Gravitational Lensing
Theories, Questions and Applications

Renkun Kuang

Tsinghua University

November 9, 2019

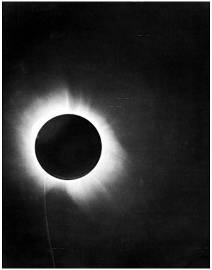
Overview

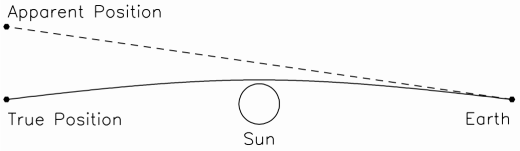
Predictions from Newtonian and GR

- First proposed by Soldner (1801) using Newtonian theory, given a deflection angle (Appendix-1 (Page 38) for derivation).
  $\alpha = \frac{2GM}{c^2 r}$, 0.85 arcsec for the Sun
- Einstein (1911) derived the same result using Equivalence principle and Euclidean metric
- Einstein (1915) derived the new result using General Relativity.
  $\alpha = \frac{4GM}{c^2 r}$, 1.7 arcsec for the Sun

Eddington's observation of the Solar Eclipse

- In 1919, Eddington measured a value close to GR's prediction using the data collected during an eclipse, stars with apparent position near the sun become visible.

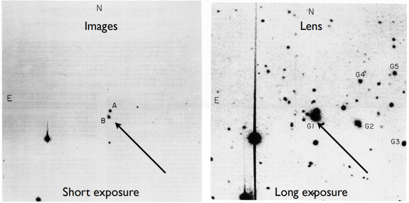




Dyson, F.W., Eddington, A.S. and Davidson, C. (1920) https://royalsocietypublishing.org/doi/pdf/10.1098/rsta.1920.0009

The first example for GravLens: 0957+561

- Eddington (1920): Multiple light paths → multi images
- Walsh et al., (1979) quasar QSO 957+561 A,B found at $z \sim 1.4$, two seen images separated by $6''$
- Lens: evidence
  1. Lensing galaxy at $z \sim 0.36$
  2. Similar spectra
  3. Ratio of optical and radio flux
  4. VLBI imaging: small scale features



Walsh, Carswell & Weymann 1979, 0957+561 A, B: twin quasistellar objects or gravitational lens? https://ui.adsabs.harvard.edu/abs/1979Natur.279..381W/abstract

General Relaticity and light deflection

- Einstein Field Equations:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

- Geodesic equations:

$$\frac{d^2x^\beta}{d\lambda^2} + \Gamma^\beta_{\mu\nu}\frac{dx^\nu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$$

  We can calculate how gravity bends light by solving geodesic eqution.

- To compute the Christoffel symbols $\Gamma^\beta_{\mu\nu}$,
  requires solving for the metric tensor $g_{\mu\nu}$,
  which requires solving the curvature equations $R_{\mu\nu} = 0$,
  $\leftarrow$ ten nonlinear partial differential equations.

General Relaticity and light deflection

Or, the velocity of the photon from the Schwarzshild metric,
$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)(dx^2+dy^2+dz^2)$,

- and Poisson Equation, $\nabla^2\Phi = 4\pi G\rho$,
- and light interval: $g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$
- which gives,

$$v = \frac{\sqrt{dx^2+dy^2+dz^2}}{dt} = \sqrt{\frac{1+2\Phi}{1-2\Phi}} \approx 1+2\Phi$$

The gravitational field decreases the speed of propagation

---

https://web.stanford.edu/ oas/SI/SRGR/notes/SchwarzschildSolution.pdf, The Schwarzschild Solution

General Relaticity and light deflection

$v = \frac{\sqrt{dx^2+dy^2+dz^2}}{dt} = \sqrt{\frac{1+2\Phi}{1-2\Phi}} \approx 1 + 2\Phi$ (natural units) $\rightarrow$
$v = c\left(1 + \frac{2}{c^2}\Phi\right)$ (SI)

- define refraction index: $n = 1 - \frac{2}{c^2}\Phi = 1 + \frac{2}{c^2}|\Phi| \geq 1$
- deflection angle: $\vec{\hat{\alpha}} = -\int \vec{\nabla}_\perp n dl = \frac{2}{c^2}\int \vec{\nabla}_\perp \Phi dl$
- which is twice of the Newtonian prediction,
- for point mass lens, $\hat{\alpha} = \frac{4GM}{bc^2}$

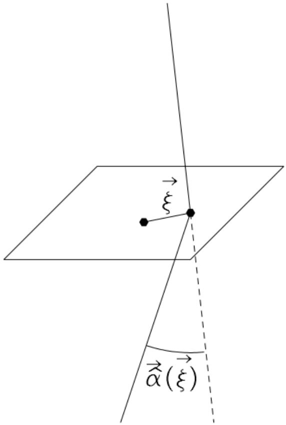
Thin screen approximation

- Most deflection occurs near to the lens ($|z| \sim b$)
  $\rightarrow$ treat all deflection as in the lens plane

- Projected surface density:
  $\Sigma(\vec{\xi}) = \int \rho(\vec{\xi}, z) dz$

- Deflection angle:
  $\vec{\hat{\alpha}}(\vec{\xi}) = \frac{4G}{c^2} \int \frac{(\vec{\xi} - \vec{\xi'})\Sigma(\vec{\xi'})}{|\vec{\xi} - \vec{\xi'}|^2} d^2\vec{\xi'}$

- In circular symmetry cases:
  $\hat{\alpha}(\xi) = \frac{4GM(\xi)}{c^2 \xi}$
  $M(\xi) = 2\pi \int_0^{\xi} \Sigma(\xi')\xi' d\xi'$

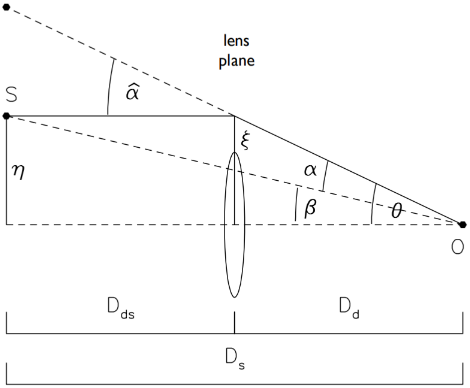


The Lens Equation

Connecting the position of images in the Lens plane and corresponding sources the Source plane.

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})$$

$$\vec{\alpha}(\vec{\theta}) = \frac{D_{ds}}{D_s}\hat{\alpha}(D_d\vec{\theta})$$

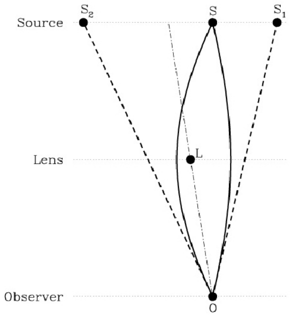


Case 1: Point mass Lens

$$\begin{cases} \hat{\alpha}(\xi) = \frac{4GM(\xi)}{c^2\xi} \\ \vec{\alpha}(\vec{\theta}) = \frac{D_{ds}}{D_s}\hat{\alpha}(D_d\vec{\theta}) \end{cases} \rightarrow \vec{\alpha}(\vec{\theta}) = \frac{4GM}{c^2}\frac{D_{ds}}{D_d D_s}\frac{\vec{\theta}}{|\vec{\theta}|^2}$$

and $\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})$,
if $\beta = 0$, gives the Einstein Radius,

$$\theta_E = \sqrt{\frac{4GM}{c^2}\frac{D_{ds}}{D_d D_s}}$$



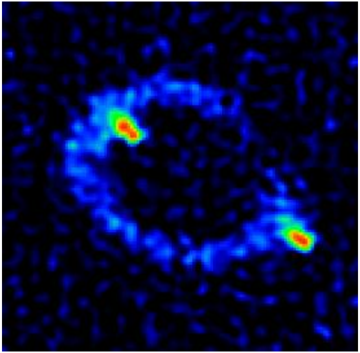
The Lens equation: $\vec{\beta} = \vec{\theta} - \theta_E^2 \frac{\vec{\theta}}{|\vec{\theta}|^2}$

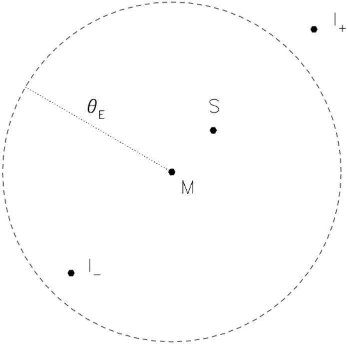
- if $\beta > \theta_E \rightarrow$, weakly lensed and weakly distorted image,
- if $\beta < \theta_E \rightarrow$, stronly lensed and multi images:

$$\theta_\pm = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right)$$

Case 1: Point mass Lens

Hewitt+ 1987, First Einstein Ring discovered in Radio Band & VLA

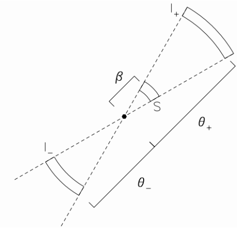


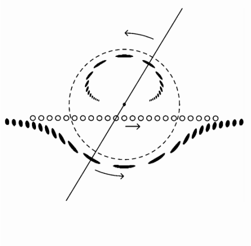


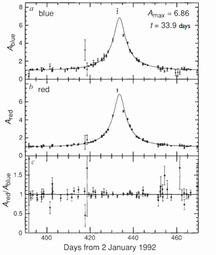
Hewitt, J., Turner, E., Schneider, D. et al. Unusual radio source MG1131+0456: a possible Einstein ring. Nature 333, 537–540 (1988) doi:10.1038/333537a0

Magnification

Gravitational lensing preserves surface brightness (Liouville's Theorem), but changes the apparent solid angle of the source $\rightarrow magnification = \frac{Area_{image}}{Area_{source}}$







Possible Gravitational Microlensing of a Star in the Large Magellanic Cloud https://arxiv.org/pdf/astro-ph/9309052v1.pdf

Magnification

Local properties of the lens mapping, described by its Jacobian matrix A

$$A \equiv \frac{\partial \vec{\beta}}{\partial \vec{\theta}} = \left(\delta_{ij} - \frac{\partial \alpha_i(\vec{\theta})}{\partial \theta_j} \right) = \left(\delta_{ij} - \frac{\partial^2 \Psi(\vec{\theta})}{\partial \theta_i \partial \theta_j} \right)$$

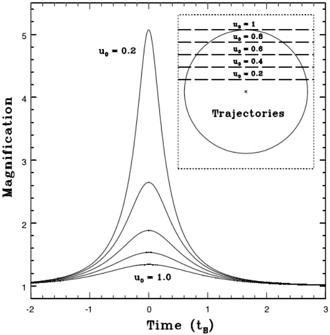
$$\mu = \left| det \left(\frac{\partial \vec{\beta}}{\partial \vec{\theta}} \right) \right|^{-1} \equiv \left| det \left(\frac{\partial \beta_i}{\partial \theta_j} \right) \right|^{-1}$$

If circularly symmetric,

$$\mu = \frac{\theta}{\beta} \frac{d\theta}{d\beta}$$

Case 1: Point mass Lens - Magnification

- Images:
  $\theta_{\pm} = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right)$
- Define $u = \beta\theta_E^{-1}$,
  Magnification:
  $$\mu_{\pm} = \left[1 - \left(\frac{\theta_E}{\theta_{\pm}}\right)^4\right]^{-1}$$
  $$= \frac{u^2+2}{2u\sqrt{u^2+4}} \pm \frac{1}{2}$$
- Total magnification:
  $\mu = |\mu_+| + |\mu_-| = \frac{u^2+2}{u\sqrt{u^2+4}}$



e.g. for source on the Einstein Ring:
$\beta = \theta_E, u = 1 \rightarrow \mu = 1.34 \rightarrow$ magnitude increase 0.32.

Shapiro time delay

- Passage through potential also leads to time delay
- without potential: $t_0 = \int \frac{dl}{c}$
- with potential:
$$t_1 = \int_{src}^{obs} \frac{dl}{v} = \int_{src}^{obs} \frac{dl}{c - \frac{2}{c}|\Phi|}$$
$$= \int_{src}^{obs} \frac{dl/c}{1 - \frac{2}{c^2}|\Phi|} = \int_{src}^{obs} \frac{dl}{c}[1 + \frac{2}{c^2}|\Phi|]$$
- so, $\Delta t = \int_{src}^{obs} \frac{2}{c^3}|\Phi| dl \rightarrow$ Shapiro delay (1964)
- Total time delay is the sum of the extra path length from the deflection and the gravitational time delay

$$t(\vec{\theta}) = \frac{1 + z_d}{c} \frac{D_d D_s}{D_{ds}} [\frac{1}{2}(\vec{\theta} - \vec{\beta})^2 - \psi \vec{\theta}] = t_{geom} + t_{grav}$$

Case 2: Singular Isothemal Sphere (SIS)

- Galaxy lenses, the distributed nature of mass
- Simple model assumes that mass $\rightarrow$ particles of ideal gas
- ideal gas:
  - equation of state: $p = \frac{\rho k T}{m}$
  - In thermal equilibrium, T is related to the 1-d velocity dispersion: $m\sigma_v^2 = kT$
  - In hydrostatic equilibrium, $\frac{p^{'}}{\rho} = -\frac{GM(r)}{r^2}, M^{'}(r) = 4\pi r^2 \rho$
  - solve the EOS $\rightarrow$ density profile: $\rho(r) = \frac{\sigma_v^2}{2\pi G}\frac{1}{r^2}$
  - so mass profile: $M(r) = \frac{2\sigma_v^2}{G}$

Case 2: Singular Isothemal Sphere (SIS)

For ideal gas, rotational velocity in circular orbit:

$$\frac{p^{'}}{\rho} = -\frac{GM(r)}{r^2}, M^{'}(r) = 4\pi r^2 \rho$$

Surface mass density:

$$v_{rot}^2 = GM/r \rightarrow$$
$$dM = \frac{v_{rot}^2}{G} dr = 4\pi r^2 \rho(r) \rightarrow$$
$$\rho(r) = \frac{v_{rot}^2}{4\pi G} \frac{dr}{r^2} \rightarrow$$
$$\Sigma(\xi) = \frac{v_{rot}^2}{4\pi G} \int_{-\infty}^{\infty} \frac{dz}{(z+\xi)^2} = \frac{v_{rot}^2}{4G\xi} = \frac{\sigma_v^2}{2G\xi}$$

where, $M(\xi) = 2\pi \int_0^{\xi} \Sigma(\xi^{'})\xi^{'} d\xi^{'}$, and $\hat{\alpha}(\xi) = \frac{4GM(\xi)}{c^2\xi}$
which gives: $\hat{\alpha} = 4\pi \frac{\sigma_v^2}{c^2}$

Case 2: Singular Isothemal Sphere (SIS)

using $\begin{cases} \hat{\alpha} = 4\pi\sigma_v^2/c^2 \\ \vec{\alpha}(\vec{\theta}) = \hat{\alpha}(D_d\vec{\theta})D_{ds}/D_s \end{cases}$

and lens equation $\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})$

we can get:

$$\alpha = \hat{\alpha}\frac{D_{ds}}{D_s} = 4\pi\frac{\sigma_v^2}{c^2}\frac{D_{ds}}{D_s} = \theta_E$$

for strong lensing, get two images as for point mass:

$$\vec{\beta} = \vec{\theta} - \theta_E\frac{\vec{\theta}}{|\vec{\theta}|}, \frac{\vec{\theta}}{|\vec{\theta}|} = \pm 1 \rightarrow$$
$$\theta_\pm = \beta \pm \theta_E$$

Magnification can be very large for source aligned with lens, from $\mu = \frac{\theta}{\beta}\frac{d\theta}{d\beta} \rightarrow$:

$$\mu_\pm = \frac{\theta_\pm}{\beta} = 1 \pm \frac{\theta_E}{\beta} = \left(1 \mp \frac{\theta_E}{\theta_\pm}\right)^{-1}$$

Separation of the two images is typically $\sim$ arcsec for galaxy lenses:

$$\theta_E = 1''.6\left(\frac{\sigma_v}{200kms^{-1}}\right)^2\left(\frac{D_{ds}}{D_s}\right)$$

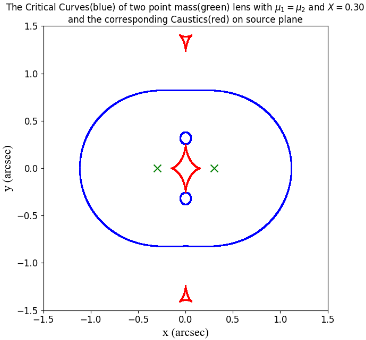
Notice: in general the core of a galaxy would not be singular

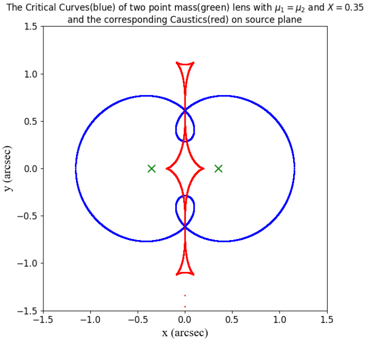
Caustics and Critical lines

- Grav Lens changes the observed brightness of the source, determined by the Jacobian matrix from the lens equation: $A \equiv \frac{\partial \vec{\beta}}{\partial \vec{\theta}} = \left(\delta_{ij} - \frac{\partial \alpha_i(\vec{\theta})}{\partial \theta_j} \right) = \left(\delta_{ij} - \frac{\partial^2 \Psi(\vec{\theta})}{\partial \theta_i \partial \theta_j} \right)$
  $\mu(\theta) = \frac{1}{detA(\theta)}$
- Image at $\vec{\theta}$ is magnified by a factor of $|\mu(\vec{\theta})|$
- Notice that $|\mu(\vec{\theta})|$ diverge at $detA(\theta) = 0 \rightarrow$ these points in the image plane form closed curves, which is so called critical lines.
- Corresponding curves in the source plane obtained via the lens equation are called caustics.

Caustics and Critical lines - Point sources

Separation $0.6'', 0.7''$:

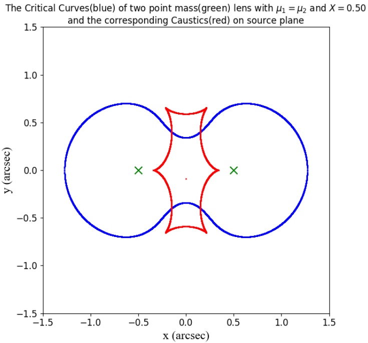


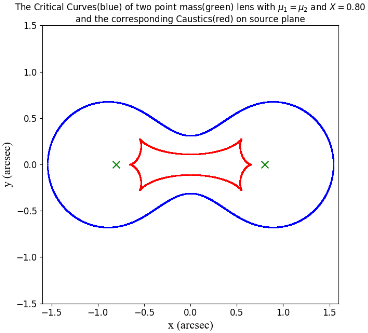


https://github.com/rkkuang/aeroastro/blob/master/gravlen/critical_and_caustics/
The two-point-mass lens - Detailed investigation of a special asymmetric gravitational lens http://adsabs.harvard.edu/abs/1986A%26A...164..237S

Caustics and Critical lines - Point sources

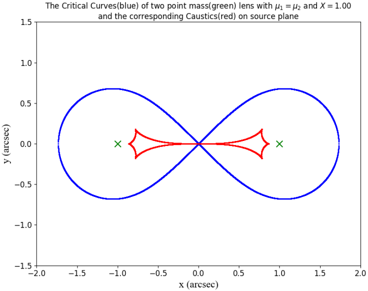
Separation $1.0'', 1.6''$:

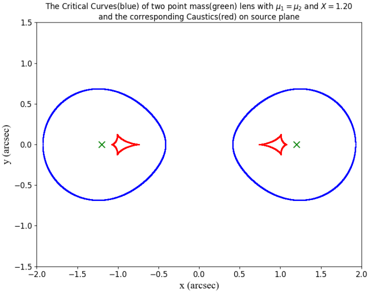




Caustics and Critical lines - Point sources

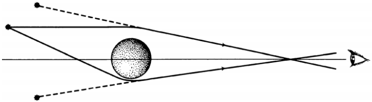
Separation $2.0'', 2.4''$:

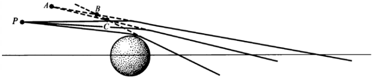




Gravitational Lens vs Genuine Lens

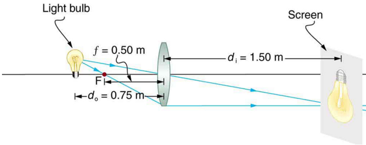
- Grav lens: the observer sees the source at two distinct locations, $\alpha \propto b^{-1}$

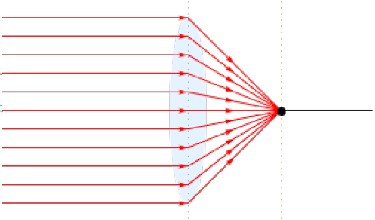




Grav lens has no well-defined focal length and cannot produce genuine images, the “images” are corresponds merely to a direction of incidence of light on the observer, not a genuine image in the sky

- Genuine lens, $\alpha \propto b$





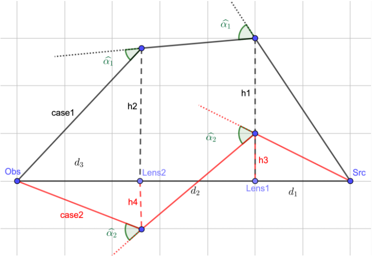
Question 1

A galaxy at redshift 0.5 can be modelled as a singular isothermal sphere; its dispersion is 200km/s. A background source at redshift 2 is lensed by the foreground galaxy into two images with a brightness ratio of 3:1, what are the angular separation and time delay between the two images? You can assume the usual cosmology.

First assume $H_0 = 69.6kms^{-1}, \Omega_m = 0.286, \Omega_{vac} = 0.714$ then the distance of z = 0.5 and z = 2 can be computed, using theories of SIS model and the time delay function we can estimate the angular separation $\sim$ 1.2 arcsec and time delay $\sim$ 45 days

Question 2

Two (N=2) galaxies are aligned perfectly with the Earth and a distant quasar. Each galaxy can be modelled as a singular isothermal sphere. How many Einstein rings are formed as a result? How will your results generalize when you have $N > 2$ galaxies?



For N galaxies, the quasar will render 2^{N-1} Einstein rings, and if we consider galaxy lensed by galaxy, will generate (at most) $\sum_{i=1}^{N-1} 2^{i-1}$ more, so the total number of Einstein rings rendered by N galaxies and a quasar will be: $\sum_{i=1}^{N} 2^{i-1} = 2^N - 1$, at most.

https://github.com/rkkuang/aeroastro/tree/master/gravlen/num_einstein_rings

Question 3

A background source is multiply-imaged, can the brightest image arrive last?

Using the relation between effective lensing potential and magnification & time delay

Effective lensing potential:

$\Psi(\vec{\theta}) = \frac{D_{ds}}{D_d D_s} \frac{2}{c^2} \int \Phi(D_d \vec{\theta}, z) dz$, $\vec{\nabla}_\theta \Psi = \vec{\alpha}$

Jacobian matrix:$A = \frac{\partial \vec{\beta}}{\partial \vec{\theta}} = \delta_{ij} - \frac{\partial^2 \Psi(\vec{\theta})}{\partial \vec{\theta}_i \partial \vec{\theta}_j} \equiv \delta_{ij} - \Psi_{ij}$

Magnification: $\mu = |detA|^{-1}$

time delay: $t(\vec{\theta}) = \frac{1+z_d}{c} \frac{D_d D_s}{D_{ds}} \left[\frac{1}{2}(\vec{\theta} - \vec{\beta})^2 - \Psi(\vec{\theta}) \right]$

$$\mu \leftarrow \Psi \rightarrow t$$

Simulation, under a potential, given a $\vec{\beta}$, solve the lens equation for $\vec{\theta_i}$ see if $\vec{\theta_k}$ which has maximum magnification also have largest time delay

Question 4

Study the caustics and critical curves of two singular isothermal spheres lensing a background quasar. Study two cases 1) Both galaxies are at the same redshift and, 2) these two galaxies are at different redshifts.

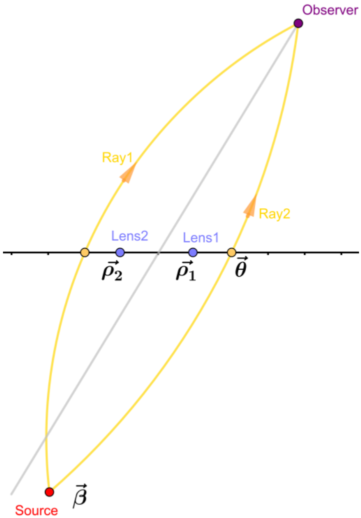
Mass density $\rightarrow$ Surfacce mass density $\rightarrow$ Effective lensing potential $\rightarrow$ Jacobian matrix $\rightarrow$ Magnification draw those points which make magnification largest $\rightarrow$ critical lens $\rightarrow$ caustics

Question 4

case 1) Both galaxies are at the same redshift

$\vec{\alpha}(\vec{\theta}) = \theta_{E1}\frac{\vec{\theta}-\vec{\rho_1}}{|\vec{\theta}-\vec{\rho_1}|} + \theta_{E2}\frac{\vec{\theta}-\vec{\rho_2}}{|\vec{\theta}-\vec{\rho_2}|}$
where $\vec{\rho_1}, \vec{\rho_2}$ are positions of Lens1, Lens2 on the lens plane. $\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})$

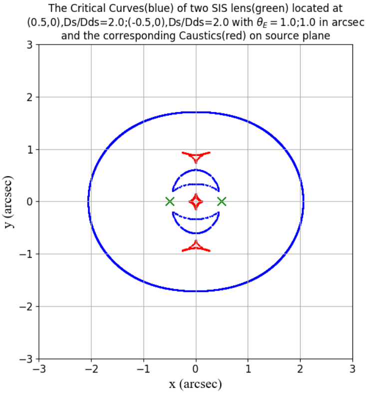
$A = \frac{\partial\vec{\beta}}{\partial\vec{\theta}} = \left(\delta_{ij} - \frac{\partial\alpha_i(\vec{\theta})}{\partial\theta_j}\right)$
$\mu(\vec{\theta}) = \frac{1}{detA(\vec{\theta})}$

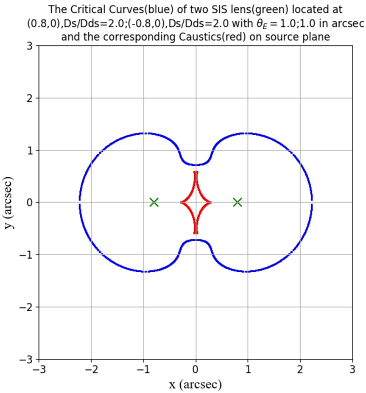


Question 4

case 1) Both galaxies are at the same redshift

Two SIS lenses ($\theta_E = 1.0''$), separation $1.0'', 1.6''$

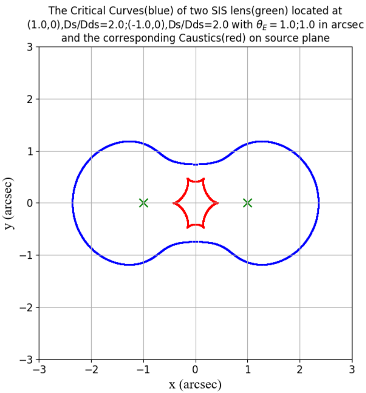


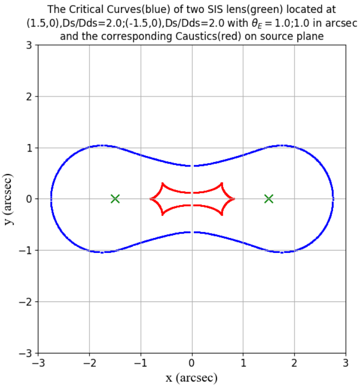


Question 4

case 1) Both galaxies are at the same redshift

Two SIS lenses ($\theta_E = 1.0''$), separation $2.0'', 3.0''$

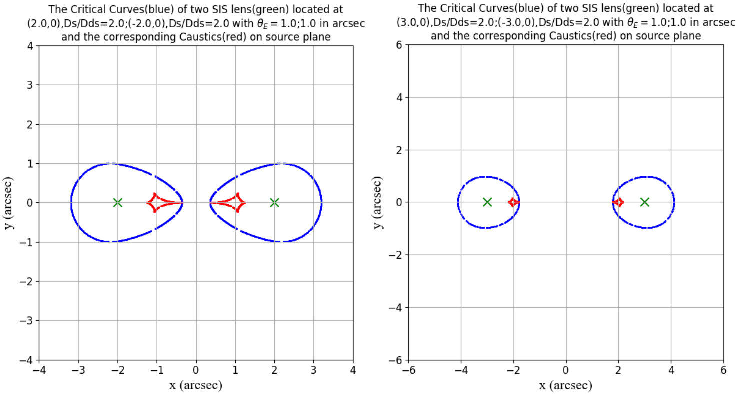




Question 4

case 1) Both galaxies are at the same redshift

Two SIS lenses ($\theta_E = 1.0''$), separation $4.0'', 6.0''$



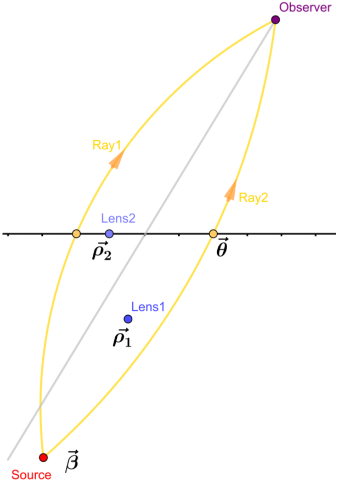
Question 4

case 2) these two galaxies are at different redshifts

Let lens1 is closer to source, for lens_i, we have,
$\vec{\beta_i} = \vec{\theta_i} - \theta_{Ei} \frac{\vec{\theta_i} - \vec{\rho_i}}{|\vec{\theta_i} - \vec{\rho_i}|}$, and
$\vec{\beta_i} = \vec{\theta_{i-1}}$

in two lenses system, $\vec{\beta_2} = \vec{\theta_1} \rightarrow$
$$\vec{\beta_1} = \vec{\theta_2} - \theta_{E2} \frac{\vec{\theta_2} - \vec{\rho_2}}{|\vec{\theta_2} - \vec{\rho_2}|} - \theta_{E1} \frac{\vec{\theta_2} - \theta_{E2} \frac{\vec{\theta_2} - \vec{\rho_2}}{|\vec{\theta_2} - \vec{\rho_2}|} - \vec{\rho_1}}{\left|\vec{\theta_2} - \theta_{E2} \frac{\vec{\theta_2} - \vec{\rho_2}}{|\vec{\theta_2} - \vec{\rho_2}|} - \vec{\rho_1}\right|}$$

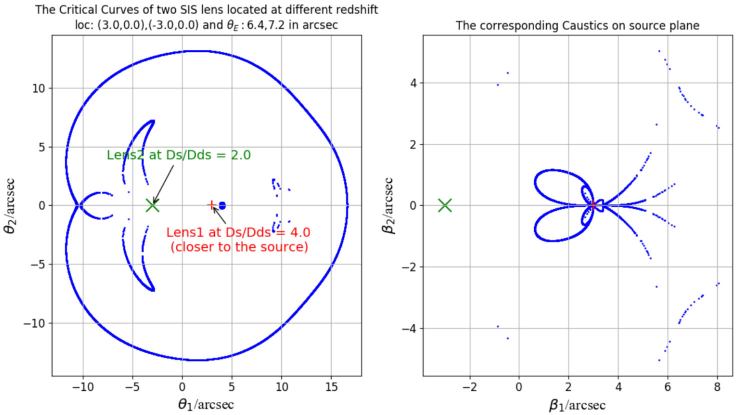
so, the Jacobian matrix can be derived using:
$$A = \frac{\partial \vec{\beta_1}}{\partial \vec{\theta_2}}$$



Question 4

case 2) these two galaxies are at different redshifts

Wrong, to be corrected:



Gravitational Lensing Applications

- Cosmic telescopes: distant, faint objects observation
- 2-d mass distribution of lenses, dark matter
- Hubble constant, cosmological constant, density parameter
- ......

The End, Thanks!

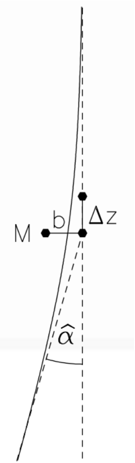
References

- https://lacosmo.com/DeflectionOfLight/index.html
- *The Mathematical Theory of Relativity*, Arthur Stanley Eddington (P101)
- http://web.mit.edu/6.055/old/S2009/notes/bending-of-light.pdf
- https://www.mathpages.com/rr/s6-03/6-03.htm *Gravitation and Spacetime*, Hans C. Ohanian, Remo Ruffini. – 3rd ed
- https://en.wikipedia.org/wiki/Gauss%27s_law_for_gravity
- *Lectures On Gravitational Lensing*, Narayan & Bartelmann
- https://www.cfa.harvard.edu/ dfabricant/huchra/ay202/lectures/lecture12.pdf
- https://web.stanford.edu/ oas/SI/SRGR/notes/SchwarzschildSolution.pdf
- ……

Appendix-1, Newtonian prediction

$$\phi(b,z) = -\frac{GM}{(b^2+z^2)^{1/2}}$$

$$\begin{aligned}
\alpha = \frac{v_b}{c} &= \frac{1}{c}\int \frac{d\Phi}{db}dt = \frac{1}{c^2}\int \frac{d\Phi}{db}dl \\
&\approx \frac{1}{c^2}\int \frac{d\Phi}{db}dz \\
&= \frac{GMb}{c^2}\int \frac{dz}{(b^2+z^2)^{3/2}} \\
&= \frac{GMb}{c^2}\left[\frac{z}{b^2\sqrt{b^2+z^2}}\Bigg|_{-\infty}^{+\infty}\right] \\
&= \frac{2GM}{c^2 b}
\end{aligned}$$



b: impact factor

Gravitational potential simplify the calculation by integrating not along the deflected ray but along z axis