

Gravitational Lensing

Theories, Questions and Applications

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Outline

- History
- Gravitational Lensing Theories
- Related Questions
- Applications

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Predictions from Newtonian and GR

- First proposed by Soldner (1801) using Newtonian theory, given a deflection angle (Appendix-1 for derivation)

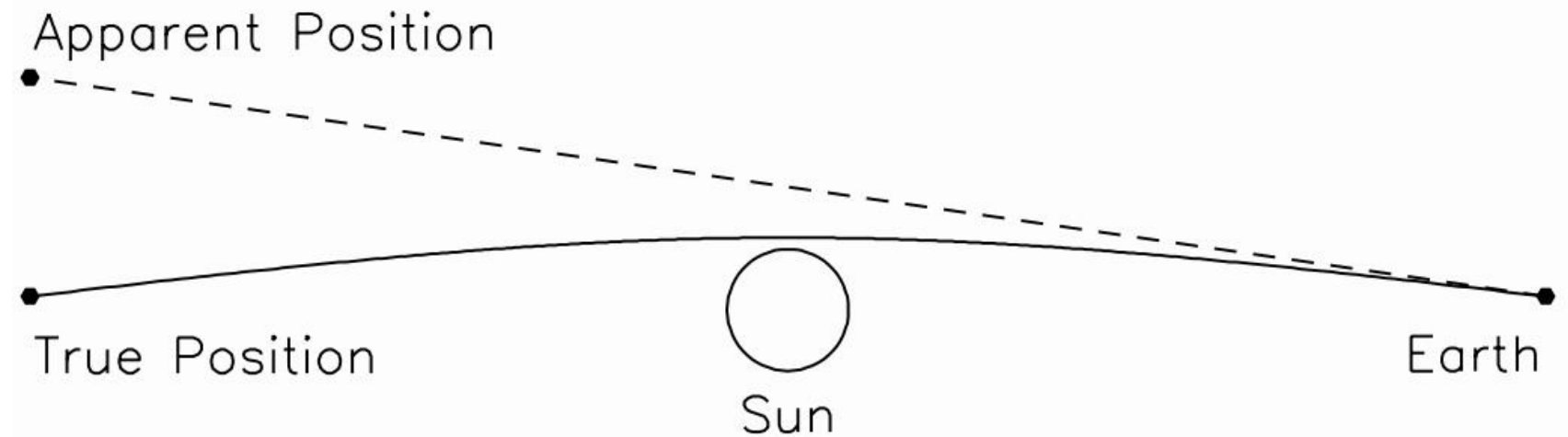
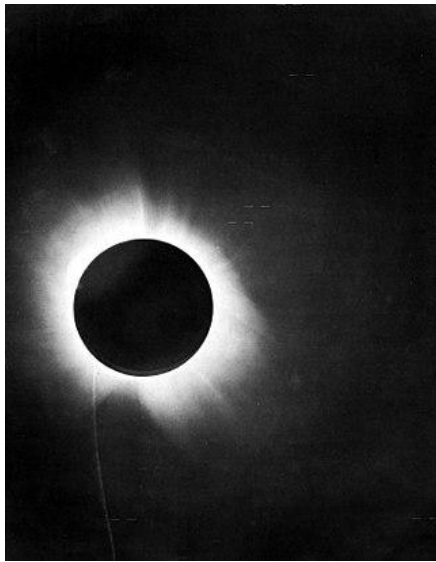
$$\alpha = \frac{2GM}{c^2 r} , 0.85 \text{ arcsec for the Sun}$$

- Einstein (1911) derived the same result using Equivalence principle and Euclidean metric
- Einstein (1915) derived the new result using General Relativity

$$\alpha = \frac{4GM}{c^2 r} , 1.7 \text{ arcsec for the Sun}$$

Eddington's obs of the Solar Eclipse

- In 1919, Eddington measured a value close to GR's prediction using the data collected during an eclipse, stars with apparent position near the sun become visible.



IX. A determination of the deflection of light by the sun's gravitational field, from observations made at the total eclipse of May 29, 1919

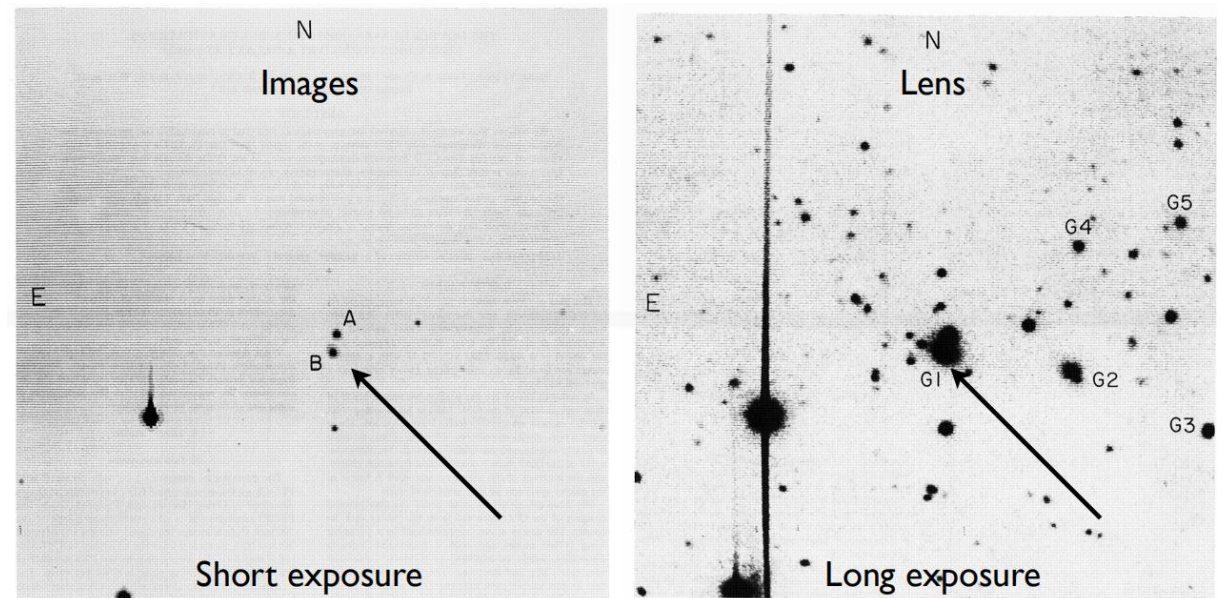
<https://royalsocietypublishing.org/doi/pdf/10.1098/rsta.1920.0009>

The first example for GravLens: 0957+561

- Eddington (1920): Multiple light paths ==> multi images
- Walsh et al., (1979) quasar QSO 957+561 A,B found at $z \sim 1.4$, two seen images separated by 6 arcsec

- Lens: evidence

1. Lensing galaxy at $z \sim 0.36$
2. Similar spectra
3. Ratio of optical and radio flux
4. VLBI imaging: small scale features



Walsh, Carswell & Weymann 1979, 0957+561 A, B: twin quasistellar objects or gravitational lens?
<https://ui.adsabs.harvard.edu/abs/1979Natur.279..381W/abstract>

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General Relativity and light deflection

- Einstein Field equations: $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$
- ==> Geodesic equations: $\frac{d^2x^\beta}{d\lambda^2} + \Gamma_{\mu\nu}^\beta \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0$
Calculate how gravity bends light by solving geodesic equation
- To compute the Christoffel symbols $\Gamma_{\mu\nu}^\beta$,
requires solving for the metric tensor $g_{\mu\nu}$,
which requires solving the curvature equations $R_{\mu\nu} = 0$,
--- ten partial-differential equations.

General Relativity and light deflection

- Or, the velocity of the photon from the Schwarzschild metric,

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)(dx^2 + dy^2 + dz^2)$$

- and Poisson Equation, $\nabla^2\Phi = 4\pi G\rho$

- and lightlike interval: $g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$

- $\Rightarrow v = \frac{\sqrt{dx^2 + dy^2 + dz^2}}{dt} = \sqrt{\frac{1 + 2\Phi}{1 - 2\Phi}} \approx 1 + 2\Phi$

- The gravitational field decreases the speed of propagation

General Relativity and light deflection

$$v = \frac{\sqrt{dx^2 + dy^2 + dz^2}}{dt} = \sqrt{\frac{1 + 2\Phi}{1 - 2\Phi}} \approx 1 + 2\Phi \text{ (natural units)} \implies v = c \left(1 + \frac{2}{c^2} \Phi \right) \text{ (SI)}$$

- define refraction index $n = 1 - \frac{2}{c^2} \Phi = 1 + \frac{2}{c^2} |\Phi| \geq 1$
- deflection angle: $\vec{\hat{\alpha}} = - \int \vec{\nabla}_{\perp} n dl = \frac{2}{c^2} \int \vec{\nabla}_{\perp} \Phi dl$
- which is twice of Newtonian prediction (Appendix-1)
- for point mass lens, $\hat{\alpha} = \frac{4GM}{bc^2}$

Thin screen approximation

- Most deflection occurs near to the lens ($|z| \sim b$)
- \Rightarrow treat all deflection as in the lens plane

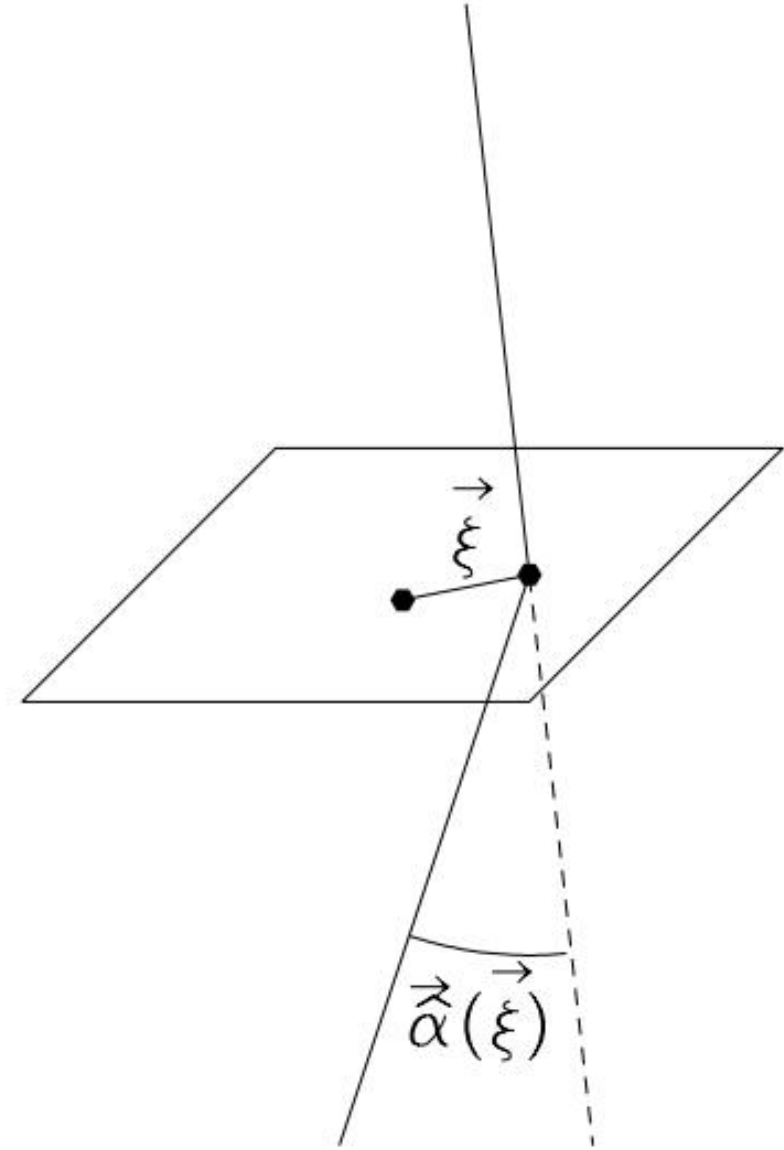
- Projected surface density: $\Sigma(\vec{\xi}) = \int \rho(\vec{\xi}, z) dz$

- Deflection angle: $\vec{\alpha}(\vec{\xi}) = \frac{4G}{c^2} \int \frac{(\vec{\xi} - \vec{\xi}') \Sigma(\vec{\xi}')}{|\vec{\xi} - \vec{\xi}'|^2} d^2 \xi'$

- In circular symmetry cases:

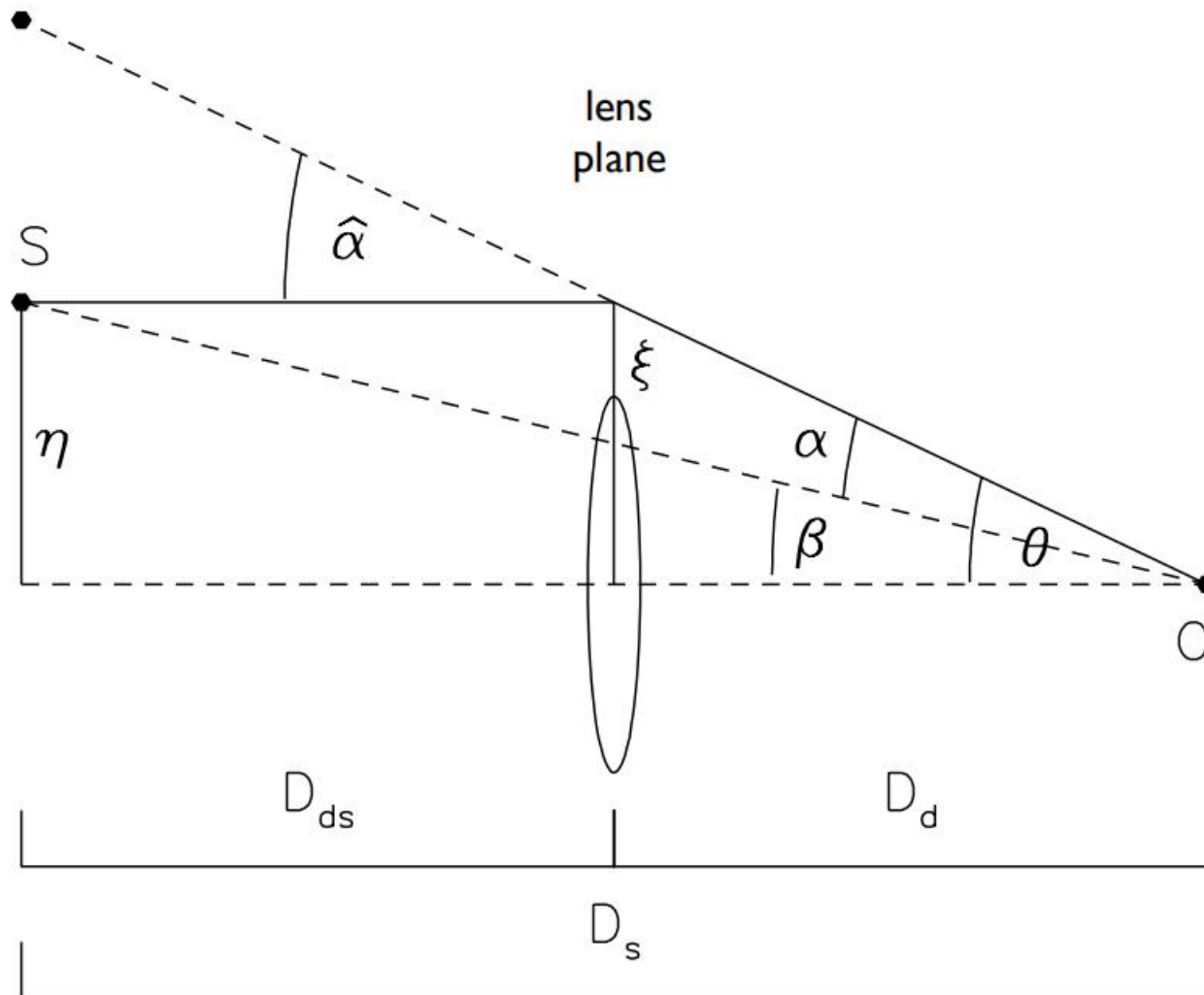
$$\hat{\alpha}(\xi) = \frac{4GM(\xi)}{c^2 \xi}$$

$$M(\xi) = 2\pi \int_0^\xi \Sigma(\xi') \xi' d\xi'$$



The Lens Equation

- connecting the Lens plane and the Source plane



$$\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})$$

$$\vec{\alpha}(\vec{\theta}) = \frac{D_{ds}}{D_s} \hat{\alpha}(D_d \vec{\theta})$$

Case 1: Point mass Lens

- $\hat{\alpha}(\xi) = \frac{4GM(\xi)}{c^2\xi} \implies \vec{\alpha}(\vec{\theta}) = \frac{4GM}{c^2} \frac{D_{LS}}{D_S D_L} \frac{\vec{\theta}}{|\vec{\theta}|^2}$
 $\vec{\alpha}(\vec{\theta}) = \frac{D_{ds}}{D_s} \hat{\alpha}(D_d \vec{\theta})$

- and $\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})$

- if $\beta = 0$, gives Einstein Radius

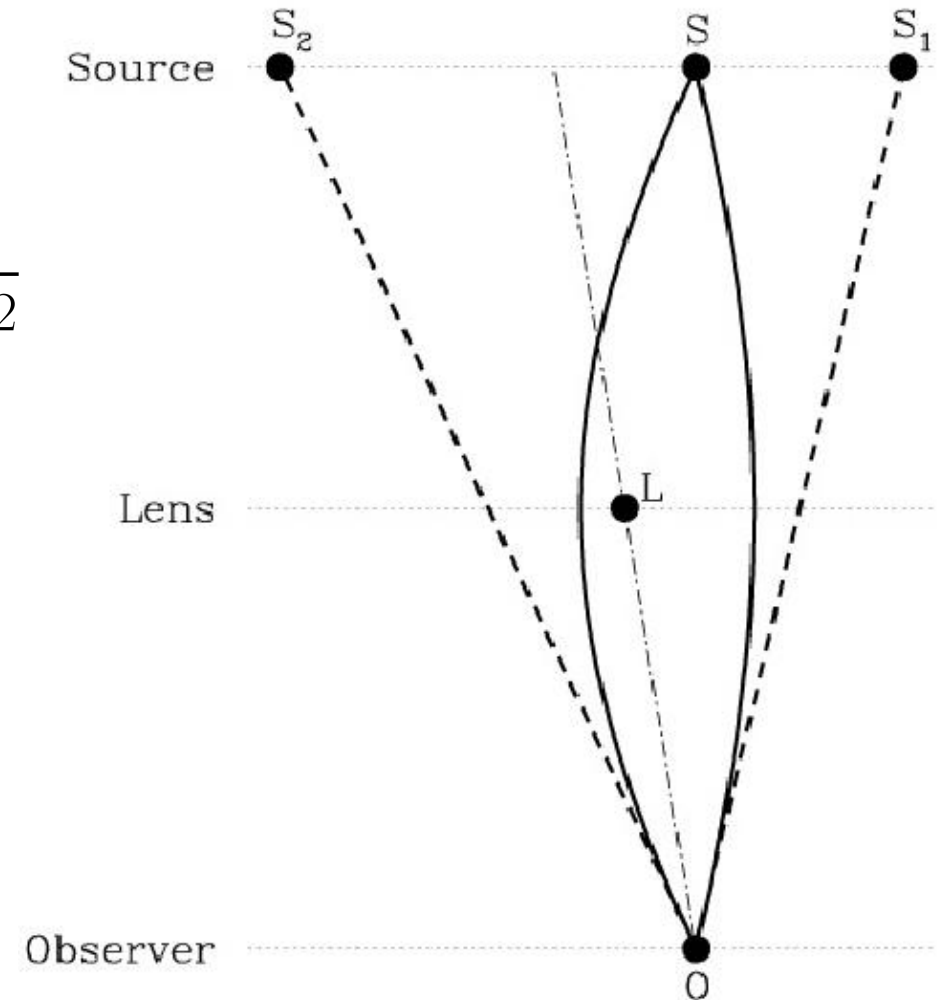
$$\theta_E = \sqrt{\frac{4GM}{c^2} \frac{D_{LS}}{D_S D_L}}$$

- Lens equation: $\vec{\beta} = \vec{\theta} - \theta_E^2 \frac{\vec{\theta}}{|\vec{\theta}|^2}$

- $\beta > \theta_E$, weakly lensed and weakly distorted image

$\beta < \theta_E$, strongly lensed and multi images

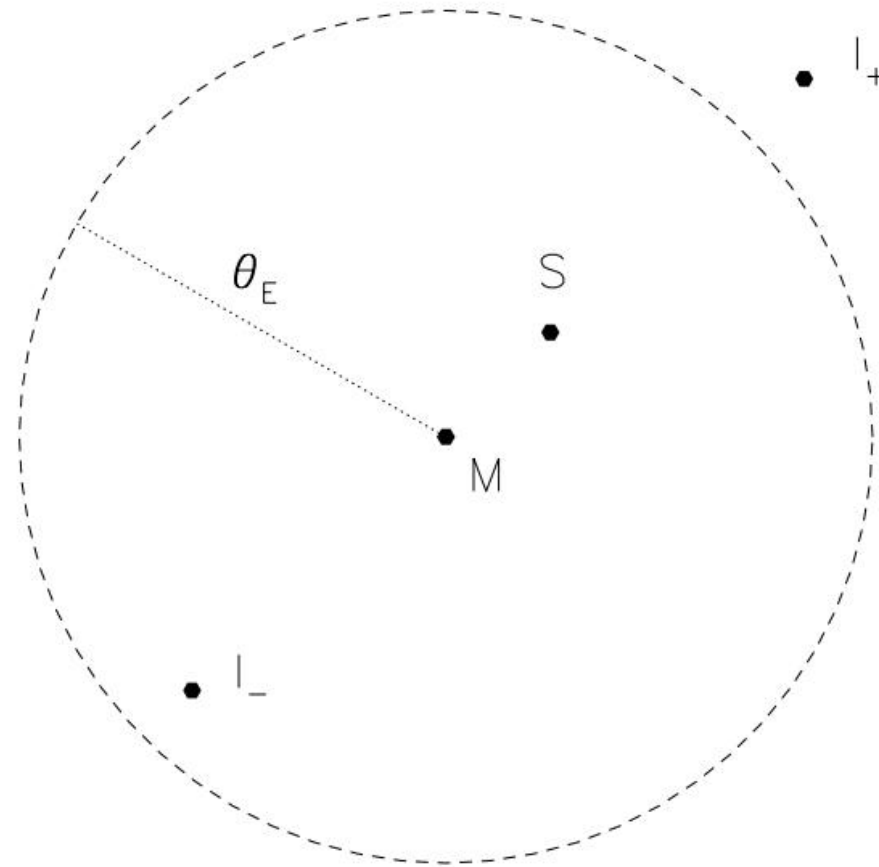
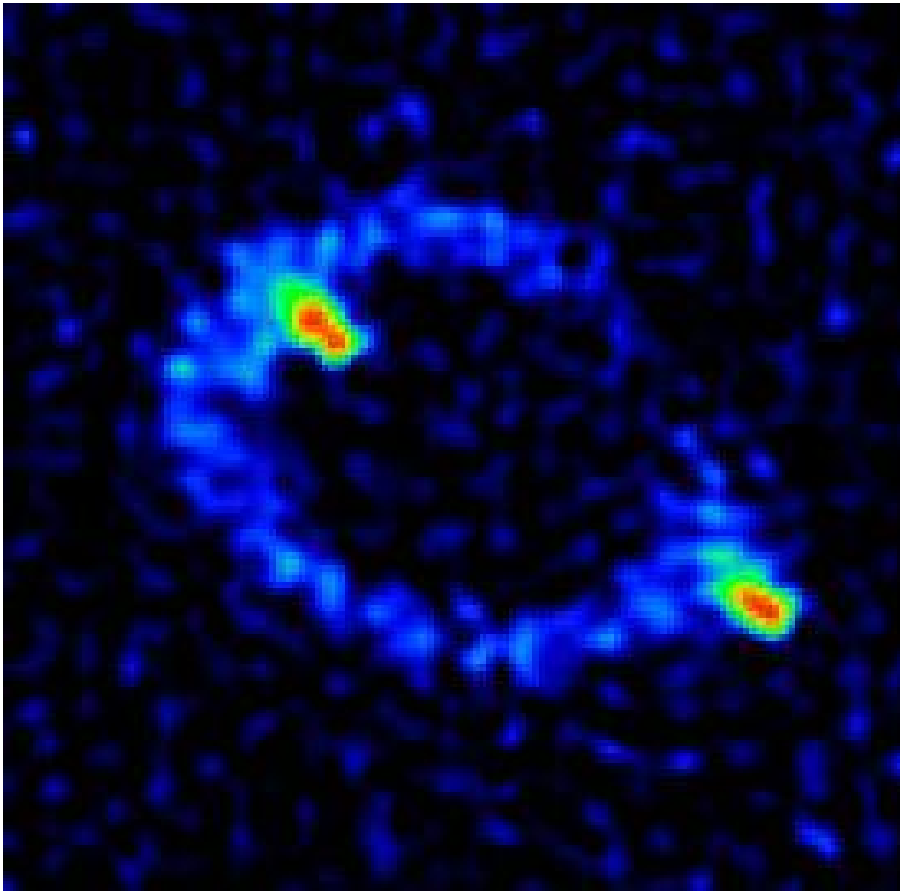
$$\theta_{\pm} = \frac{1}{2} \left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2} \right)$$



Wambsganss (1998)

Case 1: Point mass Lens

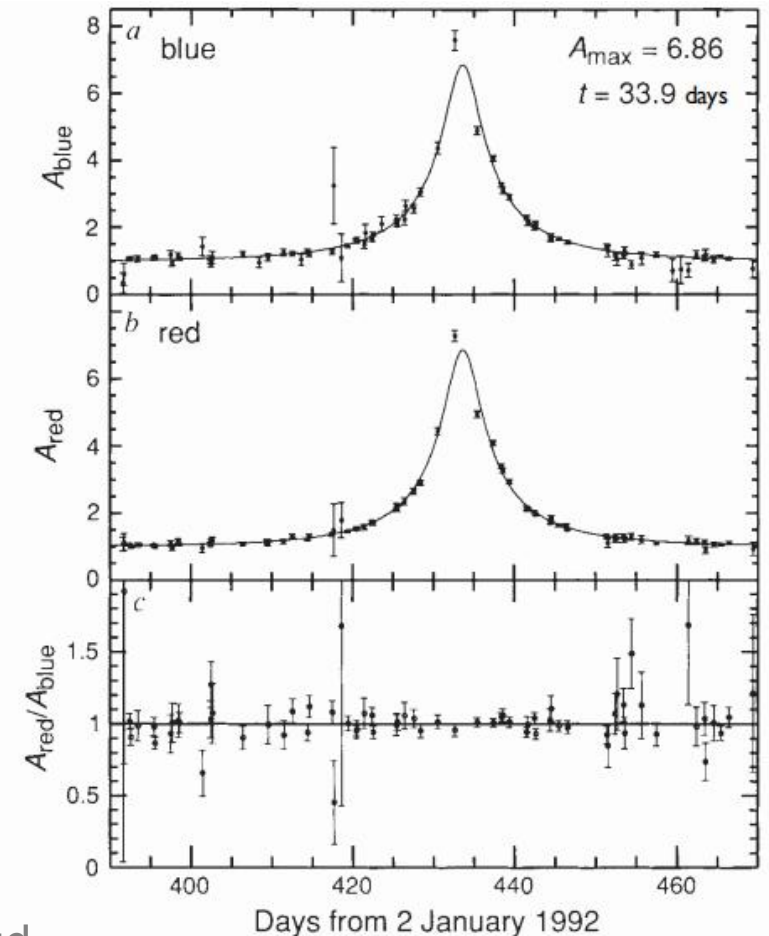
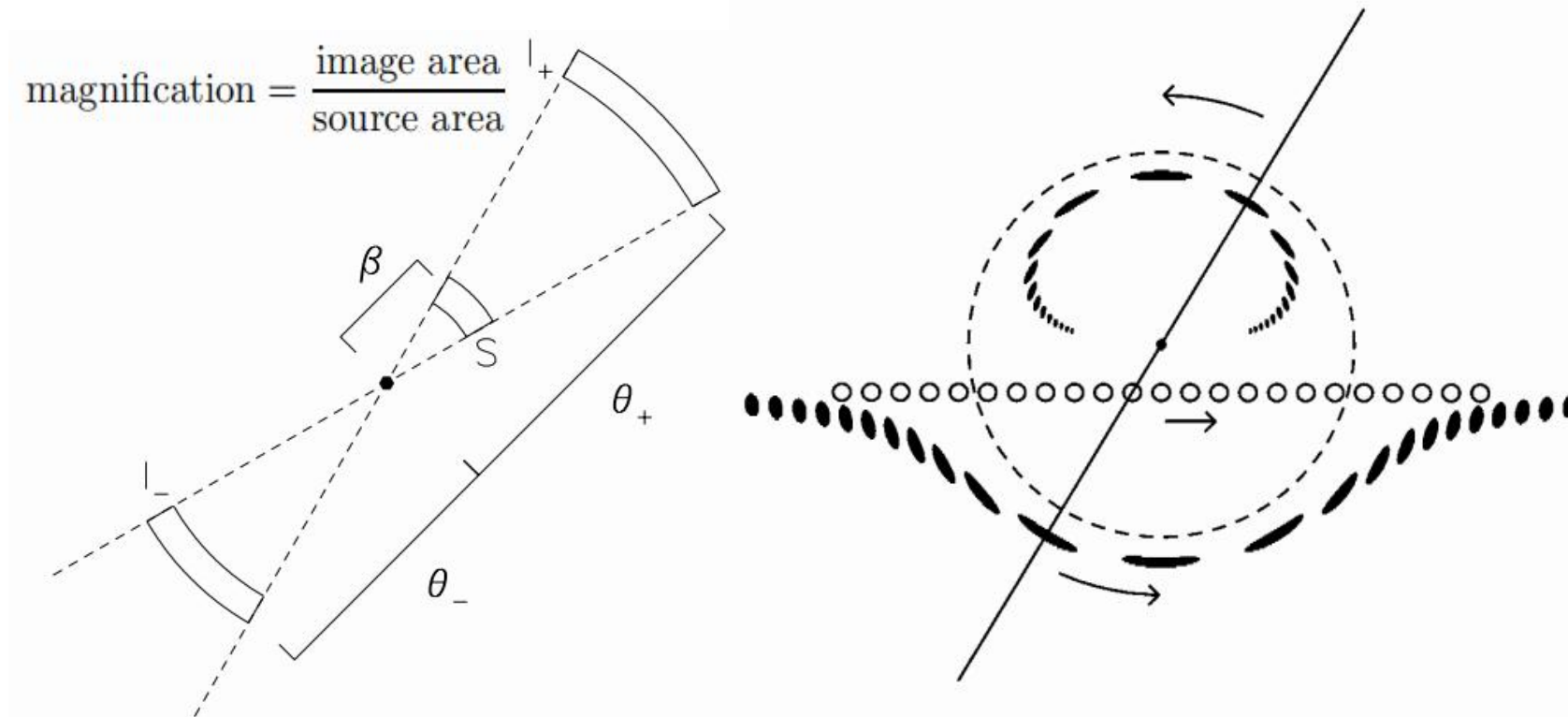
- Hewitt+ 1987, First Einstein Ring discovered in Radio Band & VLA



Hewitt, J., Turner, E., Schneider, D. et al. Unusual radio source MG1131+0456: a possible Einstein ring. Nature 333, 537–540 (1988) doi:10.1038/333537a0

Magnification

- Gravitational lensing preserves surface brightness (I), but changes the apparent solid angle of the source \Rightarrow magnification



Magnification

- Local properties of the lens mapping, described by its Jacobian matrix A

$$A \equiv \frac{\partial \vec{\beta}}{\partial \vec{\theta}} = \left(\delta_{ij} - \frac{\partial \alpha_i(\vec{\theta})}{\partial \theta_j} \right) = \left(\delta_{ij} - \frac{\partial^2 \Psi(\vec{\theta})}{\partial \theta_i \partial \theta_j} \right)$$

$$\mu = \left| \det \left(\frac{\partial \vec{\beta}}{\partial \vec{\theta}} \right) \right|^{-1} \equiv \left| \det \left(\frac{\partial \beta_i}{\partial \theta_j} \right) \right|^{-1}$$

- If circularly symmetric,

$$\mu = \frac{\theta}{\beta} \frac{d\theta}{d\beta}$$

Case 1: Point mass Lens - Magnification

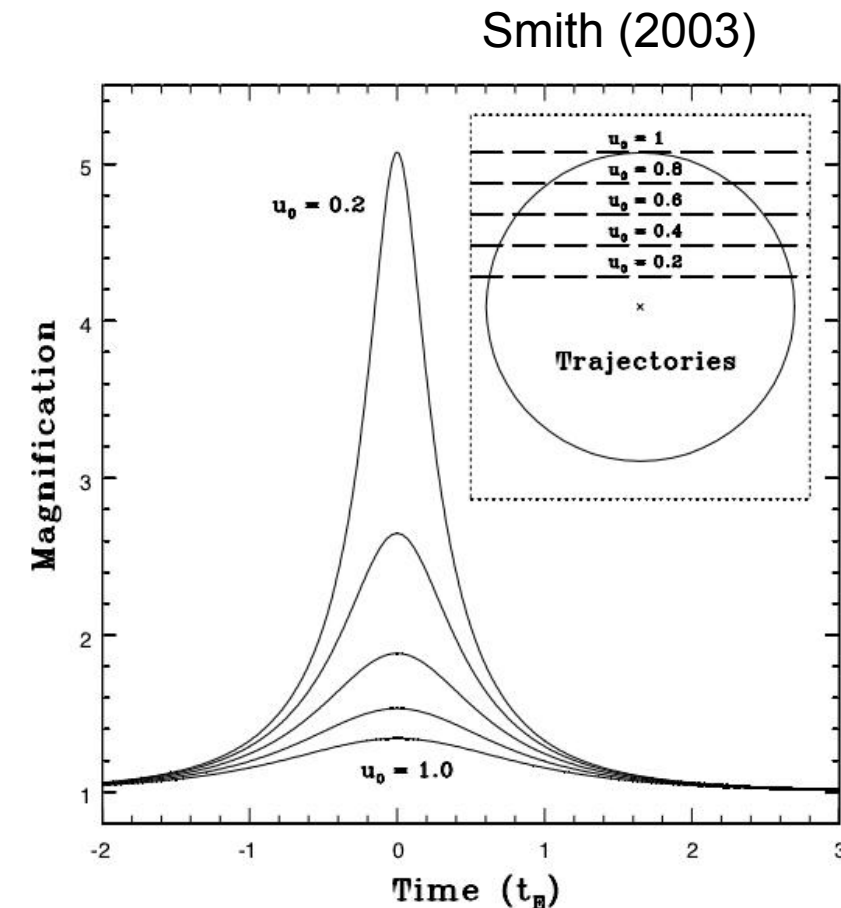
- Images: $\theta_{\pm} = \frac{1}{2}(\beta \pm \sqrt{\beta^2 + 4\theta_E^2})$
- Magnification: $\mu_{\pm} = \left[1 - \left(\frac{\theta_E}{\theta_{\pm}}\right)^4\right]^{-1} = \frac{u^2 + 2}{2u\sqrt{u^2 + 4}} \pm \frac{1}{2}$
 $u = \beta\theta_E^{-1}$
- Total magnification:

$$\mu = |\mu_+| + |\mu_-| = \frac{u^2 + 2}{u\sqrt{u^2 + 4}}$$

- e.g. for source on the Einstein Ring:

$$\beta = \theta_E, u = 1 \quad \mu = 1.17 + 0.17 = 1.34$$

==> magnitude increase 0.319



Shapiro time delay

- Passage through potential also leads to time delay

- without potential: $t_0 = \int \frac{dl}{c}$

- with potential:
$$t_1 = \int_{src}^{obs} \frac{dl}{v} = \int_{src}^{obs} \frac{dl}{c - \frac{2}{c}|\Phi|}$$

$$= \int_{src}^{obs} \frac{dl/c}{1 - \frac{2}{c^2}|\Phi|} = \int_{src}^{obs} \frac{dl}{c} \left[1 + \frac{2}{c^2}|\Phi| \right]$$

- so,
$$\Delta t = \int_{src}^{obs} \frac{2}{c^3} |\Phi| dl \quad ==> \text{Shapiro delay (1964)}$$

- Total time delay is the sum of the extra path length from the deflection and the gravitational time delay

$$t(\vec{\theta}) = \frac{1 + z_d}{c} \frac{D_d D_s}{D_{ds}} \left[\frac{1}{2} (\vec{\theta} - \vec{\beta})^2 - \psi(\vec{\theta}) \right] = t_{geom} + t_{grav}$$

Case 2: Singular Isothermal Sphere (SIS)

- Galaxy lenses, the distributed nature of mass
- Simple model assumes that mass \Rightarrow particles of ideal gas
- ideal gas:

equation of state: $p = \frac{\rho k T}{m}$

In thermal equilibrium, T is related to the 1-d velocity dispersion:

$$m\sigma_v^2 = kT$$

In hydrostatic equilibrium, $\frac{p'}{\rho} = -\frac{GM(r)}{r^2}$, $M'(r) = 4\pi r^2 \rho$

solve the EOS \Rightarrow density profile: $\rho(r) = \frac{\sigma_v^2}{2\pi G} \frac{1}{r^2}$

so mass profile: $M(r) = \frac{2\sigma_v^2 r}{G}$

Case 2: Singular Isothermal Sphere (SIS)

- ideal gas:

rotational velocity in circular orbit: $v_{rot}^2(r) = \frac{GM(r)}{r} = 2\sigma_v^2 = \text{constant}$

Surface mass density: $v_{rot}^2 = GM/r \rightarrow$

$$dM = \frac{v_{rot}^2}{G} dr = 4\pi r^2 \rho(r) \rightarrow$$

$$\rho(r) = \frac{v_{rot}^2}{4\pi G} \frac{dr}{r^2} \rightarrow$$

$$\Sigma(\xi) = \frac{v_{rot}^2}{4\pi G} \int_{-\infty}^{\infty} \frac{dz}{(z + \xi)^2} = \frac{v_{rot}^2}{4G\xi} = \frac{\sigma_v^2}{2G\xi}$$

$$\hat{\alpha}(\xi) = \frac{4GM(\xi)}{c^2\xi}$$

$$M(\xi) = 2\pi \int_0^\xi \Sigma(\xi') \xi' d\xi'$$

\Rightarrow constant deflection angle:

$$\hat{\alpha} = 4\pi \frac{\sigma_v^2}{c^2}$$

Case 2: Singular Isothermal Sphere (SIS)

using $\hat{\alpha} = 4\pi \frac{\sigma_v^2}{c^2}$ and lens equation $\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})$

$$\vec{\alpha}(\vec{\theta}) = \frac{D_{ds}}{D_s} \hat{\alpha} (D_d \vec{\theta})$$

$$\Rightarrow \alpha = \hat{\alpha} \frac{D_{ds}}{D_s} = 4\pi \frac{\sigma_v^2}{c^2} \frac{D_{ds}}{D_s} = \theta_E$$

$$\frac{\vec{\theta}}{|\vec{\theta}|} = \pm 1$$

for strong lensing, get two images as for point mass $\vec{\beta} = \vec{\theta} - \theta_E \frac{\vec{\theta}}{|\vec{\theta}|}$

$$\Rightarrow \theta_{\pm} = \beta \pm \theta_E$$

Magnification can be very large for source aligned with lens:

$$\mu = \frac{\theta}{\beta} \frac{d\theta}{d\beta} \Rightarrow \mu_{\pm} = \frac{\theta_{\pm}}{\beta} = 1 \pm \frac{\theta_E}{\beta} = \left(1 \mp \frac{\theta_E}{\theta_{\pm}}\right)^{-1}$$

Separation of the two images is typically \sim arcsec for galaxy lenses:

$$\theta_E = 1''.6 \left(\frac{\sigma_v}{200 \text{ km s}^{-1}} \right)^2 \left(\frac{D_{ds}}{D_s} \right)$$

Notice: in general the core of a galaxy would not be singular

Caustics and Critical lines

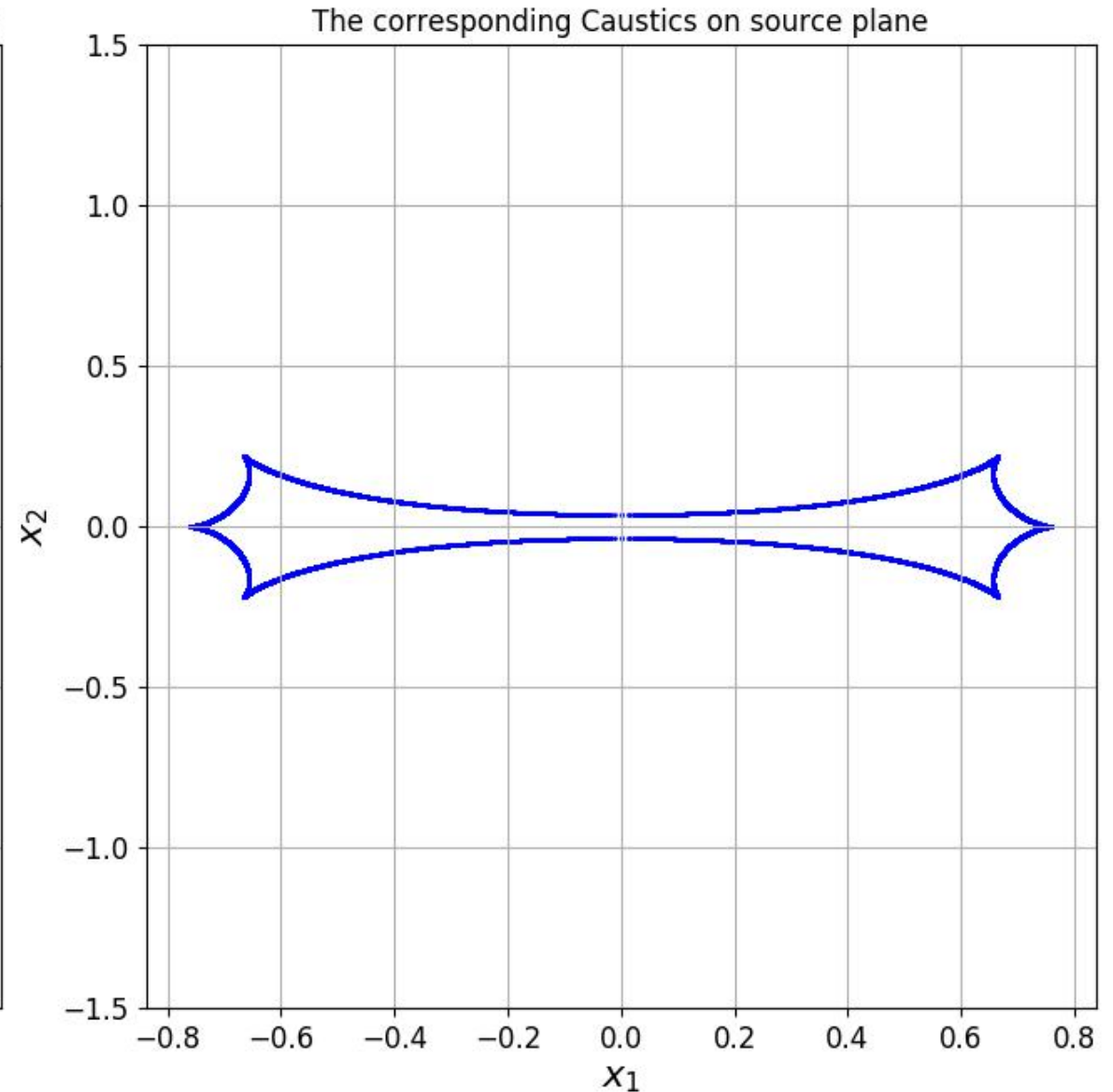
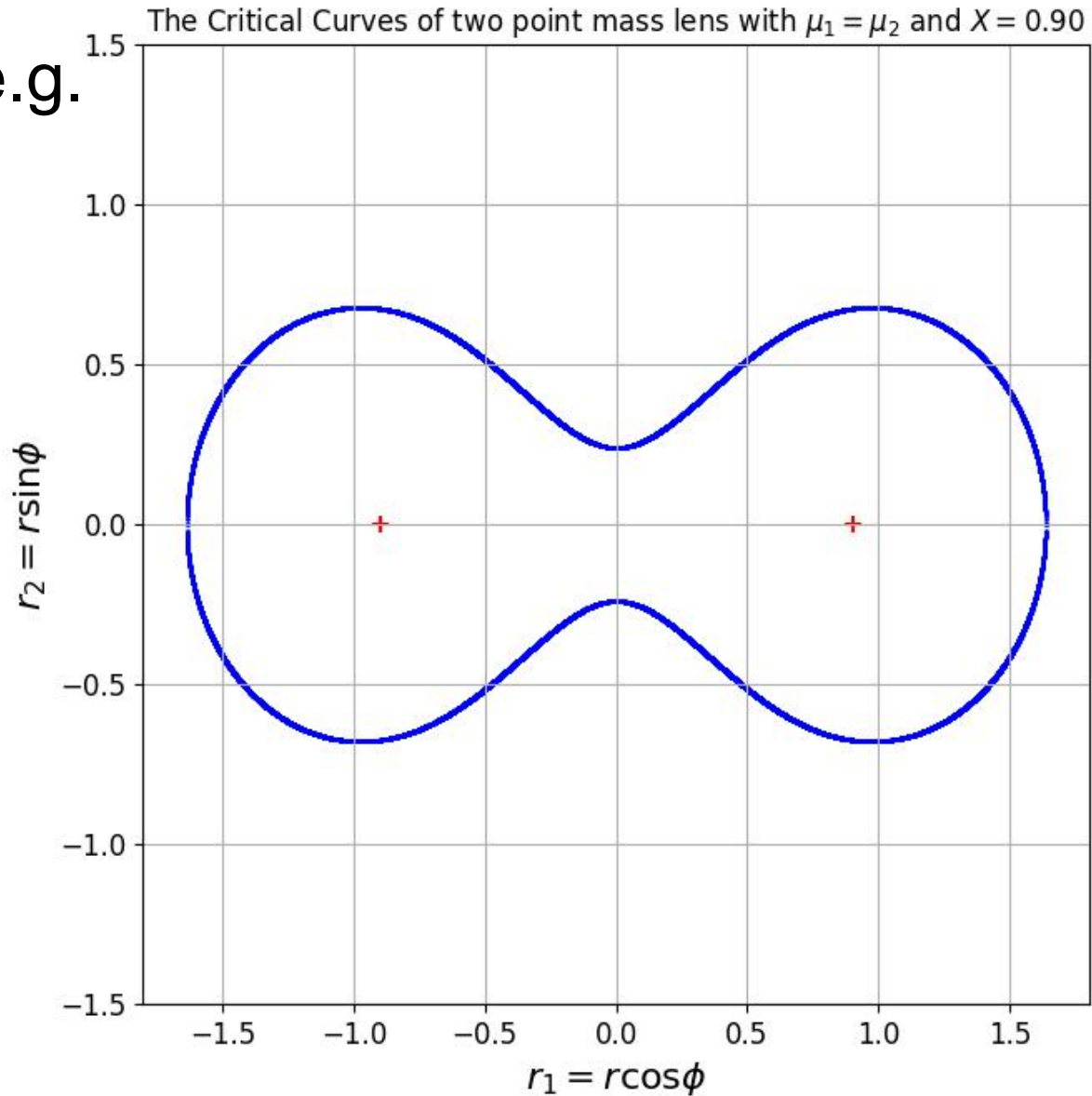
- Grav Lens changes the observed brightness of the source, determined by the Jacobian matrix from the lens equation

$$A \equiv \frac{\partial \vec{\beta}}{\partial \vec{\theta}} = \left(\delta_{ij} - \frac{\partial \alpha_i(\vec{\theta})}{\partial \theta_j} \right) = \left(\delta_{ij} - \frac{\partial^2 \Psi(\vec{\theta})}{\partial \theta_i \partial \theta_j} \right)$$
$$\mu(\theta) = \frac{1}{\det A(\theta)}$$

- Image at θ is magnified by a factor of $|\mu(\theta)|$
- Notice that $\mu(\theta)$ diverge at $\det A(\theta) = 0 \implies$ these points in the image plane form closed curves, which is so called **critical lines**.
- Corresponding curves in the source plane obtained via the lens equation are called **caustics**.

Caustics and Critical lines

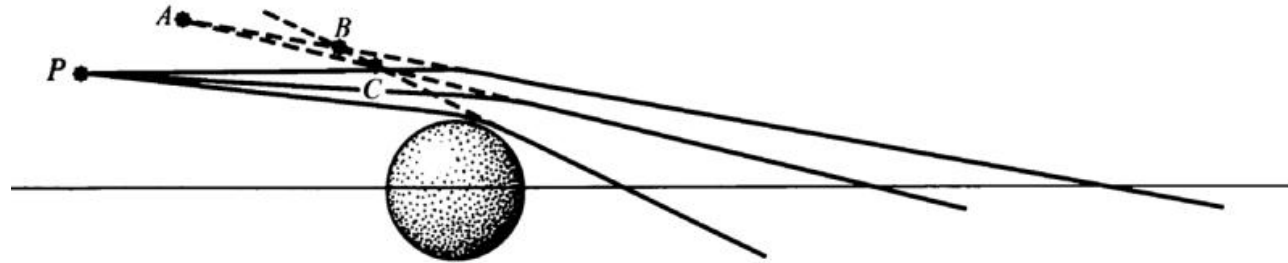
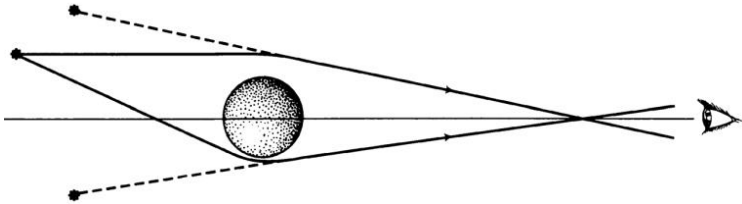
- e.g.



Gravitational Lens vs Genuine Lens

- Grav lens: the observer sees the source at two distinct locations

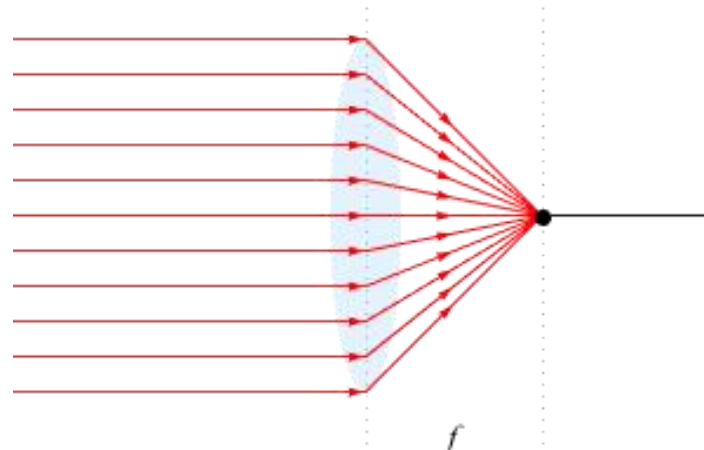
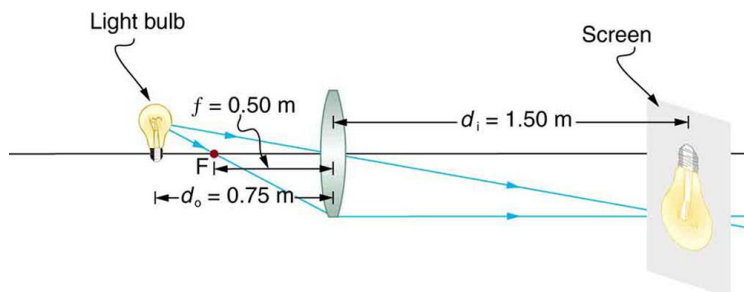
$$\alpha \propto b^{-1}$$



Grav lens has no well-defined focal length and cannot produce genuine images, the “images” are corresponds merely to a direction of incidence of light on the observer, not a genuine image in the sky

- Genuine lens:

$$\alpha \propto b$$



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Question 1

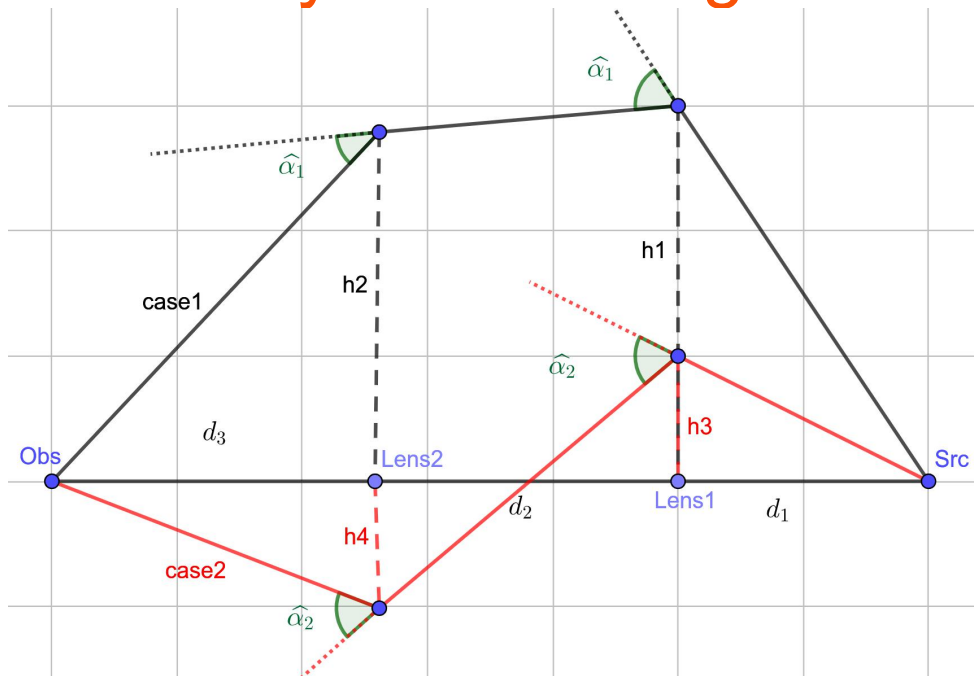
A galaxy at redshift 0.5 can be modelled as a singular isothermal sphere; its dispersion is 200km/s. A background source at redshift 2 is lensed by the foreground galaxy into two images with a brightness ratio of 3:1, what are the angular separation and time delay between the two images? You can assume the usual cosmology.

First assume $H_0 = 69.6 \text{ km s}^{-1}$, $\Omega_m = 0.286$, $\Omega_{vac} = 0.714$

then the distance of $z \sim 0.5$ and $z \sim 2$ can be computed, using theories of SIS model and the time delay function we can estimate the angular separation ~ 1.2 arcsec and time delay ~ 45 days

Question 2

Two (N=2) galaxies are aligned perfectly with the Earth and a distant quasar. Each galaxy can be modelled as a singular isothermal sphere. How many Einstein rings are formed as a result? How will your results generalize when you have $N > 2$ galaxies?



N galaxies $\Rightarrow 2^{N-1}$ rings at most

Next step could consider general situation which obs, lens, src are not aligned perfectly and galaxies are not just point mass model \Rightarrow gravitational lensing simulation

Question 3

A background source is multiply-imaged, can the brightest image arrive last?

Using the relation between effective potential and magnification and the time delay

Question 4

Study the caustics and critical curves of two singular isothermal spheres lensing a background quasar. Study two cases

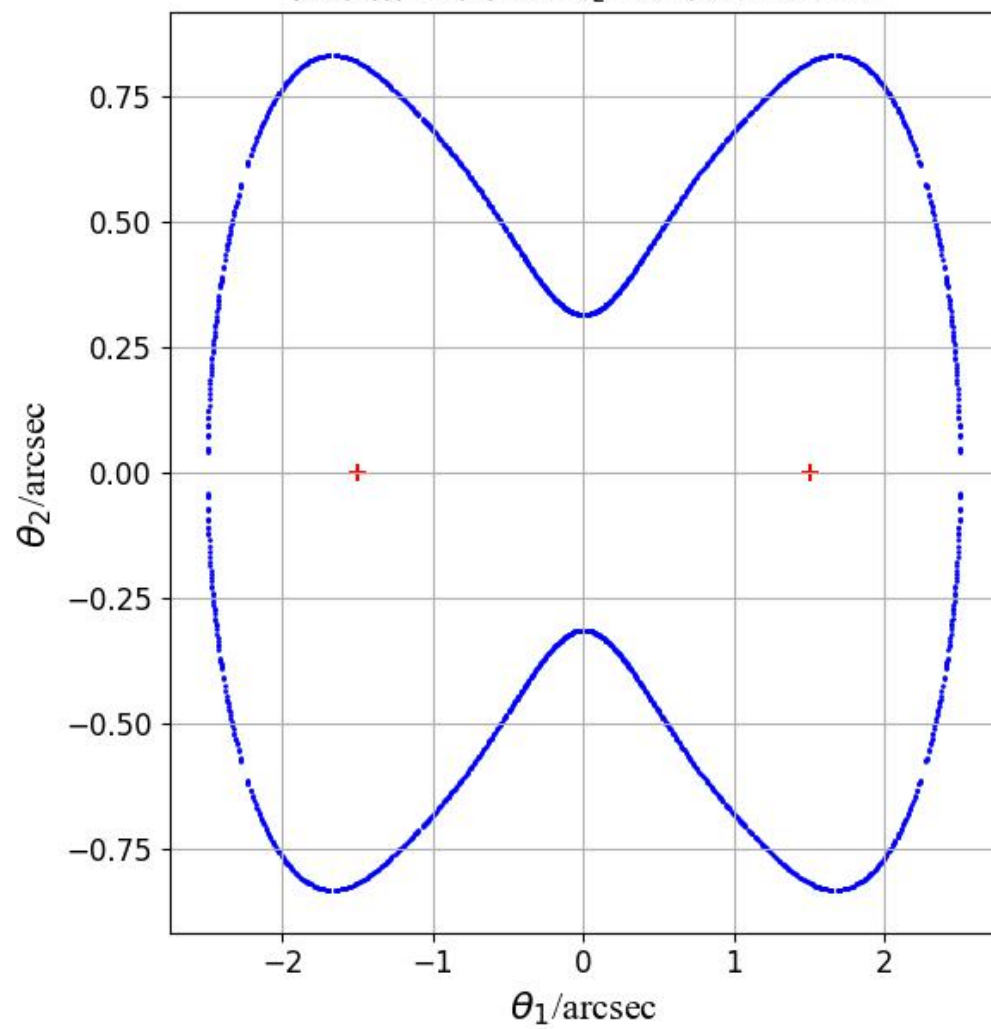
1) Both galaxies are at the same redshift and, 2) these two galaxies are at different redshifts.

Mass density \Rightarrow Jacobian matrix \Rightarrow draw those points which make magnification largest \Rightarrow critical lens \Rightarrow caustics

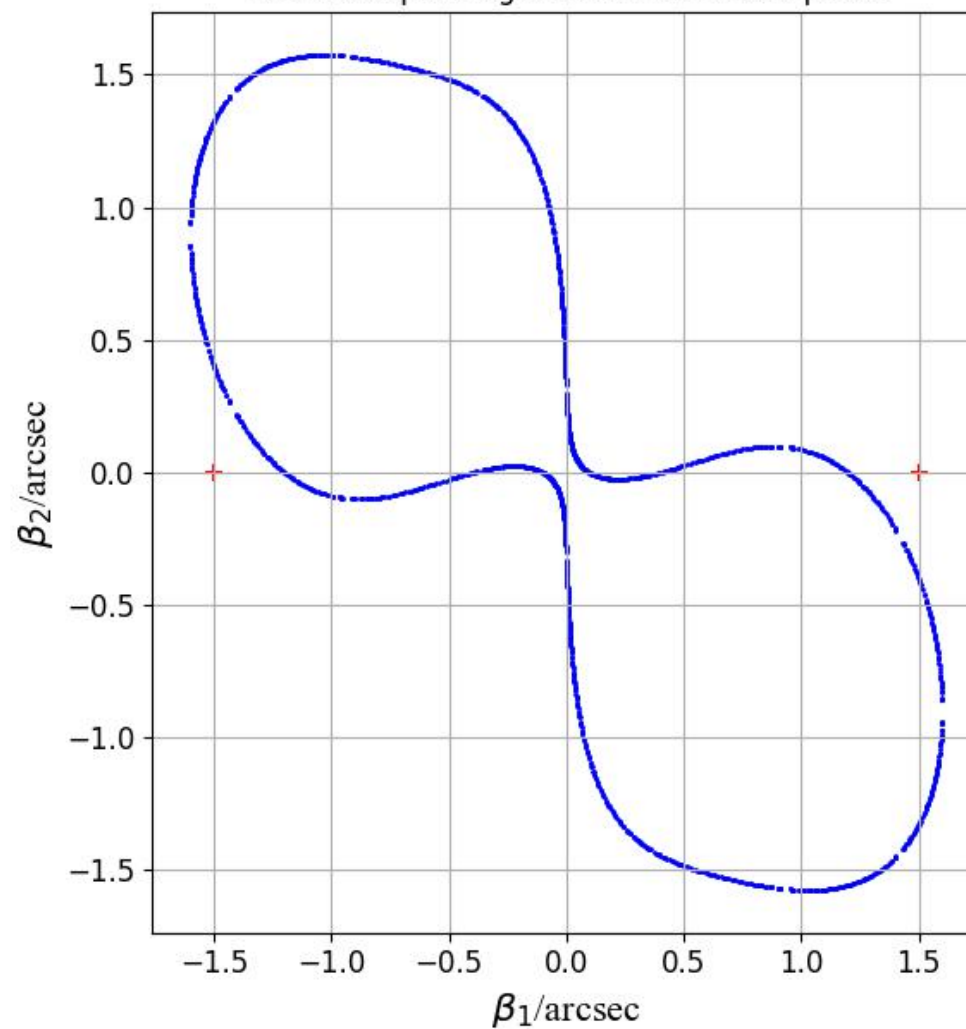
Question 4

case 1) Both galaxies are at the same redshift

The Critical Curves of two SIS lens located at the same redshift
(1.5,0),(-1.5,0) with $\theta_E = 0.8; 0.8$ in arcsec



The corresponding Caustics on source plane



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- **Applications**

Gravitational Lensing Applications

- Cosmic telescopes: distant, faint objects observation
- 2-d mass distribution of lenses, dark matter
- Hubble constant, cosmological constant, density parameter
-

References

- <https://lacosmo.com/DeflectionOfLight/index.html>
- *The Mathematical Theory of Relativity*, Arthur Stanley Eddington (P101)
- <http://web.mit.edu/6.055/old/S2009/notes/bending-of-light.pdf>
- <https://www.mathpages.com/rr/s6-03/6-03.htm>
- *Gravitation and Spacetime*, Hans C. Ohanian, Remo Ruffini. – 3rd ed
- https://en.wikipedia.org/wiki/Gauss%27s_law_for_gravity
- *Lectures On Gravitational Lensing*, Narayan & Bartelmann
- <https://www.cfa.harvard.edu/~dfabricant/huchra/ay202/lectures/lecture12.pdf>
- <https://web.stanford.edu/~oas/SI/SRGR/notes/SchwarzschildSolution.pdf>

Equations in this slide are generated by a helpful tool K_{La}TeXFormular on linux

Appendix-1

- Newtonian prediction
- b : impact factor
- gravitational potential

$$\Phi(b, z) = -\frac{GM}{(b^2 + z^2)^{1/2}}$$

simplify the calculation by integrating
not along the deflected ray but along z axis

$$\begin{aligned}\alpha &= \frac{v_b}{c} = \frac{1}{c} \int \frac{d\Phi}{db} dt = \frac{1}{c^2} \int \frac{d\Phi}{db} dl \approx \frac{1}{c^2} \int \frac{d\Phi}{db} dz \\ &= \frac{GMb}{c^2} \int \frac{dz}{(b^2 + z^2)^{3/2}} \\ &= \frac{GMb}{c^2} \left[\frac{z}{b^2 \sqrt{b^2 + z^2}} \right]_{-\infty}^{+\infty} \\ &= \frac{2GM}{c^2 b}\end{aligned}$$

