Gravitational Lensing
Theories, Questions and Applications

Renkun Kuang

Outline

- History
- Gravitational Lensing Theories
- Related Questions
- Applications

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Predictions from Newtonian and GR

- First proposed by Soldner (1801) using Newtonian theory, given a deflection angle (Appendix-1 for derivation)

$$\alpha = \frac{2GM}{c^2 r}$$, 0.85 arcsec for the Sun

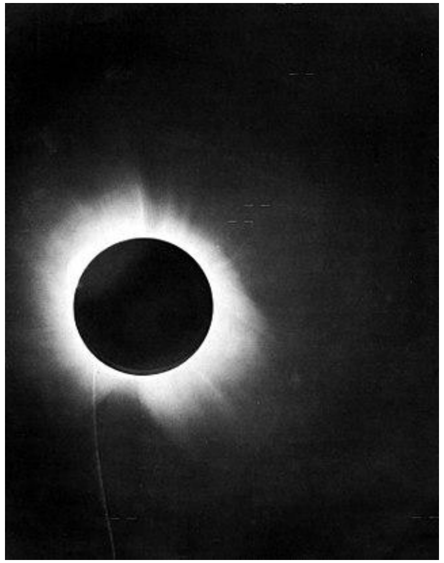
- Einstein (1911) derived the same result using Equivalence principle and Euclidean metric

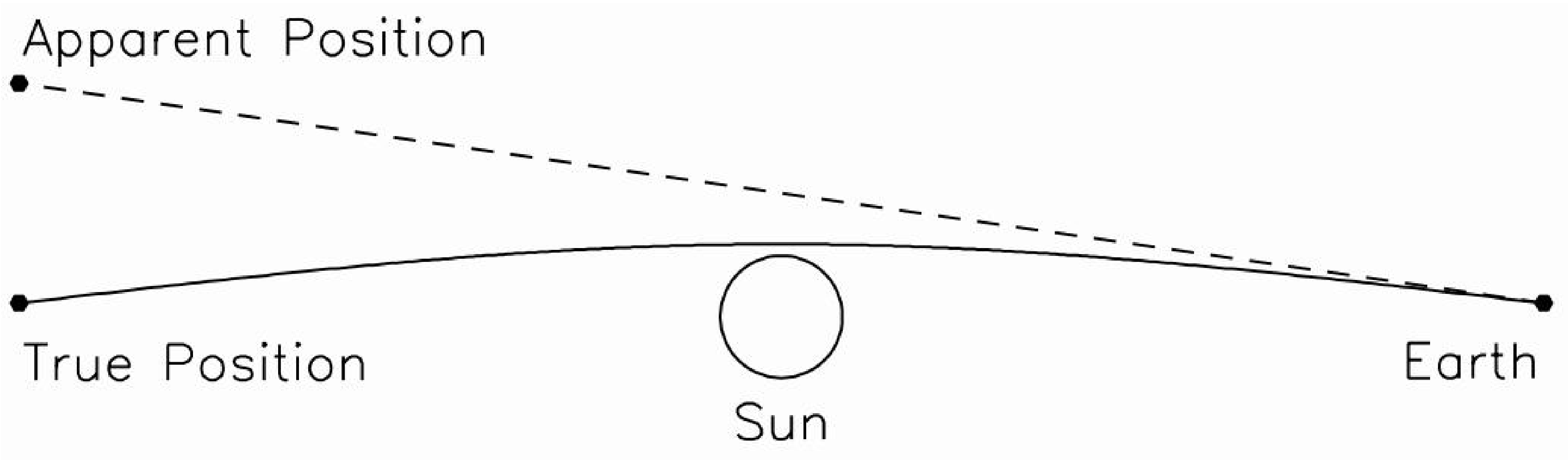
- Einstein (1915) derived the new result using General Relativity

$$\alpha = \frac{4GM}{c^2 r}$$, 1.7 arcsec for the Sun

Eddington's obs of the Solar Eclipse

- In 1919, Eddington measured a value close to GR's prediction using the data collected during an eclipse, stars with apparent position near the sun become visible.





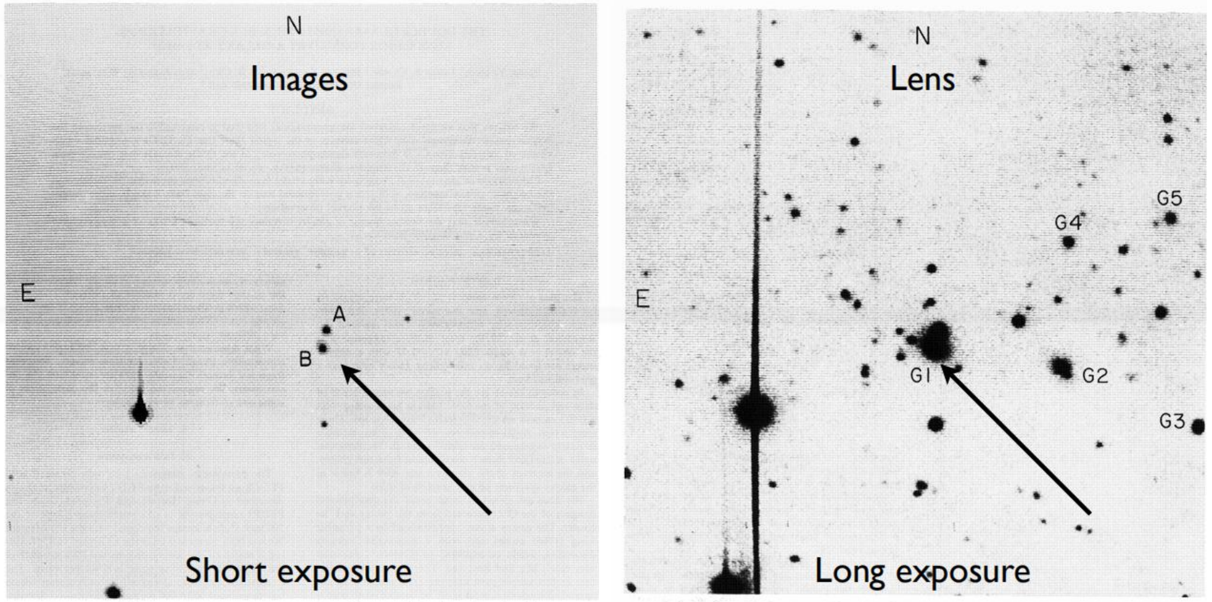
IX. A determination of the deflection of light by the sun's gravitational field, from observations made at the total eclipse of May 29, 1919
https://royalsocietypublishing.org/doi/pdf/10.1098/rsta.1920.0009

The first example for GravLens: 0957+561

- Eddington (1920): Multiple light paths ==> multi images
- Walsh et al., (1979) quasar QSO 957+561 A,B found at z~1.4, two seen images separated by 6 arcsec

- Lens: evidence

1. Lensing galaxy at z~0.36
2. Similar spectra
3. Ratio of optical and radio flux
4. VLBI imaging: small scale features



Walsh, Carswell & Weymann 1979, 0957+561 A, B: twin quasistellar objects or gravitational lens?
https://ui.adsabs.harvard.edu/abs/1979Natur.279..381W/abstract

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General Relaticity and light deflection

- Einstein Field equations: $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$

- ==> Geodesic equations: $\frac{d^2 x^\beta}{d\lambda^2} + \Gamma^\beta_{\mu\nu} \frac{dx^\nu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0$

Calculate how gravity bends light by solving geodesic eqution

- To compute the Christoffel symbols $\Gamma^\beta_{\mu\nu}$,

  requires solving for the metric tensor $g_{\mu\nu}$,

  which requires solving the curvature equations $R_{\mu\nu} = 0$,

  --- ten partial-differential equations.

General Relaticity and light deflection

- Or, the velocity of the photon from the Schwarzshild metric,

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)(dx^2 + dy^2 + dz^2)$$

- and Poisson Equation, $\nabla^2\Phi = 4\pi G\rho$

- and lightlike interval: $g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$

- ==> $v = \frac{\sqrt{dx^2 + dy^2 + dz^2}}{dt} = \sqrt{\frac{1+2\Phi}{1-2\Phi}} \approx 1 + 2\Phi$

- The gravitational field decreases the speed of propagation

https://web.stanford.edu/~oas/SI/SRGR/notes/SchwarzschildSolution.pdf

General Relaticity and light deflection

$$v = \frac{\sqrt{dx^2 + dy^2 + dz^2}}{dt} = \sqrt{\frac{1+2\Phi}{1-2\Phi}} \approx 1 + 2\Phi$$ (natural units) ==> $$v = c\left(1 + \frac{2}{c^2}\Phi\right)$$ (SI)

- define refraction index $n = 1 - \frac{2}{c^2}\Phi = 1 + \frac{2}{c^2}|\Phi| \geq 1$

- deflection angle: $\vec{\hat{\alpha}} = -\int \vec{\nabla}_\perp n dl = \frac{2}{c^2}\int \vec{\nabla}_\perp \Phi dl$

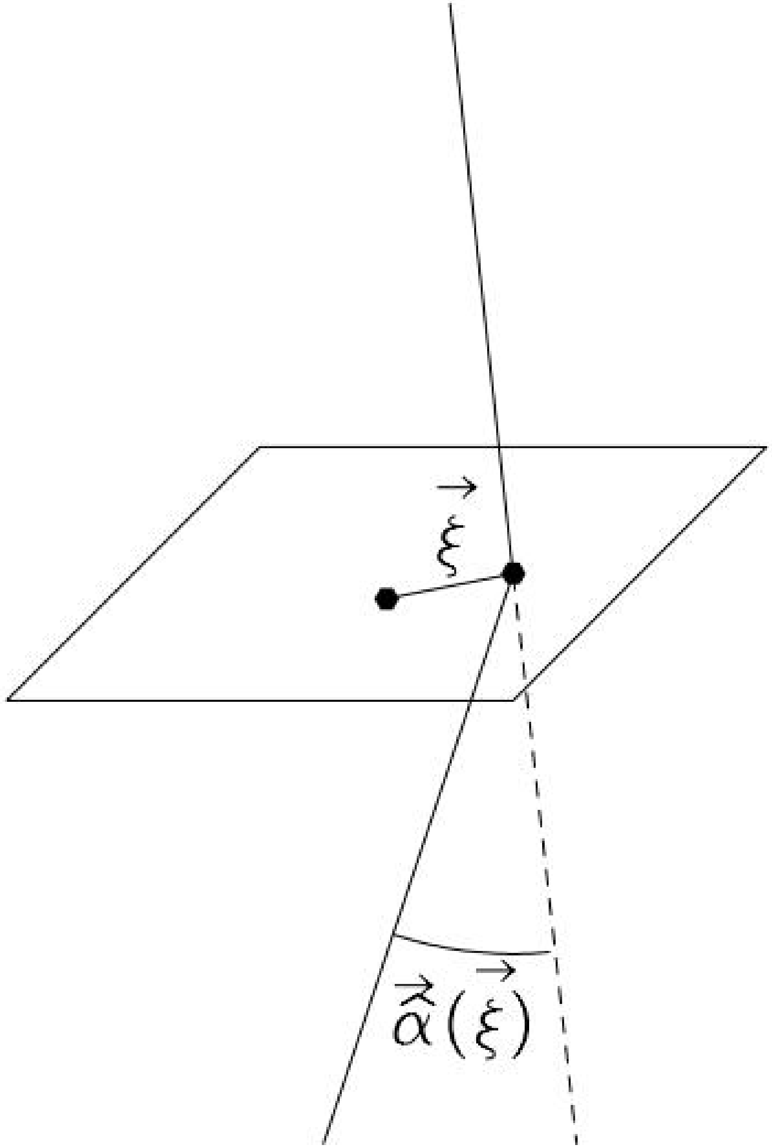
- which is twice of Newtonian prediction (Appendix-1)

- for point mass lens, $\hat{\alpha} = \frac{4GM}{bc^2}$

Thin screen approximation

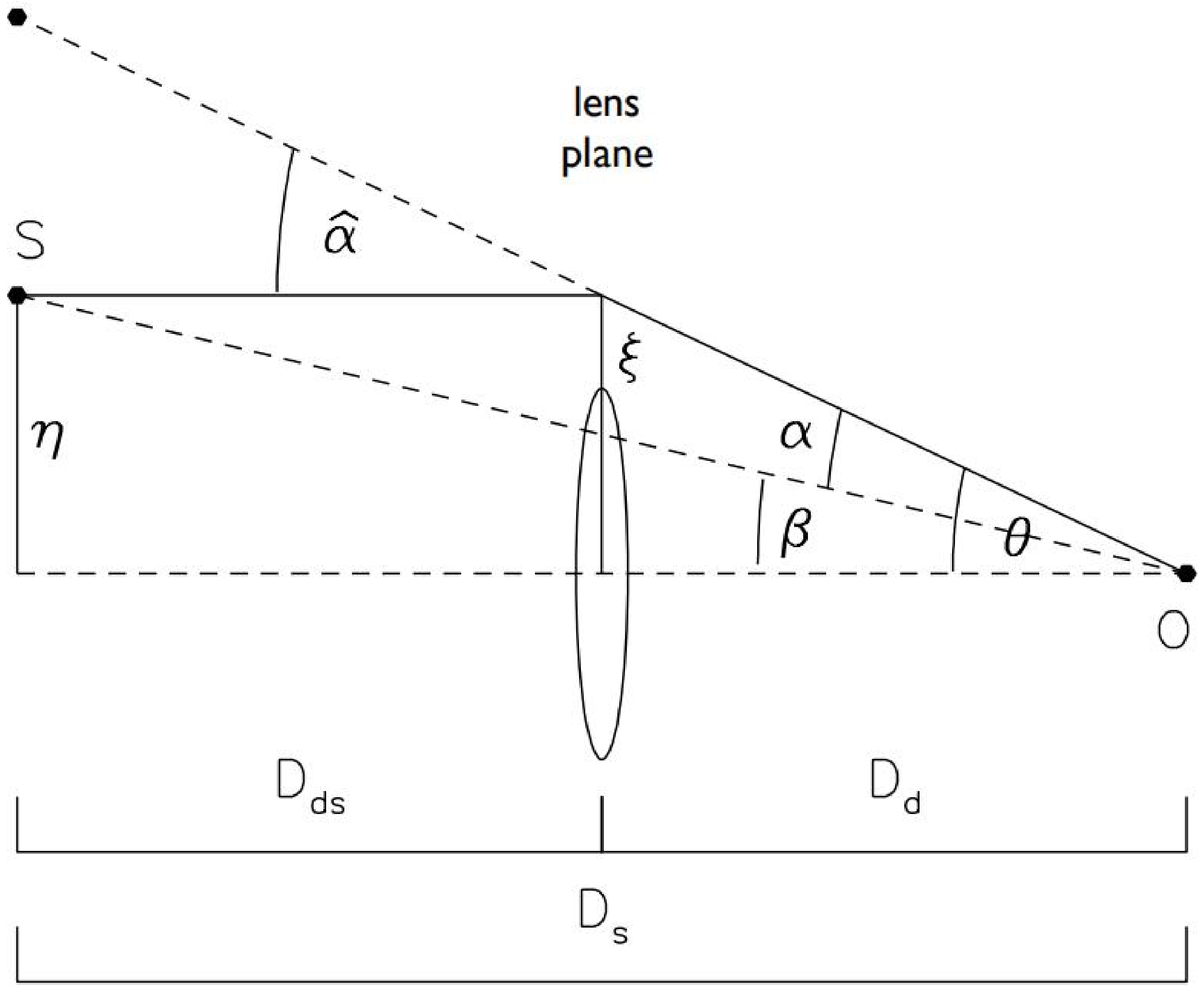
- Most deflection occurs near to the lens (|z|~b)
- ==> treat all deflection as in the lens plane

- Projected surface density: $\Sigma(\vec{\xi}) = \int \rho(\vec{\xi}, z)dz$
- Deflection angle: $$\vec{\hat{\alpha}}(\vec{\xi}) = \frac{4G}{c^2} \int \frac{(\vec{\xi} - \vec{\xi'})\Sigma(\vec{\xi'})}{|\vec{\xi} - \vec{\xi'}|^2} d^2\vec{\xi'}$$
- In citcular symmetry cases: $$\hat{\alpha}(\xi) = \frac{4GM(\xi)}{c^2\xi}$$ $$M(\xi) = 2\pi \int_0^{\xi} \Sigma(\xi')\xi' d\xi'$$



The Lens Equation

- connecting the Lens plane and the Source plane



$$\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})$$

$$\vec{\alpha}(\vec{\theta}) = \frac{D_{ds}}{D_s}\hat{\alpha}(D_d\vec{\theta})$$

Case 1: Point mass Lens

- $\hat{\alpha}(\xi) = \frac{4GM(\xi)}{c^2 \xi}$ ==> $\vec{\alpha}(\vec{\theta}) = \frac{4GM}{c^2} \frac{D_{LS}}{D_S D_L} \frac{\vec{\theta}}{|\vec{\theta}|^2}$

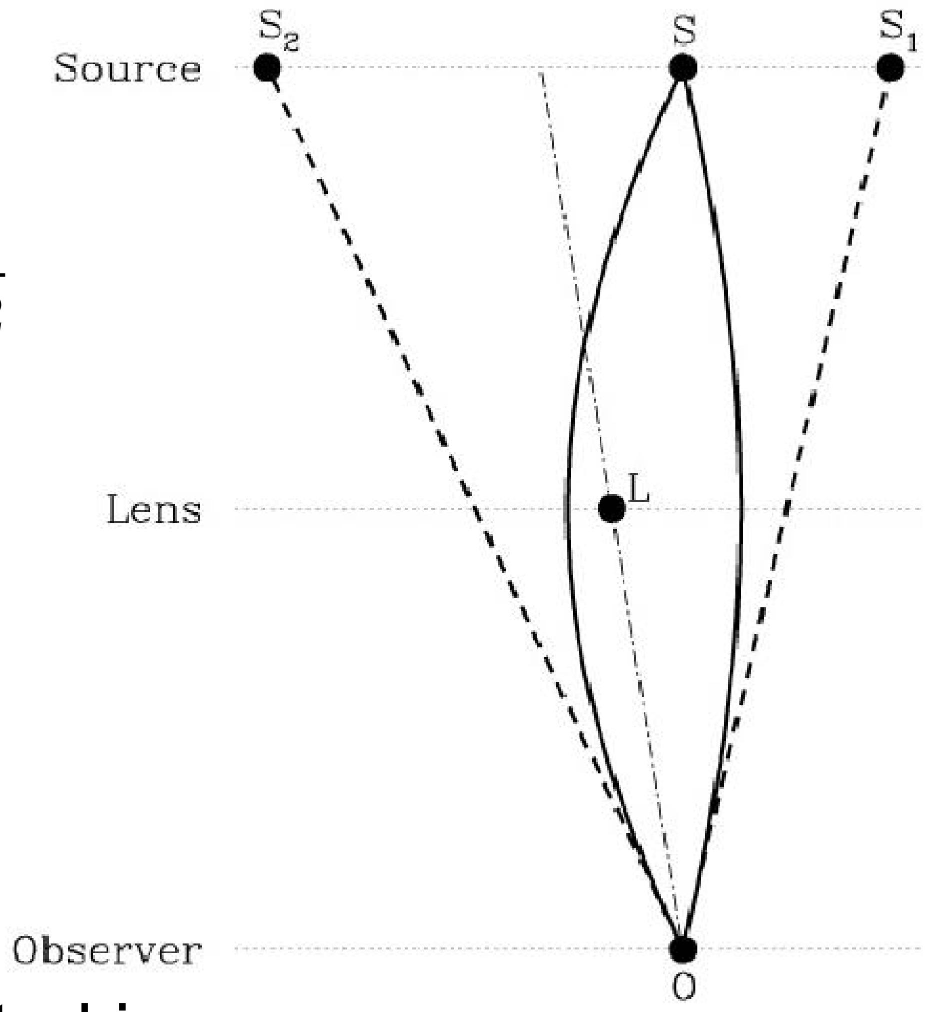
  $\vec{\alpha}(\vec{\theta}) = \frac{D_{ds}}{D_s} \hat{\alpha}(D_d \vec{\theta})$
- and $\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})$
- if $\beta = 0$, gives Einstein Radius

$$\theta_E = \sqrt{\frac{4GM}{c^2} \frac{D_{LS}}{D_S D_L}}$$

- Lens equation: $\vec{\beta} = \vec{\theta} - \theta_E^2 \frac{\vec{\theta}}{|\vec{\theta}|^2}$
- $\beta > \theta_E$, weakly lensed and weakly distorted image

  $\beta < \theta_E$, stronly lensed and multi images

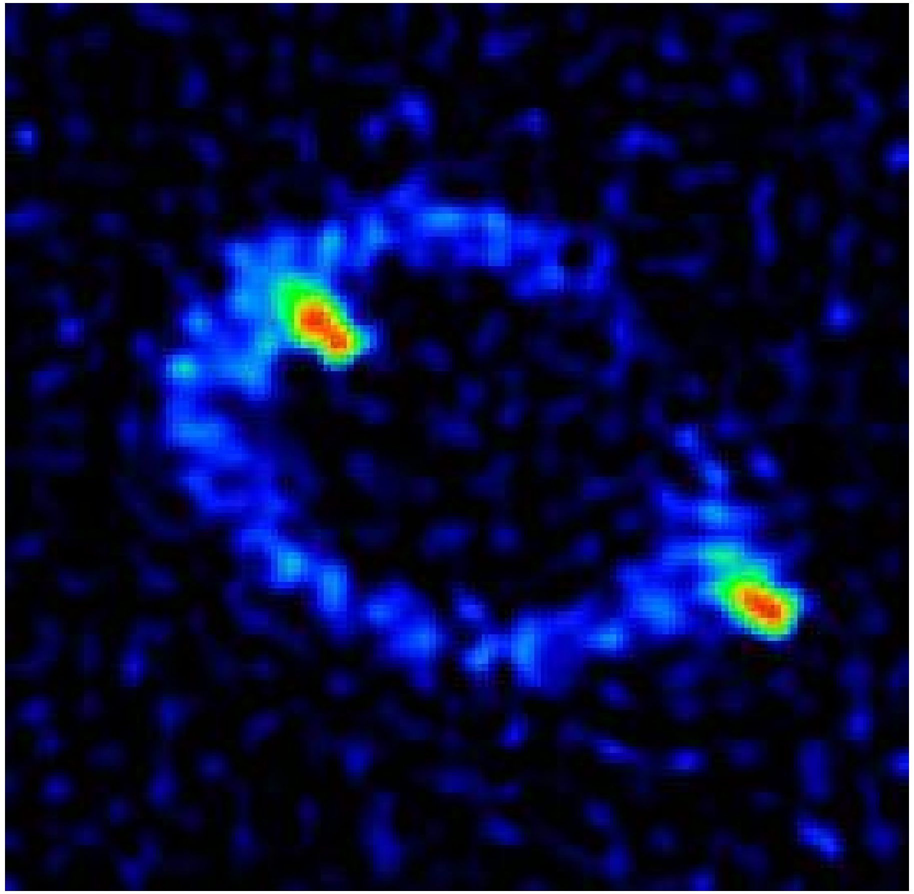
$$\theta_{\pm} = \frac{1}{2} \left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2} \right)$$

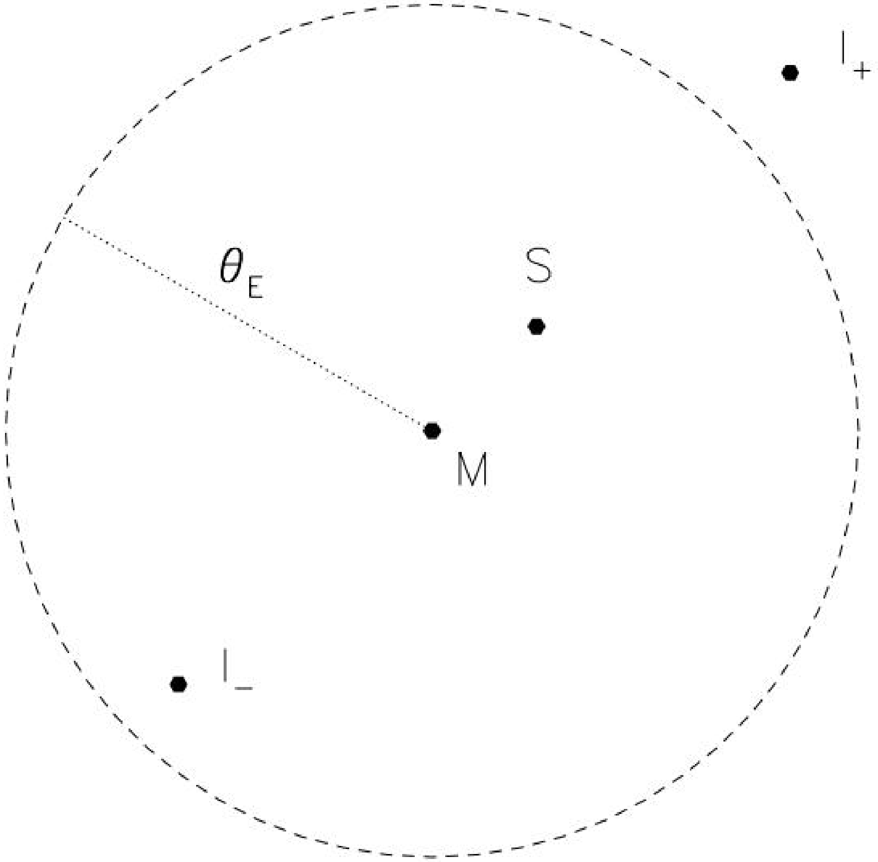


Wambsganss (1998)

Case 1: Point mass Lens

- Hewitt+ 1987, First Einstein Ring discovered in Radio Band & VLA



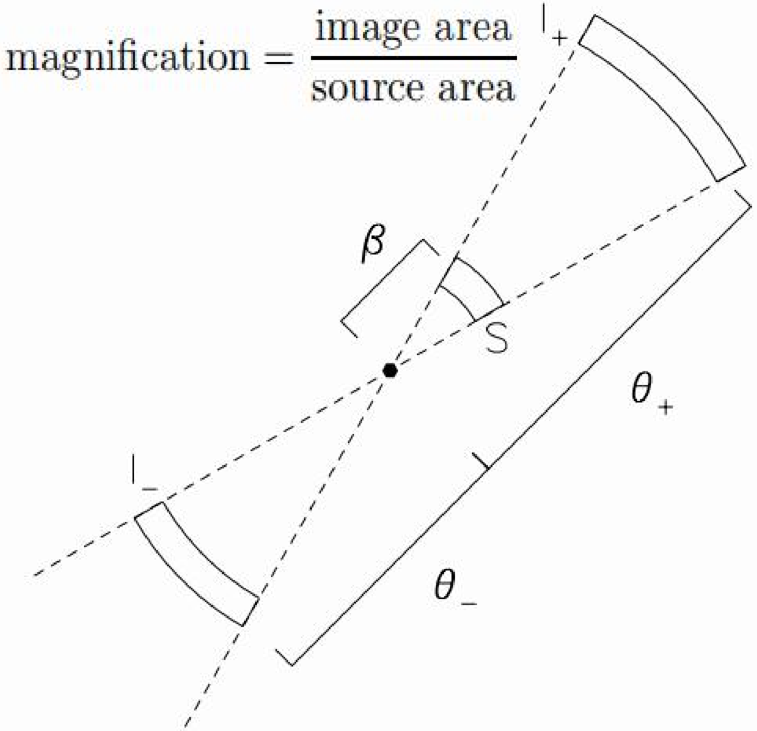


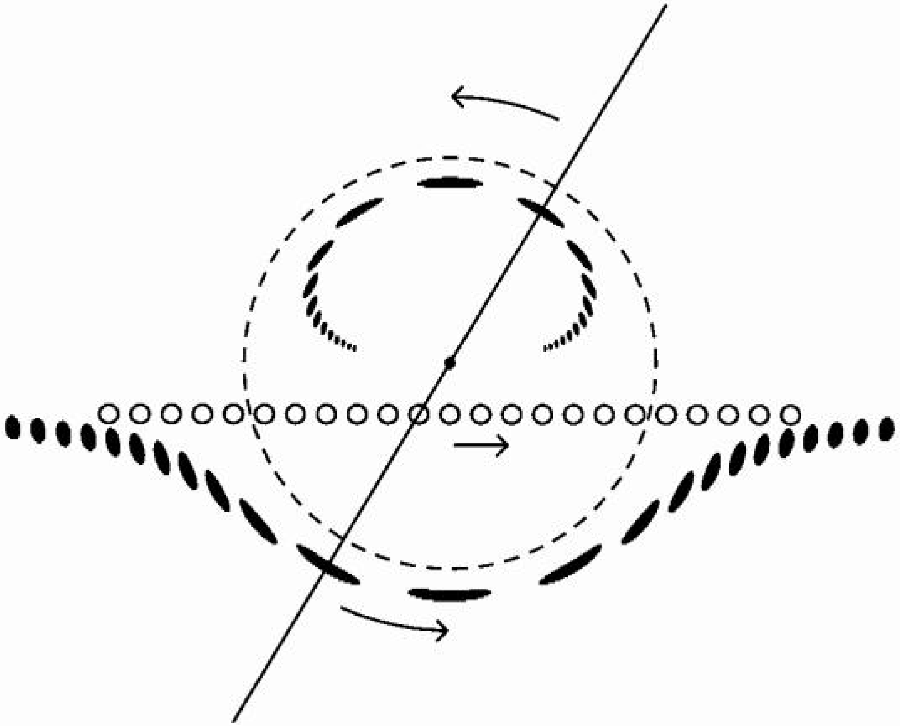
Hewitt, J., Turner, E., Schneider, D. et al. Unusual radio source MG1131+0456: a possible Einstein ring. Nature 333, 537–540 (1988) doi:10.1038/333537a0

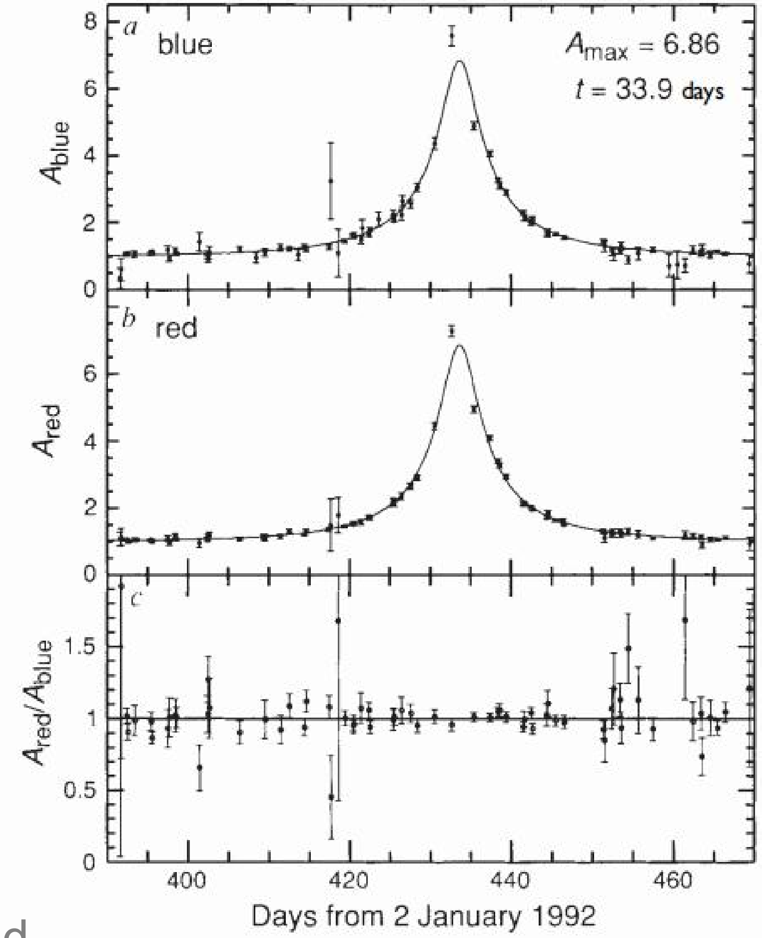
Magnification

- Gravitational lensing preserves surface brightness (), but changes the apparent solid angle of the source ==> magnification

$$\text{magnification} = \frac{\text{image area}}{\text{source area}}$$







arXiv:astro-ph/9309052v1, https://arxiv.org/pdf/astro-ph/9309052v1.pdf
Possible Gravitational Microlensing of a Star in the Large Magellanic Cloud

Magnification

- Local properties of the lens mapping, described by its Jacobian matrix A

$$A \equiv \frac{\partial \vec{\beta}}{\partial \vec{\theta}} = \left(\delta_{ij} - \frac{\partial \alpha_i(\vec{\theta})}{\partial \theta_j} \right) = \left(\delta_{ij} - \frac{\partial^2 \Psi(\vec{\theta})}{\partial \theta_i \partial \theta_j} \right)$$

$$\mu = \left| det \left(\frac{\partial \vec{\beta}}{\partial \vec{\theta}} \right) \right|^{-1} \equiv \left| det \left(\frac{\partial \beta_i}{\partial \theta_j} \right) \right|^{-1}$$

- If circularly symmetric,

$$\mu = \frac{\theta}{\beta} \frac{d\theta}{d\beta}$$

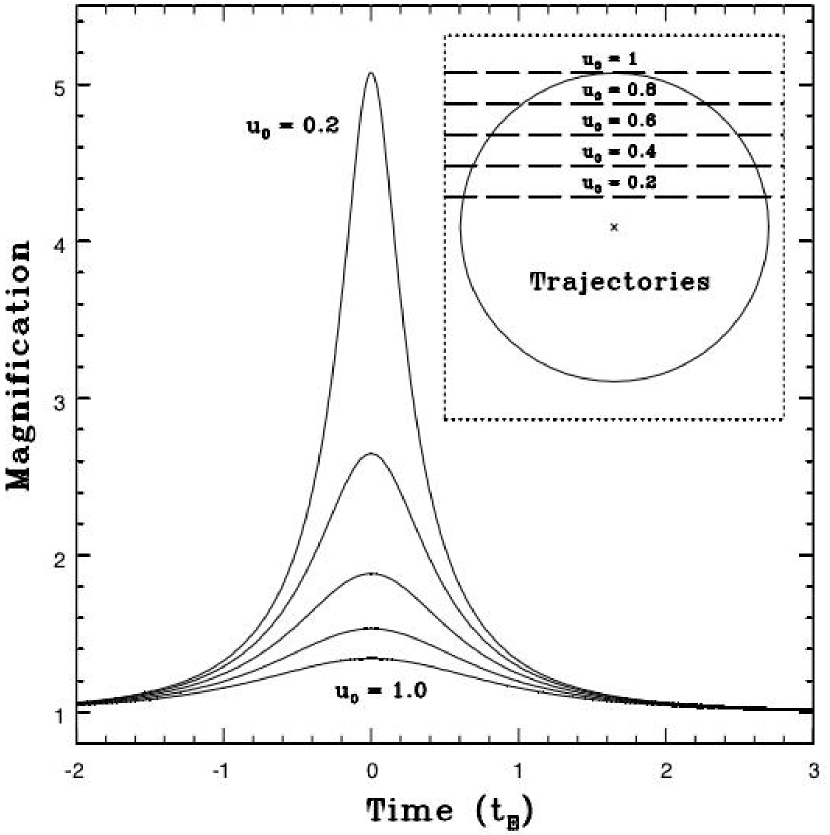
Lectures On Gravitational Lensing, Narayan & Bartelmann

Case 1: Point mass Lens - Magnification

- Images: $$\theta_\pm = \frac{1}{2}(\beta \pm \sqrt{\beta^2 + 4\theta_E^2})$$
- Magnification: $$\mu_\pm = \left[1 - \left(\frac{\theta_E}{\theta_\pm}\right)^4\right]^{-1} = \frac{u^2+2}{2u\sqrt{u^2+4}} \pm \frac{1}{2}$$
$$u = \beta\theta_E^{-1}$$
- Total magnification:
$$\mu = |\mu_+| + |\mu_-| = \frac{u^2+2}{u\sqrt{u^2+4}}$$
- e.g. for source on the Einstein Ring:
$$\beta = \theta_E, u = 1 \qquad \mu = 1.17 + 0.17 = 1.34$$

==> magnitude increase 0.319

Smith (2003)



Shapiro time delay

- Passage through potential also leads to time delay
- without potential: $t_0 = \int \frac{dl}{c}$
- with potential:

$$t_1 = \int_{src}^{obs} \frac{dl}{v} = \int_{src}^{obs} \frac{dl}{c - \frac{2}{c}|\Phi|}$$
$$= \int_{src}^{obs} \frac{dl/c}{1 - \frac{2}{c^2}|\Phi|} = \int_{src}^{obs} \frac{dl}{c}[1 + \frac{2}{c^2}|\Phi|]$$

- so, $\Delta t = \int_{src}^{obs} \frac{2}{c^3}|\Phi| dl$ ==> Shapiro delay (1964)
- Total time delay is the sum of the extra path length from the deflection and the gravitational time delay

$$t(\vec{\theta}) = \frac{1 + z_d}{c}\frac{D_d D_s}{D_{ds}}[\frac{1}{2}(\vec{\theta} - \vec{\beta})^2 - \psi(\vec{\theta})] = t_{geom} + t_{grav}$$

Case 2: Singular Isothemal Sphere (SIS)

- Galaxy lenses, the distributed nature of mass
- Simple model assumes that mass ==> particles of ideal gas
- ideal gas:

equation of state: $p = \frac{\rho k T}{m}$

In thermal equilibrium, T is related to the 1-d velocity dispersion:

$$m\sigma_v^2 = kT$$

In hydrostatic equilibrium, $\frac{p^{'}}{\rho} = -\frac{GM(r)}{r^2}, M^{'}(r) = 4\pi r^2 \rho$

solve the EOS ==> density profile: $\rho(r) = \frac{\sigma_v^2}{2\pi G}\frac{1}{r^2}$

so mass profile: $M(r) = \frac{2\sigma_v^2 r}{G}$

Case 2: Singular Isothemal Sphere (SIS)

- ideal gas:

  rotational velocity in circular orbit: $v_{rot}^2(r) = \frac{GM(r)}{r} = 2\sigma_v^2 = constant$

  Surface mass density: $v_{rot}^2 = GM/r \rightarrow$

$$dM = \frac{v_{rot}^2}{G} dr = 4\pi r^2 \rho(r) \rightarrow$$

$$\rho(r) = \frac{v_{rot}^2}{4\pi G}\frac{dr}{r^2} \rightarrow$$

$$\Sigma(\xi) = \frac{v_{rot}^2}{4\pi G}\int_{-\infty}^{\infty} \frac{dz}{(z+\xi)^2} = \frac{v_{rot}^2}{4G\xi} = \frac{\sigma_v^2}{2G\xi}$$

$$\hat{\alpha}(\xi) = \frac{4GM(\xi)}{c^2\xi}$$

$$M(\xi) = 2\pi \int_0^{\xi} \Sigma(\xi^{'})\xi^{'} d\xi^{'}$$

==> constant deflection angle:

$$\hat{\alpha} = 4\pi\frac{\sigma_v^2}{c^2}$$

Case 2: Singular Isothemal Sphere (SIS)

using $\hat{\alpha} = 4\pi \frac{\sigma_v^2}{c^2}$ and lens equation $\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})$

$$\vec{\alpha}(\vec{\theta}) = \frac{D_{ds}}{D_s} \hat{\alpha}(D_d \vec{\theta})$$

$$\frac{\vec{\theta}}{|\vec{\theta}|} = \pm 1$$

==> $\alpha = \hat{\alpha} \frac{D_{ds}}{D_s} = 4\pi \frac{\sigma_v^2}{c^2} \frac{D_{ds}}{D_s} = \theta_E$

for strong lensing, get two images as for point mass $\vec{\beta} = \vec{\theta} - \theta_E \frac{\vec{\theta}}{|\vec{\theta}|}$

==> $\theta_{\pm} = \beta \pm \theta_E$

Magnification can be very large for source aligned with lens:

$$\mu = \frac{\theta}{\beta} \frac{d\theta}{d\beta} \quad \text{==>} \quad \mu_{\pm} = \frac{\theta_{\pm}}{\beta} = 1 \pm \frac{\theta_E}{\beta} = \left(1 \mp \frac{\theta_E}{\theta_{\pm}}\right)^{-1}$$

Separation of the two images is typically ~ arcsec for galaxy lenses:

$$\theta_E = 1^{''}.6 \left(\frac{\sigma_v}{200 kms^{-1}}\right)^2 \left(\frac{D_{ds}}{D_s}\right)$$

Notice: in general the core of a galaxy would not be singular

Caustics and Critical lines

- Grav Lens changes the observed brightness of the source, determined by the Jacobian matrix from the lens equation

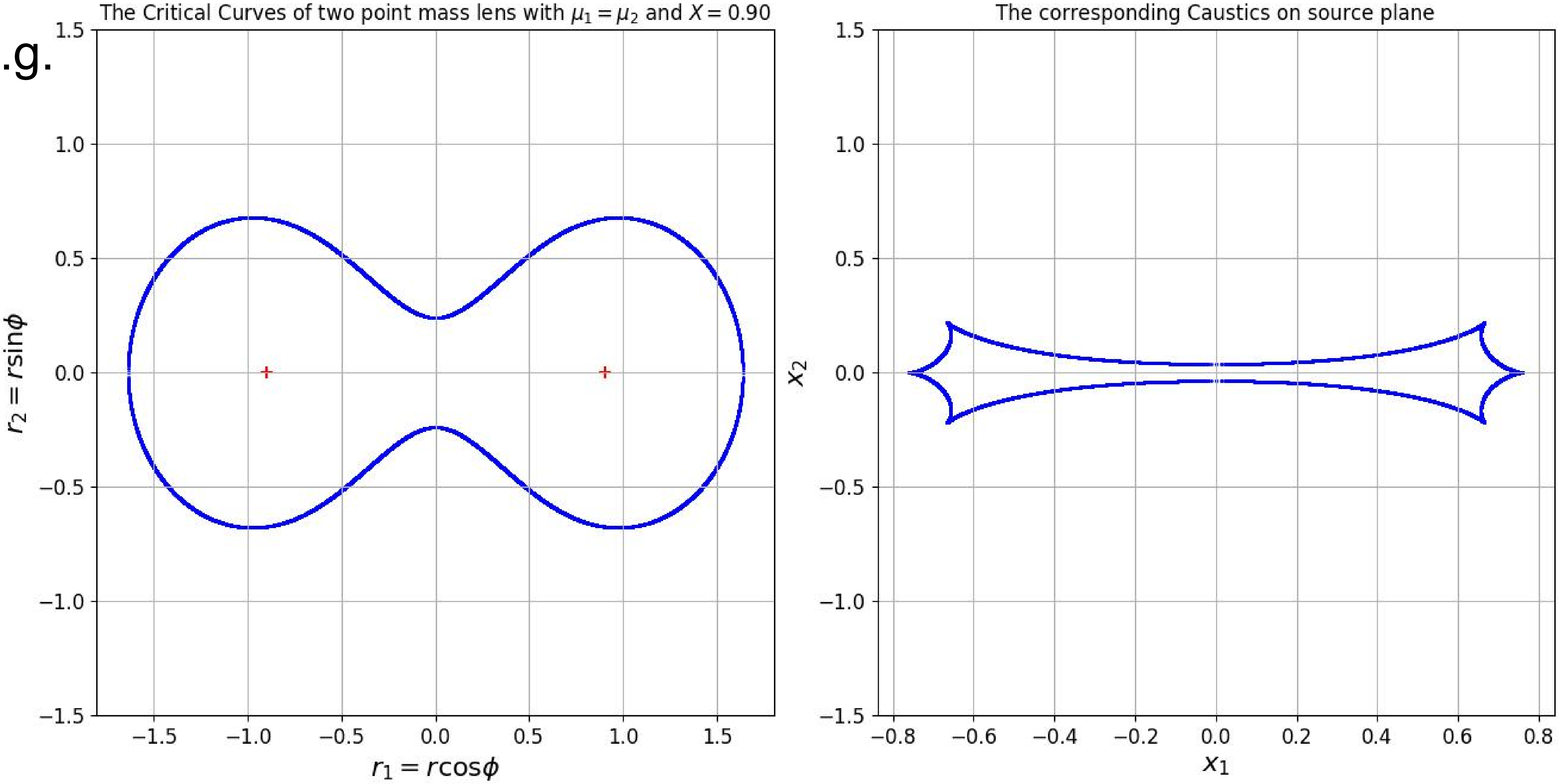
$$A \equiv \frac{\partial \vec{\beta}}{\partial \vec{\theta}} = \left(\delta_{ij} - \frac{\partial \alpha_i(\vec{\theta})}{\partial \theta_j} \right) = \left(\delta_{ij} - \frac{\partial^2 \Psi(\vec{\theta})}{\partial \theta_i \partial \theta_j} \right)$$

$$\mu(\vec{\theta}) = \frac{1}{det A(\vec{\theta})}$$

- Image at $\vec{\theta}$ is magnified by a factor of $|\mu(\vec{\theta})|$
- Notice that $|\mu(\vec{\theta})|$ diverge at $det A(\vec{\theta}) = 0$ ==> these points in the image plane form closed curves, which is so called critical lines.
- Corresponding curves in the source plane obtained via the lens equation are called caustics.

Caustics and Critical lines

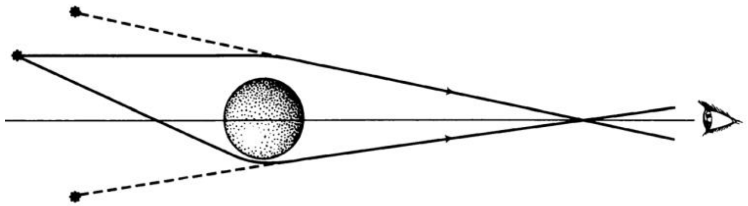
- e.g.

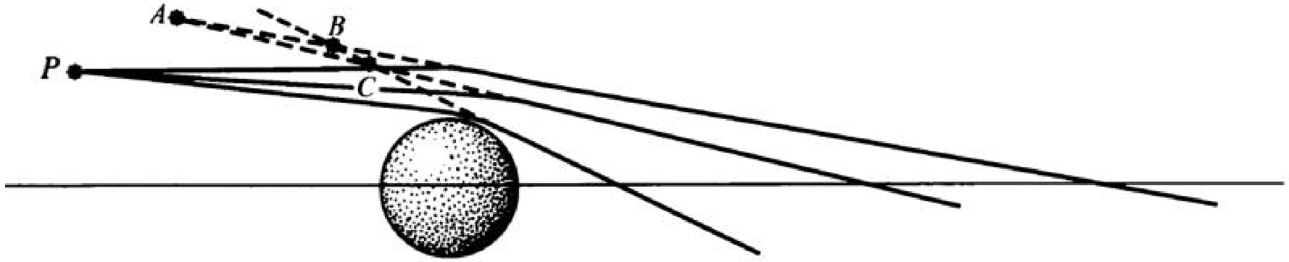


Gravitational Lens vs Genuine Lens

- Grav lens: the observer sees the source at two distinct locations

$\alpha \propto b^{-1}$

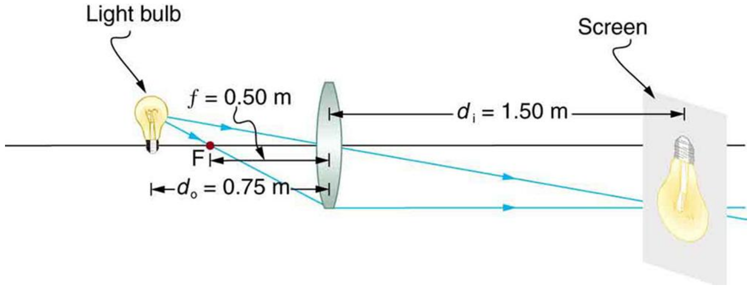


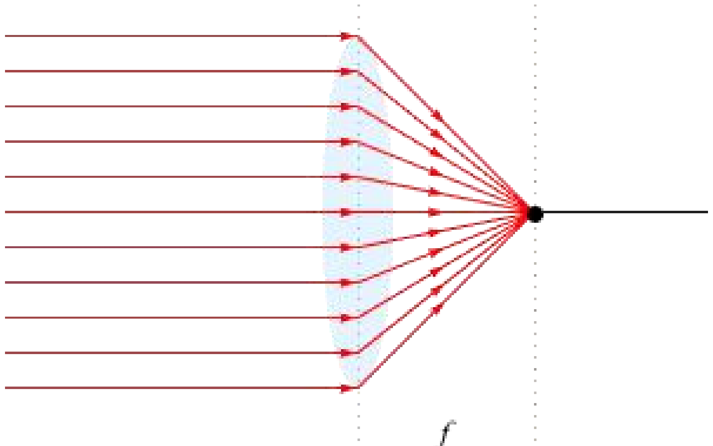


Grav lens has no well-defined focal length and cannot produce genuine images, the “images” are
corresponds merely to a direction
of incidence of light on the observer, not a genuine image in the sky

- Genuine lens:

$\alpha \propto b$





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Question 1

A galaxy at redshift 0.5 can be modelled as a singular isothermal sphere; its dispersion is 200km/s. A background

source at redshift 2 is lensed by the foreground galaxy into two images with a brightness ratio of 3:1, what are the

angular separation and time delay between the two images? You can assume the usual cosmology.

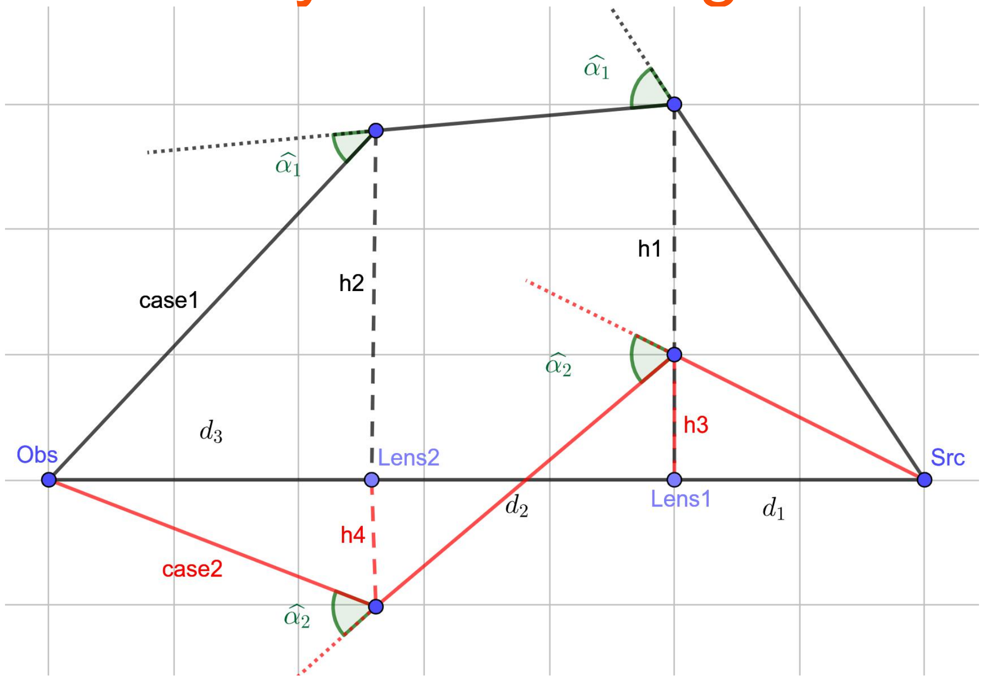
First assume $H_0 = 69.6kms^{-1}, \Omega_m = 0.286, \Omega_{vac} = 0.714$

then the distance of z ~ 0.5 and z ~ 2 can be computed,

using theories of SIS model and the time delay function we can esstimate the angular separation ~ 1.2 arcsec and time delay ~ 45 days

Question 2

Two (N=2) galaxies are aligned perfectly with the Earth and a distant quasar. Each galaxy can be modelled as a singular isothermal sphere. How many Einstein rings are formed as a result? How will your results generalize when you have N>2 galaxies?



N galaxies ==> 2^{N-1} rings at most

Next step could consider general situation which obs, lens, src are not aligned perfectly and galaxies are not just point mass model ==> gravitational lensing simulation

code: https://github.com/rkkuang/aeroastro/tree/master/gravlen/num_einstain_rings

Question 3

A background source is multiply-imaged, can the brightest image arrive last?

Using the relation between effective lensing potential and magnification & time delay

Effective lensing potential: $\Psi(\vec{\theta}) = \frac{D_{ds}}{D_d D_s}\frac{2}{c^2}\int \Phi(D_d\vec{\theta}, z)dz$ $\qquad \vec{\nabla}_\theta = \vec{\alpha}$

Jacobian matrix: $A = \frac{\partial \vec{\beta}}{\partial \vec{\theta}} = \delta_{ij} - \frac{\partial^2 \Psi(\vec{\theta})}{\partial \vec{\theta}_i \partial \vec{\theta}_j} \equiv \delta_{ij} - \Psi_{ij}$ $\qquad \mu = |detA|^{-1}$

time delay: $t(\vec{\theta}) = \frac{1+z_d}{c}\frac{D_d D_s}{D_{ds}}\left[\frac{1}{2}(\vec{\theta}-\vec{\beta})^2 - \Psi(\vec{\theta})\right]$ $\qquad \mu \leftarrow \Psi \rightarrow t$

Simulation, under a potential, given a $\vec{\beta}$, solve the lens equation for $\vec{\theta}_i$ see if $\vec{\theta}_k$ which has maximun magnification also have largest time delay

Question 4

Study the caustics and critical curves of two singular isothermal spheres lensing a background quasar. Study two cases

1) Both galaxies are at the same redshift and, 2) these two galaxies are at different redshifts.

Mass density ==> Jacobian matrix ==> draw those points which make magnification largest ==> critical lens ==> caustics

code: https://github.com/rkkuang/aeroastro/tree/master/gravlen/critical_and_caustics

Question 4

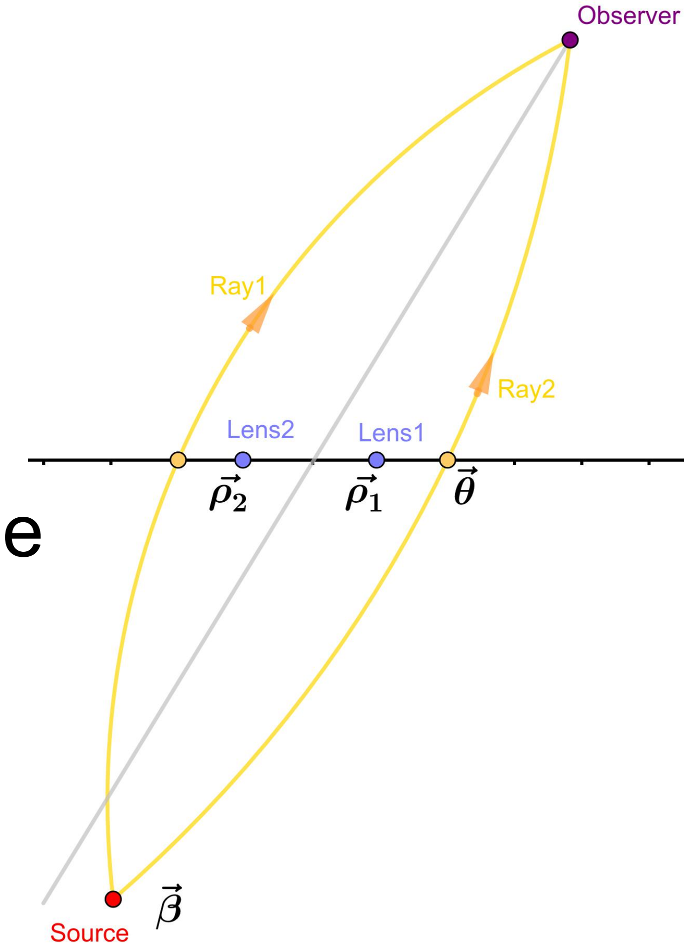
case 1) Both galaxies are at the same redshift

$$\vec{\alpha}(\vec{\theta}) = \theta_{E1}\frac{\vec{\theta}-\vec{\rho_1}}{|\vec{\theta}-\vec{\rho_1}|} + \theta_{E2}\frac{\vec{\theta}-\vec{\rho_2}}{|\vec{\theta}-\vec{\rho_2}|}$$

$\vec{\rho_1}, \vec{\rho_2}$ are positions of lens1, lens2 on the lens plane

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})$$

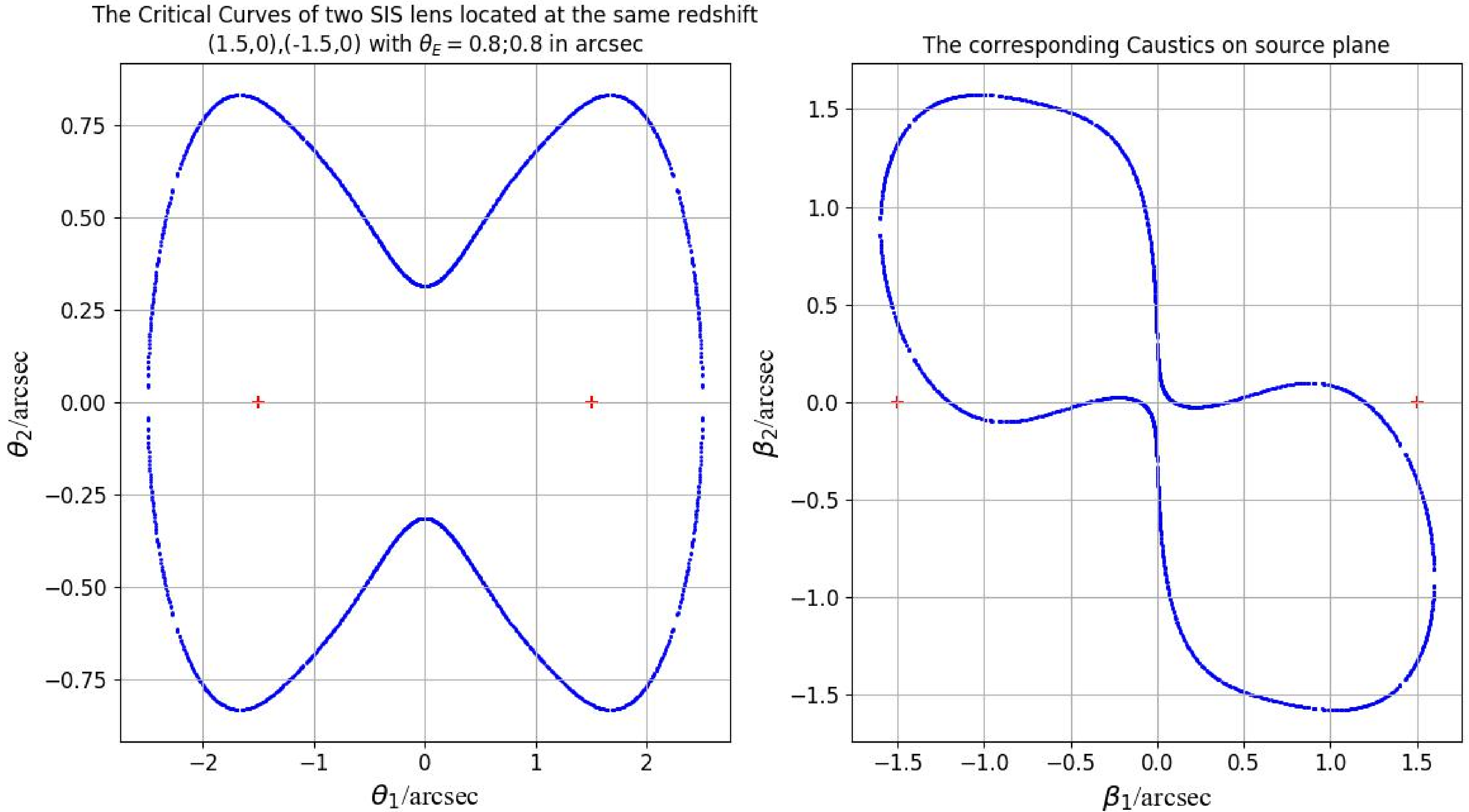
$$A = \frac{\partial \vec{\beta}}{\partial \vec{\theta}} = \left(\delta_{ij} - \frac{\partial \alpha_i(\vec{\theta})}{\partial \theta_j}\right) \qquad \mu(\vec{\theta}) = \frac{1}{det A(\vec{\theta})}$$



code: https://github.com/rkkuang/aeroastro/tree/master/gravlen/critical_and_caustics

Question 4

case 1) Both galaxies are at the same redshift



Question 4

case 2) these two galaxies are at different redshifts

Let lens1 is closer to source,

for lens_i, we have, $\vec{\beta_i} = \vec{\theta_i} - \theta_{Ei}\frac{\vec{\theta_i} - \vec{\rho_i}}{|\vec{\theta_i} - \vec{\rho_i}|}$

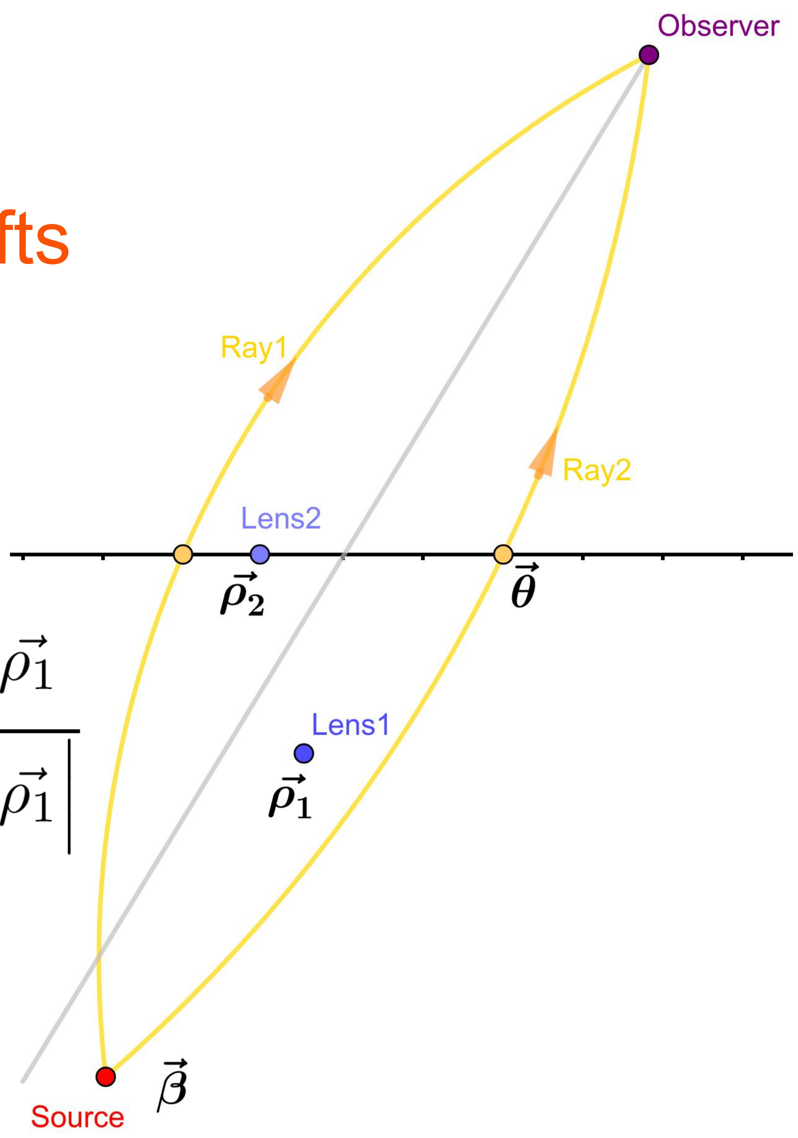
in two lenses system, $\vec{\beta_2} = \vec{\theta_1}$

==>

$$\vec{\beta_1} = \vec{\theta_2} - \theta_{E2}\frac{\vec{\theta_2} - \vec{\rho_2}}{|\vec{\theta_2} - \vec{\rho_2}|} - \theta_{E1}\frac{\vec{\theta_2} - \theta_{E2}\frac{\vec{\theta_2}-\vec{\rho_2}}{|\vec{\theta_2}-\vec{\rho_2}|} - \vec{\rho_1}}{\left|\vec{\theta_2} - \theta_{E2}\frac{\vec{\theta_2}-\vec{\rho_2}}{|\vec{\theta_2}-\vec{\rho_2}|} - \vec{\rho_1}\right|}$$

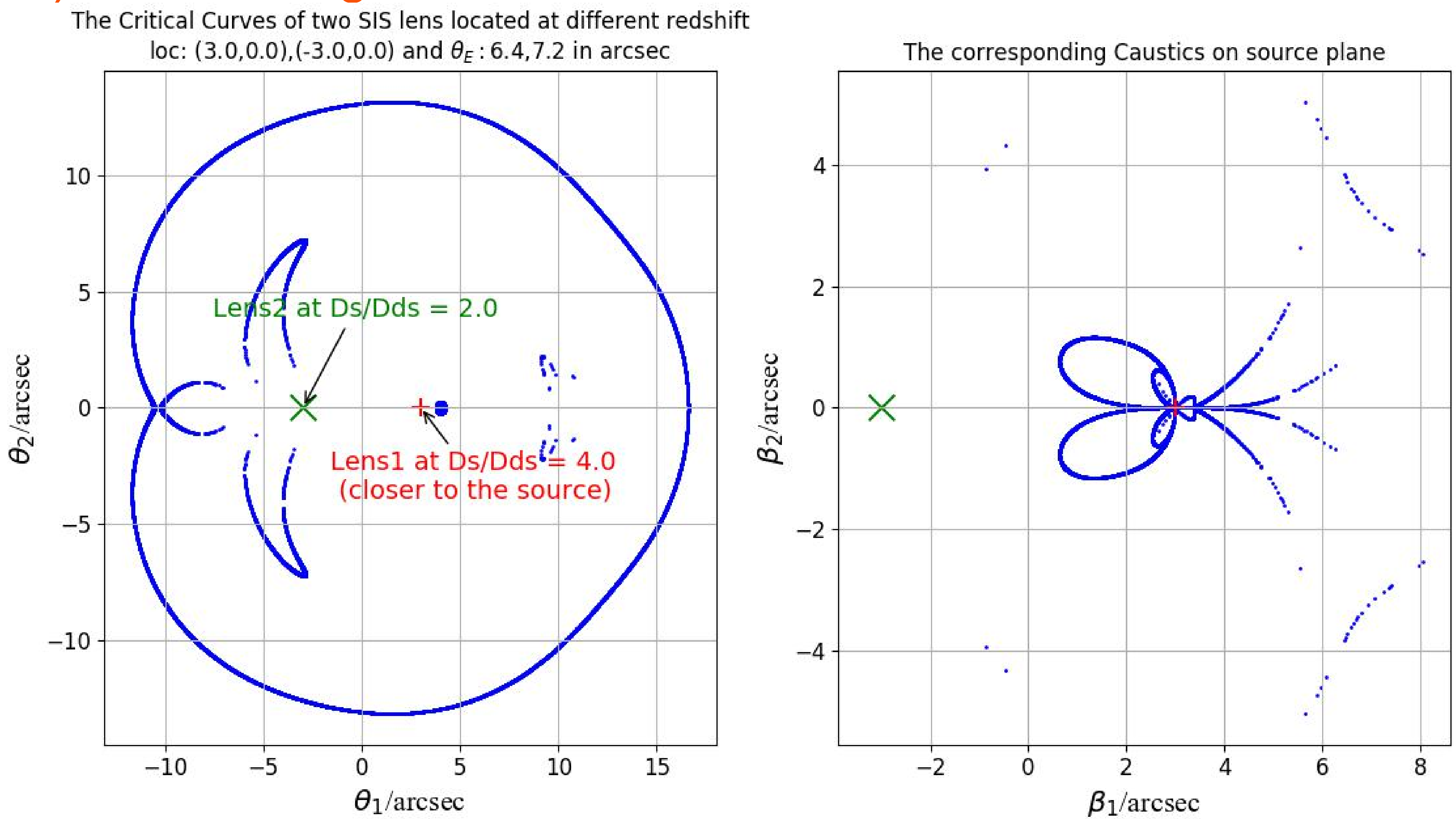
so, the Jacobian matrix can be derived:

$$A = \frac{\partial\vec{\beta_1}}{\partial\vec{\theta_2}}$$



Question 4

case 2) these two galaxies are at different redshifts



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Gravitational Lensing Applications

- Cosmic telescopes: distant, faint objects observation
- 2-d mass distribution of lenses, dark matter
- Hubble constant, cosmological constant, density parameter
- ......

References

- https://lacosmo.com/DeflectionOfLight/index.html
- *The Mathematical Theory of Relativity*, Arthur Stanley Eddington (P101)
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- https://en.wikipedia.org/wiki/Gauss%27s_law_for_gravity
- *Lectures On Gravitational Lensing,* Narayan & Bartelmann
- https://www.cfa.harvard.edu/~dfabricant/huchra/ay202/lectures/lecture12.pdf
- https://web.stanford.edu/~oas/SI/SRGR/notes/SchwarzschildSolution.pdf

Equations in this slide are generated by a helpful tool KLatexFormular on linux

Appendix-1

- Newtonial prediction
- b: impact factor
- gravitational potential

$$\Phi(b,z) = -\frac{GM}{(b^2+z^2)^{1/2}}$$

simplify the calculation by integrating
not along the deflected ray but along z axis

$$\alpha = \frac{v_b}{c} = \frac{1}{c}\int \frac{d\Phi}{db}dt = \frac{1}{c^2}\int \frac{d\Phi}{db}dl \approx \frac{1}{c^2}\int \frac{d\Phi}{db}dz$$

$$= \frac{GMb}{c^2}\int \frac{dz}{(b^2+z^2)^{3/2}}$$

$$= \frac{GMb}{c^2}\left[\frac{z}{b^2\sqrt{b^2+z^2}}\Big|_{-\infty}^{+\infty}\right]$$

$$= \frac{2GM}{c^2 b}$$