-1} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

so, plugging in $t = 1$, we get $e^A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

let's check power series:

$$e^A = I + A + \frac{A^2}{2!} + \cdots = I + A$$

since $A^2 = A^3 = \cdots = 0$

Time transfer property

for $\dot{x} = Ax$ we know

$$x(t) = \Phi(t)x(0) = e^{tA}x(0)$$

interpretation: the matrix e^{tA} propagates initial condition into state at time t

more generally we have, for *any* t and τ ,

$$x(\tau + t) = e^{tA}x(\tau)$$

(to see this, apply result above to $z(t) = x(t + \tau)$)

interpretation: the matrix e^{tA} propagates state t seconds forward in time
(backward if $t < 0$)

- recall first order (forward Euler) *approximate* state update, for small t :

$$x(\tau + t) \approx x(\tau) + t\dot{x}(\tau) = (I + tA)x(\tau)$$

- *exact* solution is

$$x(\tau + t) = e^{tA}x(\tau) = (I + tA + (tA)^2/2! + \dots)x(\tau)$$

- forward Euler is just first two terms in series

Sampling a continuous-time system

suppose $\dot{x} = Ax$

sample x at times $t_1 \leq t_2 \leq \dots$: define $z(k) = x(t_k)$

then $z(k+1) = e^{(t_{k+1}-t_k)A}z(k)$

for uniform sampling $t_{k+1} - t_k = h$, so

$$z(k+1) = e^{hA}z(k),$$

a discrete-time LDS (called *discretized version* of continuous-time system)

Piecewise constant system

consider *time-varying* LDS $\dot{x} = A(t)x$, with

$$A(t) = \begin{cases} A_0 & 0 \leq t < t_1 \\ A_1 & t_1 \leq t < t_2 \\ \vdots & \end{cases}$$

where $0 < t_1 < t_2 < \dots$ (sometimes called jump linear system)

for $t \in [t_i, t_{i+1}]$ we have

$$x(t) = e^{(t-t_i)A_i} \dots e^{(t_3-t_2)A_2} e^{(t_2-t_1)A_1} e^{t_1 A_0} x(0)$$

(matrix on righthand side is called state transition matrix for system, and denoted $\Phi(t)$)

Qualitative behavior of $x(t)$

suppose $\dot{x} = Ax$, $x(t) \in \mathbf{R}^n$

then $x(t) = e^{tA}x(0)$; $X(s) = (sI - A)^{-1}x(0)$

i th component $X_i(s)$ has form

$$X_i(s) = \frac{a_i(s)}{\mathcal{X}(s)}$$

where a_i is a polynomial of degree $< n$

thus the poles of X_i are all eigenvalues of A (but not necessarily the other way around)

first assume eigenvalues λ_i are distinct, so $X_i(s)$ cannot have repeated poles

then $x_i(t)$ has form

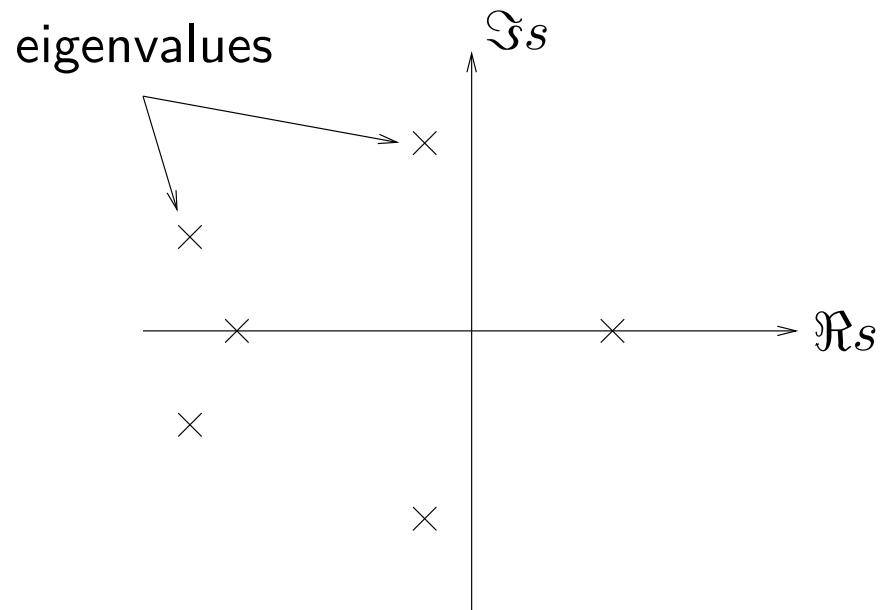
$$x_i(t) = \sum_{j=1}^n \beta_{ij} e^{\lambda_j t}$$

where β_{ij} depend on $x(0)$ (linearly)

eigenvalues determine (possible) qualitative behavior of x :

- eigenvalues give exponents that can occur in exponentials
- real eigenvalue λ corresponds to an exponentially decaying or growing term $e^{\lambda t}$ in solution
- complex eigenvalue $\lambda = \sigma + i\omega$ corresponds to decaying or growing sinusoidal term $e^{\sigma t} \cos(\omega t + \phi)$ in solution

- $\Re \lambda_j$ gives exponential growth rate (if > 0), or exponential decay rate (if < 0) of term
- $\Im \lambda_j$ gives frequency of oscillatory term (if $\neq 0$)



now suppose A has repeated eigenvalues, so X_i can have repeated poles

express eigenvalues as $\lambda_1, \dots, \lambda_r$ (distinct) with multiplicities n_1, \dots, n_r , respectively ($n_1 + \dots + n_r = n$)

then $x_i(t)$ has form

$$x_i(t) = \sum_{j=1}^r p_{ij}(t) e^{\lambda_j t}$$

where $p_{ij}(t)$ is a polynomial of degree $< n_j$ (that depends linearly on $x(0)$)

Stability

we say system $\dot{x} = Ax$ is *stable* if $e^{tA} \rightarrow 0$ as $t \rightarrow \infty$

meaning:

- state $x(t)$ converges to 0, as $t \rightarrow \infty$, no matter what $x(0)$ is
- all trajectories of $\dot{x} = Ax$ converge to 0 as $t \rightarrow \infty$

fact: $\dot{x} = Ax$ is stable if and only if all eigenvalues of A have negative real part:

$$\Re \lambda_i < 0, \quad i = 1, \dots, n$$

the ‘if’ part is clear since

$$\lim_{t \rightarrow \infty} p(t)e^{\lambda t} = 0$$

for any polynomial, if $\Re \lambda < 0$

we’ll see the ‘only if’ part next lecture

more generally, $\max_i \Re \lambda_i$ determines the maximum asymptotic logarithmic growth rate of $x(t)$ (or decay, if < 0)

Lecture 11

Eigenvectors and diagonalization

- eigenvectors
- dynamic interpretation: invariant sets
- complex eigenvectors & invariant planes
- left eigenvectors
- diagonalization
- modal form
- discrete-time stability

Eigenvectors and eigenvalues

$\lambda \in \mathbf{C}$ is an *eigenvalue* of $A \in \mathbf{C}^{n \times n}$ if

$$\chi(\lambda) = \det(\lambda I - A) = 0$$

equivalent to:

- there exists nonzero $v \in \mathbf{C}^n$ s.t. $(\lambda I - A)v = 0$, i.e.,

$$Av = \lambda v$$

any such v is called an *eigenvector* of A (associated with eigenvalue λ)

- there exists nonzero $w \in \mathbf{C}^n$ s.t. $w^T(\lambda I - A) = 0$, i.e.,

$$w^T A = \lambda w^T$$

any such w is called a *left eigenvector* of A

- if v is an eigenvector of A with eigenvalue λ , then so is αv , for any $\alpha \in \mathbf{C}$, $\alpha \neq 0$
- even when A is real, eigenvalue λ and eigenvector v can be complex
- when A and λ are real, we can always find a real eigenvector v associated with λ : if $Av = \lambda v$, with $A \in \mathbf{R}^{n \times n}$, $\lambda \in \mathbf{R}$, and $v \in \mathbf{C}^n$, then

$$A\Re v = \lambda \Re v, \quad A\Im v = \lambda \Im v$$

so $\Re v$ and $\Im v$ are real eigenvectors, if they are nonzero (and at least one is)

- *conjugate symmetry*: if A is real and $v \in \mathbf{C}^n$ is an eigenvector associated with $\lambda \in \mathbf{C}$, then \bar{v} is an eigenvector associated with $\bar{\lambda}$: taking conjugate of $Av = \lambda v$ we get $\overline{Av} = \overline{\lambda v}$, so

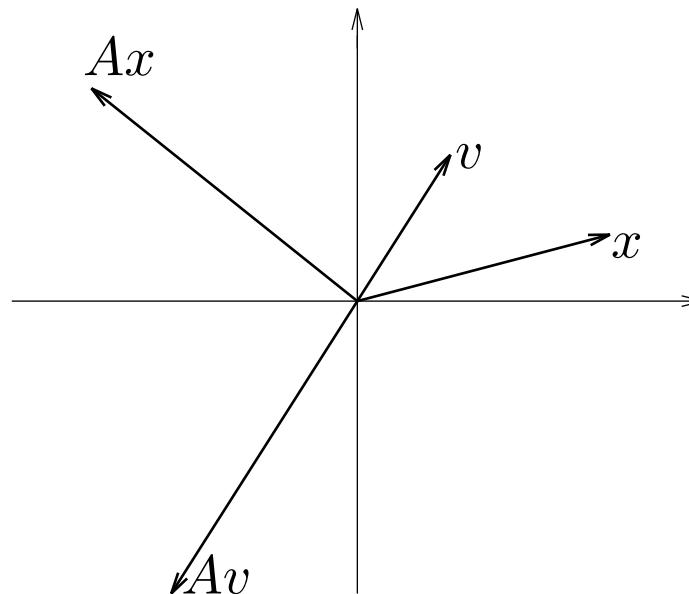
$$A\bar{v} = \bar{\lambda}\bar{v}$$

we'll assume A is real from now on . . .

Scaling interpretation

(assume $\lambda \in \mathbf{R}$ for now; we'll consider $\lambda \in \mathbf{C}$ later)

if v is an eigenvector, effect of A on v is very simple: scaling by λ



(what is λ here?)

- $\lambda \in \mathbf{R}$, $\lambda > 0$: v and Av point in same direction
- $\lambda \in \mathbf{R}$, $\lambda < 0$: v and Av point in opposite directions
- $\lambda \in \mathbf{R}$, $|\lambda| < 1$: Av smaller than v
- $\lambda \in \mathbf{R}$, $|\lambda| > 1$: Av larger than v

(we'll see later how this relates to stability of continuous- and discrete-time systems. . .)

Dynamic interpretation

suppose $Av = \lambda v$, $v \neq 0$

if $\dot{x} = Ax$ and $x(0) = v$, then $x(t) = e^{\lambda t}v$

several ways to see this, *e.g.*,

$$\begin{aligned}x(t) = e^{tA}v &= \left(I + tA + \frac{(tA)^2}{2!} + \dots \right) v \\&= v + \lambda tv + \frac{(\lambda t)^2}{2!}v + \dots \\&= e^{\lambda t}v\end{aligned}$$

(since $(tA)^k v = (\lambda t)^k v$)

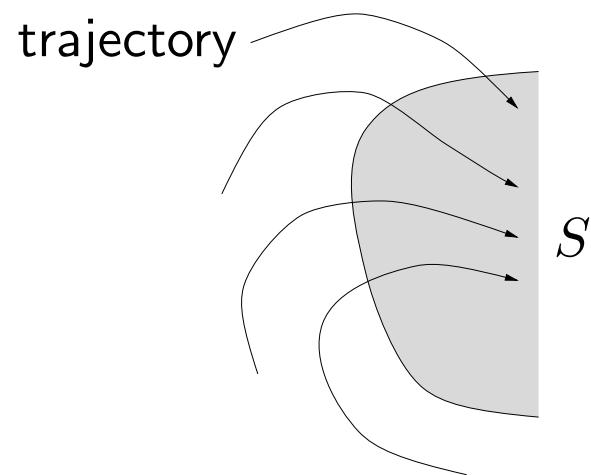
- for $\lambda \in \mathbf{C}$, solution is complex (we'll interpret later); for now, assume $\lambda \in \mathbf{R}$
- if initial state is an eigenvector v , resulting motion is very simple — always on the line spanned by v
- solution $x(t) = e^{\lambda t}v$ is called *mode* of system $\dot{x} = Ax$ (associated with eigenvalue λ)

- for $\lambda \in \mathbf{R}$, $\lambda < 0$, mode contracts or shrinks as $t \uparrow$
- for $\lambda \in \mathbf{R}$, $\lambda > 0$, mode expands or grows as $t \uparrow$

Invariant sets

a set $S \subseteq \mathbf{R}^n$ is *invariant* under $\dot{x} = Ax$ if whenever $x(t) \in S$, then $x(\tau) \in S$ for all $\tau \geq t$

i.e.: once trajectory enters S , it stays in S



vector field interpretation: trajectories only cut *into* S , never out

suppose $Av = \lambda v$, $v \neq 0$, $\lambda \in \mathbf{R}$

- line $\{ tv \mid t \in \mathbf{R} \}$ is invariant
(in fact, ray $\{ tv \mid t > 0 \}$ is invariant)
- if $\lambda < 0$, line segment $\{ tv \mid 0 \leq t \leq a \}$ is invariant

Complex eigenvectors

suppose $Av = \lambda v$, $v \neq 0$, λ is complex

for $a \in \mathbf{C}$, (complex) trajectory $ae^{\lambda t}v$ satisfies $\dot{x} = Ax$

hence so does (real) trajectory

$$\begin{aligned}x(t) &= \Re(ae^{\lambda t}v) \\&= e^{\sigma t} \begin{bmatrix} v_{\text{re}} & v_{\text{im}} \end{bmatrix} \begin{bmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{bmatrix} \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}\end{aligned}$$

where

$$v = v_{\text{re}} + iv_{\text{im}}, \quad \lambda = \sigma + i\omega, \quad a = \alpha + i\beta$$

- trajectory stays in *invariant plane* $\text{span}\{v_{\text{re}}, v_{\text{im}}\}$
- σ gives logarithmic growth/decay factor
- ω gives angular velocity of rotation in plane

Dynamic interpretation: left eigenvectors

suppose $w^T A = \lambda w^T$, $w \neq 0$

then

$$\frac{d}{dt}(w^T x) = w^T \dot{x} = w^T A x = \lambda(w^T x)$$

i.e., $w^T x$ satisfies the DE $d(w^T x)/dt = \lambda(w^T x)$

hence $w^T x(t) = e^{\lambda t} w^T x(0)$

- even if trajectory x is complicated, $w^T x$ is simple
- if, e.g., $\lambda \in \mathbf{R}$, $\lambda < 0$, halfspace $\{ z \mid w^T z \leq a \}$ is invariant (for $a \geq 0$)
- for $\lambda = \sigma + i\omega \in \mathbf{C}$, $(\Re w)^T x$ and $(\Im w)^T x$ both have form

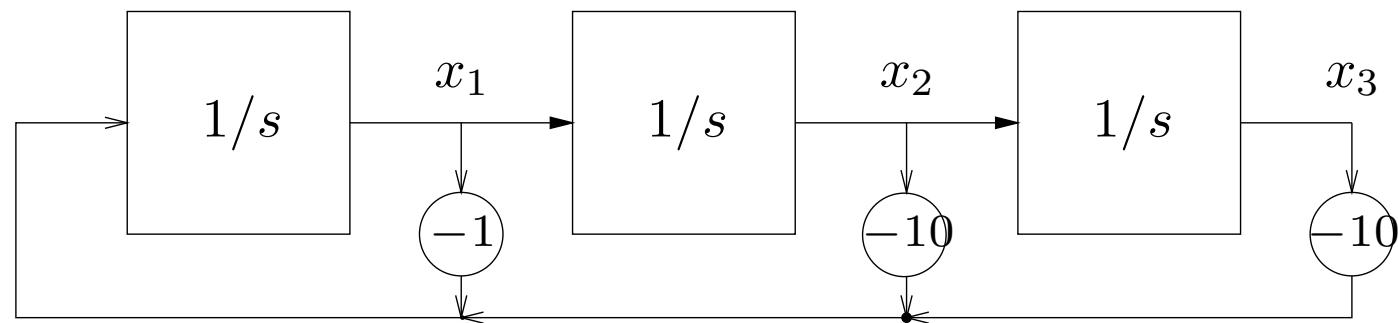
$$e^{\sigma t} (\alpha \cos(\omega t) + \beta \sin(\omega t))$$

Summary

- *right eigenvectors* are initial conditions from which resulting motion is simple (*i.e.*, remains on line or in plane)
- *left eigenvectors* give linear functions of state that are simple, for any initial condition

example 1: $\dot{x} = \begin{bmatrix} -1 & -10 & -10 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x$

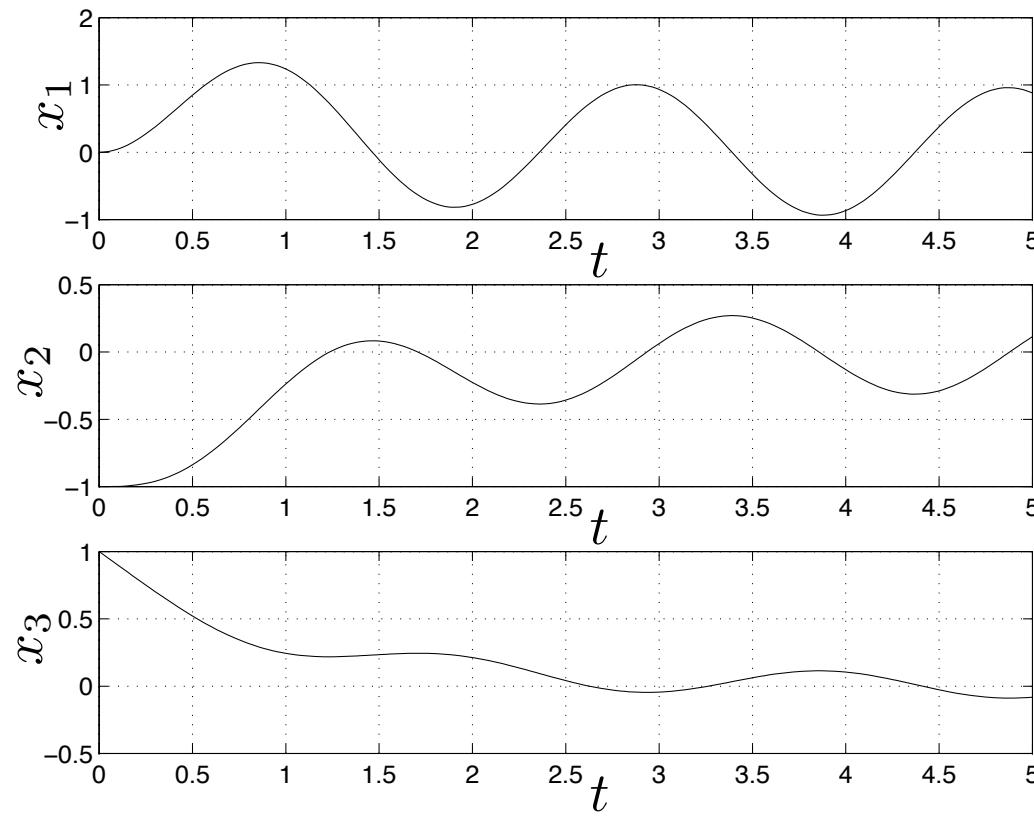
block diagram:



$$\mathcal{X}(s) = s^3 + s^2 + 10s + 10 = (s + 1)(s^2 + 10)$$

eigenvalues are $-1, \pm i\sqrt{10}$

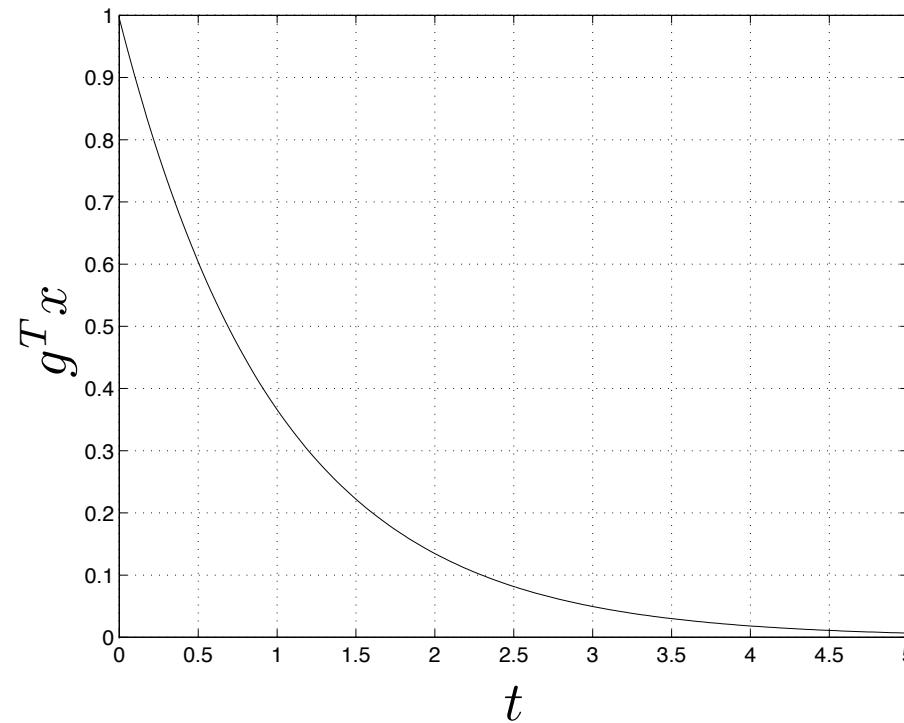
trajectory with $x(0) = (0, -1, 1)$:



left eigenvector associated with eigenvalue -1 is

$$g = \begin{bmatrix} 0.1 \\ 0 \\ 1 \end{bmatrix}$$

let's check $g^T x(t)$ when $x(0) = (0, -1, 1)$ (as above):



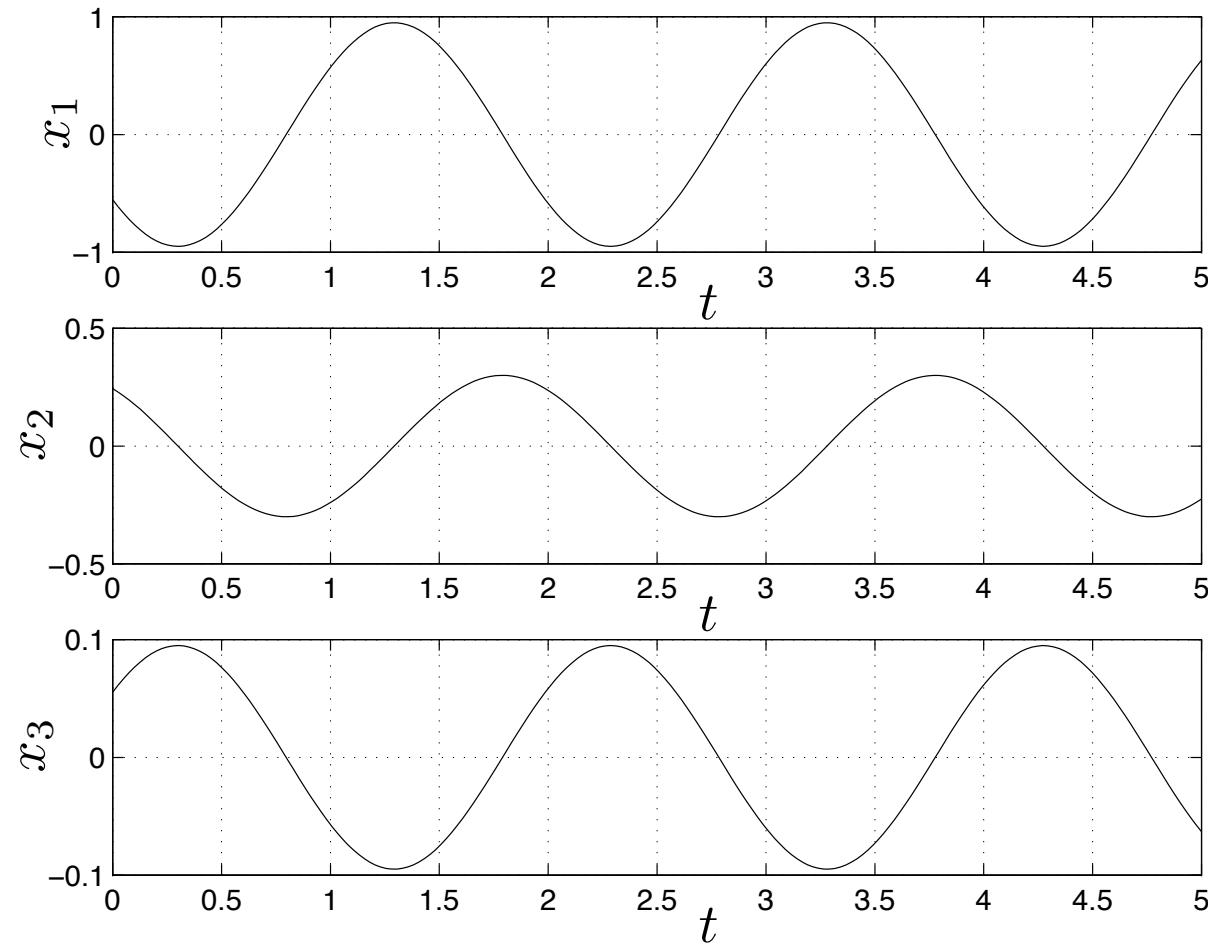
eigenvector associated with eigenvalue $i\sqrt{10}$ is

$$v = \begin{bmatrix} -0.554 + i0.771 \\ 0.244 + i0.175 \\ 0.055 - i0.077 \end{bmatrix}$$

so an invariant plane is spanned by

$$v_{\text{re}} = \begin{bmatrix} -0.554 \\ 0.244 \\ 0.055 \end{bmatrix}, \quad v_{\text{im}} = \begin{bmatrix} 0.771 \\ 0.175 \\ -0.077 \end{bmatrix}$$

for example, with $x(0) = v_{\text{re}}$ we have



Example 2: Markov chain

probability distribution satisfies $p(t + 1) = Pp(t)$

$p_i(t) = \mathbf{Prob}(z(t) = i)$ so $\sum_{i=1}^n p_i(t) = 1$

$P_{ij} = \mathbf{Prob}(z(t + 1) = i \mid z(t) = j)$, so $\sum_{i=1}^n P_{ij} = 1$
(such matrices are called *stochastic*)

rewrite as:

$$[1 \ 1 \ \dots \ 1]P = [1 \ 1 \ \dots \ 1]$$

i.e., $[1 \ 1 \ \dots \ 1]$ is a left eigenvector of P with e.v. 1

hence $\det(I - P) = 0$, so there is a right eigenvector $v \neq 0$ with $Pv = v$

it can be shown that v can be chosen so that $v_i \geq 0$, hence we can normalize v so that $\sum_{i=1}^n v_i = 1$

interpretation: v is an *equilibrium distribution*; i.e., if $p(0) = v$ then $p(t) = v$ for all $t \geq 0$

(if v is unique it is called the *steady-state distribution* of the Markov chain)

Diagonalization

suppose v_1, \dots, v_n is a *linearly independent* set of eigenvectors of $A \in \mathbf{R}^{n \times n}$:

$$Av_i = \lambda_i v_i, \quad i = 1, \dots, n$$

express as

$$A \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

define $T = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$ and $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$, so

$$AT = T\Lambda$$

and finally

$$T^{-1}AT = \Lambda$$

- T invertible since v_1, \dots, v_n linearly independent
- similarity transformation by T diagonalizes A

conversely if there is a $T = [v_1 \ \cdots \ v_n]$ s.t.

$$T^{-1}AT = \Lambda = \mathbf{diag}(\lambda_1, \dots, \lambda_n)$$

then $AT = T\Lambda$, i.e.,

$$Av_i = \lambda_i v_i, \quad i = 1, \dots, n$$

so v_1, \dots, v_n is a linearly independent set of n eigenvectors of A

we say A is *diagonalizable* if

- there exists T s.t. $T^{-1}AT = \Lambda$ is diagonal
- A has a set of n linearly independent eigenvectors

(if A is not diagonalizable, it is sometimes called *defective*)

Not all matrices are diagonalizable

example: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

characteristic polynomial is $\chi(s) = s^2$, so $\lambda = 0$ is only eigenvalue
eigenvectors satisfy $Av = 0v = 0$, i.e.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

so all eigenvectors have form $v = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$ where $v_1 \neq 0$

thus, A cannot have two independent eigenvectors

Distinct eigenvalues

fact: if A has distinct eigenvalues, i.e., $\lambda_i \neq \lambda_j$ for $i \neq j$, then A is diagonalizable

(the converse is false — A can have repeated eigenvalues but still be diagonalizable)

Diagonalization and left eigenvectors

rewrite $T^{-1}AT = \Lambda$ as $T^{-1}A = \Lambda T^{-1}$, or

$$\begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix} A = \Lambda \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}$$

where w_1^T, \dots, w_n^T are the rows of T^{-1}

thus

$$w_i^T A = \lambda_i w_i^T$$

i.e., the rows of T^{-1} are (lin. indep.) left eigenvectors, normalized so that

$$w_i^T v_j = \delta_{ij}$$

(i.e., left & right eigenvectors chosen this way are *dual bases*)

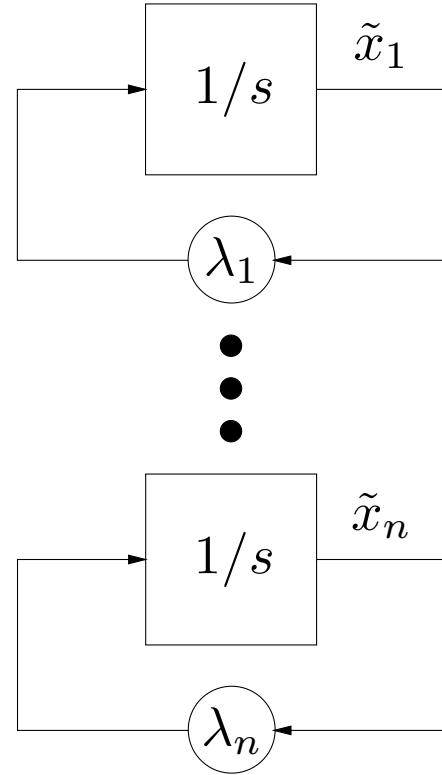
Modal form

suppose A is diagonalizable by T

define new coordinates by $x = T\tilde{x}$, so

$$T\dot{\tilde{x}} = A\tilde{x} \Leftrightarrow \dot{\tilde{x}} = T^{-1}A\tilde{x} \Leftrightarrow \dot{\tilde{x}} = \Lambda\tilde{x}$$

in new coordinate system, system is diagonal (decoupled):



trajectories consist of n independent modes, *i.e.*,

$$\tilde{x}_i(t) = e^{\lambda_i t} \tilde{x}_i(0)$$

hence the name *modal form*

Real modal form

when eigenvalues (hence T) are complex, system can be put in *real modal form*:

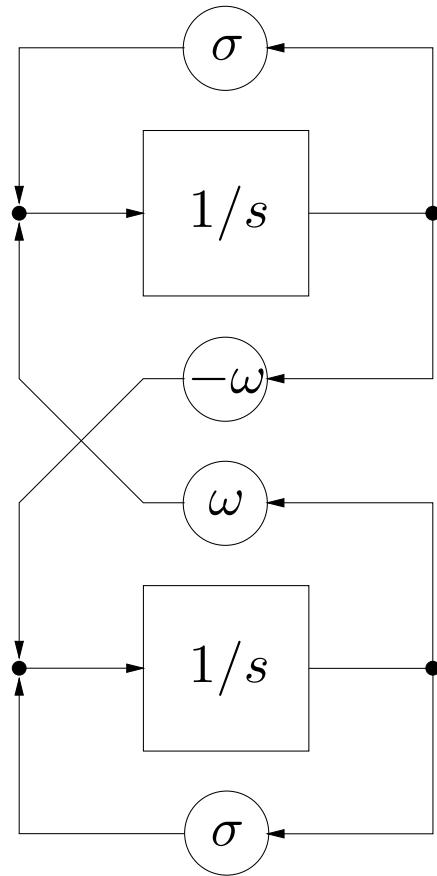
$$S^{-1}AS = \mathbf{diag}(\Lambda_r, M_{r+1}, M_{r+3}, \dots, M_{n-1})$$

where $\Lambda_r = \mathbf{diag}(\lambda_1, \dots, \lambda_r)$ are the real eigenvalues, and

$$M_j = \begin{bmatrix} \sigma_j & \omega_j \\ -\omega_j & \sigma_j \end{bmatrix}, \quad \lambda_j = \sigma_j + i\omega_j, \quad j = r+1, r+3, \dots, n$$

where λ_j are the complex eigenvalues (one from each conjugate pair)

block diagram of 'complex mode':



diagonalization simplifies many matrix expressions

e.g., resolvent:

$$\begin{aligned}(sI - A)^{-1} &= (sTT^{-1} - T\Lambda T^{-1})^{-1} \\&= (T(sI - \Lambda)T^{-1})^{-1} \\&= T(sI - \Lambda)^{-1}T^{-1} \\&= T \mathbf{diag}\left(\frac{1}{s - \lambda_1}, \dots, \frac{1}{s - \lambda_n}\right) T^{-1}\end{aligned}$$

powers (*i.e.*, discrete-time solution):

$$\begin{aligned}A^k &= (T\Lambda T^{-1})^k \\&= (T\Lambda T^{-1}) \cdots (T\Lambda T^{-1}) \\&= T\Lambda^k T^{-1} \\&= T \mathbf{diag}(\lambda_1^k, \dots, \lambda_n^k) T^{-1}\end{aligned}$$

(for $k < 0$ only if A invertible, *i.e.*, all $\lambda_i \neq 0$)

exponential (*i.e.*, continuous-time solution):

$$\begin{aligned} e^A &= I + A + A^2/2! + \dots \\ &= I + T\Lambda T^{-1} + (T\Lambda T^{-1})^2/2! + \dots \\ &= T(I + \Lambda + \Lambda^2/2! + \dots)T^{-1} \\ &= Te^\Lambda T^{-1} \\ &= T \mathbf{diag}(e^{\lambda_1}, \dots, e^{\lambda_n}) T^{-1} \end{aligned}$$

Analytic function of a matrix

for any analytic function $f : \mathbf{R} \rightarrow \mathbf{R}$, i.e., given by power series

$$f(a) = \beta_0 + \beta_1 a + \beta_2 a^2 + \beta_3 a^3 + \dots$$

we can define $f(A)$ for $A \in \mathbf{R}^{n \times n}$ (i.e., overload f) as

$$f(A) = \beta_0 I + \beta_1 A + \beta_2 A^2 + \beta_3 A^3 + \dots$$

substituting $A = T\Lambda T^{-1}$, we have

$$\begin{aligned} f(A) &= \beta_0 I + \beta_1 A + \beta_2 A^2 + \beta_3 A^3 + \dots \\ &= \beta_0 TT^{-1} + \beta_1 T\Lambda T^{-1} + \beta_2 (T\Lambda T^{-1})^2 + \dots \\ &= T (\beta_0 I + \beta_1 \Lambda + \beta_2 \Lambda^2 + \dots) T^{-1} \\ &= T \mathbf{diag}(f(\lambda_1), \dots, f(\lambda_n)) T^{-1} \end{aligned}$$

Solution via diagonalization

assume A is diagonalizable

consider LDS $\dot{x} = Ax$, with $T^{-1}AT = \Lambda$

then

$$\begin{aligned}x(t) &= e^{tA}x(0) \\&= Te^{\Lambda t}T^{-1}x(0) \\&= \sum_{i=1}^n e^{\lambda_i t}(w_i^T x(0))v_i\end{aligned}$$

thus: any trajectory can be expressed as linear combination of modes

interpretation:

- (left eigenvectors) decompose initial state $x(0)$ into modal components $w_i^T x(0)$
- $e^{\lambda_i t}$ term propagates i th mode forward t seconds
- reconstruct state as linear combination of (right) eigenvectors

application: for what $x(0)$ do we have $x(t) \rightarrow 0$ as $t \rightarrow \infty$?

divide eigenvalues into those with negative real parts

$$\Re \lambda_1 < 0, \dots, \Re \lambda_s < 0,$$

and the others,

$$\Re \lambda_{s+1} \geq 0, \dots, \Re \lambda_n \geq 0$$

from

$$x(t) = \sum_{i=1}^n e^{\lambda_i t} (w_i^T x(0)) v_i$$

condition for $x(t) \rightarrow 0$ is:

$$x(0) \in \text{span}\{v_1, \dots, v_s\},$$

or equivalently,

$$w_i^T x(0) = 0, \quad i = s + 1, \dots, n$$

(can you prove this?)

Stability of discrete-time systems

suppose A diagonalizable

consider discrete-time LDS $x(t+1) = Ax(t)$

if $A = T\Lambda T^{-1}$, then $A^k = T\Lambda^k T^{-1}$

then

$$x(t) = A^t x(0) = \sum_{i=1}^n \lambda_i^t (w_i^T x(0)) v_i \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

for all $x(0)$ if and only if

$$|\lambda_i| < 1, \quad i = 1, \dots, n.$$

we will see later that this is true even when A is not diagonalizable, so we have

fact: $x(t+1) = Ax(t)$ is stable if and only if all eigenvalues of A have magnitude less than one

Lecture 12

Jordan canonical form

- Jordan canonical form
- generalized modes
- Cayley-Hamilton theorem

Jordan canonical form

what if A cannot be diagonalized?

any matrix $A \in \mathbf{R}^{n \times n}$ can be put in *Jordan canonical form* by a similarity transformation, i.e.

$$T^{-1}AT = J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix}$$

where

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \in \mathbf{C}^{n_i \times n_i}$$

is called a *Jordan block* of size n_i with eigenvalue λ_i (so $n = \sum_{i=1}^q n_i$)

- J is upper bidiagonal
- J diagonal is the special case of n Jordan blocks of size $n_i = 1$
- Jordan form is unique (up to permutations of the blocks)
- can have multiple blocks with same eigenvalue

note: JCF is a *conceptual tool*, never used in numerical computations!

$$\mathcal{X}(s) = \det(sI - A) = (s - \lambda_1)^{n_1} \cdots (s - \lambda_q)^{n_q}$$

hence distinct eigenvalues $\Rightarrow n_i = 1 \Rightarrow A$ diagonalizable

$\dim \mathcal{N}(\lambda I - A)$ is the number of Jordan blocks with eigenvalue λ

more generally,

$$\dim \mathcal{N}(\lambda I - A)^k = \sum_{\lambda_i = \lambda} \min\{k, n_i\}$$

so from $\dim \mathcal{N}(\lambda I - A)^k$ for $k = 1, 2, \dots$ we can determine the sizes of the Jordan blocks associated with λ

- factor out T and T^{-1} , $\lambda I - A = T(\lambda I - J)T^{-1}$
- for, say, a block of size 3:

$$\lambda_i I - J_i = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad (\lambda_i I - J_i)^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\lambda_i I - J_i)^3 = 0$$

- for other blocks (say, size 3, for $k \geq 2$)

$$(\lambda_i I - J_j)^k = \begin{bmatrix} (\lambda_i - \lambda_j)^k & -k(\lambda_i - \lambda_j)^{k-1} & (k(k-1)/2)(\lambda_i - \lambda_j)^{k-2} \\ 0 & (\lambda_j - \lambda_i)^k & -k(\lambda_j - \lambda_i)^{k-1} \\ 0 & 0 & (\lambda_j - \lambda_i)^k \end{bmatrix}$$

Generalized eigenvectors

suppose $T^{-1}AT = J = \text{diag}(J_1, \dots, J_q)$

express T as

$$T = [T_1 \ T_2 \ \cdots \ T_q]$$

where $T_i \in \mathbf{C}^{n \times n_i}$ are the columns of T associated with i th Jordan block J_i

we have $AT_i = T_iJ_i$

let $T_i = [v_{i1} \ v_{i2} \ \cdots \ v_{in_i}]$

then we have:

$$Av_{i1} = \lambda_i v_{i1},$$

i.e., the first column of each T_i is an eigenvector associated with e.v. λ_i

for $j = 2, \dots, n_i$,

$$Av_{ij} = v_{i(j-1)} + \lambda_i v_{ij}$$

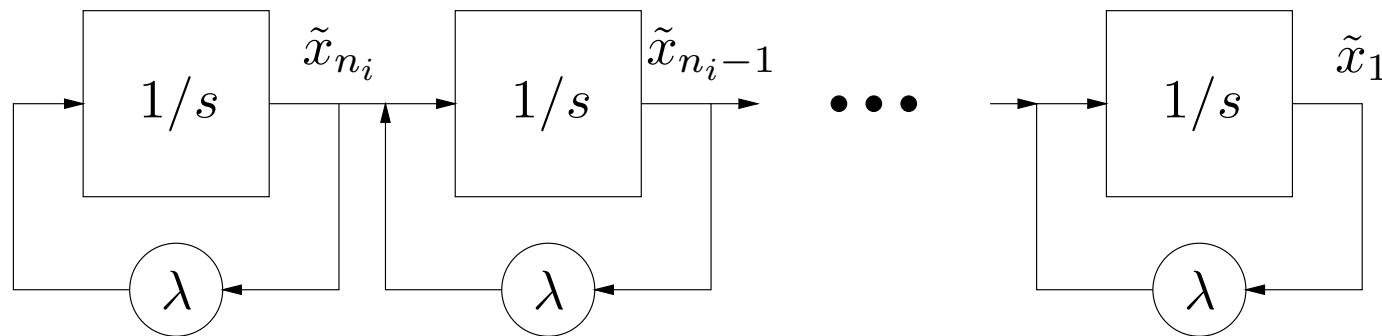
the vectors v_{i1}, \dots, v_{in_i} are sometimes called *generalized eigenvectors*

Jordan form LDS

consider LDS $\dot{x} = Ax$

by change of coordinates $x = T\tilde{x}$, can put into form $\dot{\tilde{x}} = J\tilde{x}$

system is decomposed into independent 'Jordan block systems' $\dot{\tilde{x}}_i = J_i \tilde{x}_i$



Jordan blocks are sometimes called Jordan chains
(block diagram shows why)

Resolvent, exponential of Jordan block

resolvent of $k \times k$ Jordan block with eigenvalue λ :

$$\begin{aligned}(sI - J_\lambda)^{-1} &= \begin{bmatrix} s - \lambda & -1 & & \\ & s - \lambda & \ddots & \\ & & \ddots & -1 \\ & & & s - \lambda \end{bmatrix}^{-1} \\ &= \begin{bmatrix} (s - \lambda)^{-1} & (s - \lambda)^{-2} & \cdots & (s - \lambda)^{-k} \\ & (s - \lambda)^{-1} & \cdots & (s - \lambda)^{-k+1} \\ & & \ddots & \vdots \\ & & & (s - \lambda)^{-1} \end{bmatrix} \\ &= (s - \lambda)^{-1} I + (s - \lambda)^{-2} F_1 + \cdots + (s - \lambda)^{-k} F_{k-1}\end{aligned}$$

where F_i is the matrix with ones on the i th upper diagonal

by inverse Laplace transform, exponential is:

$$\begin{aligned} e^{tJ_\lambda} &= e^{t\lambda} \left(I + tF_1 + \cdots + (t^{k-1}/(k-1)!) F_{k-1} \right) \\ &= e^{t\lambda} \begin{bmatrix} 1 & t & \cdots & t^{k-1}/(k-1)! \\ 1 & \cdots & t^{k-2}/(k-2)! & \\ \ddots & & & \vdots \\ & & & 1 \end{bmatrix} \end{aligned}$$

Jordan blocks yield:

- repeated poles in resolvent
- terms of form $t^p e^{t\lambda}$ in e^{tA}

Generalized modes

consider $\dot{x} = Ax$, with

$$x(0) = a_1 v_{i1} + \cdots + a_{n_i} v_{in_i} = T_i a$$

then $x(t) = T e^{Jt} \tilde{x}(0) = T_i e^{J_i t} a$

- trajectory stays in span of generalized eigenvectors
- coefficients have form $p(t)e^{\lambda t}$, where p is polynomial
- such solutions are called *generalized modes* of the system

with general $x(0)$ we can write

$$x(t) = e^{tA}x(0) = Te^{tJ}T^{-1}x(0) = \sum_{i=1}^q T_i e^{tJ_i} (S_i^T x(0))$$

where

$$T^{-1} = \begin{bmatrix} S_1^T \\ \vdots \\ S_q^T \end{bmatrix}$$

hence: all solutions of $\dot{x} = Ax$ are linear combinations of (generalized) modes

Cayley-Hamilton theorem

if $p(s) = a_0 + a_1s + \cdots + a_ks^k$ is a polynomial and $A \in \mathbf{R}^{n \times n}$, we define

$$p(A) = a_0I + a_1A + \cdots + a_kA^k$$

Cayley-Hamilton theorem: for any $A \in \mathbf{R}^{n \times n}$ we have $\mathcal{X}(A) = 0$, where $\mathcal{X}(s) = \det(sI - A)$

example: with $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ we have $\mathcal{X}(s) = s^2 - 5s - 2$, so

$$\begin{aligned}\mathcal{X}(A) &= A^2 - 5A - 2I \\ &= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - 5 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 2I \\ &= 0\end{aligned}$$

corollary: for every $p \in \mathbf{Z}_+$, we have

$$A^p \in \text{span} \{ I, A, A^2, \dots, A^{n-1} \}$$

(and if A is invertible, also for $p \in \mathbf{Z}$)

i.e., every power of A can be expressed as linear combination of I, A, \dots, A^{n-1}

proof: divide $\mathcal{X}(s)$ into s^p to get $s^p = q(s)\mathcal{X}(s) + r(s)$

$r = \alpha_0 + \alpha_1 s + \dots + \alpha_{n-1} s^{n-1}$ is remainder polynomial

then

$$A^p = q(A)\mathcal{X}(A) + r(A) = r(A) = \alpha_0 I + \alpha_1 A + \dots + \alpha_{n-1} A^{n-1}$$

for $p = -1$: rewrite C-H theorem

$$\mathcal{X}(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_0I = 0$$

as

$$I = A \left(-(a_1/a_0)I - (a_2/a_0)A - \cdots - (1/a_0)A^{n-1} \right)$$

(A is invertible $\Leftrightarrow a_0 \neq 0$) so

$$A^{-1} = -(a_1/a_0)I - (a_2/a_0)A - \cdots - (1/a_0)A^{n-1}$$

i.e., inverse is linear combination of A^k , $k = 0, \dots, n-1$

Proof of C-H theorem

first assume A is diagonalizable: $T^{-1}AT = \Lambda$

$$\mathcal{X}(s) = (s - \lambda_1) \cdots (s - \lambda_n)$$

since

$$\mathcal{X}(A) = \mathcal{X}(T\Lambda T^{-1}) = T\mathcal{X}(\Lambda)T^{-1}$$

it suffices to show $\mathcal{X}(\Lambda) = 0$

$$\begin{aligned}\mathcal{X}(\Lambda) &= (\Lambda - \lambda_1 I) \cdots (\Lambda - \lambda_n I) \\ &= \mathbf{diag}(0, \lambda_2 - \lambda_1, \dots, \lambda_n - \lambda_1) \cdots \mathbf{diag}(\lambda_1 - \lambda_n, \dots, \lambda_{n-1} - \lambda_n, 0) \\ &= 0\end{aligned}$$

now let's do general case: $T^{-1}AT = J$

$$\mathcal{X}(s) = (s - \lambda_1)^{n_1} \cdots (s - \lambda_q)^{n_q}$$

suffices to show $\mathcal{X}(J_i) = 0$

$$\mathcal{X}(J_i) = (J_i - \lambda_1 I)^{n_1} \cdots \underbrace{\left[\begin{array}{cccc} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ & & & \ddots \end{array} \right]^{n_i}}_{(J_i - \lambda_i I)^{n_i}} \cdots (J_i - \lambda_q I)^{n_q} = 0$$

Lecture 13

Linear dynamical systems with inputs & outputs

- inputs & outputs: interpretations
- transfer function
- impulse and step responses
- examples

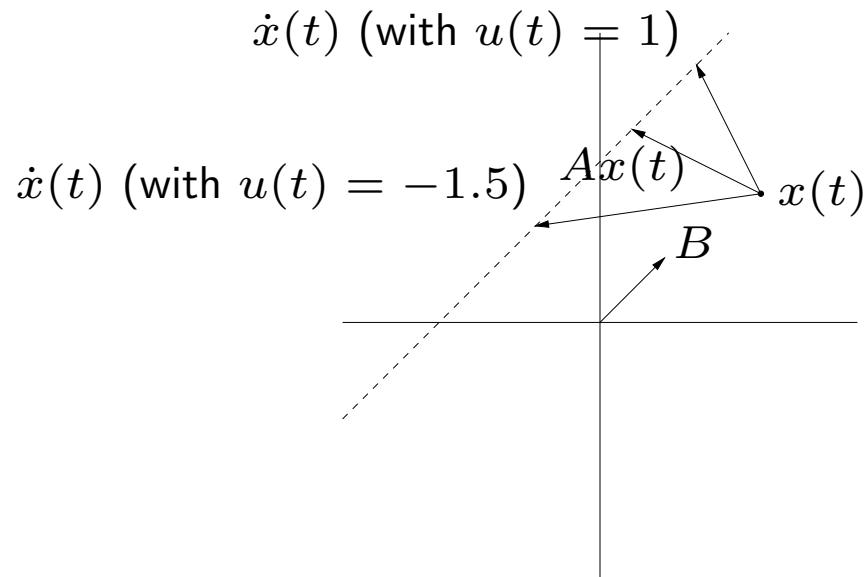
Inputs & outputs

recall continuous-time time-invariant LDS has form

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

- Ax is called the *drift term* (of \dot{x})
- Bu is called the input term (of \dot{x})

picture, with $B \in \mathbf{R}^{2 \times 1}$:



Interpretations

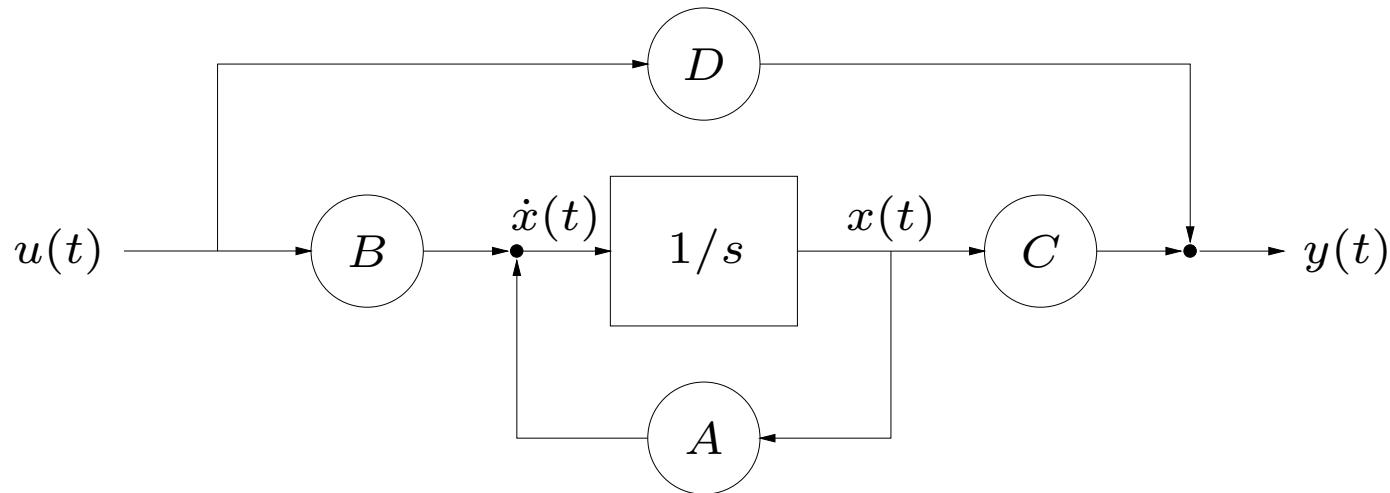
write $\dot{x} = Ax + b_1u_1 + \cdots + b_mu_m$, where $B = [b_1 \ \cdots \ b_m]$

- state derivative is sum of autonomous term (Ax) and one term per input (b_iu_i)
- each input u_i gives another degree of freedom for \dot{x} (assuming columns of B independent)

write $\dot{x} = Ax + Bu$ as $\dot{x}_i = \tilde{a}_i^T x + \tilde{b}_i^T u$, where \tilde{a}_i^T , \tilde{b}_i^T are the rows of A , B

- i th state derivative is linear function of state x and input u

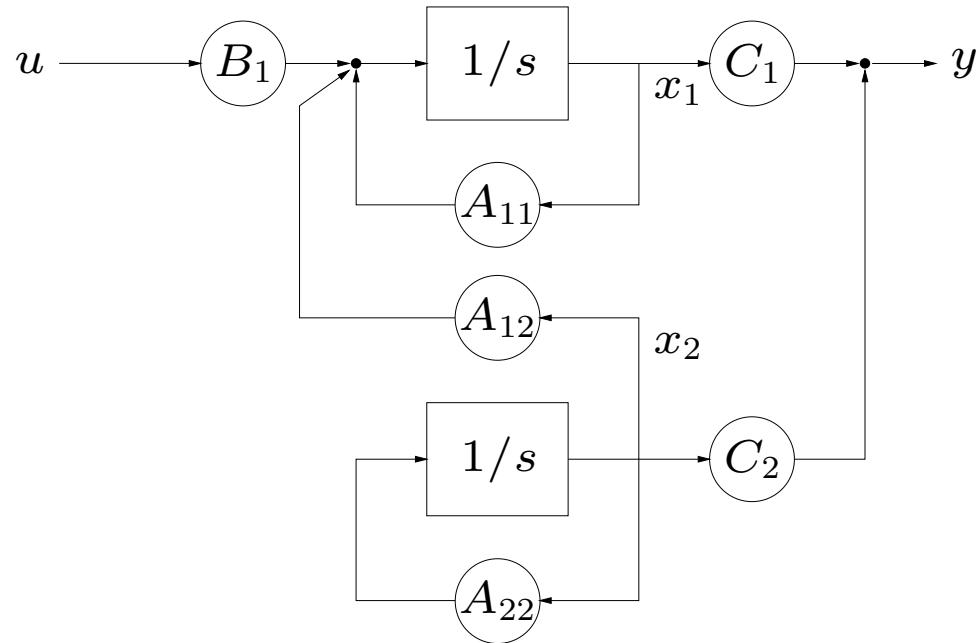
Block diagram



- A_{ij} is gain factor from state x_j into integrator i
- B_{ij} is gain factor from input u_j into integrator i
- C_{ij} is gain factor from state x_j into output y_i
- D_{ij} is gain factor from input u_j into output y_i

interesting when there is structure, *e.g.*, with $x_1 \in \mathbf{R}^{n_1}$, $x_2 \in \mathbf{R}^{n_2}$:

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



- x_2 is not affected by input u , *i.e.*, x_2 propagates autonomously
- x_2 affects y directly and through x_1

Transfer function

take Laplace transform of $\dot{x} = Ax + Bu$:

$$sX(s) - x(0) = AX(s) + BU(s)$$

hence

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$

so

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}Bu(\tau) d\tau$$

- $e^{tA}x(0)$ is the unforced or autonomous response
- $e^{tA}B$ is called the input-to-state impulse response or impulse matrix
- $(sI - A)^{-1}B$ is called the *input-to-state transfer function* or *transfer matrix*

with $y = Cx + Du$ we have:

$$Y(s) = C(sI - A)^{-1}x(0) + (C(sI - A)^{-1}B + D)U(s)$$

so

$$y(t) = Ce^{tA}x(0) + \int_0^t Ce^{(t-\tau)A}Bu(\tau) d\tau + Du(t)$$

- output term $Ce^{tA}x(0)$ due to initial condition
- $H(s) = C(sI - A)^{-1}B + D$ is called the *transfer function* or *transfer matrix*
- $h(t) = Ce^{tA}B + D\delta(t)$ is called the *impulse response* or *impulse matrix* (δ is the Dirac delta function)

with zero initial condition we have:

$$Y(s) = H(s)U(s), \quad y = h * u$$

where $*$ is convolution (of matrix valued functions)

interpretation:

- H_{ij} is transfer function from input u_j to output y_i

Impulse response

impulse response $h(t) = Ce^{tA}B + D\delta(t)$

with $x(0) = 0$, $y = h * u$, i.e.,

$$y_i(t) = \sum_{j=1}^m \int_0^t h_{ij}(t-\tau)u_j(\tau) d\tau$$

interpretations:

- $h_{ij}(t)$ is impulse response from j th input to i th output
- $h_{ij}(t)$ gives $y_i(t)$ when $u(t) = e_j\delta(t)$
- $h_{ij}(\tau)$ shows how dependent output i is, on what input j was, τ seconds ago
- i indexes output; j indexes input; τ indexes time lag

Step response

the *step response* or *step matrix* is given by

$$s(t) = \int_0^t h(\tau) d\tau$$

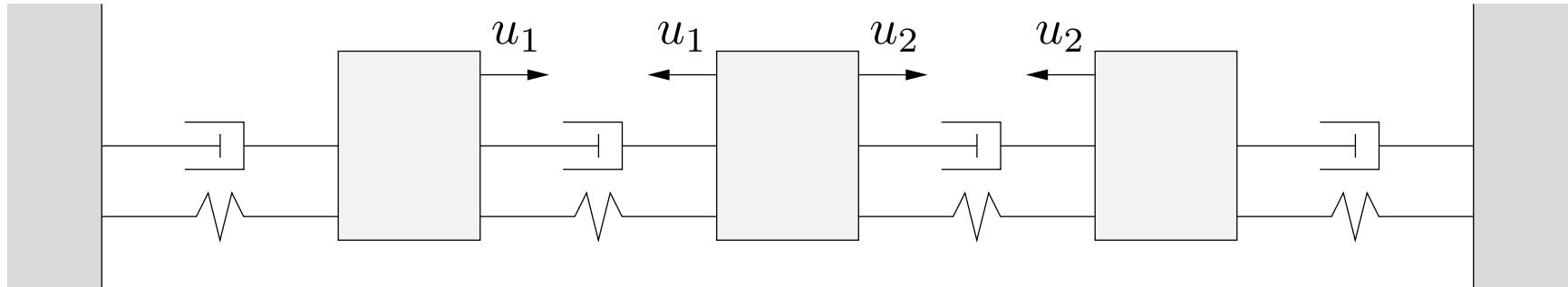
interpretations:

- $s_{ij}(t)$ is step response from j th input to i th output
- $s_{ij}(t)$ gives y_i when $u = e_j$ for $t \geq 0$

for invertible A , we have

$$s(t) = CA^{-1} (e^{tA} - I) B + D$$

Example 1



- unit masses, springs, dampers
- u_1 is tension between 1st & 2nd masses
- u_2 is tension between 2nd & 3rd masses
- $y \in \mathbb{R}^3$ is displacement of masses 1,2,3
- $x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$

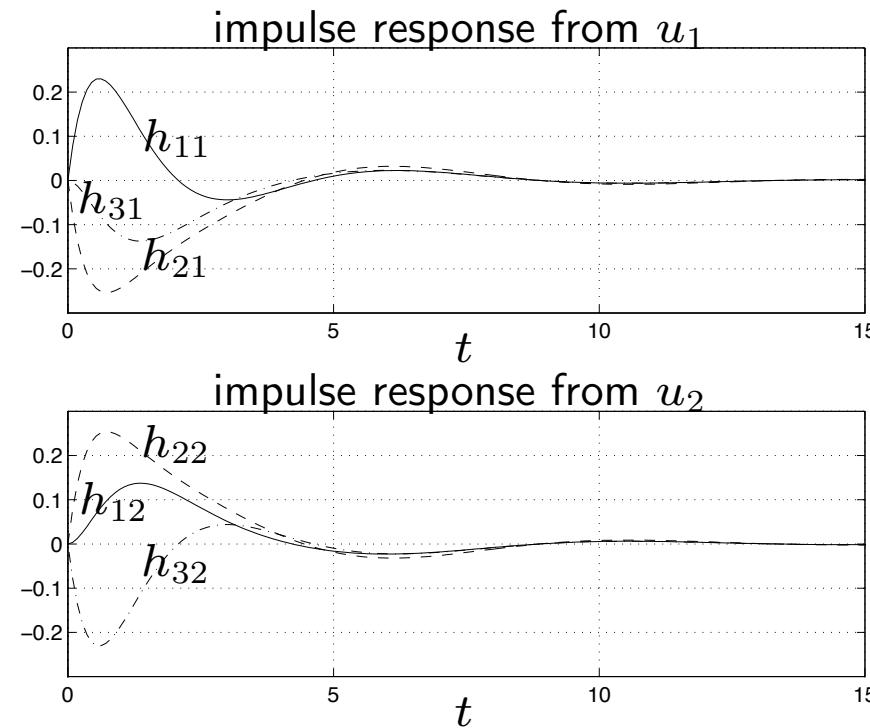
system is:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & -2 & 1 & 0 \\ 1 & -2 & 1 & 1 & -2 & 1 \\ 0 & 1 & -2 & 0 & 1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

eigenvalues of A are

$$-1.71 \pm i0.71, \quad -1.00 \pm i1.00, \quad -0.29 \pm i0.71$$

impulse response:

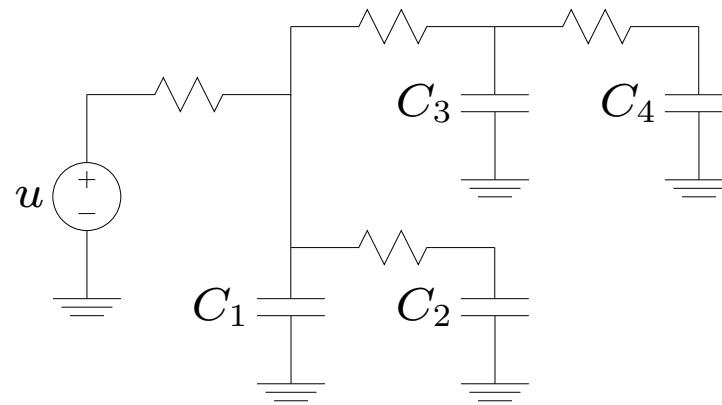


roughly speaking:

- impulse at u_1 affects third mass less than other two
- impulse at u_2 affects first mass later than other two

Example 2

interconnect circuit:



- $u(t) \in \mathbf{R}$ is input (drive) voltage
- x_i is voltage across C_i
- output is state: $y = x$
- unit resistors, unit capacitors
- step response matrix shows delay to each node

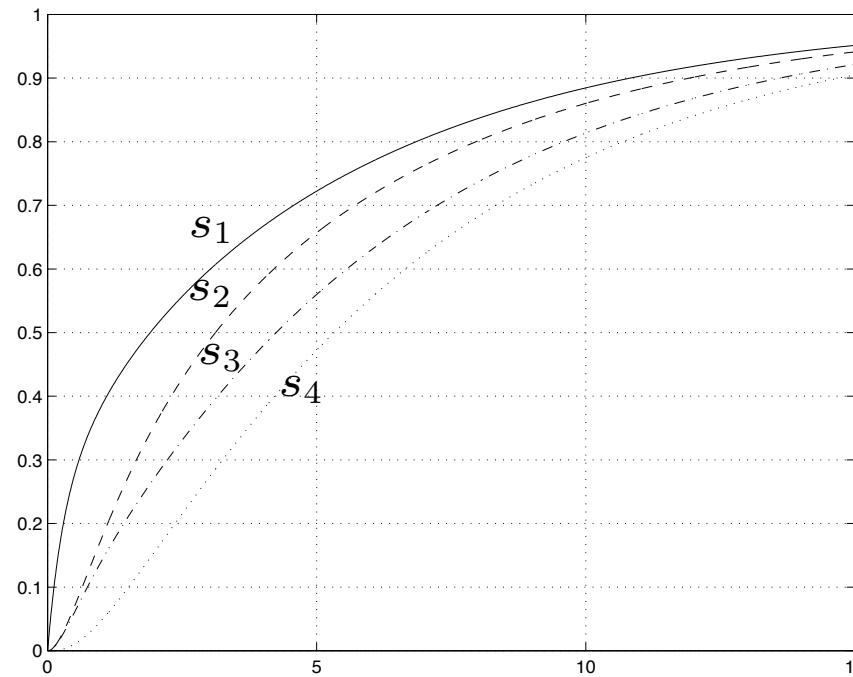
system is

$$\dot{x} = \begin{bmatrix} -3 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u, \quad y = x$$

eigenvalues of A are

$$-0.17, \quad -0.66, \quad -2.21, \quad -3.96$$

step response matrix $s(t) \in \mathbf{R}^{4 \times 1}$:



- shortest delay to x_1 ; longest delay to x_4
- delays ≈ 10 , consistent with slowest (*i.e.*, dominant) eigenvalue -0.17

DC or static gain matrix

- transfer function at $s = 0$ is $H(0) = -CA^{-1}B + D \in \mathbf{R}^{m \times p}$
- DC transfer function describes system under *static* conditions, i.e., x , u , y constant:

$$0 = \dot{x} = Ax + Bu, \quad y = Cx + Du$$

eliminate x to get $y = H(0)u$

- if system is stable,

$$H(0) = \int_0^\infty h(t) \, dt = \lim_{t \rightarrow \infty} s(t)$$

(recall: $H(s) = \int_0^\infty e^{-st} h(t) \, dt$, $s(t) = \int_0^t h(\tau) \, d\tau$)

if $u(t) \rightarrow u_\infty \in \mathbf{R}^m$, then $y(t) \rightarrow y_\infty \in \mathbf{R}^p$ where $y_\infty = H(0)u_\infty$

DC gain matrix for example 1 (springs):

$$H(0) = \begin{bmatrix} 1/4 & 1/4 \\ -1/2 & 1/2 \\ -1/4 & -1/4 \end{bmatrix}$$

DC gain matrix for example 2 (RC circuit):

$$H(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(do these make sense?)

Discretization with piecewise constant inputs

linear system $\dot{x} = Ax + Bu, y = Cx + Du$

suppose $u_d : \mathbf{Z}_+ \rightarrow \mathbf{R}^m$ is a sequence, and

$$u(t) = u_d(k) \quad \text{for } kh \leq t < (k+1)h, \quad k = 0, 1, \dots$$

define sequences

$$x_d(k) = x(kh), \quad y_d(k) = y(kh), \quad k = 0, 1, \dots$$

- $h > 0$ is called the *sample interval* (for x and y) or *update interval* (for u)
- u is piecewise constant (called *zero-order-hold*)
- x_d, y_d are sampled versions of x, y

$$\begin{aligned}
x_d(k+1) &= x((k+1)h) \\
&= e^{hA}x(kh) + \int_0^h e^{\tau A} Bu((k+1)h - \tau) d\tau \\
&= e^{hA}x_d(k) + \left(\int_0^h e^{\tau A} d\tau \right) B u_d(k)
\end{aligned}$$

x_d , u_d , and y_d satisfy discrete-time LDS equations

$$x_d(k+1) = A_d x_d(k) + B_d u_d(k), \quad y_d(k) = C_d x_d(k) + D_d u_d(k)$$

where

$$A_d = e^{hA}, \quad B_d = \left(\int_0^h e^{\tau A} d\tau \right) B, \quad C_d = C, \quad D_d = D$$

called *discretized system*

if A is invertible, we can express integral as

$$\int_0^h e^{\tau A} d\tau = A^{-1} (e^{hA} - I)$$

stability: if eigenvalues of A are $\lambda_1, \dots, \lambda_n$, then eigenvalues of A_d are $e^{h\lambda_1}, \dots, e^{h\lambda_n}$

discretization preserves stability properties since

$$\Re \lambda_i < 0 \iff |e^{h\lambda_i}| < 1$$

for $h > 0$

extensions/variations:

- *offsets*: updates for u and sampling of x, y are offset in time
- *multirate*: u_i updated, y_i sampled at different intervals
(usually integer multiples of a common interval h)

both very common in practice

Dual system

the *dual system* associated with system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

is given by

$$\dot{z} = A^T z + C^T v, \quad w = B^T z + D^T v$$

- all matrices are transposed
- role of B and C are swapped

transfer function of dual system:

$$(B^T)(sI - A^T)^{-1}(C^T) + D^T = H(s)^T$$

where $H(s) = C(sI - A)^{-1}B + D$

(for SISO case, TF of dual is same as original)

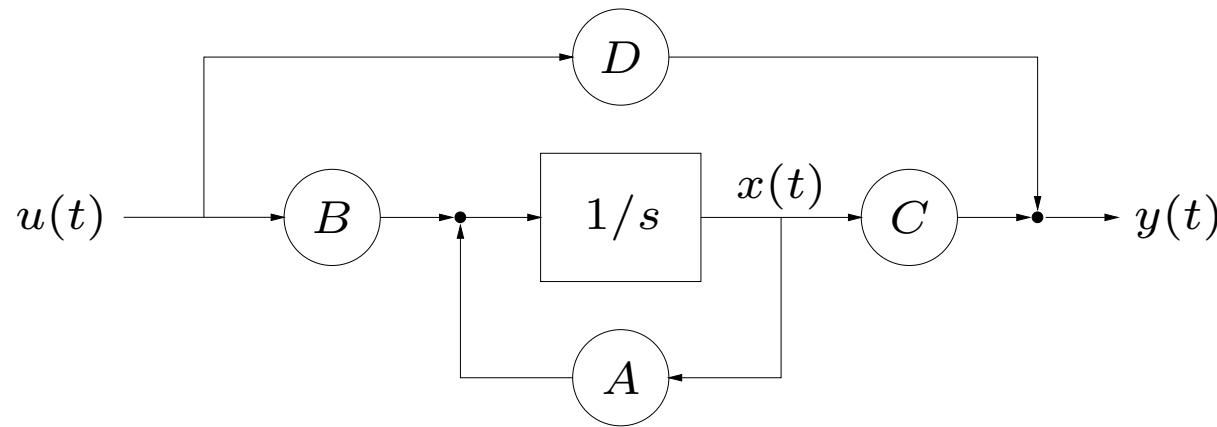
eigenvalues (hence stability properties) are the same

Dual via block diagram

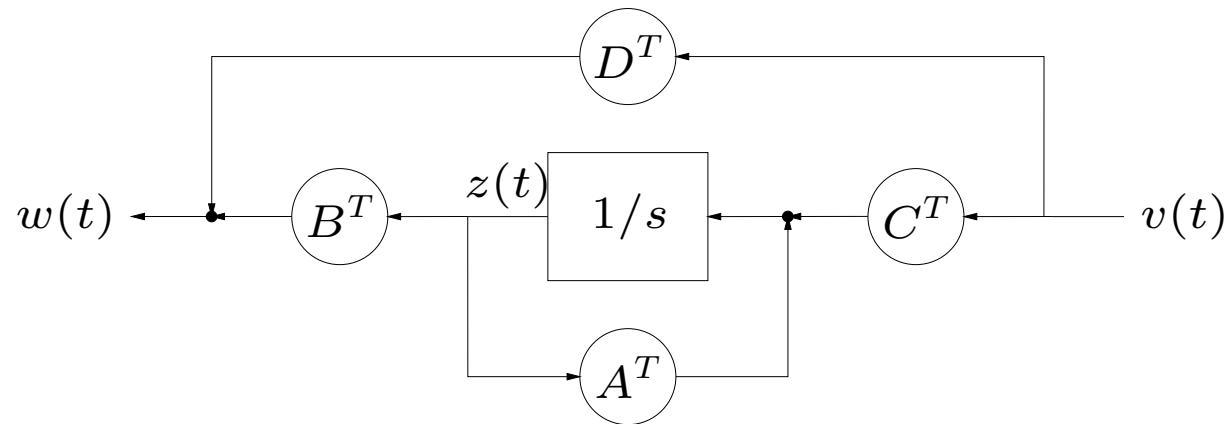
in terms of block diagrams, dual is formed by:

- transpose all matrices
- swap inputs and outputs on all boxes
- reverse directions of signal flow arrows
- swap solder joints and summing junctions

original system:



dual system:



Causality

interpretation of

$$\begin{aligned}x(t) &= e^{tA}x(0) + \int_0^t e^{(t-\tau)A}Bu(\tau) d\tau \\y(t) &= Ce^{tA}x(0) + \int_0^t Ce^{(t-\tau)A}Bu(\tau) d\tau + Du(t)\end{aligned}$$

for $t \geq 0$:

current state ($x(t)$) and output ($y(t)$) depend on *past* input ($u(\tau)$ for $\tau \leq t$)

i.e., mapping from input to state and output is *causal* (with fixed *initial* state)

now consider fixed *final* state $x(T)$: for $t \leq T$,

$$x(t) = e^{(t-T)A}x(T) + \int_T^t e^{(t-\tau)A}Bu(\tau) d\tau,$$

i.e., current state (and output) depend on future input!

so for fixed final condition, same system is anti-causal

Idea of state

$x(t)$ is called *state* of system at time t since:

- future output depends only on current state and future input
- future output depends on past input only through current state
- state summarizes effect of past inputs on future output
- state is bridge between past inputs and future outputs

Change of coordinates

start with LDS $\dot{x} = Ax + Bu, y = Cx + Du$

change coordinates in \mathbf{R}^n to \tilde{x} , with $x = T\tilde{x}$

then

$$\dot{\tilde{x}} = T^{-1}\dot{x} = T^{-1}(Ax + Bu) = T^{-1}AT\tilde{x} + T^{-1}Bu$$

hence LDS can be expressed as

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u, \quad y = \tilde{C}\tilde{x} + \tilde{D}u$$

where

$$\tilde{A} = T^{-1}AT, \quad \tilde{B} = T^{-1}B, \quad \tilde{C} = CT, \quad \tilde{D} = D$$

TF is same (since u, y aren't affected):

$$\tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D} = C(sI - A)^{-1}B + D$$

Standard forms for LDS

can change coordinates to put A in various forms (diagonal, real modal, Jordan . . .)

e.g., to put LDS in *diagonal form*, find T s.t.

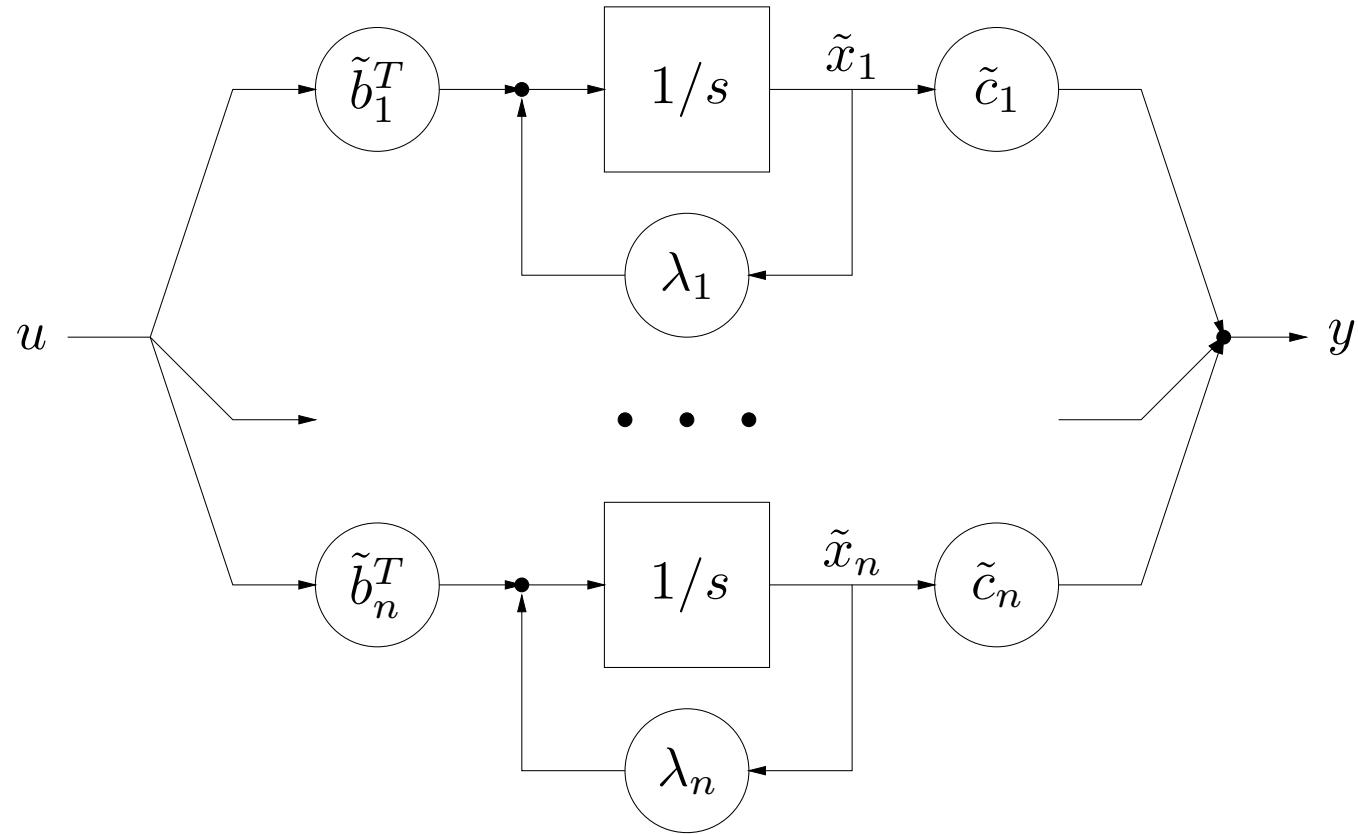
$$T^{-1}AT = \mathbf{diag}(\lambda_1, \dots, \lambda_n)$$

write

$$T^{-1}B = \begin{bmatrix} \tilde{b}_1^T \\ \vdots \\ \tilde{b}_n^T \end{bmatrix}, \quad CT = \begin{bmatrix} \tilde{c}_1 & \cdots & \tilde{c}_n \end{bmatrix}$$

so

$$\dot{\tilde{x}}_i = \lambda_i \tilde{x}_i + \tilde{b}_i^T u, \quad y = \sum_{i=1}^n \tilde{c}_i \tilde{x}_i$$

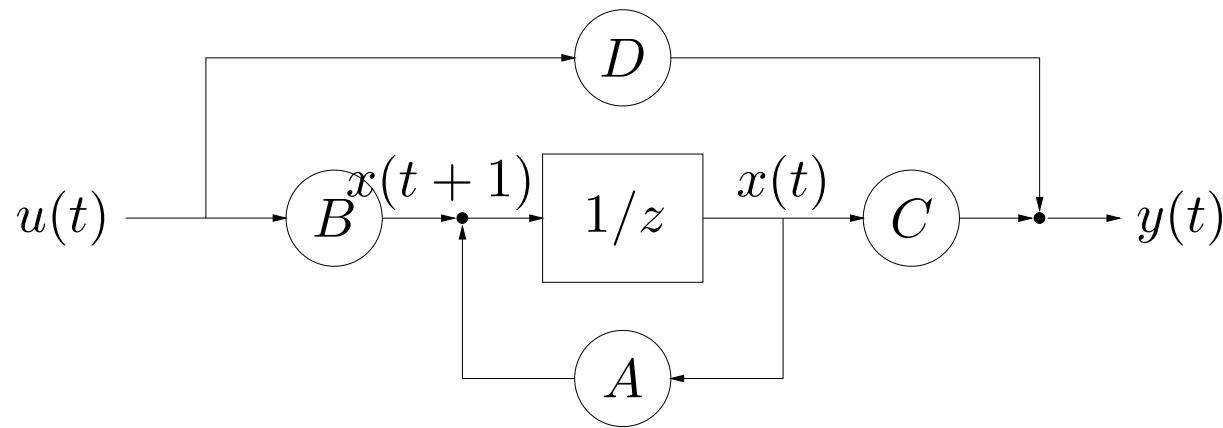


(here we assume $D = 0$)

Discrete-time systems

discrete-time LDS:

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$



- only difference w/cts-time: z instead of s
- interpretation of z^{-1} block:
 - unit delay or (shifts sequence back in time one epoch)
 - latch (plus small delay to avoid race condition)

we have:

$$x(1) = Ax(0) + Bu(0),$$

$$\begin{aligned} x(2) &= Ax(1) + Bu(1) \\ &= A^2x(0) + ABu(0) + Bu(1), \end{aligned}$$

and in general, for $t \in \mathbf{Z}_+$,

$$x(t) = A^t x(0) + \sum_{\tau=0}^{t-1} A^{(t-1-\tau)} Bu(\tau)$$

hence

$$y(t) = CA^t x(0) + h * u$$

where $*$ is discrete-time convolution and

$$h(t) = \begin{cases} D, & t = 0 \\ CA^{t-1}B, & t > 0 \end{cases}$$

is the impulse response

\mathcal{Z} -transform

suppose $w \in \mathbf{R}^{p \times q}$ is a sequence (discrete-time signal), i.e.,

$$w : \mathbf{Z}_+ \rightarrow \mathbf{R}^{p \times q}$$

recall \mathcal{Z} -transform $W = \mathcal{Z}(w)$:

$$W(z) = \sum_{t=0}^{\infty} z^{-t} w(t)$$

where $W : D \subseteq \mathbf{C} \rightarrow \mathbf{C}^{p \times q}$ (D is domain of W)

time-advanced or shifted signal v :

$$v(t) = w(t+1), \quad t = 0, 1, \dots$$

\mathcal{Z} -transform of time-advanced signal:

$$\begin{aligned} V(z) &= \sum_{t=0}^{\infty} z^{-t} w(t+1) \\ &= z \sum_{t=1}^{\infty} z^{-t} w(t) \\ &= zW(z) - zw(0) \end{aligned}$$

Discrete-time transfer function

take \mathcal{Z} -transform of system equations

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

yields

$$zX(z) - zx(0) = AX(z) + BU(z), \quad Y(z) = CX(z) + DU(z)$$

solve for $X(z)$ to get

$$X(z) = (zI - A)^{-1}zx(0) + (zI - A)^{-1}BU(z)$$

(note extra z in first term!)

hence

$$Y(z) = H(z)U(z) + C(zI - A)^{-1}zx(0)$$

where $H(z) = C(zI - A)^{-1}B + D$ is the *discrete-time transfer function*

note power series expansion of resolvent:

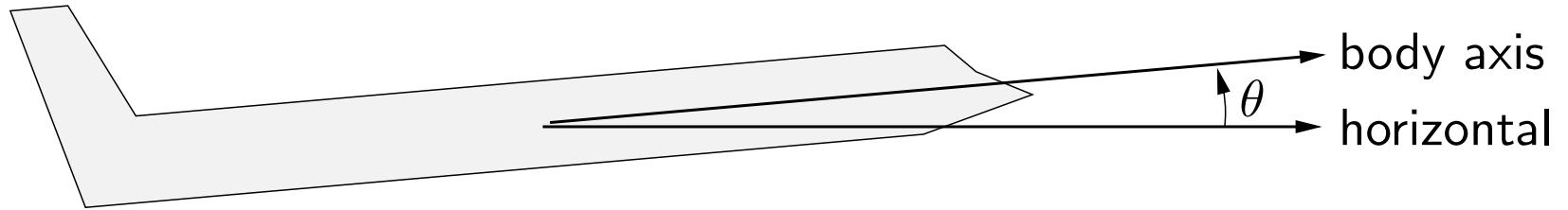
$$(zI - A)^{-1} = z^{-1}I + z^{-2}A + z^{-3}A^2 + \dots$$

Lecture 14

Example: Aircraft dynamics

- longitudinal aircraft dynamics
- wind gust & control inputs
- linearized dynamics
- steady-state analysis
- eigenvalues & modes
- impulse matrices

Longitudinal aircraft dynamics



variables are (small) deviations from operating point or *trim conditions* state (components):

- u : velocity of aircraft along body axis
- v : velocity of aircraft perpendicular to body axis
(down is positive)
- θ : angle between body axis and horizontal
(up is positive)
- $\dot{\theta} = q$: angular velocity of aircraft (pitch rate)

Inputs

disturbance inputs:

- u_w : velocity of wind along body axis
- v_w : velocity of wind perpendicular to body axis

control or actuator inputs:

- δ_e : elevator angle ($\delta_e > 0$ is down)
- δ_t : thrust

Linearized dynamics

for 747, level flight, 40000 ft, 774 ft/sec,

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -.003 & .039 & 0 & -.322 \\ -.065 & -.319 & 7.74 & 0 \\ .020 & -.101 & -.429 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u - u_w \\ v - v_w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} .01 & 1 \\ -.18 & -.04 \\ -1.16 & .598 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_e \\ \delta_t \end{bmatrix}$$

- units: ft, sec, crad ($= 0.01\text{rad} \approx 0.57^\circ$)
- matrix coefficients are called *stability derivatives*

outputs of interest:

- aircraft speed u (deviation from trim)
- climb rate $\dot{h} = -v + 7.74\theta$

Steady-state analysis

DC gain from $(u_w, v_w, \delta_e, \delta_t)$ to (u, \dot{h}) :

$$H(0) = -CA^{-1}B + D = \begin{bmatrix} 1 & 0 & 27.2 & -15.0 \\ 0 & -1 & -1.34 & 24.9 \end{bmatrix}$$

gives steady-state change in speed & climb rate due to wind, elevator & thrust changes

solve for control variables in terms of wind velocities, desired speed & climb rate

$$\begin{bmatrix} \delta_e \\ \delta_t \end{bmatrix} = \begin{bmatrix} .0379 & .0229 \\ .0020 & .0413 \end{bmatrix} \begin{bmatrix} u - u_w \\ \dot{h} + v_w \end{bmatrix}$$

- level flight, increase in speed is obtained mostly by increasing elevator (*i.e.*, downwards)
- constant speed, increase in climb rate is obtained by increasing thrust and increasing elevator (*i.e.*, downwards)
(thrust on 747 gives strong pitch up torque)

Eigenvalues and modes

eigenvalues are

$$-0.3750 \pm 0.8818i, \quad -0.0005 \pm 0.0674i$$

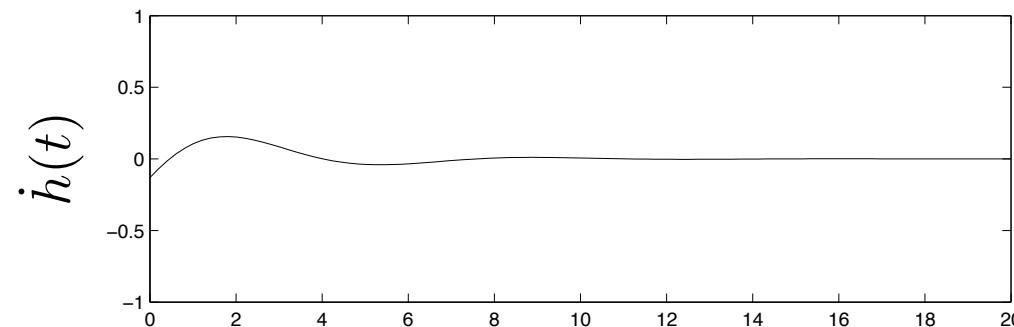
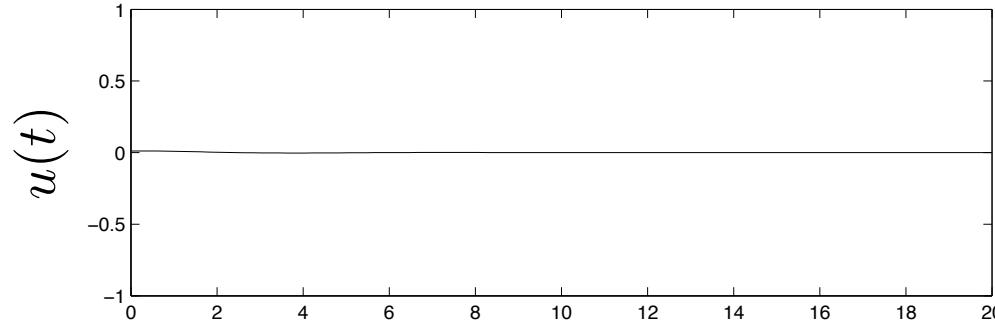
- two complex modes, called *short-period* and *phugoid*, respectively
- system is stable (but lightly damped)
- hence step responses converge (eventually) to DC gain matrix

eigenvectors are

$$x_{\text{short}} = \begin{bmatrix} 0.0005 \\ -0.5433 \\ -0.0899 \\ -0.0283 \end{bmatrix} \pm i \begin{bmatrix} 0.0135 \\ 0.8235 \\ -0.0677 \\ 0.1140 \end{bmatrix},$$
$$x_{\text{phug}} = \begin{bmatrix} -0.7510 \\ -0.0962 \\ -0.0111 \\ 0.1225 \end{bmatrix} \pm i \begin{bmatrix} 0.6130 \\ 0.0941 \\ 0.0082 \\ 0.1637 \end{bmatrix}$$

Short-period mode

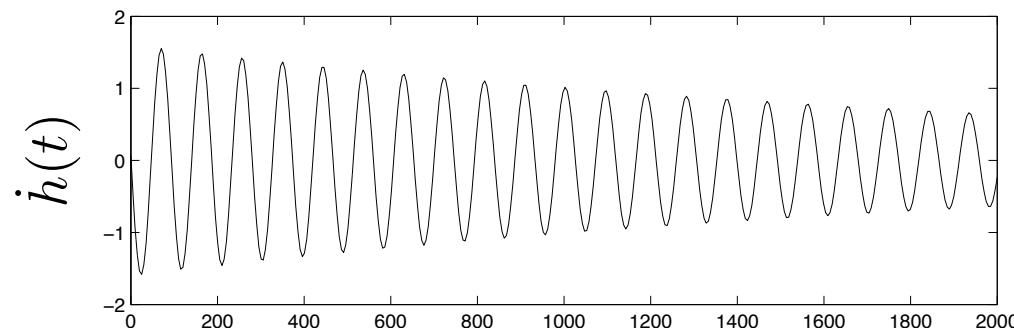
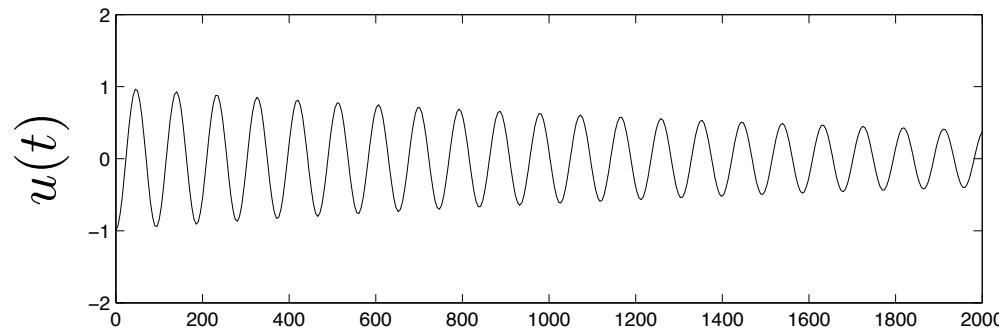
$$y(t) = Ce^{tA}(\Re x_{\text{short}}) \text{ (pure short-period mode motion)}$$



- only small effect on speed u
- period ≈ 7 sec, decays in ≈ 10 sec

Phugoid mode

$$y(t) = Ce^{tA}(\Re x_{\text{phug}}) \text{ (pure phugoid mode motion)}$$

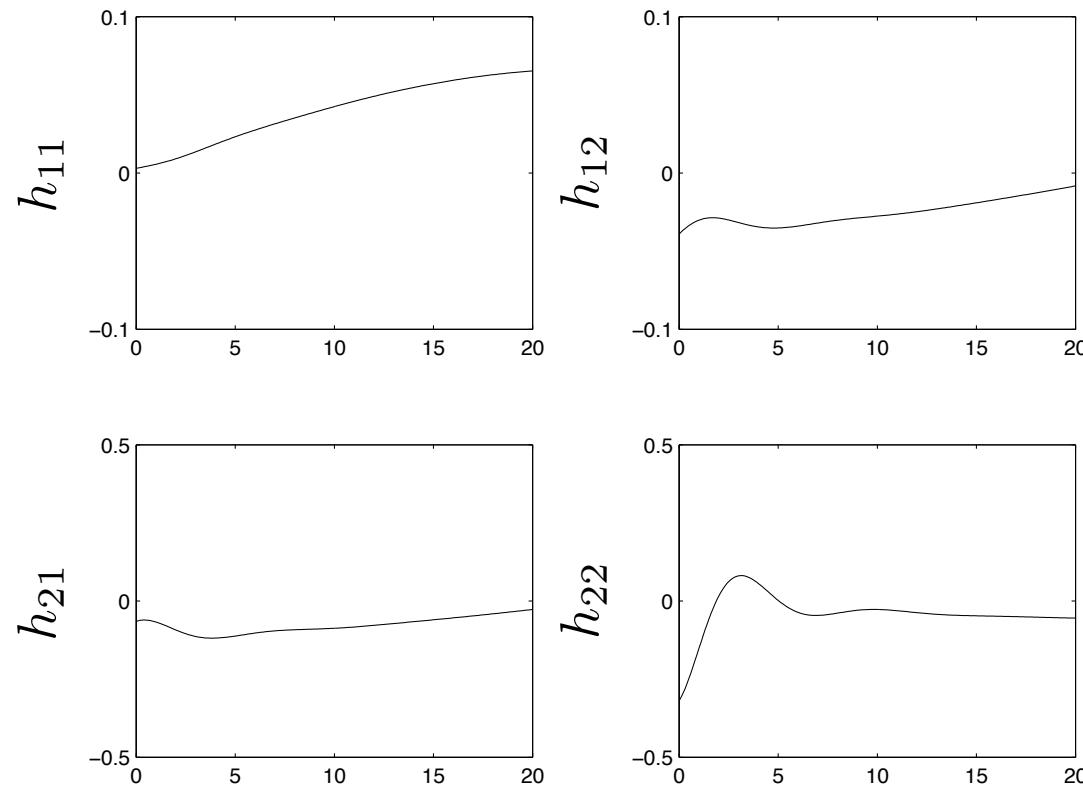


- affects both speed and climb rate
- period ≈ 100 sec; decays in ≈ 5000 sec

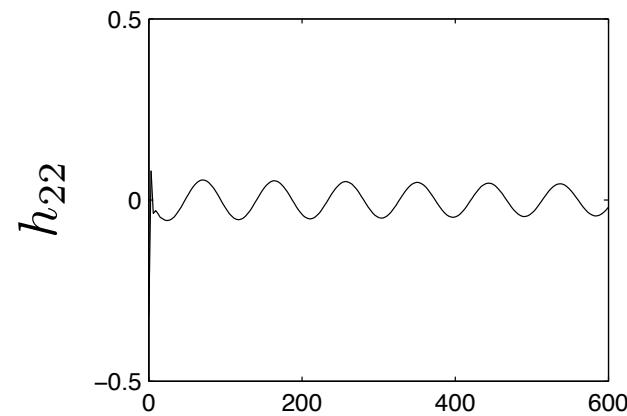
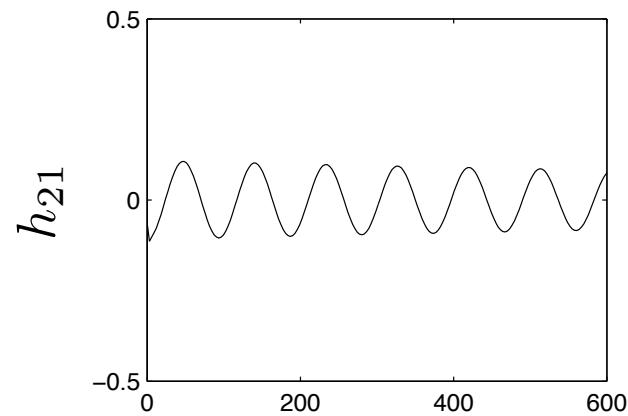
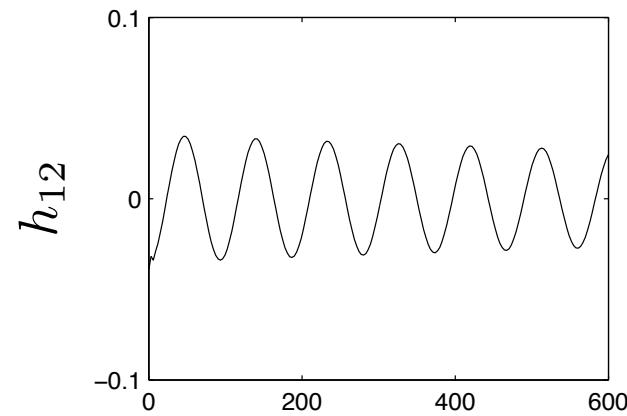
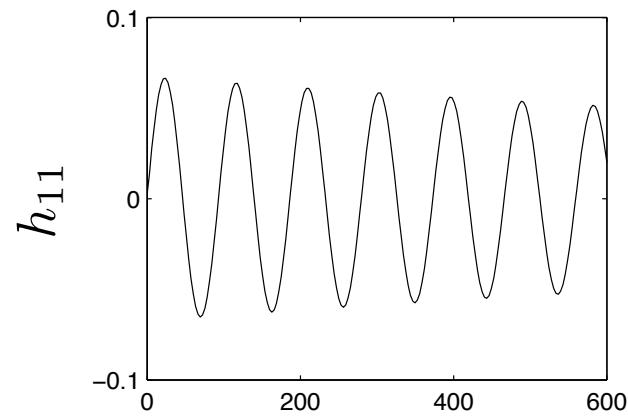
Dynamic response to wind gusts

impulse response matrix from (u_w, v_w) to (u, \dot{h}) (gives response to short wind bursts)

over time period $[0, 20]$:



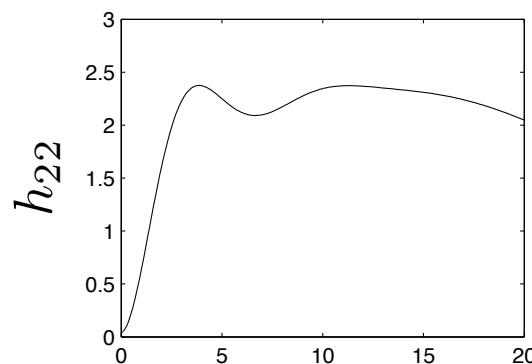
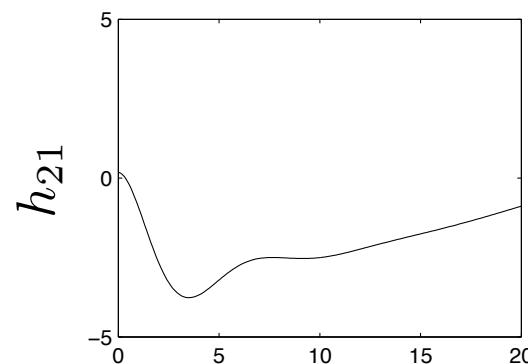
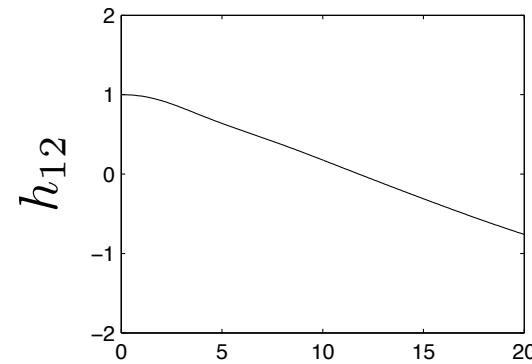
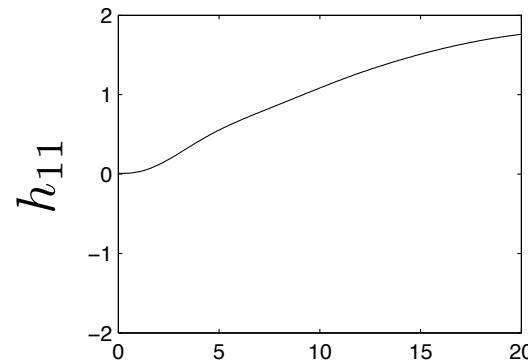
over time period $[0, 600]$:



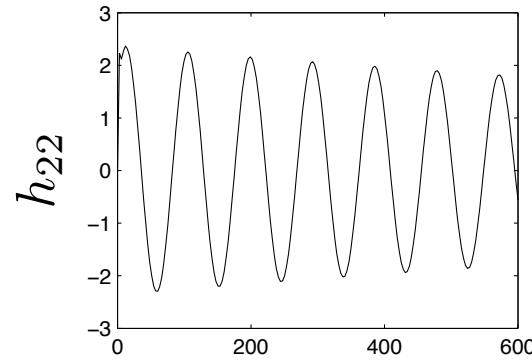
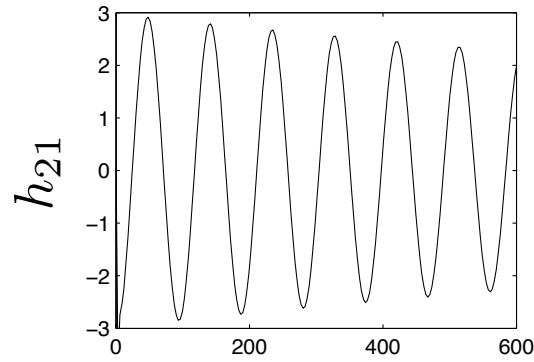
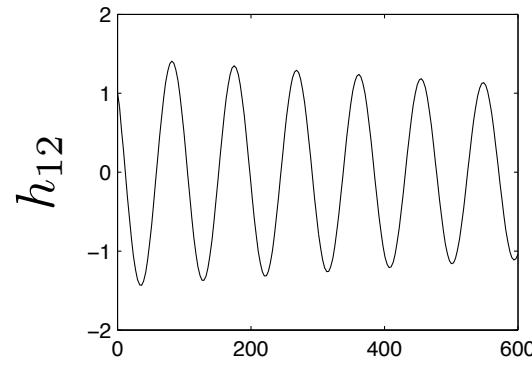
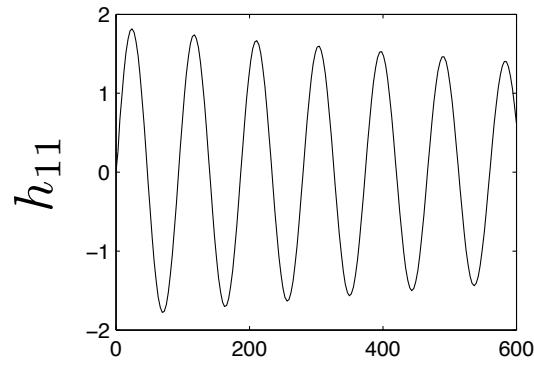
Dynamic response to actuators

impulse response matrix from (δ_e, δ_t) to (u, \dot{h})

over time period $[0, 20]$:



over time period $[0, 600]$:



Lecture 15

Symmetric matrices, quadratic forms, matrix norm, and SVD

- eigenvectors of symmetric matrices
- quadratic forms
- inequalities for quadratic forms
- positive semidefinite matrices
- norm of a matrix
- singular value decomposition

Eigenvalues of symmetric matrices

suppose $A \in \mathbf{R}^{n \times n}$ is symmetric, i.e., $A = A^T$

fact: the eigenvalues of A are real

to see this, suppose $Av = \lambda v$, $v \neq 0$, $v \in \mathbf{C}^n$

then

$$\bar{v}^T A v = \bar{v}^T (A v) = \lambda \bar{v}^T v = \lambda \sum_{i=1}^n |v_i|^2$$

but also

$$\bar{v}^T A v = \overline{(A v)}^T v = \overline{(\lambda v)}^T v = \bar{\lambda} \sum_{i=1}^n |v_i|^2$$

so we have $\lambda = \bar{\lambda}$, i.e., $\lambda \in \mathbf{R}$ (hence, can assume $v \in \mathbf{R}^n$)

Eigenvectors of symmetric matrices

fact: there is a set of orthonormal eigenvectors of A , i.e., q_1, \dots, q_n s.t.
 $Aq_i = \lambda_i q_i$, $q_i^T q_j = \delta_{ij}$

in matrix form: there is an orthogonal Q s.t.

$$Q^{-1}AQ = Q^T AQ = \Lambda$$

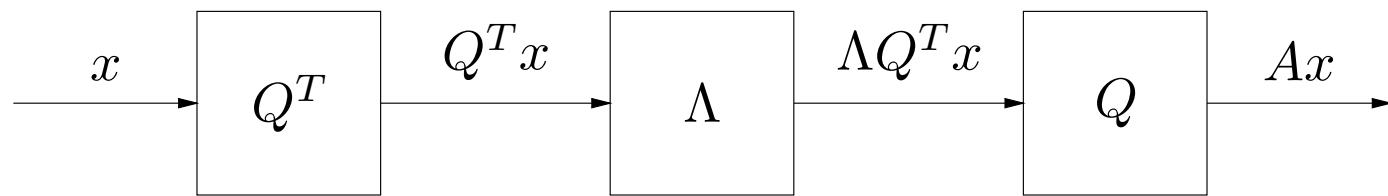
hence we can express A as

$$A = Q\Lambda Q^T = \sum_{i=1}^n \lambda_i q_i q_i^T$$

in particular, q_i are both left and right eigenvectors

Interpretations

$$A = Q\Lambda Q^T$$



linear mapping $y = Ax$ can be decomposed as

- resolve into q_i coordinates
- scale coordinates by λ_i
- reconstitute with basis q_i

or, geometrically,

- rotate by Q^T
- diagonal real scale ('dilation') by Λ
- rotate back by Q

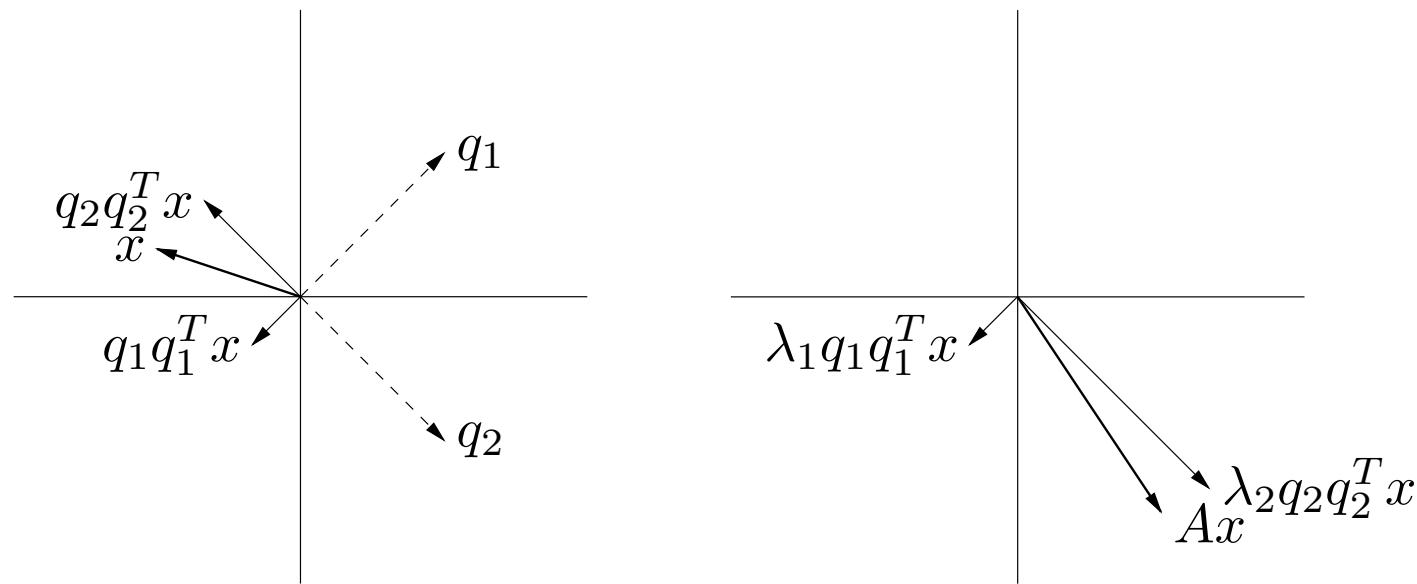
decomposition

$$A = \sum_{i=1}^n \lambda_i q_i q_i^T$$

expresses A as linear combination of 1-dimensional projections

example:

$$\begin{aligned} A &= \begin{bmatrix} -1/2 & 3/2 \\ 3/2 & -1/2 \end{bmatrix} \\ &= \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)^T \end{aligned}$$



proof (case of λ_i distinct)

since λ_i distinct, can find v_1, \dots, v_n , a set of linearly independent eigenvectors of A :

$$Av_i = \lambda_i v_i, \quad \|v_i\| = 1$$

then we have

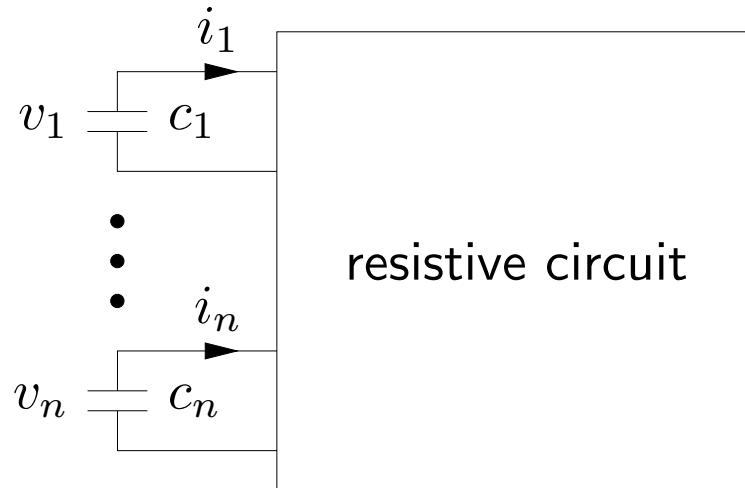
$$v_i^T (Av_j) = \lambda_j v_i^T v_j = (Av_i)^T v_j = \lambda_i v_i^T v_j$$

$$\text{so } (\lambda_i - \lambda_j)v_i^T v_j = 0$$

for $i \neq j$, $\lambda_i \neq \lambda_j$, hence $v_i^T v_j = 0$

- in this case we can say: eigenvectors *are* orthogonal
- in general case (λ_i not distinct) we must say: eigenvectors *can be chosen* to be orthogonal

Example: RC circuit



$$c_k \dot{v}_k = -i_k, \quad i = Gv$$

$G = G^T \in \mathbb{R}^{n \times n}$ is conductance matrix of resistive circuit

thus $\dot{v} = -C^{-1}Gv$ where $C = \text{diag}(c_1, \dots, c_n)$

note $-C^{-1}G$ is not symmetric

use state $x_i = \sqrt{c_i}v_i$, so

$$\dot{x} = C^{1/2}\dot{v} = -C^{-1/2}GC^{-1/2}x$$

where $C^{1/2} = \text{diag}(\sqrt{c_1}, \dots, \sqrt{c_n})$

we conclude:

- eigenvalues $\lambda_1, \dots, \lambda_n$ of $-C^{-1/2}GC^{-1/2}$ (hence, $-C^{-1}G$) are real
- eigenvectors q_i (in x_i coordinates) can be chosen orthogonal
- eigenvectors in voltage coordinates, $s_i = C^{-1/2}q_i$, satisfy

$$-C^{-1}Gs_i = \lambda_i s_i, \quad s_i^T C s_i = \delta_{ij}$$

Quadratic forms

a function $f : \mathbf{R}^n \rightarrow \mathbf{R}$ of the form

$$f(x) = x^T A x = \sum_{i,j=1}^n A_{ij} x_i x_j$$

is called a *quadratic form*

in a quadratic form we may as well assume $A = A^T$ since

$$x^T A x = x^T ((A + A^T)/2) x$$

(($A + A^T$)/2 is called the *symmetric part* of A)

uniqueness: if $x^T A x = x^T B x$ for all $x \in \mathbf{R}^n$ and $A = A^T$, $B = B^T$, then $A = B$

Examples

- $\|Bx\|^2 = x^T B^T B x$

- $\sum_{i=1}^{n-1} (x_{i+1} - x_i)^2$

- $\|F x\|^2 - \|G x\|^2$

sets defined by quadratic forms:

- $\{ x \mid f(x) = a \}$ is called a *quadratic surface*
- $\{ x \mid f(x) \leq a \}$ is called a *quadratic region*

Inequalities for quadratic forms

suppose $A = A^T$, $A = Q\Lambda Q^T$ with eigenvalues sorted so $\lambda_1 \geq \dots \geq \lambda_n$

$$\begin{aligned} x^T Ax &= x^T Q\Lambda Q^T x \\ &= (Q^T x)^T \Lambda (Q^T x) \\ &= \sum_{i=1}^n \lambda_i (q_i^T x)^2 \\ &\leq \lambda_1 \sum_{i=1}^n (q_i^T x)^2 \\ &= \lambda_1 \|x\|^2 \end{aligned}$$

i.e., we have $x^T Ax \leq \lambda_1 x^T x$

similar argument shows $x^T Ax \geq \lambda_n \|x\|^2$, so we have

$$\lambda_n x^T x \leq x^T Ax \leq \lambda_1 x^T x$$

sometimes λ_1 is called λ_{\max} , λ_n is called λ_{\min}

note also that

$$q_1^T A q_1 = \lambda_1 \|q_1\|^2, \quad q_n^T A q_n = \lambda_n \|q_n\|^2,$$

so the inequalities are tight

Positive semidefinite and positive definite matrices

suppose $A = A^T \in \mathbf{R}^{n \times n}$

we say A is *positive semidefinite* if $x^T Ax \geq 0$ for all x

- denoted $A \geq 0$ (and sometimes $A \succeq 0$)
- $A \geq 0$ if and only if $\lambda_{\min}(A) \geq 0$, i.e., all eigenvalues are nonnegative
- **not** the same as $A_{ij} \geq 0$ for all i, j

we say A is *positive definite* if $x^T Ax > 0$ for all $x \neq 0$

- denoted $A > 0$
- $A > 0$ if and only if $\lambda_{\min}(A) > 0$, i.e., all eigenvalues are positive

Matrix inequalities

- we say A is *negative semidefinite* if $-A \geq 0$
- we say A is *negative definite* if $-A > 0$
- otherwise, we say A is *indefinite*

matrix inequality: if $B = B^T \in \mathbf{R}^n$ we say $A \geq B$ if $A - B \geq 0$, $A < B$ if $B - A > 0$, etc.

for example:

- $A \geq 0$ means A is positive semidefinite
- $A > B$ means $x^T Ax > x^T Bx$ for all $x \neq 0$

many properties that you'd guess hold actually do, *e.g.*,

- if $A \geq B$ and $C \geq D$, then $A + C \geq B + D$
- if $B \leq 0$ then $A + B \leq A$
- if $A \geq 0$ and $\alpha \geq 0$, then $\alpha A \geq 0$
- $A^2 \geq 0$
- if $A > 0$, then $A^{-1} > 0$

matrix inequality is only a *partial order*: we can have

$$A \not\geq B, \quad B \not\geq A$$

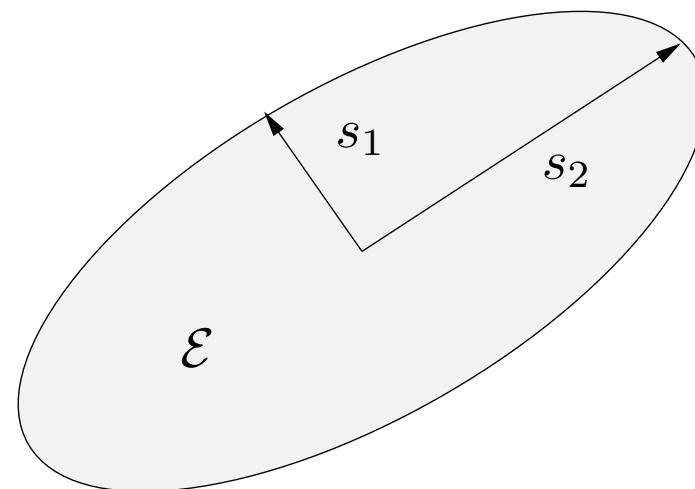
(such matrices are called *incomparable*)

Ellipsoids

if $A = A^T > 0$, the set

$$\mathcal{E} = \{ x \mid x^T A x \leq 1 \}$$

is an *ellipsoid* in \mathbf{R}^n , centered at 0



semi-axes are given by $s_i = \lambda_i^{-1/2} q_i$, i.e.:

- eigenvectors determine directions of semiaxes
- eigenvalues determine lengths of semiaxes

note:

- in direction q_1 , $x^T A x$ is *large*, hence ellipsoid is *thin* in direction q_1
- in direction q_n , $x^T A x$ is *small*, hence ellipsoid is *fat* in direction q_n
- $\sqrt{\lambda_{\max}/\lambda_{\min}}$ gives maximum eccentricity

if $\tilde{\mathcal{E}} = \{ x \mid x^T B x \leq 1 \}$, where $B > 0$, then $\mathcal{E} \subseteq \tilde{\mathcal{E}} \iff A \geq B$

Gain of a matrix in a direction

suppose $A \in \mathbf{R}^{m \times n}$ (not necessarily square or symmetric)

for $x \in \mathbf{R}^n$, $\|Ax\|/\|x\|$ gives the *amplification factor* or *gain* of A in the direction x

obviously, gain varies with direction of input x

questions:

- what is maximum gain of A
(and corresponding maximum gain direction)?
- what is minimum gain of A
(and corresponding minimum gain direction)?
- how does gain of A vary with direction?

Matrix norm

the maximum gain

$$\max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

is called the *matrix norm* or *spectral norm* of A and is denoted $\|A\|$

$$\max_{x \neq 0} \frac{\|Ax\|^2}{\|x\|^2} = \max_{x \neq 0} \frac{x^T A^T A x}{\|x\|^2} = \lambda_{\max}(A^T A)$$

so we have $\|A\| = \sqrt{\lambda_{\max}(A^T A)}$

similarly the minimum gain is given by

$$\min_{x \neq 0} \|Ax\|/\|x\| = \sqrt{\lambda_{\min}(A^T A)}$$

note that

- $A^T A \in \mathbf{R}^{n \times n}$ is symmetric and $A^T A \geq 0$ so $\lambda_{\min}, \lambda_{\max} \geq 0$
- ‘max gain’ input direction is $x = q_1$, eigenvector of $A^T A$ associated with λ_{\max}
- ‘min gain’ input direction is $x = q_n$, eigenvector of $A^T A$ associated with λ_{\min}

example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

$$\begin{aligned} A^T A &= \begin{bmatrix} 35 & 44 \\ 44 & 56 \end{bmatrix} \\ &= \begin{bmatrix} 0.620 & -0.785 \\ 0.785 & 0.620 \end{bmatrix} \begin{bmatrix} 90.7 & 0 \\ 0 & 0.265 \end{bmatrix} \begin{bmatrix} 0.620 & -0.785 \\ 0.785 & 0.620 \end{bmatrix}^T \end{aligned}$$

then $\|A\| = \sqrt{\lambda_{\max}(A^T A)} = 9.53$:

$$\left\| \begin{bmatrix} 0.620 \\ 0.785 \end{bmatrix} \right\| = 1, \quad \left\| A \begin{bmatrix} 0.620 \\ 0.785 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 2.19 \\ 5.00 \\ 7.81 \end{bmatrix} \right\| = 9.53$$

min gain is $\sqrt{\lambda_{\min}(A^T A)} = 0.514$:

$$\left\| \begin{bmatrix} -0.785 \\ 0.620 \end{bmatrix} \right\| = 1, \quad \left\| A \begin{bmatrix} -0.785 \\ 0.620 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0.45 \\ 0.12 \\ -0.21 \end{bmatrix} \right\| = 0.514$$

for all $x \neq 0$, we have

$$0.514 \leq \frac{\|Ax\|}{\|x\|} \leq 9.53$$

Properties of matrix norm

- consistent with vector norm: matrix norm of $a \in \mathbf{R}^{n \times 1}$ is
$$\sqrt{\lambda_{\max}(a^T a)} = \sqrt{a^T a}$$
- for any x , $\|Ax\| \leq \|A\| \|x\|$
- scaling: $\|aA\| = |a| \|A\|$
- triangle inequality: $\|A + B\| \leq \|A\| + \|B\|$
- definiteness: $\|A\| = 0 \iff A = 0$
- norm of product: $\|AB\| \leq \|A\| \|B\|$

Singular value decomposition

more complete picture of gain properties of A given by *singular value decomposition* (SVD) of A :

$$A = U\Sigma V^T$$

where

- $A \in \mathbf{R}^{m \times n}$, $\text{Rank}(A) = r$
- $U \in \mathbf{R}^{m \times r}$, $U^T U = I$
- $V \in \mathbf{R}^{n \times r}$, $V^T V = I$
- $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_r)$, where $\sigma_1 \geq \dots \geq \sigma_r > 0$

with $U = [u_1 \cdots u_r]$, $V = [v_1 \cdots v_r]$,

$$A = U\Sigma V^T = \sum_{i=1}^r \sigma_i u_i v_i^T$$

- σ_i are the (nonzero) *singular values* of A
- v_i are the *right* or *input singular vectors* of A
- u_i are the *left* or *output singular vectors* of A

$$A^T A = (U \Sigma V^T)^T (U \Sigma V^T) = V \Sigma^2 V^T$$

hence:

- v_i are eigenvectors of $A^T A$ (corresponding to nonzero eigenvalues)
- $\sigma_i = \sqrt{\lambda_i(A^T A)}$ (and $\lambda_i(A^T A) = 0$ for $i > r$)
- $\|A\| = \sigma_1$

similarly,

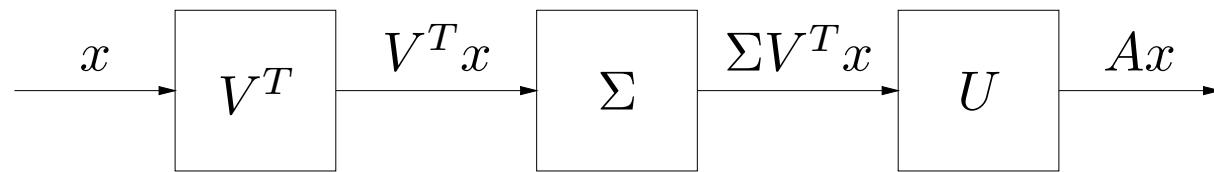
$$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = U\Sigma^2 U^T$$

hence:

- u_i are eigenvectors of AA^T (corresponding to nonzero eigenvalues)
- $\sigma_i = \sqrt{\lambda_i(AA^T)}$ (and $\lambda_i(AA^T) = 0$ for $i > r$)
- u_1, \dots, u_r are orthonormal basis for $\text{range}(A)$
- v_1, \dots, v_r are orthonormal basis for $\mathcal{N}(A)^\perp$

Interpretations

$$A = U\Sigma V^T = \sum_{i=1}^r \sigma_i u_i v_i^T$$



linear mapping $y = Ax$ can be decomposed as

- compute coefficients of x along input directions v_1, \dots, v_r
- scale coefficients by σ_i
- reconstitute along output directions u_1, \dots, u_r

difference with eigenvalue decomposition for symmetric A : input and output directions are *different*

- v_1 is most sensitive (highest gain) input direction
- u_1 is highest gain output direction
- $Av_1 = \sigma_1 u_1$

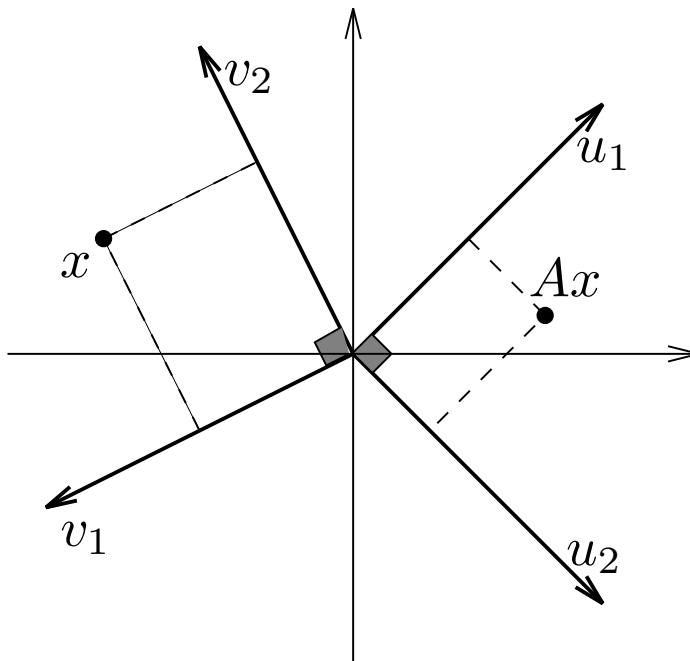
SVD gives clearer picture of gain as function of input/output directions

example: consider $A \in \mathbb{R}^{4 \times 4}$ with $\Sigma = \text{diag}(10, 7, 0.1, 0.05)$

- input components along directions v_1 and v_2 are amplified (by about 10) and come out mostly along plane spanned by u_1, u_2
- input components along directions v_3 and v_4 are attenuated (by about 10)
- $\|Ax\|/\|x\|$ can range between 10 and 0.05
- A is nonsingular
- for some applications you might say A is *effectively* rank 2

example: $A \in \mathbb{R}^{2 \times 2}$, with $\sigma_1 = 1$, $\sigma_2 = 0.5$

- resolve x along v_1 , v_2 : $v_1^T x = 0.5$, $v_2^T x = 0.6$, i.e., $x = 0.5v_1 + 0.6v_2$
- now form $Ax = (v_1^T x)\sigma_1 u_1 + (v_2^T x)\sigma_2 u_2 = (0.5)(1)u_1 + (0.6)(0.5)u_2$



Lecture 16

SVD Applications

- general pseudo-inverse
- full SVD
- image of unit ball under linear transformation
- SVD in estimation/inversion
- sensitivity of linear equations to data error
- low rank approximation via SVD

General pseudo-inverse

if $A \neq 0$ has SVD $A = U\Sigma V^T$,

$$A^\dagger = V\Sigma^{-1}U^T$$

is the *pseudo-inverse* or *Moore-Penrose inverse* of A

if A is skinny and full rank,

$$A^\dagger = (A^T A)^{-1} A^T$$

gives the least-squares approximate solution $x_{\text{ls}} = A^\dagger y$

if A is fat and full rank,

$$A^\dagger = A^T (A A^T)^{-1}$$

gives the least-norm solution $x_{\text{ln}} = A^\dagger y$

in general case:

$$X_{\text{ls}} = \{ z \mid \|Az - y\| = \min_w \|Aw - y\| \}$$

is set of least-squares approximate solutions

$x_{\text{pinv}} = A^\dagger y \in X_{\text{ls}}$ has minimum norm on X_{ls} , i.e., x_{pinv} is the minimum-norm, least-squares approximate solution

Pseudo-inverse via regularization

for $\mu > 0$, let x_μ be (unique) minimizer of

$$\|Ax - y\|^2 + \mu\|x\|^2$$

i.e.,

$$x_\mu = (A^T A + \mu I)^{-1} A^T y$$

here, $A^T A + \mu I > 0$ and so is invertible

then we have $\lim_{\mu \rightarrow 0} x_\mu = A^\dagger y$

in fact, we have $\lim_{\mu \rightarrow 0} (A^T A + \mu I)^{-1} A^T = A^\dagger$

(check this!)

Full SVD

SVD of $A \in \mathbf{R}^{m \times n}$ with $\text{Rank}(A) = r$:

$$A = U_1 \Sigma_1 V_1^T = \begin{bmatrix} u_1 & \cdots & u_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_r^T \end{bmatrix}$$

- find $U_2 \in \mathbf{R}^{m \times (m-r)}$, $V_2 \in \mathbf{R}^{n \times (n-r)}$ s.t. $U = [U_1 \ U_2] \in \mathbf{R}^{m \times m}$ and $V = [V_1 \ V_2] \in \mathbf{R}^{n \times n}$ are orthogonal
- add zero rows/cols to Σ_1 to form $\Sigma \in \mathbf{R}^{m \times n}$:

$$\Sigma = \left[\begin{array}{c|c} \Sigma_1 & 0_{r \times (n-r)} \\ \hline 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{array} \right]$$

then we have

$$A = U_1 \Sigma_1 V_1^T = \left[\begin{array}{c|c} U_1 & U_2 \end{array} \right] \left[\begin{array}{c|c} \Sigma_1 & 0_{r \times (n-r)} \\ \hline 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{array} \right] \left[\begin{array}{c} V_1^T \\ V_2^T \end{array} \right]$$

i.e.:

$$A = U \Sigma V^T$$

called *full SVD* of A

(SVD with positive singular values only called *compact SVD*)

Image of unit ball under linear transformation

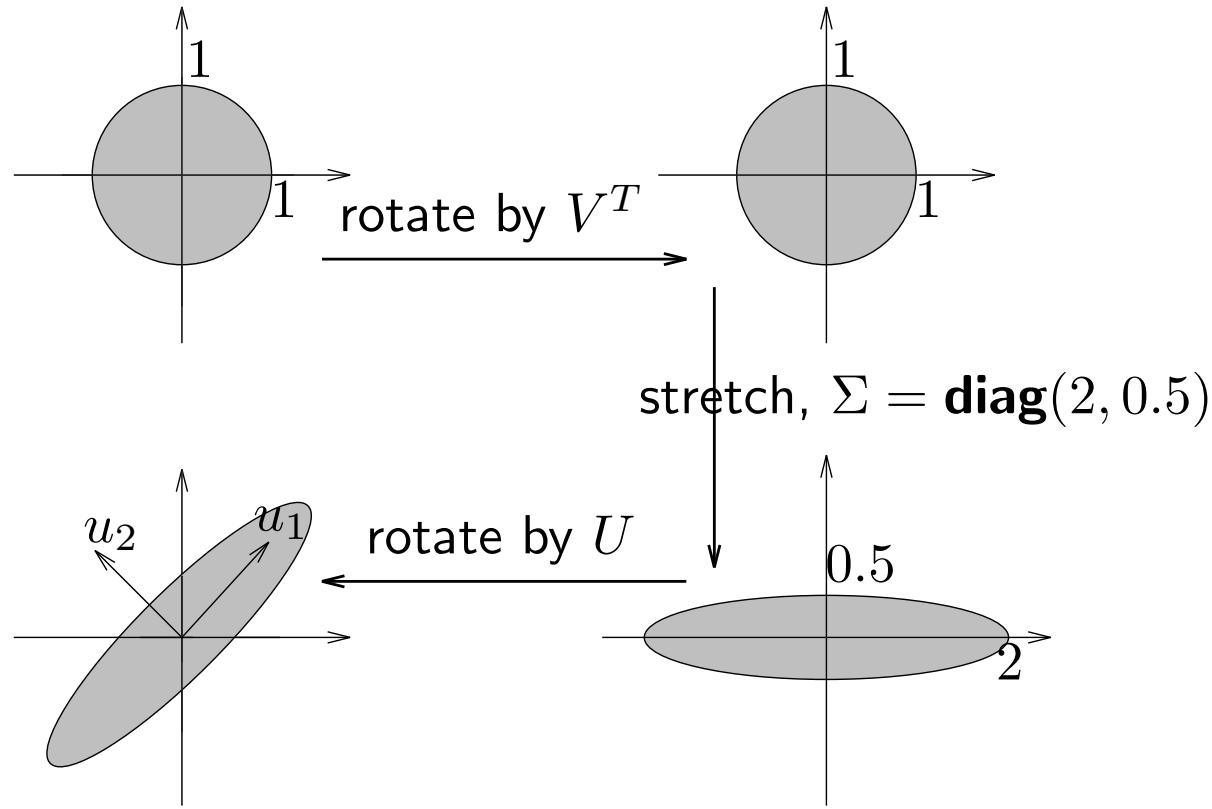
full SVD:

$$A = U\Sigma V^T$$

gives interpretation of $y = Ax$:

- rotate (by V^T)
- stretch along axes by σ_i ($\sigma_i = 0$ for $i > r$)
- zero-pad (if $m > n$) or truncate (if $m < n$) to get m -vector
- rotate (by U)

Image of unit ball under A



$\{Ax \mid \|x\| \leq 1\}$ is *ellipsoid* with principal axes $\sigma_i u_i$.

SVD in estimation/inversion

suppose $y = Ax + v$, where

- $y \in \mathbf{R}^m$ is measurement
- $x \in \mathbf{R}^n$ is vector to be estimated
- v is a measurement noise or error

‘norm-bound’ model of noise: we assume $\|v\| \leq \alpha$ but otherwise know nothing about v (α gives max norm of noise)

- consider estimator $\hat{x} = By$, with $BA = I$ (*i.e.*, unbiased)
- estimation or inversion error is $\tilde{x} = \hat{x} - x = Bv$
- set of possible estimation errors is ellipsoid

$$\tilde{x} \in \mathcal{E}_{\text{unc}} = \{ Bv \mid \|v\| \leq \alpha \}$$

- $x = \hat{x} - \tilde{x} \in \hat{x} - \mathcal{E}_{\text{unc}} = \hat{x} + \mathcal{E}_{\text{unc}}$, *i.e.*:
true x lies in *uncertainty ellipsoid* \mathcal{E}_{unc} , centered at estimate \hat{x}
- ‘good’ estimator has ‘small’ \mathcal{E}_{unc} (with $BA = I$, of course)

semiaxes of \mathcal{E}_{unc} are $\alpha\sigma_i u_i$ (singular values & vectors of B)

e.g., maximum norm of error is $\alpha\|B\|$, i.e., $\|\hat{x} - x\| \leq \alpha\|B\|$

optimality of least-squares: suppose $BA = I$ is any estimator, and $B_{\text{ls}} = A^\dagger$ is the least-squares estimator

then:

- $B_{\text{ls}}B_{\text{ls}}^T \leq BB^T$
- $\mathcal{E}_{\text{ls}} \subseteq \mathcal{E}$
- in particular $\|B_{\text{ls}}\| \leq \|B\|$

i.e., the least-squares estimator gives the *smallest* uncertainty ellipsoid

Example: navigation using range measurements (lect. 4)

we have

$$y = - \begin{bmatrix} k_1^T \\ k_2^T \\ k_3^T \\ k_4^T \end{bmatrix} x + v$$

where $k_i \in \mathbf{R}^2$

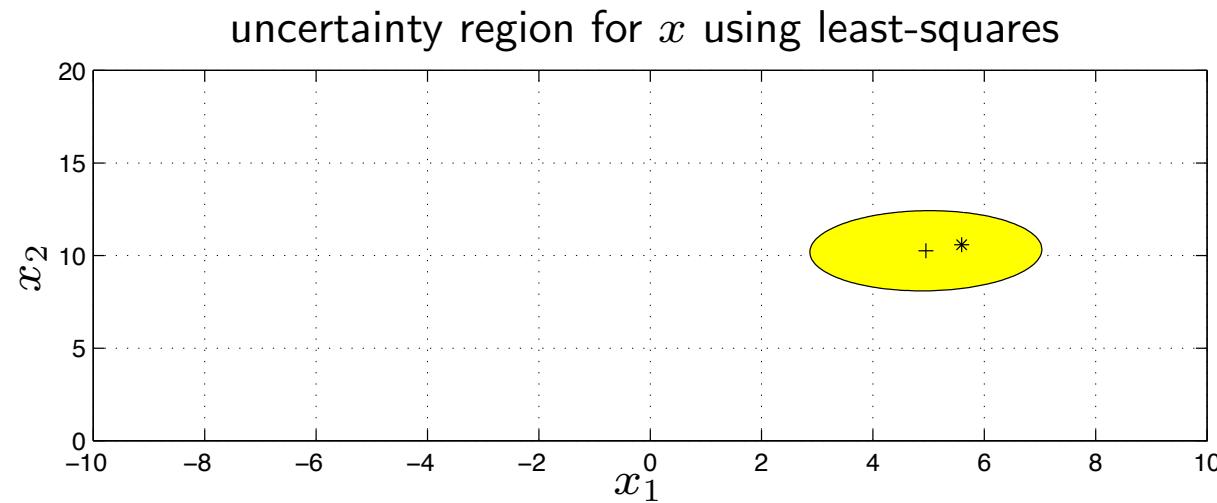
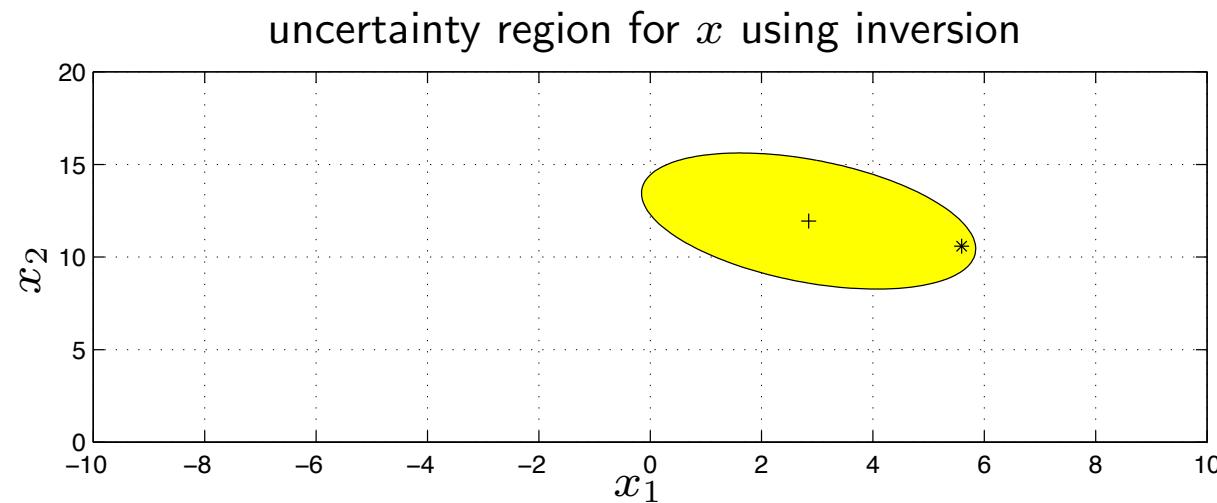
using first two measurements and inverting:

$$\hat{x} = - \begin{bmatrix} \begin{bmatrix} k_1^T \\ k_2^T \end{bmatrix}^{-1} & 0_{2 \times 2} \end{bmatrix} y$$

using all four measurements and least-squares:

$$\hat{x} = A^\dagger y$$

uncertainty regions (with $\alpha = 1$):



Proof of optimality property

suppose $A \in \mathbf{R}^{m \times n}$, $m > n$, is full rank

SVD: $A = U\Sigma V^T$, with V orthogonal

$B_{\text{ls}} = A^\dagger = V\Sigma^{-1}U^T$, and B satisfies $BA = I$

define $Z = B - B_{\text{ls}}$, so $B = B_{\text{ls}} + Z$

then $ZA = ZU\Sigma V^T = 0$, so $ZU = 0$ (multiply by $V\Sigma^{-1}$ on right)

therefore

$$\begin{aligned} BB^T &= (B_{\text{ls}} + Z)(B_{\text{ls}} + Z)^T \\ &= B_{\text{ls}}B_{\text{ls}}^T + B_{\text{ls}}Z^T + ZB_{\text{ls}}^T + ZZ^T \\ &= B_{\text{ls}}B_{\text{ls}}^T + ZZ^T \\ &\geq B_{\text{ls}}B_{\text{ls}}^T \end{aligned}$$

using $ZB_{\text{ls}}^T = (ZU)\Sigma^{-1}V^T = 0$

Sensitivity of linear equations to data error

consider $y = Ax$, $A \in \mathbf{R}^{n \times n}$ invertible; of course $x = A^{-1}y$

suppose we have an error or noise in y , i.e., y becomes $y + \delta y$

then x becomes $x + \delta x$ with $\delta x = A^{-1}\delta y$

hence we have $\|\delta x\| = \|A^{-1}\delta y\| \leq \|A^{-1}\| \|\delta y\|$

if $\|A^{-1}\|$ is large,

- small errors in y can lead to large errors in x
- can't solve for x given y (with small errors)
- hence, A can be considered singular in practice

a more refined analysis uses *relative* instead of *absolute* errors in x and y
since $y = Ax$, we also have $\|y\| \leq \|A\|\|x\|$, hence

$$\frac{\|\delta x\|}{\|x\|} \leq \|A\|\|A^{-1}\| \frac{\|\delta y\|}{\|y\|}$$

$$\kappa(A) = \|A\|\|A^{-1}\| = \sigma_{\max}(A)/\sigma_{\min}(A)$$

is called the *condition number* of A

we have:

relative error in solution $x \leq$ condition number \cdot relative error in data y

or, in terms of # bits of guaranteed accuracy:

$$\# \text{ bits accuracy in solution} \approx \# \text{ bits accuracy in data} - \log_2 \kappa$$

we say

- A is well conditioned if κ is small
- A is poorly conditioned if κ is large

(definition of ‘small’ and ‘large’ depend on application)

same analysis holds for least-squares approximate solutions with A nonsquare, $\kappa = \sigma_{\max}(A)/\sigma_{\min}(A)$

Low rank approximations

suppose $A \in \mathbf{R}^{m \times n}$, $\text{Rank}(A) = r$, with SVD $A = U\Sigma V^T = \sum_{i=1}^r \sigma_i u_i v_i^T$

we seek matrix \hat{A} , $\text{Rank}(\hat{A}) \leq p < r$, s.t. $\hat{A} \approx A$ in the sense that $\|A - \hat{A}\|$ is minimized

solution: optimal rank p approximator is

$$\hat{A} = \sum_{i=1}^p \sigma_i u_i v_i^T$$

- hence $\|A - \hat{A}\| = \left\| \sum_{i=p+1}^r \sigma_i u_i v_i^T \right\| = \sigma_{p+1}$
- interpretation: SVD dyads $u_i v_i^T$ are ranked in order of ‘importance’; take p to get rank p approximant

proof: suppose $\text{Rank}(B) \leq p$

then $\dim \mathcal{N}(B) \geq n - p$

also, $\dim \text{span}\{v_1, \dots, v_{p+1}\} = p + 1$

hence, the two subspaces intersect, *i.e.*, there is a unit vector $z \in \mathbf{R}^n$ s.t.

$$Bz = 0, \quad z \in \text{span}\{v_1, \dots, v_{p+1}\}$$

$$(A - B)z = Az = \sum_{i=1}^{p+1} \sigma_i u_i v_i^T z$$

$$\|(A - B)z\|^2 = \sum_{i=1}^{p+1} \sigma_i^2 (v_i^T z)^2 \geq \sigma_{p+1}^2 \|z\|^2$$

hence $\|A - B\| \geq \sigma_{p+1} = \|A - \hat{A}\|$

Distance to singularity

another interpretation of σ_i :

$$\sigma_i = \min\{ \|A - B\| \mid \text{Rank}(B) \leq i - 1 \}$$

i.e., the distance (measured by matrix norm) to the nearest rank $i - 1$ matrix

for example, if $A \in \mathbf{R}^{n \times n}$, $\sigma_n = \sigma_{\min}$ is distance to nearest singular matrix

hence, small σ_{\min} means A is near to a singular matrix

application: model simplification

suppose $y = Ax + v$, where

- $A \in \mathbf{R}^{100 \times 30}$ has SVs

$$10, 7, 2, 0.5, 0.01, \dots, 0.0001$$

- $\|x\|$ is on the order of 1
- unknown error or noise v has norm on the order of 0.1

then the terms $\sigma_i u_i v_i^T x$, for $i = 5, \dots, 30$, are substantially smaller than the noise term v

simplified model:

$$y = \sum_{i=1}^4 \sigma_i u_i v_i^T x + v$$

Lecture 17

Example: Quantum mechanics

- wave function and Schrodinger equation
- discretization
- preservation of probability
- eigenvalues & eigenstates
- example

Quantum mechanics

- single particle in interval $[0, 1]$, mass m
- potential $V : [0, 1] \rightarrow \mathbf{R}$

$\Psi : [0, 1] \times \mathbf{R}_+ \rightarrow \mathbf{C}$ is (complex-valued) *wave function*

interpretation: $|\Psi(x, t)|^2$ is probability density of particle at position x , time t

(so $\int_0^1 |\Psi(x, t)|^2 dx = 1$ for all t)

evolution of Ψ governed by *Schrodinger* equation:

$$i\hbar \dot{\Psi} = \left(V - \frac{\hbar^2}{2m} \nabla_x^2 \right) \Psi = H\Psi$$

where H is *Hamiltonian* operator, $i = \sqrt{-1}$

Discretization

let's discretize position x into N discrete points, k/N , $k = 1, \dots, N$

wave function is approximated as *vector* $\Psi(t) \in \mathbf{C}^N$

∇_x^2 operator is approximated as *matrix*

$$\nabla^2 = N^2 \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & 1 & -2 \end{bmatrix}$$

so $w = \nabla^2 v$ means

$$w_k = \frac{(v_{k+1} - v_k)/(1/N) - (v_k - v_{k-1})(1/N)}{1/N}$$

(which approximates $w = \partial^2 v / \partial x^2$)

discretized Schrodinger equation is (complex) linear dynamical system

$$\dot{\Psi} = (-i/\hbar)(V - (\hbar/2m)\nabla^2)\Psi = (-i/\hbar)H\Psi$$

where V is a diagonal matrix with $V_{kk} = V(k/N)$

hence we analyze using linear dynamical system theory (with complex vectors & matrices):

$$\dot{\Psi} = (-i/\hbar)H\Psi$$

solution of Shrodinger equation: $\Psi(t) = e^{(-i/\hbar)tH}\Psi(0)$

matrix $e^{(-i/\hbar)tH}$ propagates wave function forward in time t seconds
(backward if $t < 0$)

Preservation of probability

$$\begin{aligned}\frac{d}{dt} \|\Psi\|^2 &= \frac{d}{dt} \Psi^* \Psi \\&= \dot{\Psi}^* \Psi + \Psi^* \dot{\Psi} \\&= ((-i/\hbar)H\Psi)^* \Psi + \Psi^*((-i/\hbar)H\Psi) \\&= (i/\hbar)\Psi^* H \Psi + (-i/\hbar)\Psi^* H \Psi \\&= 0\end{aligned}$$

(using $H = H^T \in \mathbf{R}^{N \times N}$)

hence, $\|\Psi(t)\|^2$ is constant; our discretization preserves probability *exactly*

$U = e^{-(i/\hbar)tH}$ is *unitary*, meaning $U^*U = I$

unitary is extension of *orthogonal* for complex matrix: if $U \in \mathbf{C}^{N \times N}$ is unitary and $z \in \mathbf{C}^N$, then

$$\|Uz\|^2 = (Uz)^*(Uz) = z^*U^*Uz = z^*z = \|z\|^2$$

Eigenvalues & eigenstates

H is symmetric, so

- its eigenvalues $\lambda_1, \dots, \lambda_N$ are real ($\lambda_1 \leq \dots \leq \lambda_N$)
- its eigenvectors v_1, \dots, v_N can be chosen to be orthogonal (and real)

from $Hv = \lambda v \Leftrightarrow (-i/\hbar)Hv = (-i/\hbar)\lambda v$ we see:

- eigenvectors of $(-i/\hbar)H$ are same as eigenvectors of H , i.e., v_1, \dots, v_N
- eigenvalues of $(-i/\hbar)H$ are $(-i/\hbar)\lambda_1, \dots, (-i/\hbar)\lambda_N$ (which are pure imaginary)

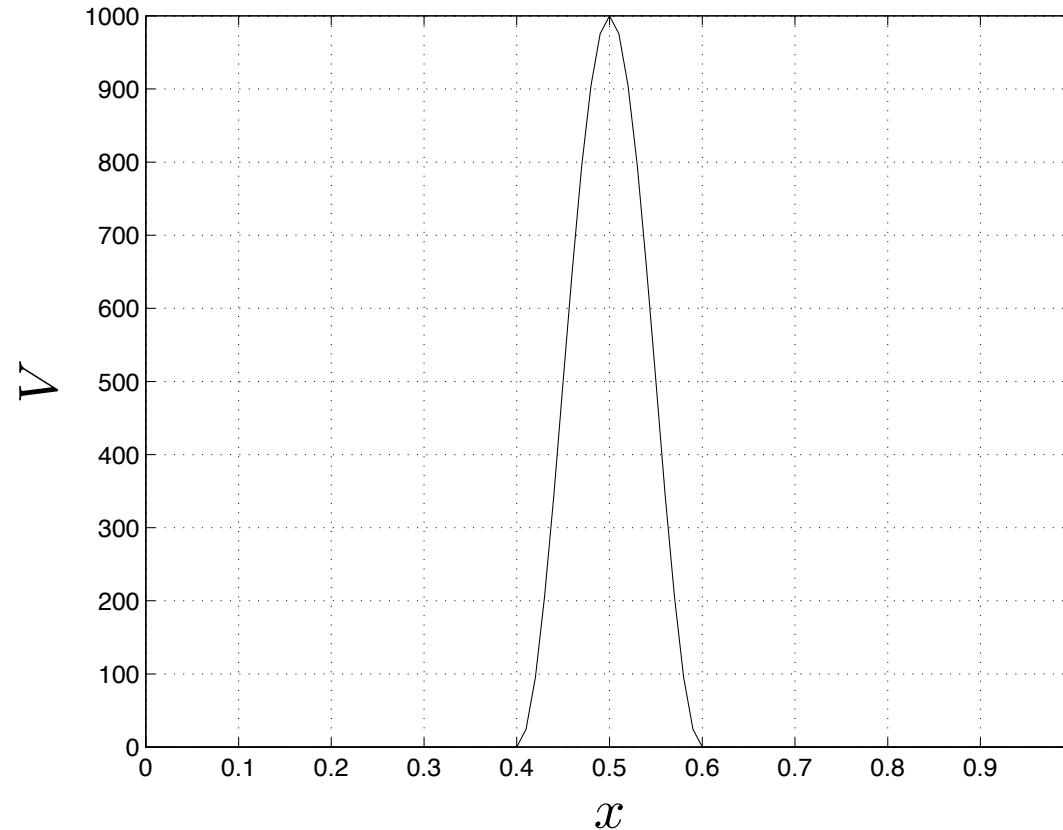
- eigenvectors v_k are called *eigenstates* of system
- eigenvalue λ_k is *energy* of eigenstate v_k
- for mode $\Psi(t) = e^{(-i/\hbar)\lambda_k t} v_k$, probability density

$$|\Psi_m(t)|^2 = \left| e^{(-i/\hbar)\lambda_k t} v_k \right|^2 = |v_{mk}|^2$$

doesn't change with time (v_{mk} is m th entry of v_k)

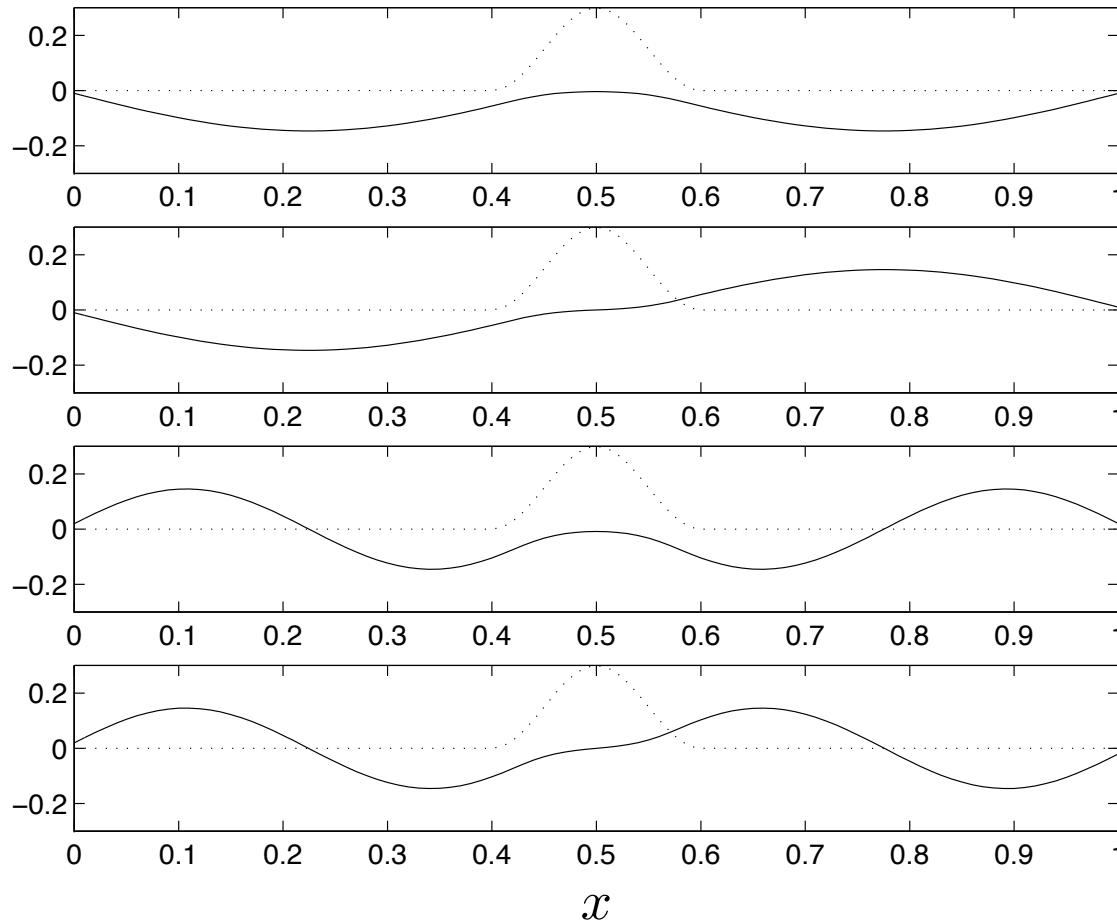
Example

Potential Function $V(x)$



- potential bump in middle of infinite potential well
- (for this example, we set $\hbar = 1$, $m = 1 \dots$)

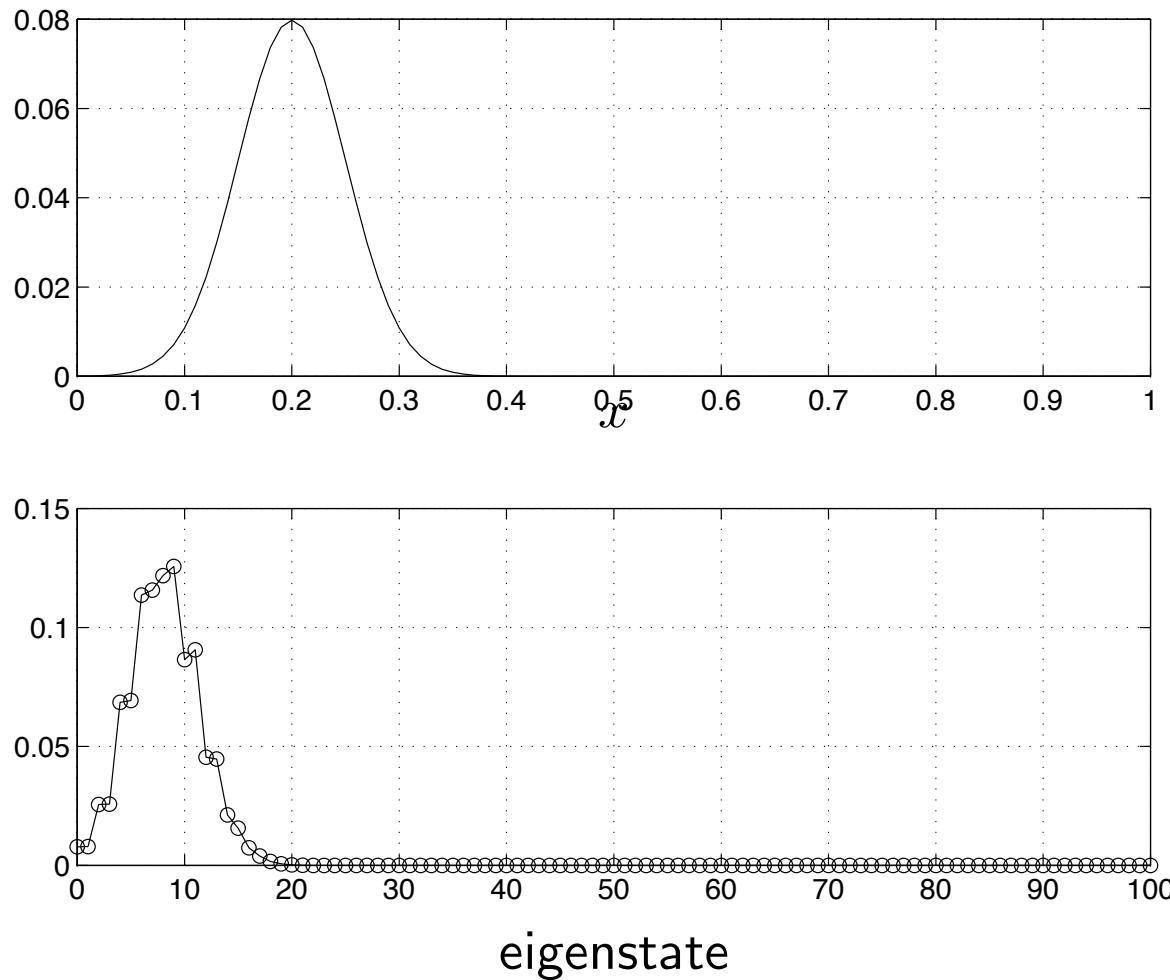
lowest energy eigenfunctions



- potential V shown as dotted line (scaled to fit plot)
- four eigenstates with lowest energy shown (*i.e.*, v_1, v_2, v_3, v_4)

now let's look at a trajectory of Ψ , with initial wave function $\Psi(0)$

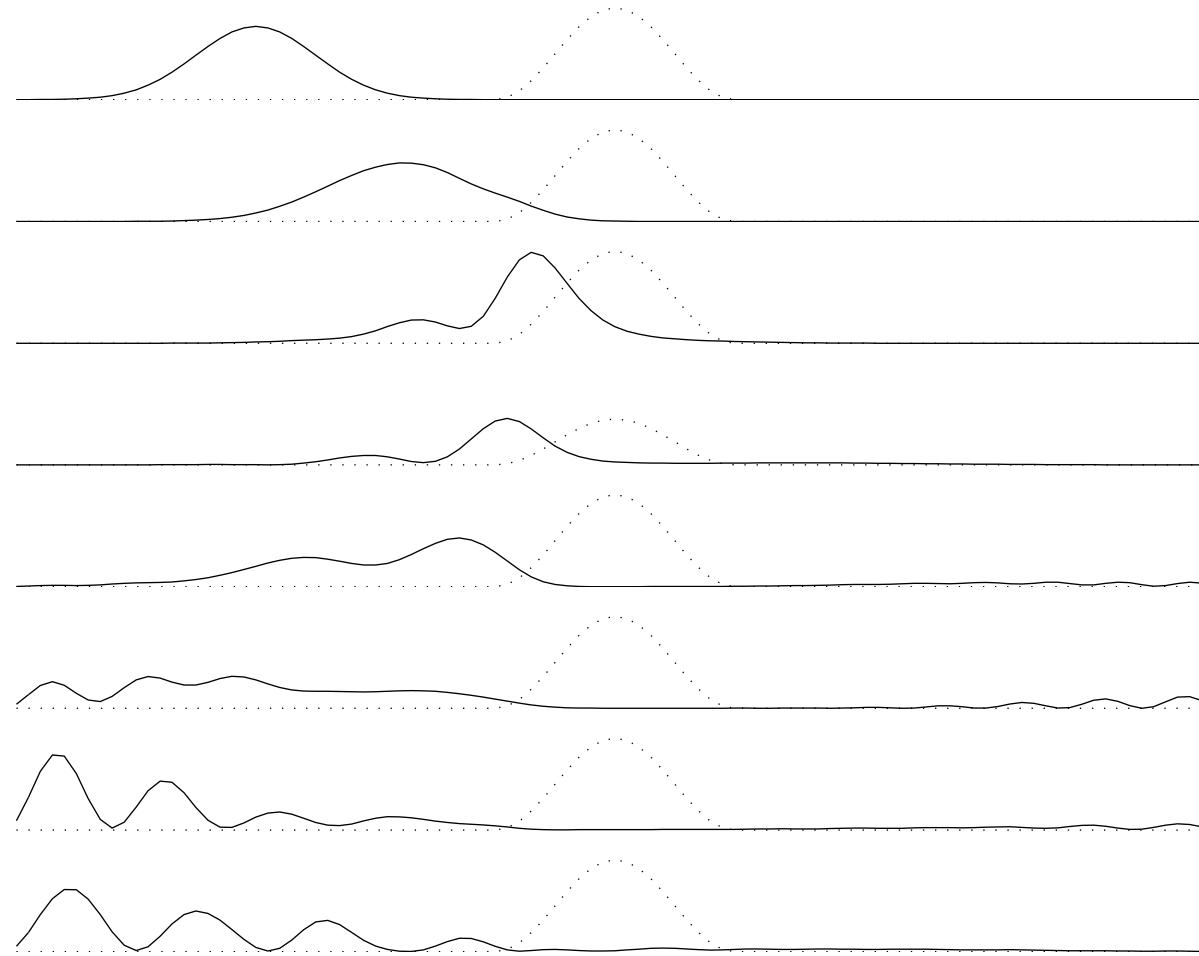
- particle near $x = 0.2$
- with momentum to right (can't see in plot of $|\Psi|^2$)
- (expected) kinetic energy half potential bump height



- top plot shows initial probability density $|\Psi(0)|^2$
- bottom plot shows $|v_k^* \Psi(0)|^2$, i.e., resolution of $\Psi(0)$ into eigenstates

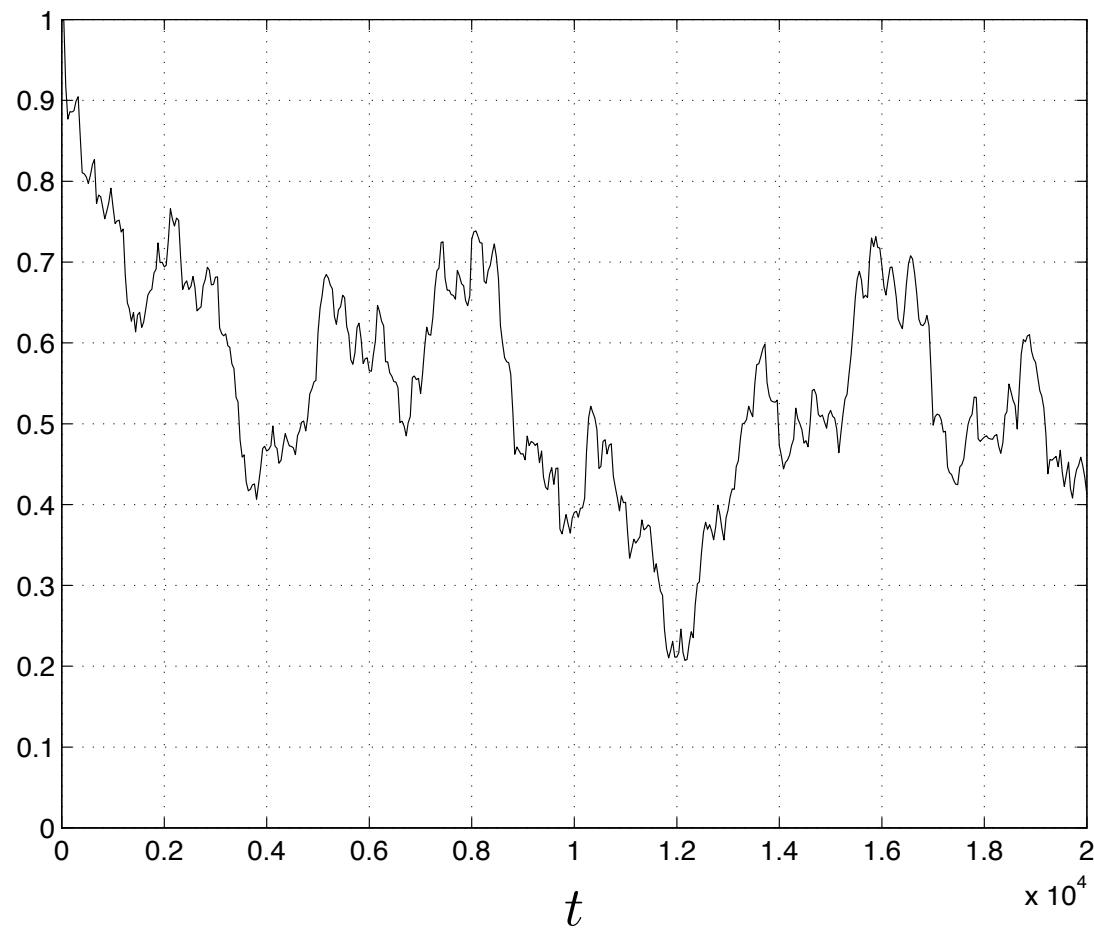
time evolution, for $t = 0, 40, 80, \dots, 320$:

$$|\Psi(t)|^2$$



cf. classical solution:

- particle rolls half way up potential bump, stops, then rolls back down
- reverses velocity when it hits the wall at left
(perfectly elastic collision)
- then repeats



plot shows probability that particle is in left half of well, i.e., $\sum_{k=1}^{N/2} |\Psi_k(t)|^2$, versus time t

Lecture 18

Controllability and state transfer

- state transfer
- reachable set, controllability matrix
- minimum norm inputs
- infinite-horizon minimum norm transfer

State transfer

consider $\dot{x} = Ax + Bu$ (or $x(t+1) = Ax(t) + Bu(t)$) over time interval $[t_i, t_f]$

we say input $u : [t_i, t_f] \rightarrow \mathbf{R}^m$ steers or transfers state from $x(t_i)$ to $x(t_f)$ (over time interval $[t_i, t_f]$)

(subscripts stand for *initial* and *final*)

questions:

- where can $x(t_i)$ be transferred to at $t = t_f$?
- how quickly can $x(t_i)$ be transferred to some x_{target} ?
- how do we find a u that transfers $x(t_i)$ to $x(t_f)$?
- how do we find a ‘small’ or ‘efficient’ u that transfers $x(t_i)$ to $x(t_f)$?

Reachability

consider state transfer from $x(0) = 0$ to $x(t)$

we say $x(t)$ is *reachable* (in t seconds or epochs)

we define $\mathcal{R}_t \subseteq \mathbf{R}^n$ as the set of points reachable in t seconds or epochs

for CT system $\dot{x} = Ax + Bu$,

$$\mathcal{R}_t = \left\{ \int_0^t e^{(t-\tau)A} Bu(\tau) d\tau \mid u : [0, t] \rightarrow \mathbf{R}^m \right\}$$

and for DT system $x(t+1) = Ax(t) + Bu(t)$,

$$\mathcal{R}_t = \left\{ \sum_{\tau=0}^{t-1} A^{t-1-\tau} Bu(\tau) \mid u(0), \dots, u(t-1) \in \mathbf{R}^m \right\}$$

- \mathcal{R}_t is a subspace of \mathbf{R}^n
- $\mathcal{R}_t \subseteq \mathcal{R}_s$ if $t \leq s$
(i.e., can reach more points given more time)

we define the *reachable set* \mathcal{R} as the set of points reachable for some t :

$$\mathcal{R} = \bigcup_{t \geq 0} \mathcal{R}_t$$

Reachability for discrete-time LDS

DT system $x(t+1) = Ax(t) + Bu(t)$, $x(t) \in \mathbf{R}^n$

$$x(t) = \mathcal{C}_t \begin{bmatrix} u(t-1) \\ \vdots \\ u(0) \end{bmatrix}$$

where $\mathcal{C}_t = [B \ AB \ \cdots \ A^{t-1}B]$

so reachable set at t is $\mathcal{R}_t = \text{range}(\mathcal{C}_t)$

by C-H theorem, we can express each A^k for $k \geq n$ as linear combination of A^0, \dots, A^{n-1}

hence for $t \geq n$, $\text{range}(\mathcal{C}_t) = \text{range}(\mathcal{C}_n)$

thus we have

$$\mathcal{R}_t = \begin{cases} \text{range}(\mathcal{C}_t) & t < n \\ \text{range}(\mathcal{C}) & t \geq n \end{cases}$$

where $\mathcal{C} = \mathcal{C}_n$ is called the *controllability matrix*

- any state that can be reached can be reached by $t = n$
- the reachable set is $\mathcal{R} = \text{range}(\mathcal{C})$

Controllable system

system is called *reachable* or *controllable* if all states are reachable (*i.e.*, $\mathcal{R} = \mathbf{R}^n$)

system is reachable if and only if $\text{Rank}(\mathcal{C}) = n$

example: $x(t+1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$

controllability matrix is $\mathcal{C} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

hence system is not controllable; reachable set is

$$\mathcal{R} = \text{range}(\mathcal{C}) = \{ x \mid x_1 = x_2 \}$$

General state transfer

with $t_f > t_i$,

$$x(t_f) = A^{t_f - t_i} x(t_i) + \mathcal{C}_{t_f - t_i} \begin{bmatrix} u(t_f - 1) \\ \vdots \\ u(t_i) \end{bmatrix}$$

hence can transfer $x(t_i)$ to $x(t_f) = x_{\text{des}}$

$$\Leftrightarrow x_{\text{des}} - A^{t_f - t_i} x(t_i) \in \mathcal{R}_{t_f - t_i}$$

- general state transfer reduces to reachability problem
- if system is controllable any state transfer can be achieved in $\leq n$ steps
- important special case: driving state to zero (sometimes called regulating or controlling state)

Least-norm input for reachability

assume system is reachable, $\text{Rank}(\mathcal{C}_t) = n$

to steer $x(0) = 0$ to $x(t) = x_{\text{des}}$, inputs $u(0), \dots, u(t-1)$ must satisfy

$$x_{\text{des}} = \mathcal{C}_t \begin{bmatrix} u(t-1) \\ \vdots \\ u(0) \end{bmatrix}$$

among all u that steer $x(0) = 0$ to $x(t) = x_{\text{des}}$, the one that minimizes

$$\sum_{\tau=0}^{t-1} \|u(\tau)\|^2$$

is given by

$$\begin{bmatrix} u_{\ln}(t-1) \\ \vdots \\ u_{\ln}(0) \end{bmatrix} = \mathcal{C}_t^T (\mathcal{C}_t \mathcal{C}_t^T)^{-1} x_{\text{des}}$$

u_{\ln} is called *least-norm* or *minimum energy* input that effects state transfer

can express as

$$u_{\ln}(\tau) = B^T (A^T)^{(t-1-\tau)} \left(\sum_{s=0}^{t-1} A^s B B^T (A^T)^s \right)^{-1} x_{\text{des}},$$

for $\tau = 0, \dots, t-1$

\mathcal{E}_{\min} , the minimum value of $\sum_{\tau=0}^{t-1} \|u(\tau)\|^2$ required to reach $x(t) = x_{\text{des}}$, is sometimes called *minimum energy* required to reach $x(t) = x_{\text{des}}$

$$\begin{aligned}
 \mathcal{E}_{\min} &= \sum_{\tau=0}^{t-1} \|u_{\ln}(\tau)\|^2 \\
 &= (\mathcal{C}_t^T (\mathcal{C}_t \mathcal{C}_t^T)^{-1} x_{\text{des}})^T \mathcal{C}_t^T (\mathcal{C}_t \mathcal{C}_t^T)^{-1} x_{\text{des}} \\
 &= x_{\text{des}}^T (\mathcal{C}_t \mathcal{C}_t^T)^{-1} x_{\text{des}} \\
 &= x_{\text{des}}^T \left(\sum_{\tau=0}^{t-1} A^\tau B B^T (A^T)^\tau \right)^{-1} x_{\text{des}}
 \end{aligned}$$

- $\mathcal{E}_{\min}(x_{\text{des}}, t)$ gives measure of how hard it is to reach $x(t) = x_{\text{des}}$ from $x(0) = 0$ (*i.e.*, how large a u is required)
- $\mathcal{E}_{\min}(x_{\text{des}}, t)$ gives practical measure of controllability/reachability (as function of x_{des}, t)
- ellipsoid $\{ z \mid \mathcal{E}_{\min}(z, t) \leq 1 \}$ shows points in state space reachable at t with one unit of energy
(shows directions that can be reached with small inputs, and directions that can be reached only with large inputs)

\mathcal{E}_{\min} as function of t :

if $t \geq s$ then

$$\sum_{\tau=0}^{t-1} A^\tau BB^T (A^T)^\tau \geq \sum_{\tau=0}^{s-1} A^\tau BB^T (A^T)^\tau$$

hence

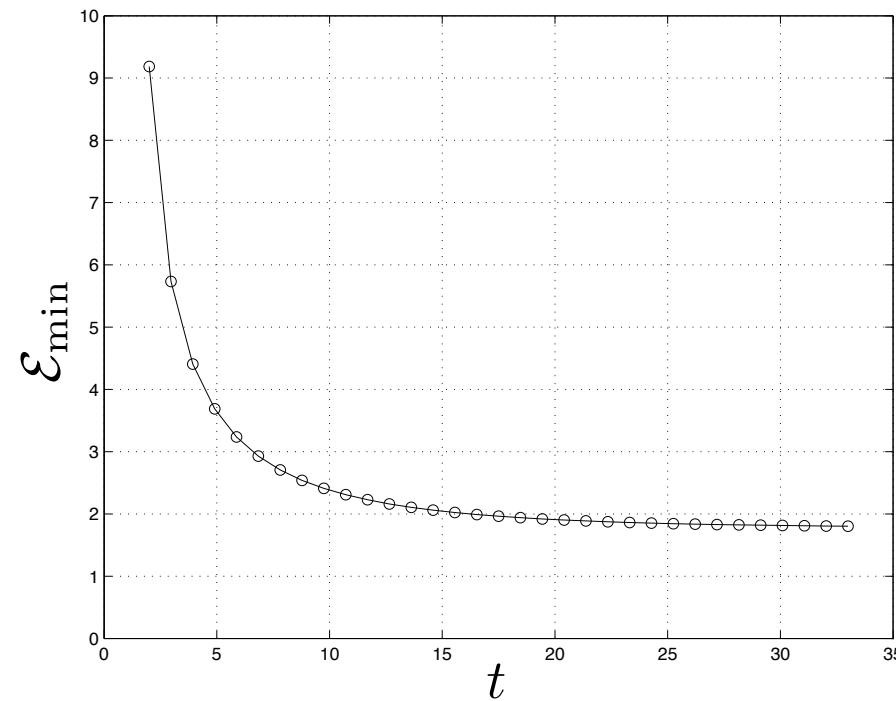
$$\left(\sum_{\tau=0}^{t-1} A^\tau BB^T (A^T)^\tau \right)^{-1} \leq \left(\sum_{\tau=0}^{s-1} A^\tau BB^T (A^T)^\tau \right)^{-1}$$

so $\mathcal{E}_{\min}(x_{\text{des}}, t) \leq \mathcal{E}_{\min}(x_{\text{des}}, s)$

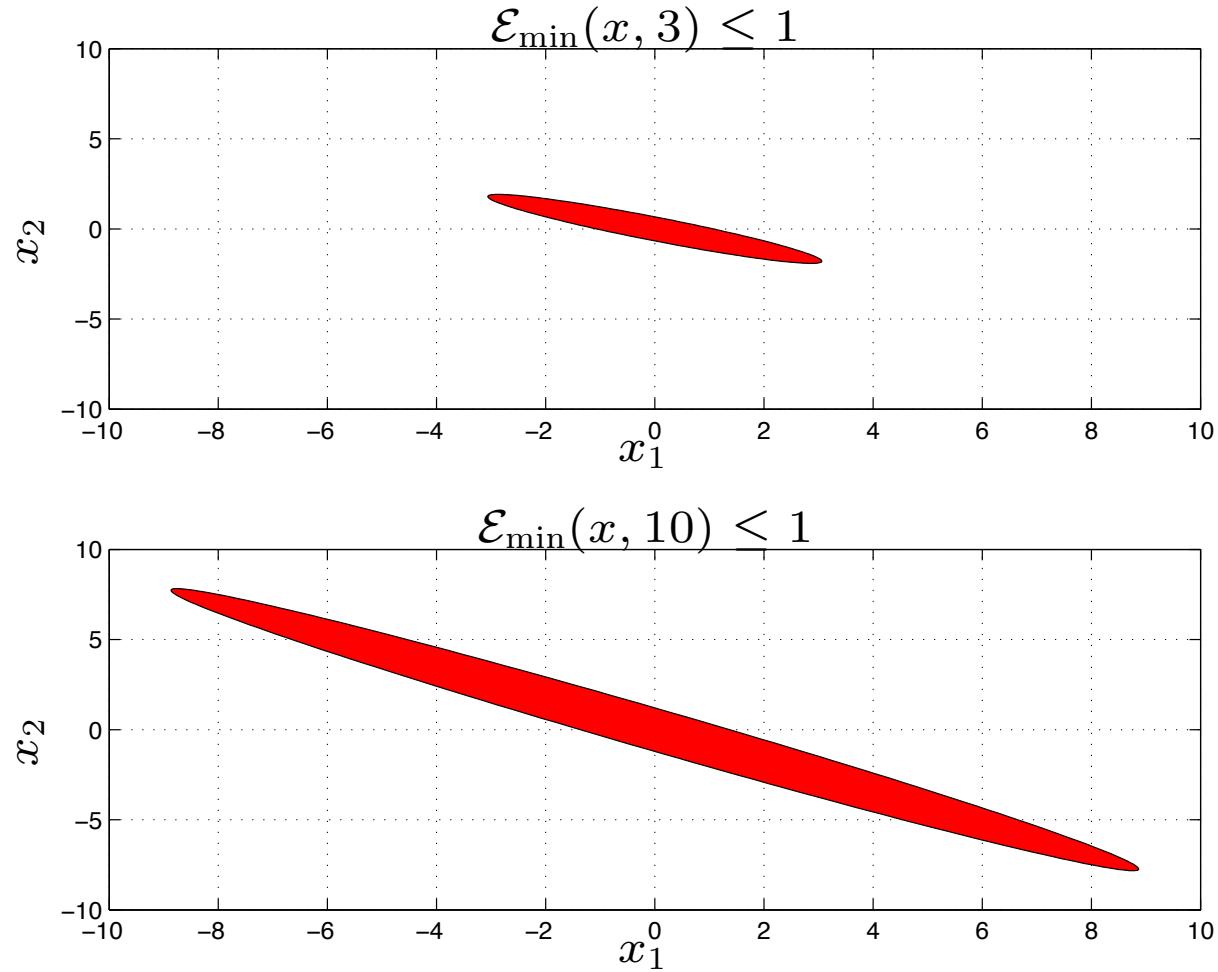
i.e.: takes less energy to get somewhere more leisurely

example: $x(t+1) = \begin{bmatrix} 1.75 & 0.8 \\ -0.95 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$

$\mathcal{E}_{\min}(z, t)$ for $z = [1 \ 1]^T$:



ellipsoids $\mathcal{E}_{\min} \leq 1$ for $t = 3$ and $t = 10$:



Minimum energy over infinite horizon

the matrix

$$P = \lim_{t \rightarrow \infty} \left(\sum_{\tau=0}^{t-1} A^\tau B B^T (A^T)^\tau \right)^{-1}$$

always exists, and gives the minimum energy required to reach a point x_{des} (with no limit on t):

$$\min \left\{ \sum_{\tau=0}^{t-1} \|u(\tau)\|^2 \mid x(0) = 0, x(t) = x_{\text{des}} \right\} = x_{\text{des}}^T P x_{\text{des}}$$

if A is stable, $P > 0$ (*i.e.*, can't get anywhere for free)

if A is not stable, then P can have nonzero nullspace

- $Pz = 0, z \neq 0$ means can get to z using u 's with energy as small as you like
(u just gives a little kick to the state; the instability carries it out to z efficiently)
- basis of highly maneuverable, unstable aircraft

Continuous-time reachability

consider now $\dot{x} = Ax + Bu$ with $x(t) \in \mathbf{R}^n$

reachable set at time t is

$$\mathcal{R}_t = \left\{ \int_0^t e^{(t-\tau)A} Bu(\tau) d\tau \mid u : [0, t] \rightarrow \mathbf{R}^m \right\}$$

fact: for $t > 0$, $\mathcal{R}_t = \mathcal{R} = \text{range}(\mathcal{C})$, where

$$\mathcal{C} = [B \quad AB \quad \dots \quad A^{n-1}B]$$

is the controllability matrix of (A, B)

- same \mathcal{R} as discrete-time system
- for continuous-time system, any reachable point can be reached as fast as you like (with large enough u)

first let's show for *any* u (and $x(0) = 0$) we have $x(t) \in \text{range}(\mathcal{C})$

write e^{tA} as power series:

$$e^{tA} = I + \frac{t}{1!}A + \frac{t^2}{2!}A^2 + \dots$$

by C-H, express A^n, A^{n+1}, \dots in terms of A^0, \dots, A^{n-1} and collect powers of A :

$$e^{tA} = \alpha_0(t)I + \alpha_1(t)A + \dots + \alpha_{n-1}(t)A^{n-1}$$

therefore

$$\begin{aligned} x(t) &= \int_0^t e^{\tau A} Bu(t-\tau) d\tau \\ &= \int_0^t \left(\sum_{i=0}^{n-1} \alpha_i(\tau) A^i \right) Bu(t-\tau) d\tau \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^{n-1} A^i B \int_0^t \alpha_i(\tau) u(t - \tau) d\tau \\
&= \mathcal{C}z
\end{aligned}$$

where $z_i = \int_0^t \alpha_i(\tau) u(t - \tau) d\tau$

hence, $x(t)$ is always in $\text{range}(\mathcal{C})$

need to show converse: every point in $\text{range}(\mathcal{C})$ can be reached

Impulsive inputs

suppose $x(0_-) = 0$ and we apply input $u(t) = \delta^{(k)}(t)f$, where $\delta^{(k)}$ denotes k th derivative of δ and $f \in \mathbf{R}^m$

then $U(s) = s^k f$, so

$$\begin{aligned} X(s) &= (sI - A)^{-1}Bs^k f \\ &= (s^{-1}I + s^{-2}A + \dots)Bs^k f \\ &= (\underbrace{s^{k-1} + \dots + sA^{k-2} + A^{k-1}}_{\text{impulsive terms}} + s^{-1}A^k + \dots)Bf \end{aligned}$$

hence

$$x(t) = \text{impulsive terms} + A^k B f + A^{k+1} B f \frac{t}{1!} + A^{k+2} B f \frac{t^2}{2!} + \dots$$

in particular, $x(0_+) = A^k B f$

thus, input $u = \delta^{(k)} f$ transfers state from $x(0_-) = 0$ to $x(0_+) = A^k B f$

now consider input of form

$$u(t) = \delta(t)f_0 + \cdots + \delta^{(n-1)}(t)f_{n-1}$$

where $f_i \in \mathbf{R}^m$

by linearity we have

$$x(0_+) = Bf_0 + \cdots + A^{n-1}Bf_{n-1} = \mathcal{C} \begin{bmatrix} f_0 \\ \vdots \\ f_{n-1} \end{bmatrix}$$

hence we can reach any point in $\text{range}(\mathcal{C})$
(at least, using impulse inputs)

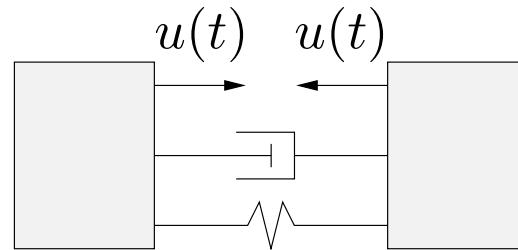
can also be shown that any point in $\text{range}(\mathcal{C})$ can be reached for any $t > 0$ using *nonimpulsive* inputs

fact: if $x(0) \in \mathcal{R}$, then $x(t) \in \mathcal{R}$ for all t (no matter what u is)

to show this, need to show $e^{tA}x(0) \in \mathcal{R}$ if $x(0) \in \mathcal{R} \dots$

Example

- unit masses at y_1, y_2 , connected by unit springs, dampers
- input is tension between masses
- state is $x = [y^T \dot{y}^T]^T$



system is

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} u$$

- can we maneuver state anywhere, starting from $x(0) = 0$?
- if not, where can we maneuver state?

controllability matrix is

$$\mathcal{C} = [\ B \ AB \ A^2B \ A^3B] = \begin{bmatrix} 0 & 1 & -2 & 2 \\ 0 & -1 & 2 & -2 \\ 1 & -2 & 2 & 0 \\ -1 & 2 & -2 & 0 \end{bmatrix}$$

hence reachable set is

$$\mathcal{R} = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

we can reach states with $y_1 = -y_2$, $\dot{y}_1 = -\dot{y}_2$, i.e., precisely the differential motions

it's obvious — internal force does not affect center of mass position or total momentum!

Least-norm input for reachability

(also called *minimum energy input*)

assume that $\dot{x} = Ax + Bu$ is reachable

we seek u that steers $x(0) = 0$ to $x(t) = x_{\text{des}}$ and minimizes

$$\int_0^t \|u(\tau)\|^2 d\tau$$

let's discretize system with interval $h = t/N$

(we'll let $N \rightarrow \infty$ later)

thus u is piecewise constant:

$$u(\tau) = u_d(k) \quad \text{for } kh \leq \tau < (k+1)h, \quad k = 0, \dots, N-1$$

so

$$x(t) = \begin{bmatrix} B_d & A_d B_d & \cdots & A_d^{N-1} B_d \end{bmatrix} \begin{bmatrix} u_d(N-1) \\ \vdots \\ u_d(0) \end{bmatrix}$$

where

$$A_d = e^{hA}, \quad B_d = \int_0^h e^{\tau A} d\tau B$$

least-norm u_d that yields $x(t) = x_{\text{des}}$ is

$$u_{\text{dln}}(k) = B_d^T (A_d^T)^{(N-1-k)} \left(\sum_{i=0}^{N-1} A_d^i B_d B_d^T (A_d^T)^i \right)^{-1} x_{\text{des}}$$

let's express in terms of A :

$$B_d^T (A_d^T)^{(N-1-k)} = B_d^T e^{(t-\tau)A^T}$$

where $\tau = t(k + 1)/N$

for N large, $B_d \approx (t/N)B$, so this is approximately

$$(t/N)B^T e^{(t-\tau)A^T}$$

similarly

$$\begin{aligned} \sum_{i=0}^{N-1} A_d^i B_d B_d^T (A_d^T)^i &= \sum_{i=0}^{N-1} e^{(ti/N)A} B_d B_d^T e^{(ti/N)A^T} \\ &\approx (t/N) \int_0^t e^{\bar{t}A} B B^T e^{\bar{t}A^T} d\bar{t} \end{aligned}$$

for large N

hence least-norm discretized input is approximately

$$u_{\text{ln}}(\tau) = B^T e^{(t-\tau)A^T} \left(\int_0^t e^{\bar{t}A} BB^T e^{\bar{t}A^T} d\bar{t} \right)^{-1} x_{\text{des}}, \quad 0 \leq \tau \leq t$$

for large N

hence, this is the least-norm continuous input

- can make t small, but get larger u
- cf. DT solution: sum becomes integral

min energy is

$$\int_0^t \|u_{\ln}(\tau)\|^2 d\tau = x_{\text{des}}^T Q(t)^{-1} x_{\text{des}}$$

where

$$Q(t) = \int_0^t e^{\tau A} B B^T e^{\tau A^T} d\tau$$

can show

$$\begin{aligned} (A, B) \text{ controllable} &\Leftrightarrow Q(t) > 0 \text{ for all } t > 0 \\ &\Leftrightarrow Q(s) > 0 \text{ for some } s > 0 \end{aligned}$$

in fact, $\text{range}(Q(t)) = \mathcal{R}$ for any $t > 0$

Minimum energy over infinite horizon

the matrix

$$P = \lim_{t \rightarrow \infty} \left(\int_0^t e^{\tau A} B B^T e^{\tau A^T} d\tau \right)^{-1}$$

always exists, and gives minimum energy required to reach a point x_{des} (with no limit on t):

$$\min \left\{ \int_0^t \|u(\tau)\|^2 d\tau \mid x(0) = 0, x(t) = x_{\text{des}} \right\} = x_{\text{des}}^T P x_{\text{des}}$$

- if A is stable, $P > 0$ (*i.e.*, can't get anywhere for free)
- if A is not stable, then P can have nonzero nullspace
- $Pz = 0$, $z \neq 0$ means can get to z using u 's with energy as small as you like (u just gives a little kick to the state; the instability carries it out to z efficiently)

General state transfer

consider state transfer from $x(t_i)$ to $x(t_f) = x_{\text{des}}$, $t_f > t_i$

since

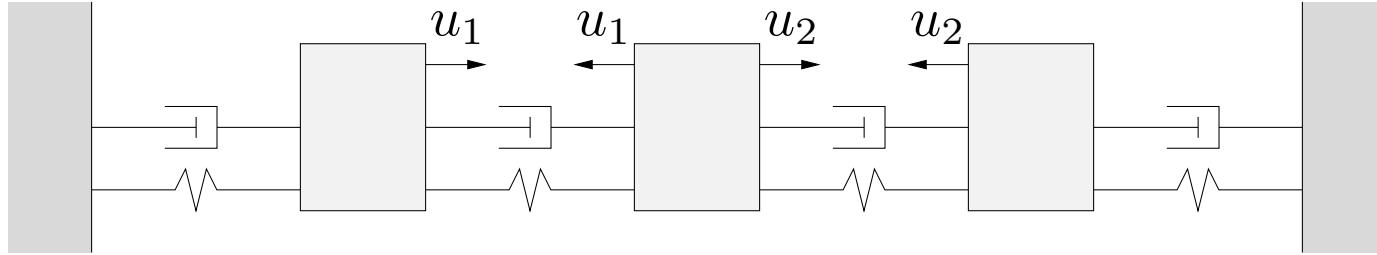
$$x(t_f) = e^{(t_f - t_i)A} x(t_i) + \int_{t_i}^{t_f} e^{(t_f - \tau)A} B u(\tau) d\tau$$

u steers $x(t_i)$ to $x(t_f) = x_{\text{des}} \Leftrightarrow$

u (shifted by t_i) steers $x(0) = 0$ to $x(t_f - t_i) = x_{\text{des}} - e^{(t_f - t_i)A} x(t_i)$

- general state transfer reduces to reachability problem
- if system is controllable, any state transfer can be effected
 - in ‘zero’ time with impulsive inputs
 - in any positive time with non-impulsive inputs

Example



- unit masses, springs, dampers
- u_1 is force between 1st & 2nd masses
- u_2 is force between 2nd & 3rd masses
- $y \in \mathbb{R}^3$ is displacement of masses 1,2,3
- $x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$

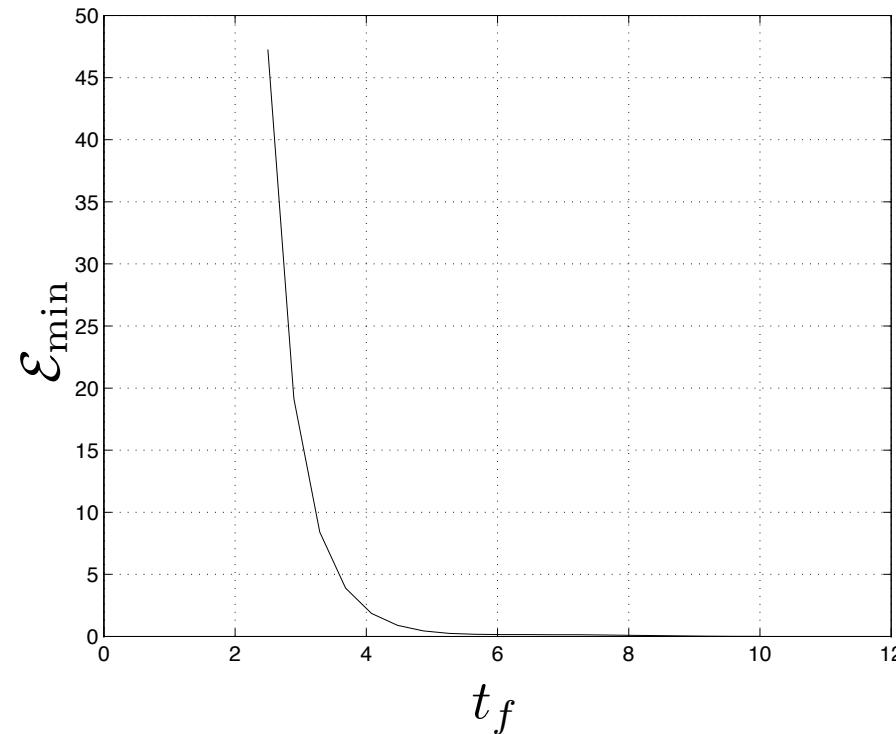
system is:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & -2 & 1 & 0 \\ 1 & -2 & 1 & 1 & -2 & 1 \\ 0 & 1 & -2 & 0 & 1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

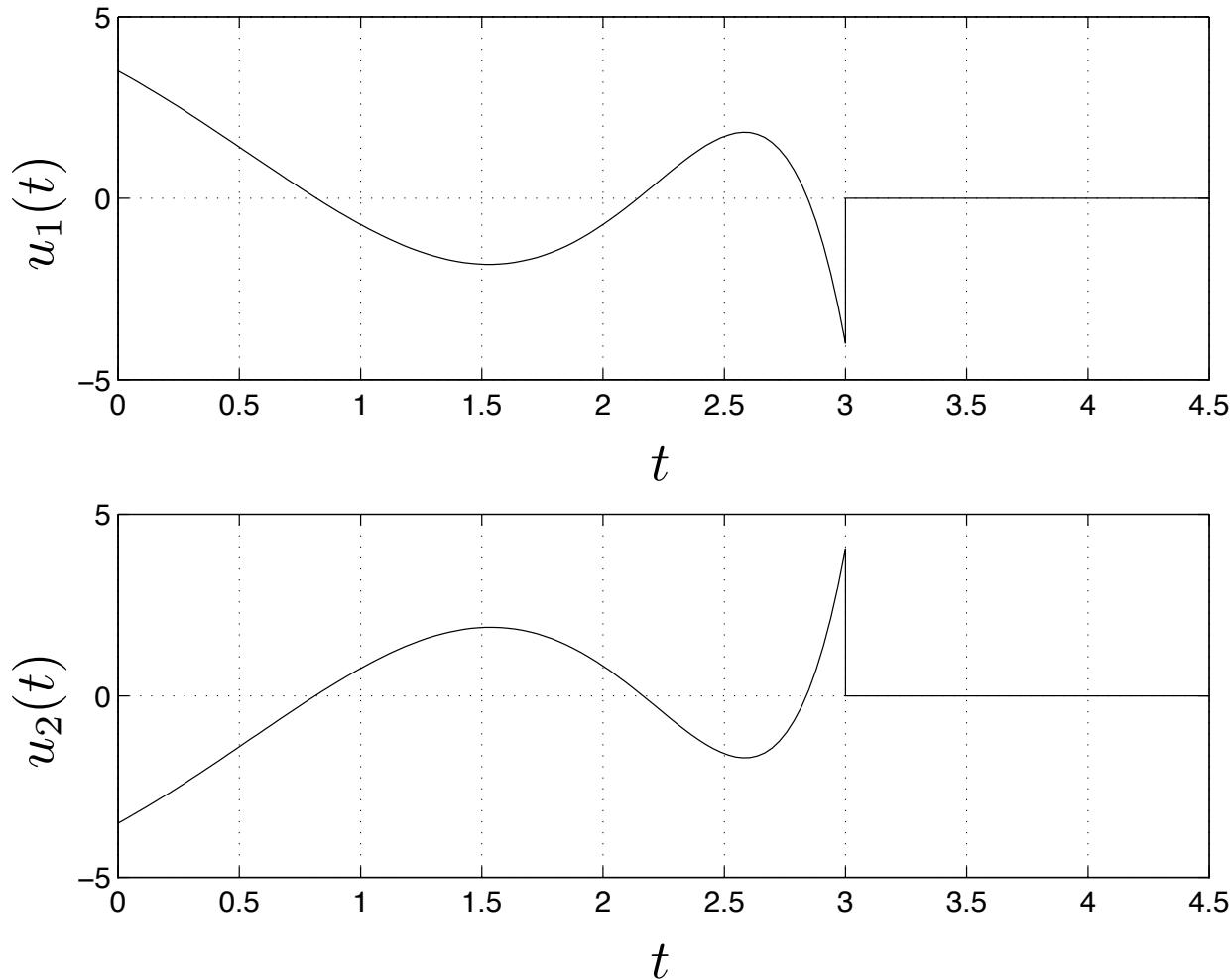
steer state from $x(0) = e_1$ to $x(t_f) = 0$

i.e., control initial state e_1 to zero at $t = t_f$

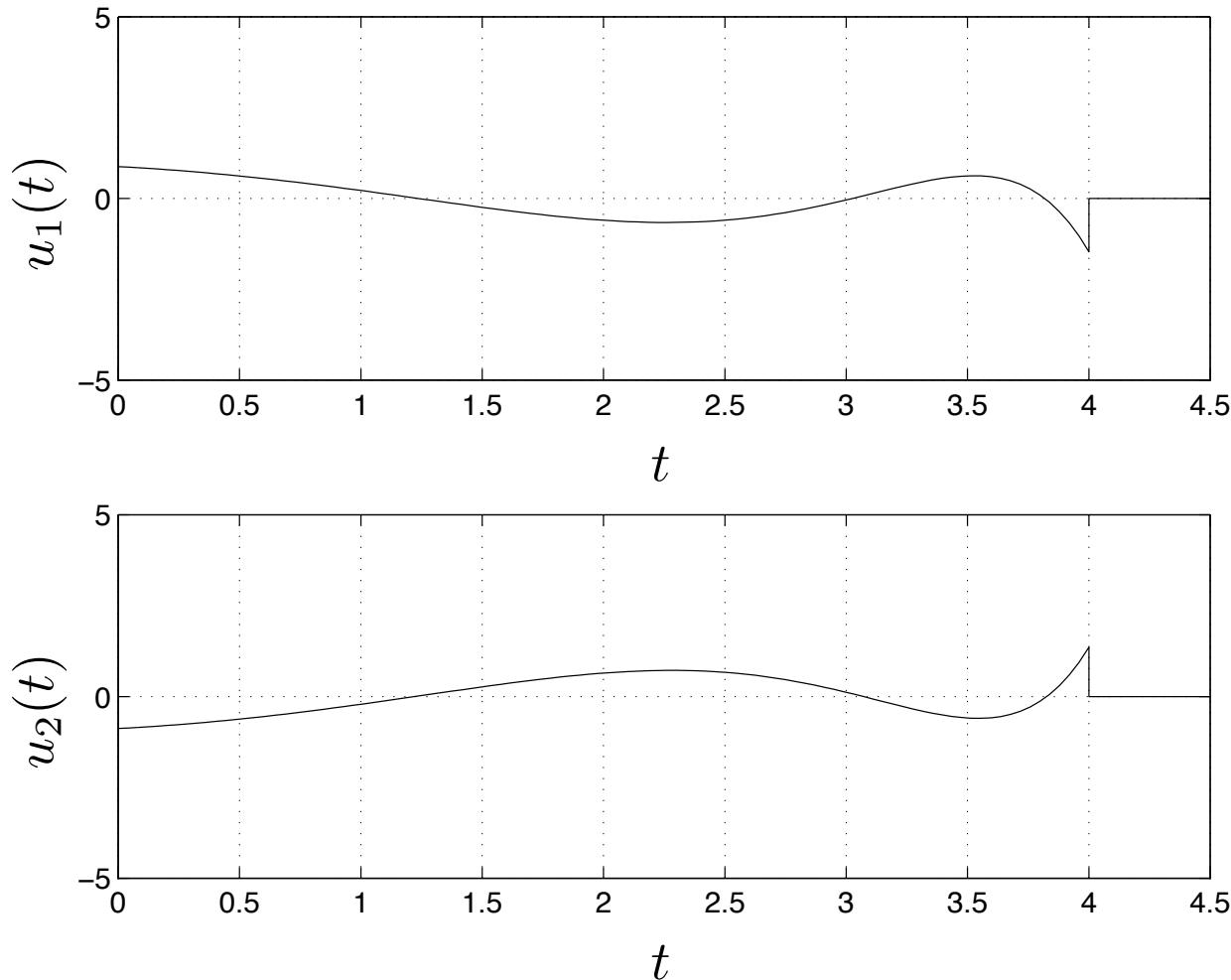
$$\mathcal{E}_{\min} = \int_0^{t_f} \|u_{\ln}(\tau)\|^2 d\tau \text{ vs. } t_f:$$



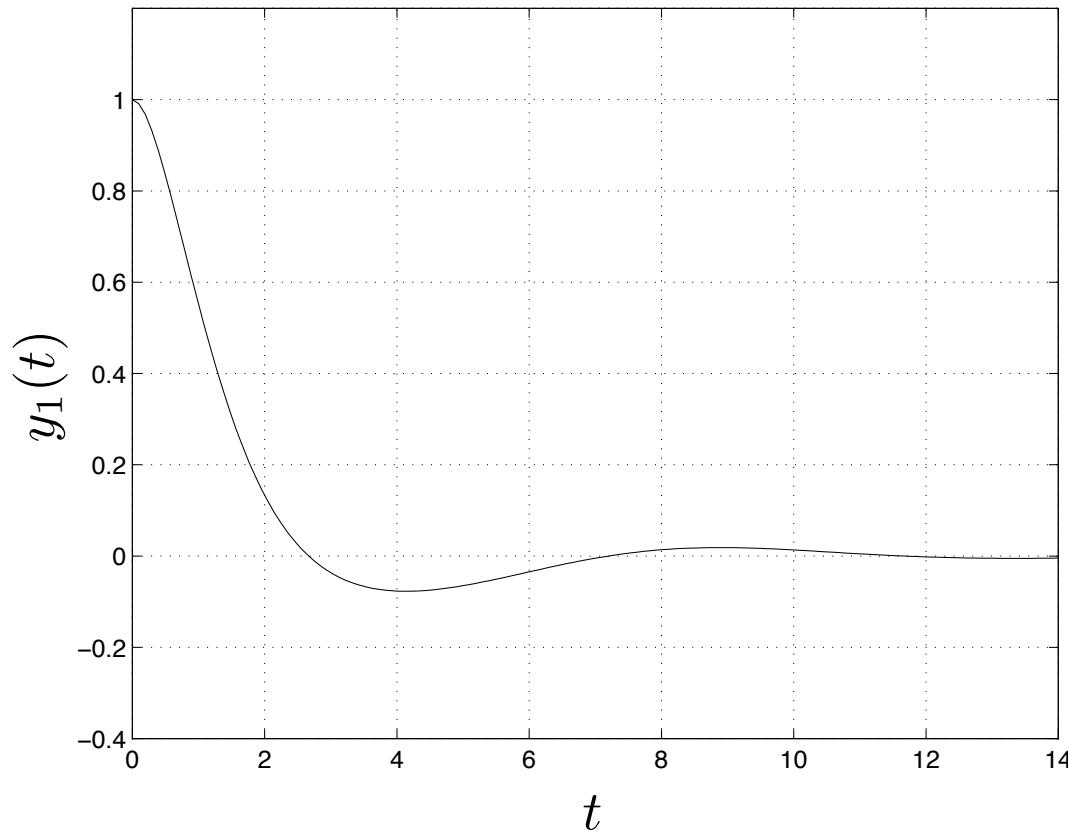
for $t_f = 3$, $u = u_{\ln}$ is:



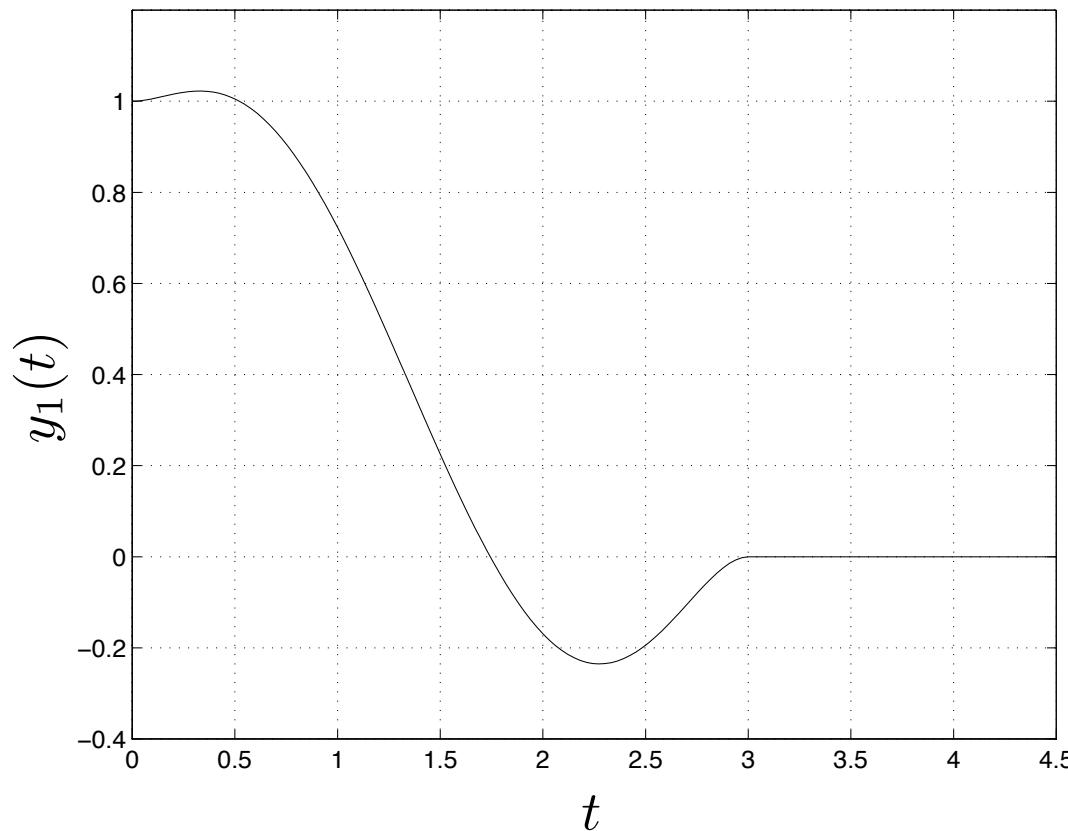
and for $t_f = 4$:



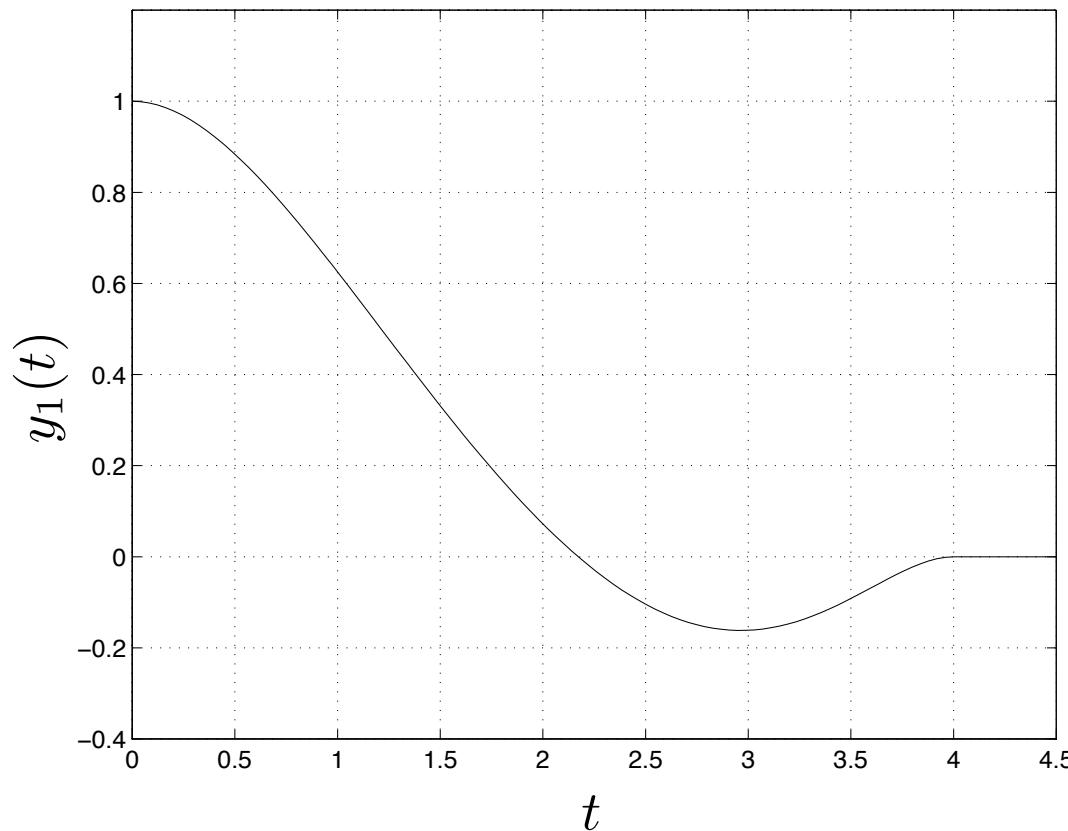
output y_1 for $u = 0$:



output y_1 for $u = u_{\ln}$ with $t_f = 3$:



output y_1 for $u = u_{\ln}$ with $t_f = 4$:



Lecture 19

Observability and state estimation

- state estimation
- discrete-time observability
- observability – controllability duality
- observers for noiseless case
- continuous-time observability
- least-squares observers
- example

State estimation set up

we consider the discrete-time system

$$x(t+1) = Ax(t) + Bu(t) + w(t), \quad y(t) = Cx(t) + Du(t) + v(t)$$

- w is state *disturbance* or *noise*
- v is sensor *noise* or *error*
- A, B, C , and D are known
- u and y are observed over time interval $[0, t - 1]$
- w and v are not known, but can be described statistically, or assumed small (*e.g.*, in RMS value)

State estimation problem

state estimation problem: estimate $x(s)$ from

$$u(0), \dots, u(t-1), y(0), \dots, y(t-1)$$

- $s = 0$: estimate initial state
- $s = t - 1$: estimate current state
- $s = t$: estimate (*i.e.*, predict) next state

an algorithm or system that yields an estimate $\hat{x}(s)$ is called an *observer* or *state estimator*

$\hat{x}(s)$ is denoted $\hat{x}(s|t-1)$ to show what information estimate is based on
(read, “ $\hat{x}(s)$ given $t-1$ ”)

Noiseless case

let's look at finding $x(0)$, with no state or measurement noise:

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

with $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$, $y(t) \in \mathbf{R}^p$

then we have

$$\begin{bmatrix} y(0) \\ \vdots \\ y(t-1) \end{bmatrix} = \mathcal{O}_t x(0) + \mathcal{T}_t \begin{bmatrix} u(0) \\ \vdots \\ u(t-1) \end{bmatrix}$$

where

$$\mathcal{O}_t = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{t-1} \end{bmatrix}, \quad \mathcal{T}_t = \begin{bmatrix} D & 0 & \cdots & & \\ CB & D & 0 & \cdots & \\ \vdots & & & & \\ CA^{t-2}B & CA^{t-3}B & \cdots & CB & D \end{bmatrix}$$

- \mathcal{O}_t maps initial state into resulting output over $[0, t - 1]$
- \mathcal{T}_t maps input to output over $[0, t - 1]$

hence we have

$$\mathcal{O}_t x(0) = \begin{bmatrix} y(0) \\ \vdots \\ y(t-1) \end{bmatrix} - \mathcal{T}_t \begin{bmatrix} u(0) \\ \vdots \\ u(t-1) \end{bmatrix}$$

RHS is known, $x(0)$ is to be determined

hence:

- can uniquely determine $x(0)$ if and only if $\mathcal{N}(\mathcal{O}_t) = \{0\}$
- $\mathcal{N}(\mathcal{O}_t)$ gives ambiguity in determining $x(0)$
- if $x(0) \in \mathcal{N}(\mathcal{O}_t)$ and $u = 0$, output is zero over interval $[0, t - 1]$
- input u does not affect ability to determine $x(0)$;
its effect can be subtracted out

Observability matrix

by C-H theorem, each A^k is linear combination of A^0, \dots, A^{n-1}

hence for $t \geq n$, $\mathcal{N}(\mathcal{O}_t) = \mathcal{N}(\mathcal{O})$ where

$$\mathcal{O} = \mathcal{O}_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is called the *observability matrix*

if $x(0)$ can be deduced from u and y over $[0, t - 1]$ for any t , then $x(0)$ can be deduced from u and y over $[0, n - 1]$

$\mathcal{N}(\mathcal{O})$ is called *unobservable subspace*; describes ambiguity in determining state from input and output

system is called *observable* if $\mathcal{N}(\mathcal{O}) = \{0\}$, i.e., $\text{Rank}(\mathcal{O}) = n$

Observability – controllability duality

let $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ be dual of system (A, B, C, D) , i.e.,

$$\tilde{A} = A^T, \quad \tilde{B} = C^T, \quad \tilde{C} = B^T, \quad \tilde{D} = D^T$$

controllability matrix of dual system is

$$\begin{aligned}\tilde{\mathcal{C}} &= [\tilde{B} \ \tilde{A}\tilde{B} \cdots \tilde{A}^{n-1}\tilde{B}] \\ &= [C^T \ A^T C^T \cdots (A^T)^{n-1} C^T] \\ &= \mathcal{O}^T,\end{aligned}$$

transpose of observability matrix

similarly we have $\tilde{\mathcal{O}} = \mathcal{C}^T$

thus, system is observable (controllable) if and only if dual system is controllable (observable)

in fact,

$$\mathcal{N}(\mathcal{O}) = \text{range}(\mathcal{O}^T)^\perp = \text{range}(\tilde{\mathcal{C}})^\perp$$

i.e., unobservable subspace is orthogonal complement of controllable subspace of dual

Observers for noiseless case

suppose $\text{Rank}(\mathcal{O}_t) = n$ (*i.e.*, system is observable) and let F be any left inverse of \mathcal{O}_t , *i.e.*, $F\mathcal{O}_t = I$

then we have the observer

$$x(0) = F \left(\begin{bmatrix} y(0) \\ \vdots \\ y(t-1) \end{bmatrix} - \mathcal{T}_t \begin{bmatrix} u(0) \\ \vdots \\ u(t-1) \end{bmatrix} \right)$$

which deduces $x(0)$ (exactly) from u, y over $[0, t-1]$

in fact we have

$$x(\tau - t + 1) = F \left(\begin{bmatrix} y(\tau - t + 1) \\ \vdots \\ y(\tau) \end{bmatrix} - \mathcal{T}_t \begin{bmatrix} u(\tau - t + 1) \\ \vdots \\ u(\tau) \end{bmatrix} \right)$$

i.e., our observer estimates what state was $t - 1$ epochs ago, given past $t - 1$ inputs & outputs

observer is (multi-input, multi-output) *finite impulse response* (FIR) filter, with inputs u and y , and output \hat{x}

Invariance of unobservable set

fact: the unobservable subspace $\mathcal{N}(\mathcal{O})$ is invariant, i.e., if $z \in \mathcal{N}(\mathcal{O})$, then $Az \in \mathcal{N}(\mathcal{O})$

proof: suppose $z \in \mathcal{N}(\mathcal{O})$, i.e., $CA^k z = 0$ for $k = 0, \dots, n-1$

evidently $CA^k(Az) = 0$ for $k = 0, \dots, n-2$;

$$CA^{n-1}(Az) = CA^n z = - \sum_{i=0}^{n-1} \alpha_i CA^i z = 0$$

(by C-H) where

$$\det(sI - A) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_0$$

Continuous-time observability

continuous-time system with no sensor or state noise:

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

can we deduce state x from u and y ?

let's look at derivatives of y :

$$\begin{aligned}y &= Cx + Du \\ \dot{y} &= C\dot{x} + D\dot{u} = CAx + CBu + D\dot{u} \\ \ddot{y} &= CA^2x + CABu + CB\dot{u} + D\ddot{u}\end{aligned}$$

and so on

hence we have

$$\begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \mathcal{O}x + \mathcal{T} \begin{bmatrix} u \\ \dot{u} \\ \vdots \\ u^{(n-1)} \end{bmatrix}$$

where \mathcal{O} is the observability matrix and

$$\mathcal{T} = \begin{bmatrix} D & 0 & \cdots \\ CB & D & 0 & \cdots \\ \vdots & & & \\ CA^{n-2}B & CA^{n-3}B & \cdots & CB & D \end{bmatrix}$$

(same matrices we encountered in discrete-time case!)

rewrite as

$$\mathcal{O}x = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix} - \mathcal{T} \begin{bmatrix} u \\ \dot{u} \\ \vdots \\ u^{(n-1)} \end{bmatrix}$$

RHS is known; x is to be determined

hence if $\mathcal{N}(\mathcal{O}) = \{0\}$ we can deduce $x(t)$ from derivatives of $u(t)$, $y(t)$ up to order $n - 1$

in this case we say system is observable

can construct an observer using any left inverse F of \mathcal{O} :

$$x = F \left(\begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix} - \mathcal{T} \begin{bmatrix} u \\ \dot{u} \\ \vdots \\ u^{(n-1)} \end{bmatrix} \right)$$

- reconstructs $x(t)$ (exactly and instantaneously) from

$$u(t), \dots, u^{(n-1)}(t), y(t), \dots, y^{(n-1)}(t)$$

- derivative-based state reconstruction is dual of state transfer using impulsive inputs

A converse

suppose $z \in \mathcal{N}(\mathcal{O})$ (the unobservable subspace), and u is any input, with x, y the corresponding state and output, *i.e.*,

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

then state trajectory $\tilde{x} = x + e^{tA}z$ satisfies

$$\dot{\tilde{x}} = A\tilde{x} + Bu, \quad y = C\tilde{x} + Du$$

i.e., input/output signals u, y consistent with both state trajectories x, \tilde{x}
hence if system is unobservable, no signal processing of any kind applied to
 u and y can deduce x

unobservable subspace $\mathcal{N}(\mathcal{O})$ gives fundamental ambiguity in deducing x
from u, y

Least-squares observers

discrete-time system, with sensor noise:

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t) + v(t)$$

we assume $\text{Rank}(\mathcal{O}_t) = n$ (hence, system is observable)

least-squares observer uses pseudo-inverse:

$$\hat{x}(0) = \mathcal{O}_t^\dagger \left(\begin{bmatrix} y(0) \\ \vdots \\ y(t-1) \end{bmatrix} - \mathcal{T}_t \begin{bmatrix} u(0) \\ \vdots \\ u(t-1) \end{bmatrix} \right)$$

where $\mathcal{O}_t^\dagger = (\mathcal{O}_t^T \mathcal{O}_t)^{-1} \mathcal{O}_t^T$

interpretation: $\hat{x}_{ls}(0)$ minimizes discrepancy between

- output \hat{y} that *would be* observed, with input u and initial state $x(0)$ (and no sensor noise), and
- output y that *was* observed,

measured as $\sum_{\tau=0}^{t-1} \|\hat{y}(\tau) - y(\tau)\|^2$

can express least-squares initial state estimate as

$$\hat{x}_{ls}(0) = \left(\sum_{\tau=0}^{t-1} (A^T)^\tau C^T C A^\tau \right)^{-1} \sum_{\tau=0}^{t-1} (A^T)^\tau C^T \tilde{y}(\tau)$$

where \tilde{y} is observed output with portion due to input subtracted:
 $\tilde{y} = y - h * u$ where h is impulse response

Least-squares observer uncertainty ellipsoid

since $\mathcal{O}_t^\dagger \mathcal{O}_t = I$, we have

$$\tilde{x}(0) = \hat{x}_{\text{ls}}(0) - x(0) = \mathcal{O}_t^\dagger \begin{bmatrix} v(0) \\ \vdots \\ v(t-1) \end{bmatrix}$$

where $\tilde{x}(0)$ is the estimation error of the initial state

in particular, $\hat{x}_{\text{ls}}(0) = x(0)$ if sensor noise is zero
(*i.e.*, observer recovers exact state in noiseless case)

now assume sensor noise is unknown, but has RMS value $\leq \alpha$,

$$\frac{1}{t} \sum_{\tau=0}^{t-1} \|v(\tau)\|^2 \leq \alpha^2$$

set of possible estimation errors is ellipsoid

$$\tilde{x}(0) \in \mathcal{E}_{\text{unc}} = \left\{ \mathcal{O}_t^\dagger \begin{bmatrix} v(0) \\ \vdots \\ v(t-1) \end{bmatrix} \mid \frac{1}{t} \sum_{\tau=0}^{t-1} \|v(\tau)\|^2 \leq \alpha^2 \right\}$$

\mathcal{E}_{unc} is ‘uncertainty ellipsoid’ for $x(0)$ (least-square gives best \mathcal{E}_{unc})

shape of uncertainty ellipsoid determined by matrix

$$(\mathcal{O}_t^T \mathcal{O}_t)^{-1} = \left(\sum_{\tau=0}^{t-1} (A^T)^\tau C^T C A^\tau \right)^{-1}$$

maximum norm of error is

$$\|\hat{x}_{\text{ls}}(0) - x(0)\| \leq \alpha \sqrt{t} \|\mathcal{O}_t^\dagger\|$$

Infinite horizon uncertainty ellipsoid

the matrix

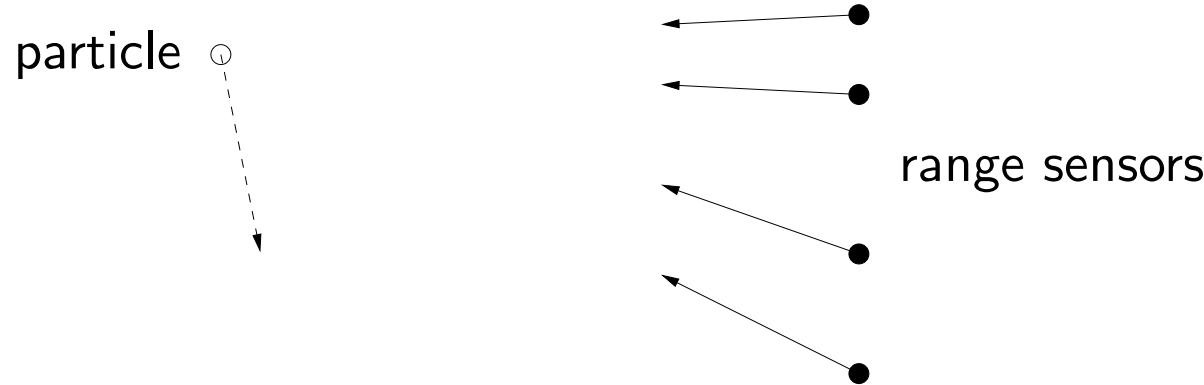
$$P = \lim_{t \rightarrow \infty} \left(\sum_{\tau=0}^{t-1} (A^T)^\tau C^T C A^\tau \right)^{-1}$$

always exists, and gives the limiting uncertainty in estimating $x(0)$ from u , y over longer and longer periods:

- if A is stable, $P > 0$
i.e., can't estimate initial state perfectly even with infinite number of measurements $u(t)$, $y(t)$, $t = 0, \dots$ (since memory of $x(0)$ fades . . .)
- if A is not stable, then P can have nonzero nullspace
i.e., initial state estimation error gets arbitrarily small (at least in some directions) as more and more of signals u and y are observed

Example

- particle in \mathbb{R}^2 moves with uniform velocity
- (linear, noisy) range measurements from directions $-15^\circ, 0^\circ, 20^\circ, 30^\circ$, once per second
- range noises IID $\mathcal{N}(0, 1)$; can assume RMS value of v is not much more than 2
- no assumptions about initial position & velocity



problem: estimate initial position & velocity from range measurements

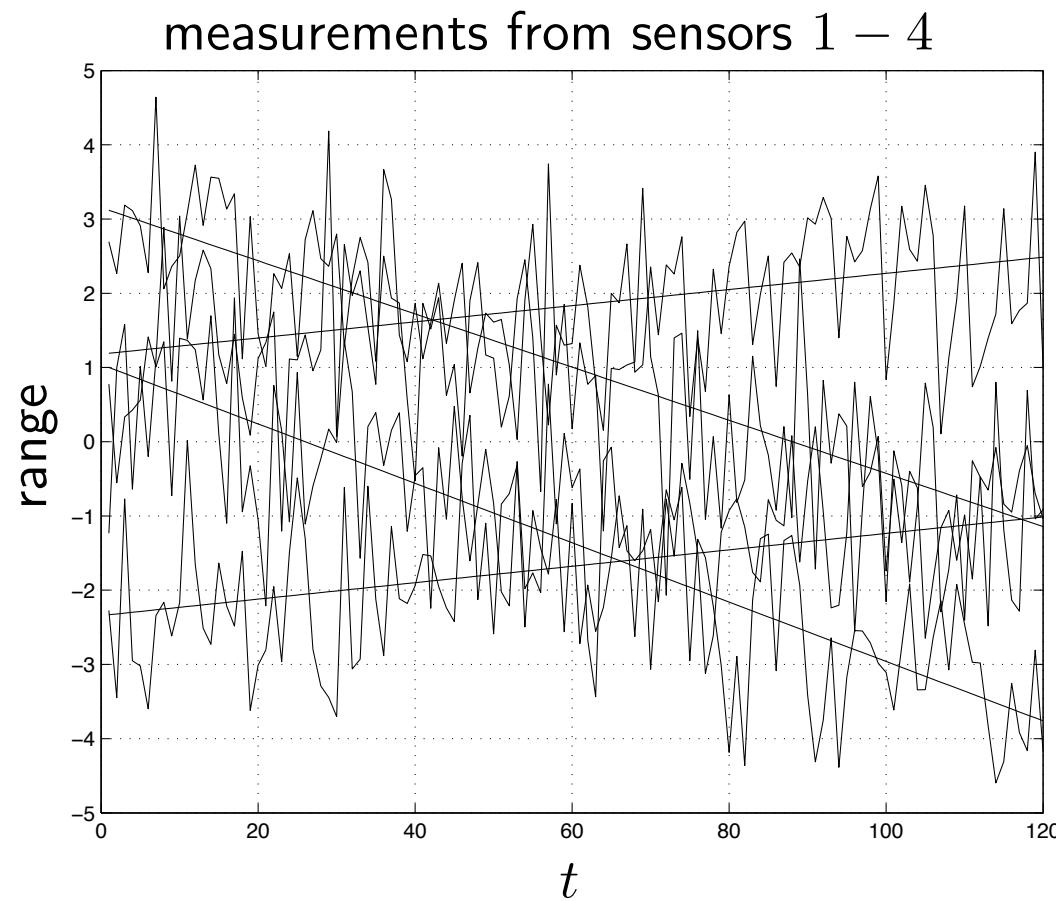
express as linear system

$$x(t+1) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(t), \quad y(t) = \begin{bmatrix} k_1^T \\ \vdots \\ k_4^T \end{bmatrix} x(t) + v(t)$$

- $(x_1(t), x_2(t))$ is position of particle
- $(x_3(t), x_4(t))$ is velocity of particle
- can assume RMS value of v is around 2
- k_i is unit vector from sensor i to origin

true initial position & velocities: $x(0) = (1 \ -3 \ -0.04 \ 0.03)$

range measurements (& noiseless versions):

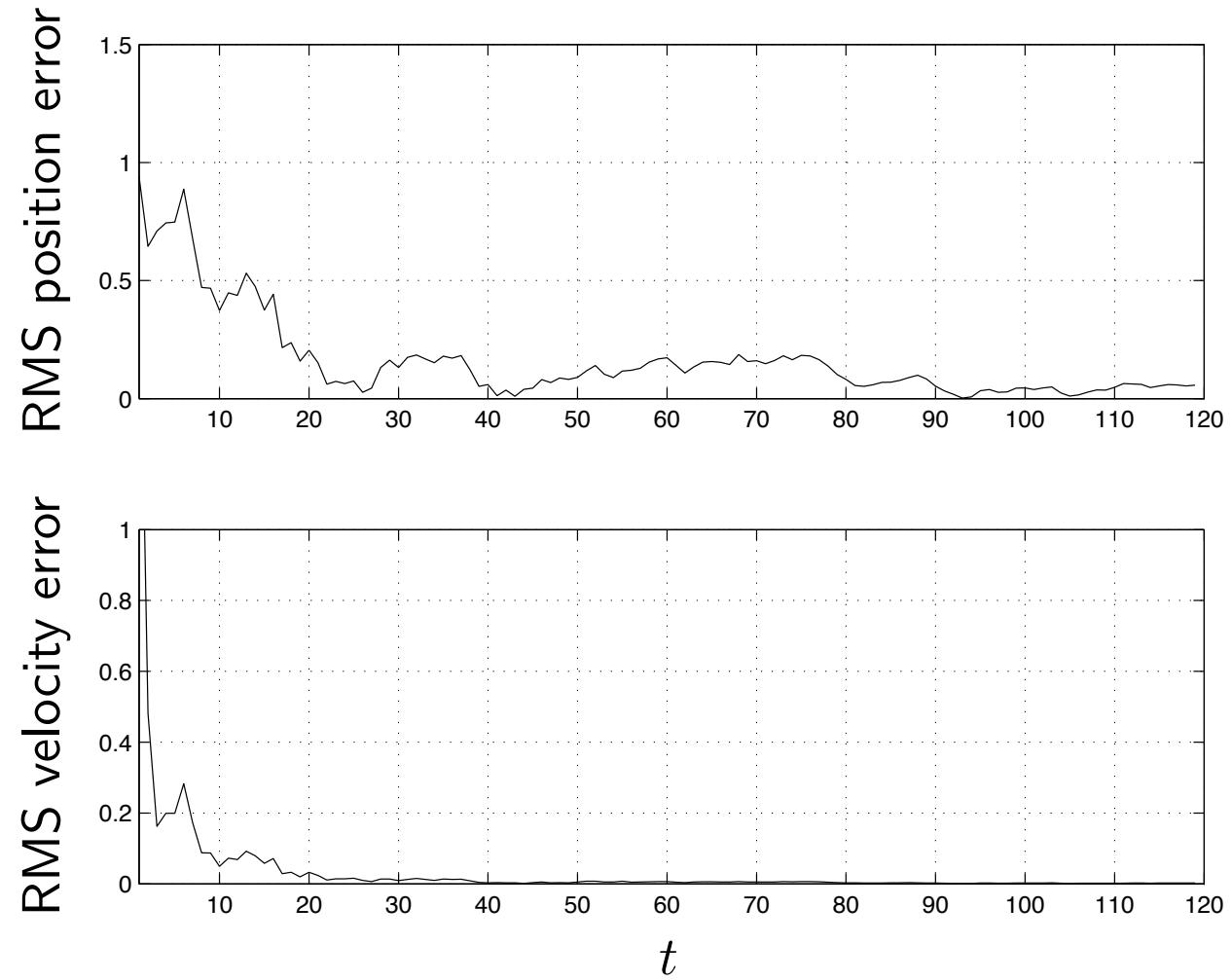


- estimate based on $(y(0), \dots, y(t))$ is $\hat{x}(0|t)$

- actual RMS position error is

$$\sqrt{(\hat{x}_1(0|t) - x_1(0))^2 + (\hat{x}_2(0|t) - x_2(0))^2}$$

(similarly for actual RMS velocity error)



Continuous-time least-squares state estimation

assume $\dot{x} = Ax + Bu$, $y = Cx + Du + v$ is observable

least-squares estimate of initial state $x(0)$, given $u(\tau)$, $y(\tau)$, $0 \leq \tau \leq t$:
choose $\hat{x}_{ls}(0)$ to minimize integral square residual

$$J = \int_0^t \|\tilde{y}(\tau) - Ce^{\tau A}x(0)\|^2 d\tau$$

where $\tilde{y} = y - h * u$ is observed output minus part due to input

let's expand as $J = x(0)^T Q x(0) + 2r^T x(0) + s$,

$$Q = \int_0^t e^{\tau A^T} C^T C e^{\tau A} d\tau, \quad r = \int_0^t e^{\tau A^T} C^T \tilde{y}(\tau) d\tau,$$

$$s = \int_0^t \tilde{y}(\tau)^T \tilde{y}(\tau) d\tau$$

setting $\nabla_{x(0)} J$ to zero, we obtain the least-squares observer

$$\hat{x}_{\text{ls}}(0) = Q^{-1}r = \left(\int_0^t e^{\tau A^T} C^T C e^{\tau A} d\tau \right)^{-1} \int_0^t e^{A^T \tau} C^T \tilde{y}(\tau) d\tau$$

estimation error is

$$\tilde{x}(0) = \hat{x}_{\text{ls}}(0) - x(0) = \left(\int_0^t e^{\tau A^T} C^T C e^{\tau A} d\tau \right)^{-1} \int_0^t e^{\tau A^T} C^T v(\tau) d\tau$$

therefore if $v = 0$ then $\hat{x}_{\text{ls}}(0) = x(0)$

Lecture 20

Some parting thoughts . . .

- linear algebra
- levels of understanding
- what's next?

Linear algebra

- comes up in *many* practical contexts (EE, ME, CE, AA, OR, Econ, . . .)
- nowadays is readily *done*
cf. 10 yrs ago (when it was mostly *talked about*)
- Matlab or equiv for fooling around
- real codes (*e.g.*, LAPACK) widely available
- current level of linear algebra technology:
 - 500 – 1000 vbles: easy with general purpose codes
 - much more possible with special structure, special codes (*e.g.*, sparse, convolution, banded, . . .)

Levels of understanding

Simple, intuitive view:

- 17 vbles, 17 eqns: usually has unique solution
- 80 vbles, 60 eqns: 20 extra degrees of freedom

Platonic view:

- singular, rank, range, nullspace, Jordan form, controllability
- everything is precise & unambiguous
- gives insight & deeper understanding
- sometimes misleading in practice

Quantitative view:

- based on ideas like least-squares, SVD
- gives numerical measures for ideas like singularity, rank, etc.
- interpretation depends on (practical) context
- very useful in practice

- must have understanding at one level before moving to next
- **never forget** which level you are operating in

What's next?

- EE364a — convex optimization I (Win 12-13)
- EE364b — convex optimization II

(plus lots of other EE, CS, CME, MS&E, Stat, ME, AA courses on signal processing, control, graphics & vision, machine learning, computational geometry, numerical linear algebra, . . .)