

# Gravitational Lensing Theories, Questions and Applications

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# Overview

GravLens

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History

Gravitational  
Lensing Theories

Related Questions

Applications

History

Gravitational Lensing Theories

Related Questions

Applications

# Predictions from Newtonian and GR

- ▶ First proposed by Soldner (1801) using Newtonian theory, given a deflection angle (Appendix-1 (Page 71) for derivation).

$$\alpha = \frac{2GM}{c^2 r}, 0.85 \text{ arcsec for the Sun}$$

- ▶ Einstein (1911) derived the same result using Equivalence principle and Euclidean metric
- ▶ Einstein (1915) derived the new result using General Relativity.

$$\alpha = \frac{4GM}{c^2 r}, 1.7 \text{ arcsec for the Sun}$$

# Eddington's observation of the Solar Eclipse

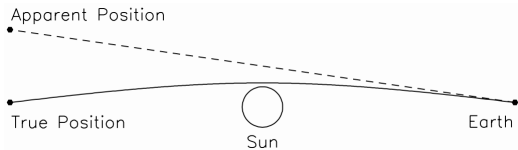
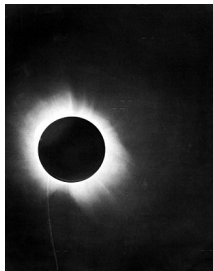
## History

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- ▶ In 1919, Eddington measured a value close to GR's prediction using the data collected during an eclipse, stars with apparent position near the sun become visible.

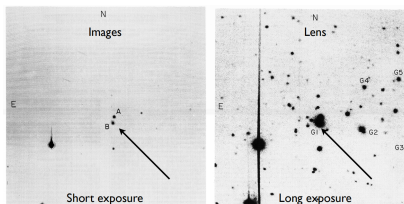


# The first example for GravLens: 0957+561

- ▶ Eddington (1920): Multiple light paths  $\rightarrow$  multi images
- ▶ Walsh et al., (1979) quasar QSO 957+561 A,B found at  $z \sim 1.4$ , two seen images separated by  $6''$

- ▶ Lens: evidence

1. Lensing galaxy at  $z \sim 0.36$
2. Similar spectra
3. Ratio of optical and radio flux
4. VLBI imaging: small scale features



- ▶ Einstein Field Equations:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

- ▶ Geodesic equations:

$$\frac{d^2x^\beta}{d\lambda^2} + \Gamma_{\mu\nu}^\beta \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0$$

We can calculate how gravity bends light by solving geodesic equation.

- ▶ To compute the Christoffel symbols  $\Gamma_{\mu\nu}^\beta$ , requires solving for the metric tensor  $g_{\mu\nu}$ , which requires solving the curvature equations  $R_{\mu\nu} = 0$ ,  
← ten nonlinear partial differential equations.

# General Relativity and light deflection

Or, the velocity of the photon from the Schwarzschild metric,

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)(dx^2 + dy^2 + dz^2),$$

- ▶ and Poisson Equation,  $\nabla^2\Phi = 4\pi G\rho$ ,
- ▶ and light interval:  $g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$
- ▶ which gives,

$$v = \frac{\sqrt{dx^2 + dy^2 + dz^2}}{dt} = \sqrt{\frac{1 + 2\Phi}{1 - 2\Phi}} \approx 1 + 2\Phi$$

The gravitational field decreases the speed of propagation

# General Relativity and light deflection

$$v = \frac{\sqrt{dx^2+dy^2+dz^2}}{dt} = \sqrt{\frac{1+2\Phi}{1-2\Phi}} \approx 1 + 2\Phi \text{ (natural units)} \rightarrow$$
$$v = c \left(1 + \frac{2}{c^2}\Phi\right) \text{ (SI)}$$

- ▶ define refraction index:  $n = 1 - \frac{2}{c^2}\Phi = 1 + \frac{2}{c^2}|\Phi| \geq 1$
- ▶ deflection angle:  $\vec{\hat{\alpha}} = - \int \vec{\nabla}_{\perp} n dl = \frac{2}{c^2} \int \vec{\nabla}_{\perp} \Phi dl$
- ▶ which is twice of the Newtonian prediction,
- ▶ for point mass lens,  $\hat{\alpha} = \frac{4GM}{bc^2}$



# Thin screen approximation

- ▶ Most deflection occurs near to the lens ( $|z| \sim b$ )  
→ treat all deflection as in the lens plane

- ▶ Projected surface density:

$$\Sigma(\vec{\xi}) = \int \rho(\vec{\xi}, z) dz$$

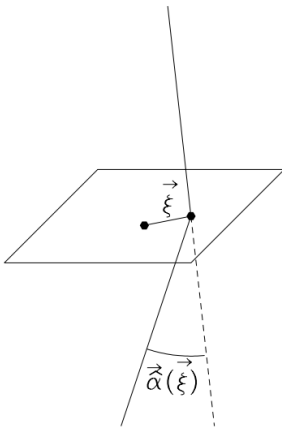
- ▶ Deflection angle:

$$\vec{\alpha}(\vec{\xi}) = \frac{4G}{c^2} \int \frac{(\vec{\xi} - \vec{\xi}') \Sigma(\vec{\xi}')}{|\vec{\xi} - \vec{\xi}'|^2} d^2 \xi'$$

- ▶ In circular symmetry cases:

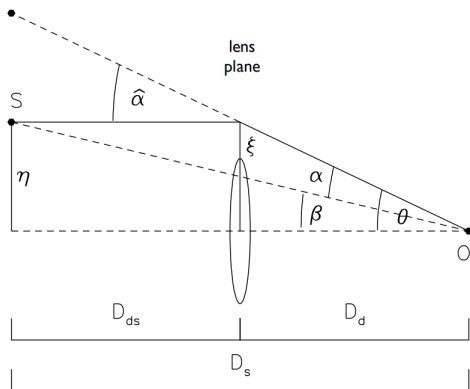
$$\hat{\alpha}(\xi) = \frac{4GM(\xi)}{c^2 \xi}$$

$$M(\xi) = 2\pi \int_0^\xi \Sigma(\xi') \xi' d\xi'$$



# The Lens Equation

Connecting the position of images in the Lens plane and corresponding sources the Source plane.



$$\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})$$

$$\vec{\alpha}(\vec{\theta}) = \frac{D_{ds}}{D_s} \hat{\alpha}(D_d \vec{\theta})$$

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## Case 1: Point mass Lens

$$\begin{cases} \hat{\alpha}(\xi) = \frac{4GM(\xi)}{c^2\xi} \\ \vec{\alpha}(\vec{\theta}) = \frac{D_{ds}}{D_s}\hat{\alpha}(D_d\vec{\theta}) \end{cases} \rightarrow \vec{\alpha}(\vec{\theta}) = \frac{4GM}{c^2} \frac{D_{ds}}{D_d D_s} \frac{\vec{\theta}}{|\vec{\theta}|^2}$$

and  $\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})$ ,

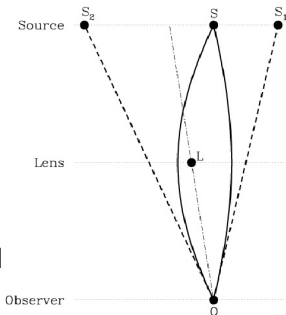
if  $\beta = 0$ , gives the Einstein Radius,

$$\theta_E = \sqrt{\frac{4GM}{c^2} \frac{D_{ds}}{D_d D_s}}$$

The Lens equation:  $\vec{\beta} = \vec{\theta} - \theta_E^2 \frac{\vec{\theta}}{|\vec{\theta}|^2}$

- ▶ if  $\beta > \theta_E \rightarrow$ , weakly lensed and weakly distorted image,
- ▶ if  $\beta < \theta_E \rightarrow$ , strongly lensed and multi images:

$$\theta_{\pm} = \frac{1}{2} \left( \beta \pm \sqrt{\beta^2 + 4\theta_E^2} \right)$$



# Case 1: Point mass Lens

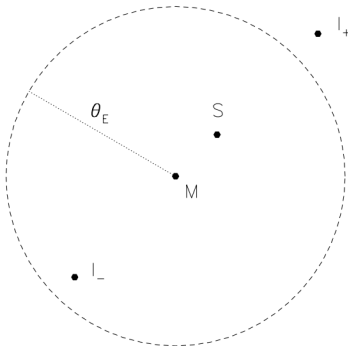
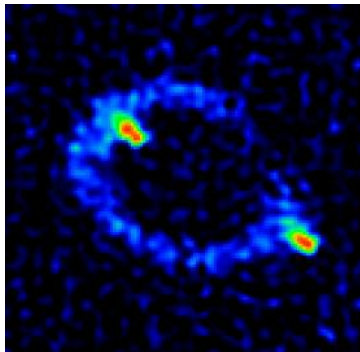
Hewitt+ 1987, First Einstein Ring discovered in Radio Band & VLA

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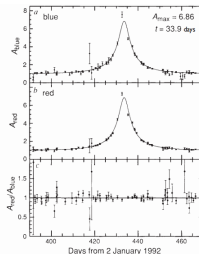
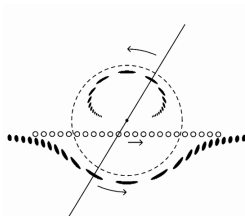
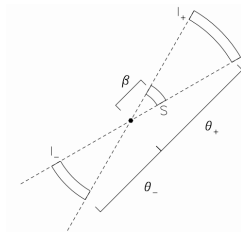


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Hewitt, J., Turner, E., Schneider, D. et al. Unusual radio source MG1131+0456: a possible Einstein ring. Nature 333, 537–540 (1988) doi:10.1038/333537a0

# Magnification

Gravitational lensing preserves surface brightness (**Liouville's Theorem**), but changes the apparent solid angle of the source  $\rightarrow$  *magnification* =  $\frac{Area_{image}}{Area_{source}}$



Local properties of the lens mapping, described by its Jacobian matrix  $A$

$$A \equiv \frac{\partial \vec{\beta}}{\partial \vec{\theta}} = \left( \delta_{ij} - \frac{\partial \alpha_i(\vec{\theta})}{\partial \theta_j} \right) = \left( \delta_{ij} - \frac{\partial^2 \Psi(\vec{\theta})}{\partial \theta_i \partial \theta_j} \right)$$

$$\mu = \left| \det \left( \frac{\partial \vec{\beta}}{\partial \vec{\theta}} \right) \right|^{-1} \equiv \left| \det \left( \frac{\partial \beta_i}{\partial \theta_j} \right) \right|^{-1}$$

If circularly symmetric,

$$\mu = \frac{\theta}{\beta} \frac{d\theta}{d\beta}$$

# Case 1: Point mass Lens - Magnification

► Images:

$$\theta_{\pm} = \frac{1}{2} \left( \beta \pm \sqrt{\beta^2 + 4\theta_E^2} \right)$$

► Define  $u = \beta\theta_E^{-1}$ ,

Magnification:

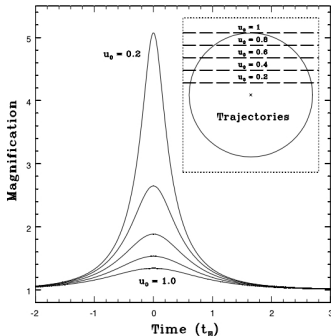
$$\begin{aligned} \mu_{\pm} &= \left[ 1 - \left( \frac{\theta_E}{\theta_{\pm}} \right)^4 \right]^{-1} \\ &= \frac{u^2 + 2}{2u\sqrt{u^2 + 4}} \pm \frac{1}{2} \end{aligned}$$

► Total magnification:

$$\mu = |\mu_+| + |\mu_-| = \frac{u^2 + 2}{u\sqrt{u^2 + 4}}$$

e.g. for source on the Einstein Ring:

$\beta = \theta_E, u = 1 \rightarrow \mu = 1.34 \rightarrow$  magnitude increase 0.32.



# Shapiro time delay

- ▶ Passage through potential also leads to time delay

- ▶ without potential:  $t_0 = \int \frac{dl}{c}$

- ▶ with potential:

$$\begin{aligned} t_1 &= \int_{src}^{obs} \frac{dl}{v} = \int_{src}^{obs} \frac{dl}{c - \frac{2}{c}|\Phi|} \\ &= \int_{src}^{obs} \frac{dl/c}{1 - \frac{2}{c^2}|\Phi|} = \int_{src}^{obs} \frac{dl}{c} \left[ 1 + \frac{2}{c^2}|\Phi| \right] \end{aligned}$$

- ▶ so,  $\Delta t = \int_{src}^{obs} \frac{2}{c^3}|\Phi|dl \rightarrow$  Shapiro delay (1964)
- ▶ Total time delay is the sum of the extra path length from the deflection and the gravitational time delay

$$t(\vec{\theta}) = \frac{1+z_d}{c} \frac{D_d D_s}{D_{ds}} \left[ \frac{1}{2}(\vec{\theta} - \vec{\beta})^2 - \psi(\vec{\theta}) \right] = t_{geom} + t_{grav}$$



## Case 2: Singular Isothermal Sphere (SIS)

- ▶ Galaxy lenses, the distributed nature of mass
- ▶ Simple model assumes that mass  $\rightarrow$  particles of ideal gas
- ▶ ideal gas:
  - ▶ equation of state:  $p = \frac{\rho k T}{m}$
  - ▶ In thermal equilibrium,  $T$  is related to the 1-d velocity dispersion:  $m \sigma_v^2 = k T$
  - ▶ In hydrostatic equilibrium,
 
$$\frac{p'}{\rho} = -\frac{GM(r)}{r^2}, M'(r) = 4\pi r^2 \rho$$
  - ▶ solve the EOS  $\rightarrow$  density profile:  $\rho(r) = \frac{\sigma_v^2}{2\pi G} \frac{1}{r^2}$
  - ▶ so mass profile:  $M(r) = \frac{2\sigma_v^2}{G} r$

## Case 2: Singular Isothermal Sphere (SIS)

For ideal gas, rotational velocity in circular orbit:

$$\frac{v'}{\rho} = -\frac{GM(r)}{r^2}, M'(r) = 4\pi r^2 \rho$$

Surface mass density:

$$v_{rot}^2 = GM/r \rightarrow$$

$$dM = \frac{v_{rot}^2}{G} dr = 4\pi r^2 \rho(r) \rightarrow$$

$$\rho(r) = \frac{v_{rot}^2}{4\pi G} \frac{dr}{r^2} \rightarrow$$

$$\Sigma(\xi) = \frac{v_{rot}^2}{4\pi G} \int_{-\infty}^{\infty} \frac{dz}{(z + \xi)^2} = \frac{v_{rot}^2}{4G\xi} = \frac{\sigma_v^2}{2G\xi}$$

where,  $M(\xi) = 2\pi \int_0^\xi \Sigma(\xi') \xi' d\xi'$ , and  $\hat{\alpha}(\xi) = \frac{4GM(\xi)}{c^2 \xi}$

which gives:  $\hat{\alpha} = 4\pi \frac{\sigma_v^2}{c^2}$

## Case 2: Singular Isothermal Sphere (SIS)

using  $\begin{cases} \hat{\alpha} = 4\pi\sigma_v^2/c^2 \\ \vec{\alpha}(\vec{\theta}) = \hat{\alpha}(D_d\vec{\theta})D_{ds}/D_s \end{cases}$

and lens equation  $\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})$

we can get:

$$\alpha = \hat{\alpha} \frac{D_{ds}}{D_s} = 4\pi \frac{\sigma_v^2}{c^2} \frac{D_{ds}}{D_s} = \theta_E$$

for strong lensing, get two images as for point mass:

$$\vec{\beta} = \vec{\theta} - \theta_E \frac{\vec{\theta}}{|\vec{\theta}|}, \frac{\vec{\theta}}{|\vec{\theta}|} = \pm 1 \rightarrow$$

$$\theta_{\pm} = \beta \pm \theta_E$$

Magnification can be very large for source aligned with lens,

from  $\mu = \frac{\theta}{\beta} \frac{d\theta}{d\beta} \rightarrow$ :

$$\mu_{\pm} = \frac{\theta_{\pm}}{\beta} = 1 \pm \frac{\theta_E}{\beta} = \left(1 \mp \frac{\theta_E}{\theta_{\pm}}\right)^{-1}$$

Separation of the two images is typically  $\sim$  arcsec for galaxy lenses:

$$\theta_E = 1''.6 \left( \frac{\sigma_v}{200 \text{ km s}^{-1}} \right)^2 \left( \frac{D_{ds}}{D_s} \right)$$

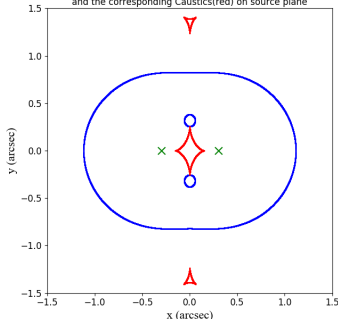
# Caustics and Critical lines

- ▶ Grav Lens changes the observed brightness of the source, determined by the Jacobian matrix from the lens equation:  $A \equiv \frac{\partial \vec{\beta}}{\partial \vec{\theta}} = \left( \delta_{ij} - \frac{\partial \alpha_i(\vec{\theta})}{\partial \theta_j} \right) = \left( \delta_{ij} - \frac{\partial^2 \Psi(\vec{\theta})}{\partial \theta_i \partial \theta_j} \right)$   
 $\mu(\theta) = \frac{1}{\det A(\theta)}$
- ▶ Image at  $\vec{\theta}$  is magnified by a factor of  $|\mu(\vec{\theta})|$
- ▶ Notice that  $|\mu(\vec{\theta})|$  diverge at  $\det A(\theta) = 0 \rightarrow$  these points in the image plane form closed curves, which is so called **critical lines**.
- ▶ Corresponding curves in the source plane obtained via the lens equation are called **caustics**.

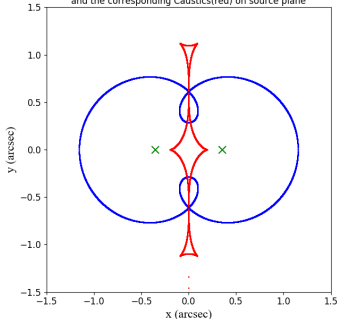
# Caustics and Critical lines - Point sources

Separation  $0.6''$ ,  $0.7''$ :

The Critical Curves(blue) of two point mass(green) lens with  $\mu_1 = \mu_2$  and  $X = 0.30$  and the corresponding Caustics(red) on source plane



The Critical Curves(blue) of two point mass(green) lens with  $\mu_1 = \mu_2$  and  $X = 0.35$  and the corresponding Caustics(red) on source plane

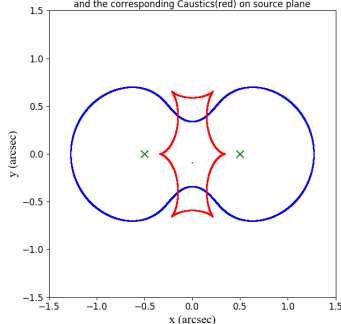


[https://github.com/rkkuang/aeroastro/blob/master/gravlen/critical\\_and\\_caustics/](https://github.com/rkkuang/aeroastro/blob/master/gravlen/critical_and_caustics/)  
The two-point-mass lens - Detailed investigation of a special asymmetric gravitational lens  
<http://adsabs.harvard.edu/abs/1986A%26A...164..237S>

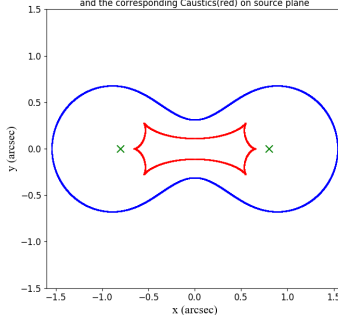
# Caustics and Critical lines - Point sources

Separation  $1.0''$ ,  $1.6''$ :

The Critical Curves(blue) of two point mass(green) lens with  $\mu_1 = \mu_2$  and  $X = 0.50$  and the corresponding Caustics(red) on source plane

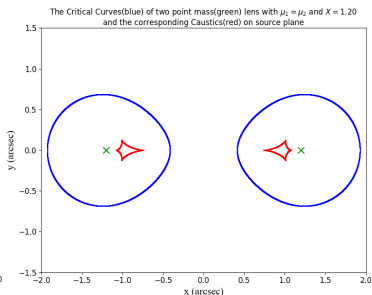
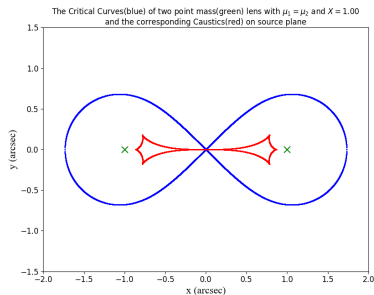


The Critical Curves(blue) of two point mass(green) lens with  $\mu_1 = \mu_2$  and  $X = 0.80$  and the corresponding Caustics(red) on source plane



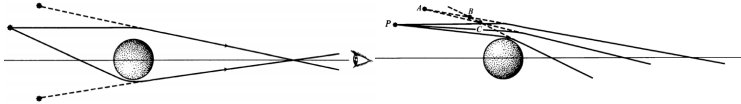
# Caustics and Critical lines - Point sources

Separation  $2.0''$ ,  $2.4''$ :



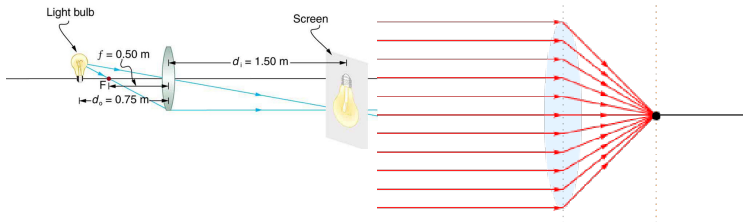
# Gravitational Lens vs Genuine Lens

- Grav lens: the observer sees the source at two distinct locations,  $\alpha \propto b^{-1}$



Grav lens has no well-defined focal length and cannot produce genuine images, the “images” are corresponds merely to a direction of incidence of light on the observer, not a genuine image in the sky

- Genuine lens,  $\alpha \propto b$





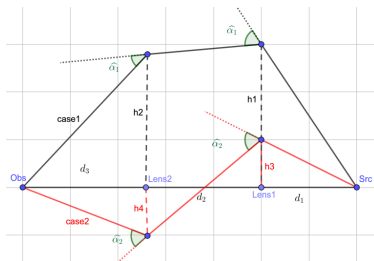
## Question 1

A galaxy at redshift 0.5 can be modelled as a singular isothermal sphere; its dispersion is 200km/s. A background source at redshift 2 is lensed by the foreground galaxy into two images with a brightness ratio of 3:1, what are the angular separation and time delay between the two images? You can assume the usual cosmology.

First assume  $H_0 = 69.6 \text{ km s}^{-1}$ ,  $\Omega_m = 0.286$ ,  $\Omega_{vac} = 0.714$  then the distance of  $z = 0.5$  and  $z = 2$  can be computed, using theories of SIS model and the time delay function we can estimate the angular separation  $\sim 1.2$  arcsec and time delay  $\sim 45$  days

## Question 2

Two ( $N=2$ ) galaxies are aligned perfectly with the Earth and a distant quasar. Each galaxy can be modelled as a singular isothermal sphere. How many Einstein rings are formed as a result? How will your results generalize when you have  $N > 2$  galaxies?



For  $N$  galaxies, the quasar will render  $2^{N-1}$  Einstein rings, and if we consider galaxy lensed by galaxy, will generate (at most)  $\sum_{i=1}^{N-1} 2^{i-1}$  more, so the total number of Einstein rings rendered by  $N$  galaxies and a quasar will be:  $\sum_{i=1}^N 2^{i-1} = 2^N - 1$ , at most.

## Question 3

A background source is multiply-imaged, can the brightest image arrive last?

Using the relation between effective lensing potential and magnification & time delay

Effective lensing potential:

$$\Psi(\vec{\theta}) = \frac{D_{ds}}{D_d D_s} \frac{2}{c^2} \int \Phi(D_d \vec{\theta}, z) dz, \quad \vec{\nabla}_{\vec{\theta}} \Psi = \vec{\alpha}$$

$$\text{Jacobian matrix: } A = \frac{\partial \vec{\beta}}{\partial \vec{\theta}} = \delta_{ij} - \frac{\partial^2 \Psi(\vec{\theta})}{\partial \theta_i \partial \theta_j} \equiv \delta_{ij} - \Psi_{ij}$$

$$\text{Magnification: } \mu = |\det A|^{-1}$$

$$\text{time delay: } t(\vec{\theta}) = \frac{1+z_d}{c} \frac{D_d D_s}{D_{ds}} \left[ \frac{1}{2} (\vec{\theta} - \vec{\beta})^2 - \Psi(\vec{\theta}) \right]$$

$$\mu \leftarrow \Psi \rightarrow t$$

Simulation, under a potential, given a  $\vec{\beta}$ , solve the lens equation for  $\vec{\theta}_i$  see if  $\vec{\theta}_k$  which has maximum magnification also have largest time delay

## Question 3

A background source is multiply-imaged, can the brightest image arrive last?

► N point mass, same redshift (single plane).

Effective lensing potential:

$$\psi(\vec{\theta}) = \frac{D_{ds}}{D_s} \frac{4G}{D_d c^2} \sum_{i=1}^N M_i \ln |\vec{\theta} - \vec{p}_i|$$

The deflection angle:

$$\alpha(\vec{\theta}) = \frac{D_{ds}}{D_s} \frac{4G}{D_d c^2} \sum_{i=1}^N \frac{M_i}{|\vec{\theta} - \vec{p}_i|}$$

where  $M_i, \vec{p}_i$  are the mass and position of the  $i_{th}$  point mass.

Above equations can be rewritten with the Einstein Radius

$\theta_{Ei}$  of the  $i_{th}$  point mass as:

$$\begin{aligned} \psi(\vec{\theta}) &= \sum_{i=1}^N \theta_{Ei}^2 \ln |\vec{\theta} - \vec{p}_i| \\ \alpha(\vec{\theta}) &= \sum_{i=1}^N \frac{\theta_{Ei}^2 (\vec{\theta} - \vec{p}_i)}{|\vec{\theta} - \vec{p}_i|^2} \end{aligned}$$

The lens equation:

$$\vec{\beta} = \vec{\theta} - \alpha(\vec{\theta})$$

## Question 3

A background source is multiply-imaged, can the brightest image arrive last?

► N point mass, same redshift (single plane).

Components of the Jacobian matrix A:

$$A_{11} = 1 - \sum_{i=1}^N \theta_{Ei}^2 [r^{-2} - 2(\theta_x - p_{ix})^2 r^{-4}]$$

$$A_{12} = A_{21} = 2 \sum_{i=1}^N \theta_{Ei}^2 [(\theta_x - p_{ix})(\theta_y - p_{iy}) r^{-4}]$$

$$A_{22} = 1 - \sum_{i=1}^N \theta_{Ei}^2 [r^{-2} - 2(\theta_y - p_{iy})^2 r^{-4}]$$

where  $r = |\vec{\theta} - \vec{p}_i|$

Thus the magnification can be written as:

$$\mu = (A_{11}A_{22} - A_{12}A_{21})^{-1} \quad \text{Time delay:}$$

$$t(\vec{\theta}) \propto \frac{1}{2}(\vec{\theta} - \vec{\beta})^2 - \psi(\vec{\theta})$$

So, after solve the lens equation:  $\vec{\beta} = \vec{\theta} - \sum_{i=1}^N \frac{\theta_{Ei}^2(\vec{\theta} - \vec{p}_i)}{|\vec{\theta} - \vec{p}_i|^2}$ ,

we can compare the resultant images' magnification and time delay, see if the brightest image arrive last.

## Question 3

A background source is multiply-imaged, can the brightest image arrive last?

► If  $N = 1$ ,

for  $\beta > 0$ , there will be two images,

$\theta_{\pm} = \frac{1}{2} \left( \beta \pm \sqrt{\beta^2 + 4\theta_E^2} \right)$ , we can obtain the time delay

difference  $t_- - t_+ = \frac{1}{2}\beta\sqrt{\beta^2 + 4\theta_E^2} + \ln \frac{(\sqrt{\beta^2 + 4\theta_E^2} + \beta)^2}{4\theta_E^2} > 0$

always holds, thus the brightest image always arrive first.

## Question 3

A background source is multiply-imaged, can the brightest image arrive last?

- If  $N > 1 \rightarrow$  test by coding:

def  $a_i = \theta_{Ei}^2$ , the system of lens equations:

$$\begin{cases} \beta_x = \theta_x - \sum_{i=1}^N a_i(\theta_x - p_{ix})/r^2 \\ \beta_y = \theta_y - \sum_{i=1}^N a_i(\theta_y - p_{iy})/r^2 \\ r^2 = (\theta_x - p_{ix})^2 + (\theta_y - p_{iy})^2 \end{cases}$$

- Outer loop: For a lens system of  $N$  point mass with fixed  $\theta_{Ei}, (p_{ix}, p_{iy})$
- Inner loop: Given a source position  $(\beta_x, \beta_y)$ , solve above system of equations for  $M$  images:  $(\theta_{xj}, \theta_{yj})$  and compute corresponding  $\mu_j, t_j$ , check if there exists a  $k$  which satisfies  $\mu_k = \max(\mu)$  &  $t_k = \max(t)$

## Question 3

A background source is multiply-imaged, can the brightest image arrive last?

► If  $N > 1 \rightarrow$  test by coding:

Yes, the brightest image can arrive last when merely  $N = 2$ .



## Question 4

Study the caustics and critical curves of two singular isothermal spheres lensing a background quasar. Study two cases 1) Both galaxies are at the same redshift and, 2) these two galaxies are at different redshifts.

Mass density  $\rightarrow$  Surface mass density  $\rightarrow$  Effective lensing potential  $\rightarrow$  Jacobian matrix  $\rightarrow$  Magnification draw those points which make magnification largest  $\rightarrow$  critical lens  $\rightarrow$  caustics

## Question 4

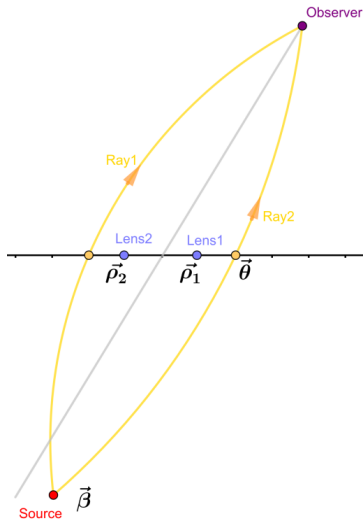
case 1) Both galaxies are at the same redshift

$$\vec{\alpha}(\vec{\theta}) = \theta_{E1} \frac{\vec{\theta} - \vec{\rho}_1}{|\vec{\theta} - \vec{\rho}_1|} + \theta_{E2} \frac{\vec{\theta} - \vec{\rho}_2}{|\vec{\theta} - \vec{\rho}_2|}$$

where  $\vec{\rho}_1, \vec{\rho}_2$  are positions of Lens1, Lens2 on the lens plane.  $\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})$

$$A = \frac{\partial \vec{\beta}}{\partial \vec{\theta}} = \left( \delta_{ij} - \frac{\partial \alpha_i(\vec{\theta})}{\partial \theta_j} \right)$$

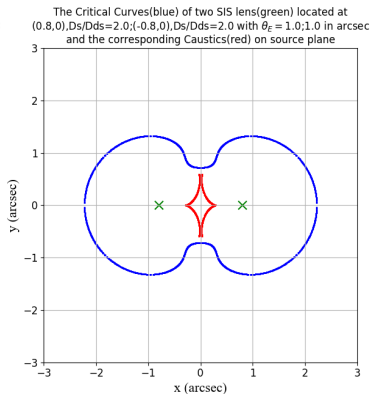
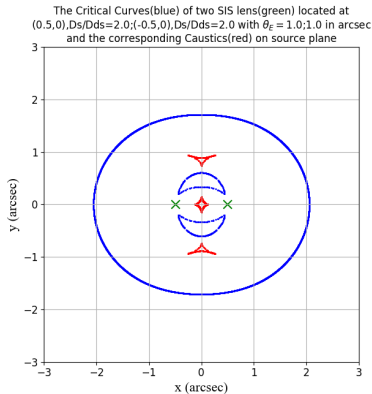
$$\mu(\vec{\theta}) = \frac{1}{\det A(\vec{\theta})}$$



## Question 4

case 1) Both galaxies are at the same redshift

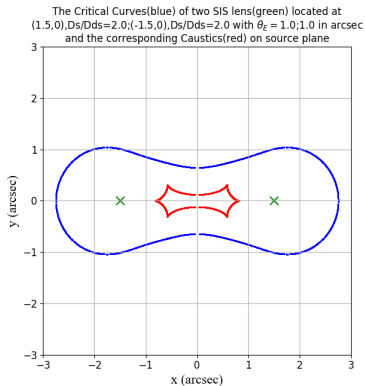
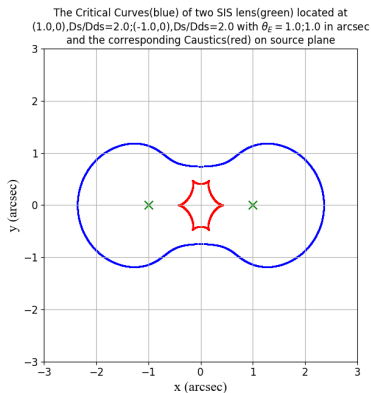
Two SIS lenses ( $\theta_E = 1.0''$ ), separation  $1.0'', 1.6''$



## Question 4

case 1) Both galaxies are at the same redshift

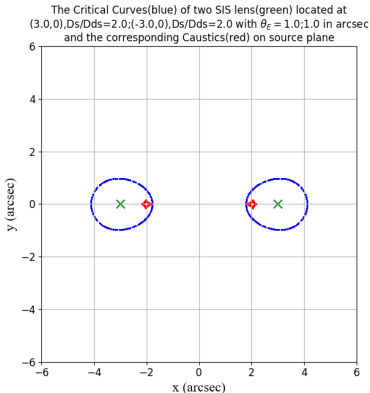
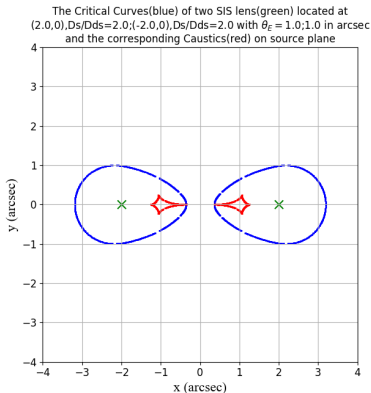
Two SIS lenses ( $\theta_E = 1.0''$ ), separation  $2.0''$ ,  $3.0''$



## Question 4

case 1) Both galaxies are at the same redshift

Two SIS lenses ( $\theta_E = 1.0''$ ), separation  $4.0''$ ,  $6.0''$



## Question 4

case 2) these two galaxies are at different redshifts

Lens equations of two lens planes system:

$$\vec{\eta} = \frac{D_s}{D_1} \vec{\xi}^{(1)} - D_{1s} \vec{\alpha}^{(1)} \left( \vec{\xi}^{(1)} \right) - D_{2s} \vec{\alpha}^{(2)} \left( \vec{\xi}^{(2)} \right)$$

$$\vec{\xi}^{(2)} = \frac{D_2}{D_1} \vec{\xi}^{(1)} - D_{12} \vec{\alpha}^{(1)} \left( \vec{\xi}^{(1)} \right)$$

For two point mass lenses:

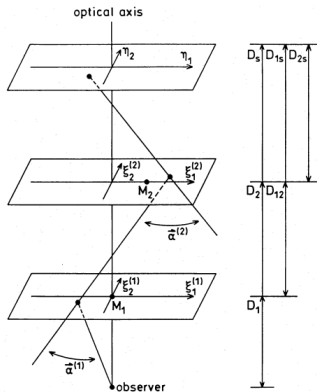
$$\vec{\alpha}^{(1)} \left( \vec{\xi}^{(1)} \right) = \frac{4GM_1}{c^2} \frac{\vec{\xi}^{(1)}}{|\vec{\xi}^{(1)}|^2},$$

$$\vec{\alpha}^{(2)} \left( \vec{\xi}^{(2)} \right) = \frac{4GM_2}{c^2} \frac{\vec{\xi}^{(2)} - \vec{\xi}_{M_2}^{(2)}}{|\vec{\xi}^{(2)} - \vec{\xi}_{M_2}^{(2)}|^2}$$

For two SIS lenses:

$$\vec{\alpha}^{(1)} \left( \vec{\xi}^{(1)} \right) = \frac{4\pi\sigma_{v1}^2}{c^2} \frac{\vec{\xi}^{(1)}}{|\vec{\xi}^{(1)}|},$$

$$\vec{\alpha}^{(2)} \left( \vec{\xi}^{(2)} \right) = \frac{4\pi\sigma_{v2}^2}{c^2} \frac{\vec{\xi}^{(2)} - \vec{\xi}_{M_2}^{(2)}}{|\vec{\xi}^{(2)} - \vec{\xi}_{M_2}^{(2)}|}$$



Classification of the multiple deflection two point-mass gravitational lens models and application of catastrophe theory in lensing. Erdl, Helmut and Schneider, Peter  
<https://ui.adsabs.harvard.edu/abs/1993A%26A...268..453E/abstract>

## Question 4

case 2) these two galaxies are at different redshifts

For two SIS lenses:

$$\vec{\alpha}^{(1)} \left( \vec{\xi}^{(1)} \right) = \frac{4\pi\sigma_{v1}^2}{c^2} \frac{\vec{\xi}^{(1)}}{|\vec{\xi}^{(1)}|},$$

$$\vec{\alpha}^{(2)} \left( \vec{\xi}^{(2)} \right) = \frac{4\pi\sigma_{v2}^2}{c^2} \frac{\vec{\xi}^{(2)} - \vec{\xi}_{M_2}^{(2)}}{|\vec{\xi}^{(2)} - \vec{\xi}_{M_2}^{(2)}|}$$

Rearrange these equations:

$$\vec{y} = \vec{x} - m_1 \frac{\vec{x}}{|\vec{x}|} - m_2 \frac{\vec{t} - \vec{\rho}}{|\vec{t} - \vec{\rho}|}$$

$$\vec{t} = \vec{x} - m_1 \beta \frac{\vec{x}}{|\vec{x}|}$$

where  $m_1 = D_{1s}\theta_1$ ,  $m_2 = D_{2s}\theta_2$ ,  $\beta = D_{12}/(D_2D_{1s})$ ,

$$\vec{\xi}^{(2)} = D_2\vec{t}, \quad \vec{\xi}_{M_2}^{(2)} = D_2\vec{\rho}, \quad \vec{\xi}^{(1)} = D_1\vec{x}, \quad \vec{\eta} = D_s\vec{y},$$

$$\theta_i = \frac{4\pi\sigma_{vi}^2}{c^2} = 1'' \cdot 6 \left( \frac{\sigma_{vi}}{200 \text{ km/s}} \right)^2$$

Thus the Jacobian matrix of this two lenses system:

$$A = \frac{\partial \vec{y}}{\partial \vec{x}}$$

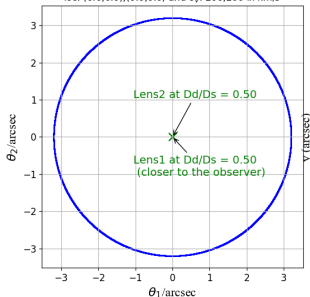
$$\mu(\vec{x}) = \frac{1}{\det A(\vec{x})}$$

## Question 4

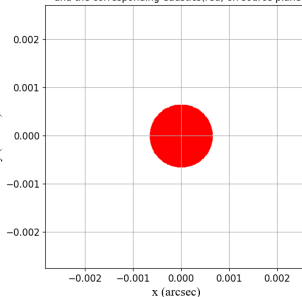
case 2) these two galaxies are at different redshifts

- Obesever, Lens1, Lens2 are on a line, e.g.  $\vec{\rho} = (0, 0)$ , and  $D_1 = 1, D_2 = 1, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc: (0.0,0.0),(0.0,0.0) and  $\sigma_v$ : 200,200 in km/s



and the corresponding Caustics(red) on source plane



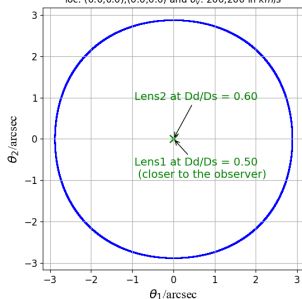


## Question 4

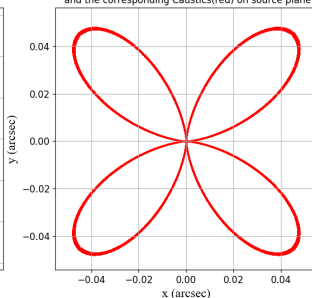
case 2) these two galaxies are at different redshifts

- Obesever, Lens1, Lens2 are on a line, e.g.  $\vec{\rho} = (0, 0)$ ,  
and  $D_1 = 1, D_2 = 1.2, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc: (0,0,0,0),(0,0,0,0) and  $\sigma_v$ : 200,200 in km/s



and the corresponding Caustics(red) on source plane

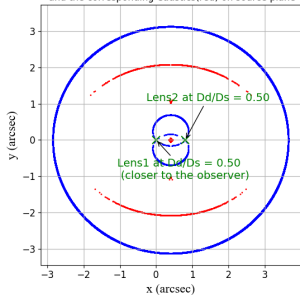


## Question 4

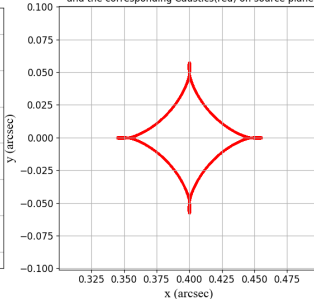
case 2) these two galaxies are at different redshifts

►  $\vec{\rho} = (0.8, 0)$ , and  $D_1 = 1, D_2 = 1, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc:  $(0.0, 0.0), (0.8, 0.0)$  and  $\sigma_v: 200, 200$  in  $km/s$   
and the corresponding Caustics(red) on source plane



and the corresponding Caustics(red) on source plane

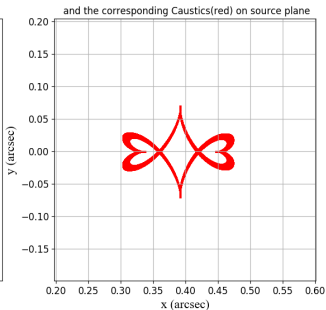
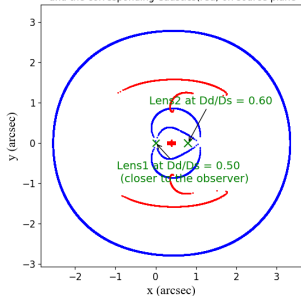


## Question 4

case 2) these two galaxies are at different redshifts

- $\vec{\rho} = (0.8, 0)$ , and  $D_1 = 1, D_2 = 1.2, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc:  $(0.0, 0.0), (0.8, 0.0)$  and  $\sigma_v: 200, 200$  in  $\text{km/s}$   
and the corresponding Caustics(red) on source plane

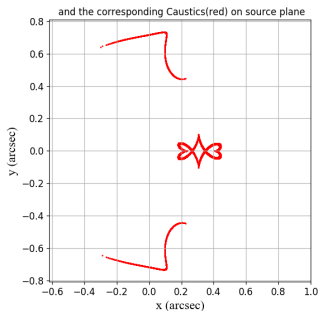
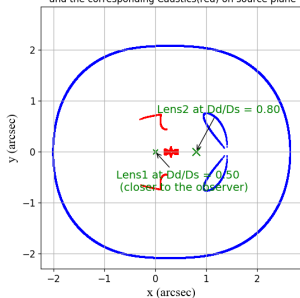


## Question 4

case 2) these two galaxies are at different redshifts

- $\vec{\rho} = (0.8, 0)$ , and  $D_1 = 1, D_2 = 1.6, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc:  $(0.0, 0.0), (0.8, 0.0)$  and  $\sigma_v: 200, 200$  in  $km/s$   
and the corresponding Caustics(red) on source plane

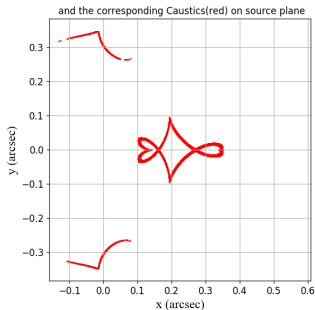
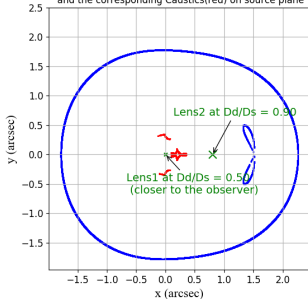


## Question 4

case 2) these two galaxies are at different redshifts

- $\vec{\rho} = (0.8, 0)$ , and  $D_1 = 1, D_2 = 1.8, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc:  $(0.0, 0.0), (0.8, 0.0)$  and  $\sigma_v: 200, 200$  in  $km/s$   
and the corresponding Caustics(red) on source plane

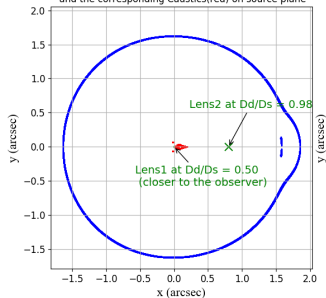


## Question 4

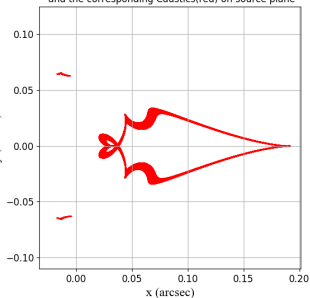
case 2) these two galaxies are at different redshifts

- $\vec{\rho} = (0.8, 0)$ , and  $D_1 = 1, D_2 = 1.96, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc:  $(0.0, 0.0), (0.8, 0.0)$  and  $\sigma_v: 200, 200$  in  $km/s$   
and the corresponding Caustics(red) on source plane



and the corresponding Caustics(red) on source plane

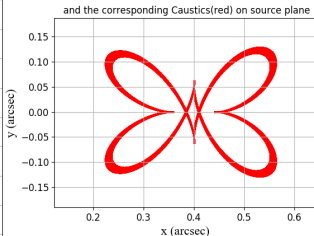
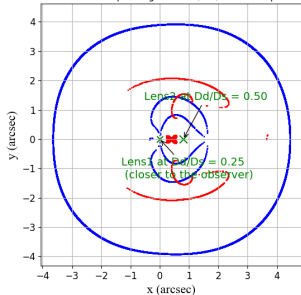


# Question 4

case 2) these two galaxies are at different redshifts

- $\vec{\rho} = (0.8, 0)$ , and  $D_1 = 0.5, D_2 = 1, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc:  $(0.0, 0.0), (0.8, 0.0)$  and  $\sigma_v: 200, 200$  in  $km/s$   
and the corresponding Caustics(red) on source plane

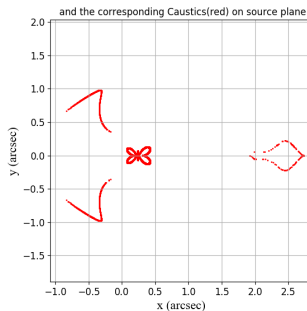
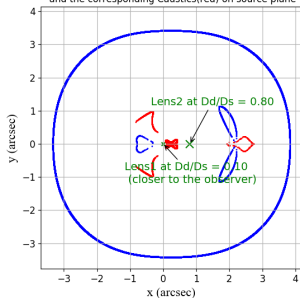


## Question 4

case 2) these two galaxies are at different redshifts

- $\vec{\rho} = (0.8, 0)$ , and  $D_1 = 0.2, D_2 = 1.6, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc:  $(0.0, 0.0), (0.8, 0.0)$  and  $\sigma_v: 200, 200$  in  $km/s$   
and the corresponding Caustics(red) on source plane



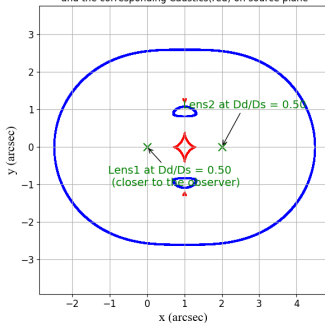


## Question 4

case 2) these two galaxies are at different redshifts

►  $\vec{\rho} = (2, 0)$ , and  $D_1 = 1, D_2 = 1, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc: (0.0,0.0),(2.0,0.0) and  $\sigma_v$ : 200,200 in km/s  
and the corresponding Caustics(red) on source plane

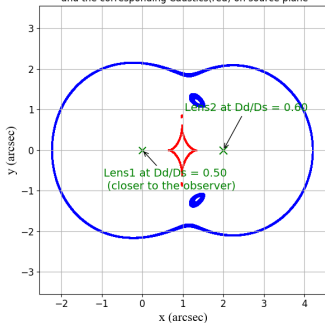


## Question 4

case 2) these two galaxies are at different redshifts

- $\vec{\rho} = (2, 0)$ , and  $D_1 = 1, D_2 = 1.2, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc: (0.0,0.0),(2.0,0.0) and  $\sigma_v$ : 200,200 in km/s  
and the corresponding Caustics(red) on source plane

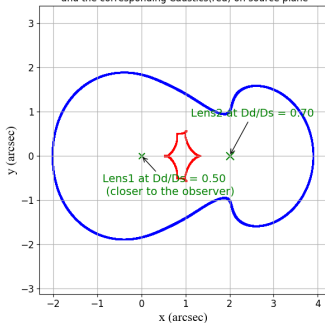


## Question 4

case 2) these two galaxies are at different redshifts

- $\vec{\rho} = (2, 0)$ , and  $D_1 = 1, D_2 = 1.4, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc: (0.0,0.0),(2.0,0.0) and  $\sigma_v$ : 200,200 in km/s  
and the corresponding Caustics(red) on source plane

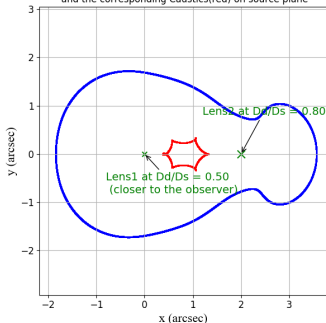


## Question 4

case 2) these two galaxies are at different redshifts

- $\vec{\rho} = (2, 0)$ , and  $D_1 = 1, D_2 = 1.6, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc: (0.0,0.0),(2.0,0.0) and  $\sigma_v$ : 200,200 in km/s  
and the corresponding Caustics(red) on source plane

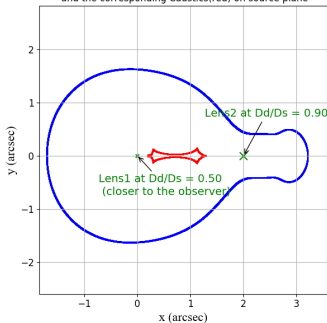


## Question 4

case 2) these two galaxies are at different redshifts

- $\vec{\rho} = (2, 0)$ , and  $D_1 = 1, D_2 = 1.8, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc: (0.0,0.0),(2.0,0.0) and  $\sigma_v$ : 200,200 in km/s  
and the corresponding Caustics(red) on source plane

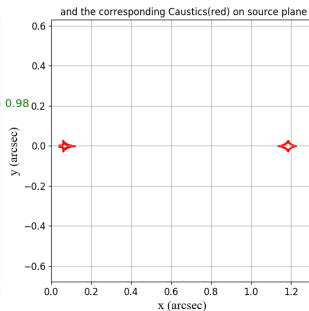
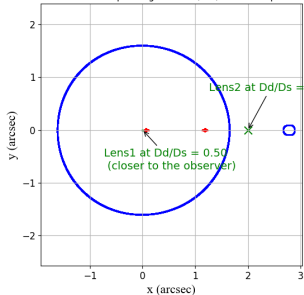


## Question 4

case 2) these two galaxies are at different redshifts

►  $\vec{\rho} = (2, 0)$ , and  $D_1 = 1, D_2 = 1.96, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc:  $(0.0, 0.0), (2.0, 0.0)$  and  $\sigma_v: 200, 200$  in  $km/s$   
and the corresponding Caustics(red) on source plane

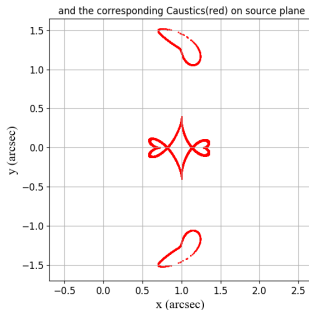
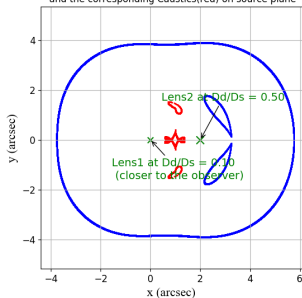


## Question 4

case 2) these two galaxies are at different redshifts

- $\vec{\rho} = (2, 0)$ , and  $D_1 = 0.2, D_2 = 1, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc:  $(0.0, 0.0), (2.0, 0.0)$  and  $\sigma_v$ : 200, 200 in km/s  
and the corresponding Caustics(red) on source plane

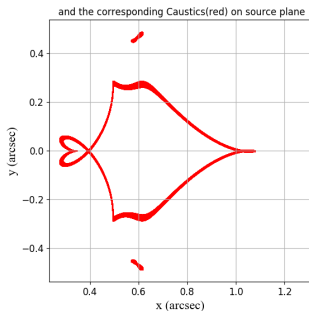
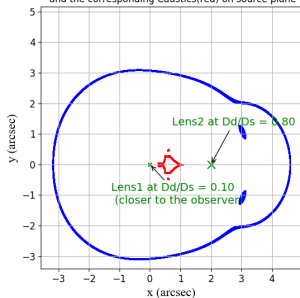


## Question 4

case 2) these two galaxies are at different redshifts

►  $\vec{\rho} = (2, 0)$ , and  $D_1 = 0.2, D_2 = 1.6, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc:  $(0, 0, 0.0), (2.0, 0.0)$  and  $\sigma_v: 200, 200$  in  $km/s$   
and the corresponding Caustics(red) on source plane



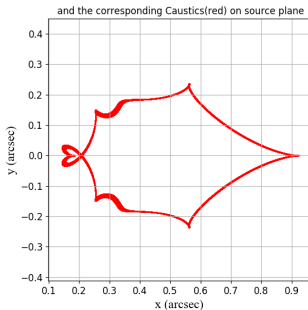
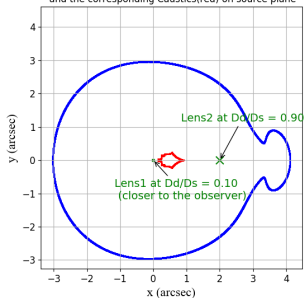


## Question 4

case 2) these two galaxies are at different redshifts

►  $\vec{\rho} = (2, 0)$ , and  $D_1 = 0.2, D_2 = 1.8, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc: (0.0,0.0), (2.0,0.0) and  $\sigma_v$ : 200,200 in km/s  
and the corresponding Caustics(red) on source plane

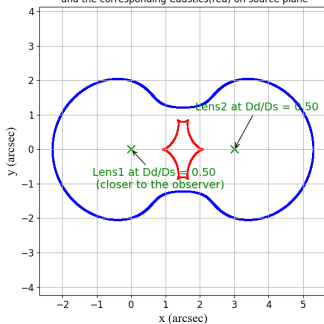


## Question 4

case 2) these two galaxies are at different redshifts

- $\vec{\rho} = (3, 0)$ , and  $D_1 = 1, D_2 = 1, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc: (0.0,0.0),(3.0,0.0) and  $\sigma_v$ : 200,200 in km/s  
and the corresponding Caustics(red) on source plane

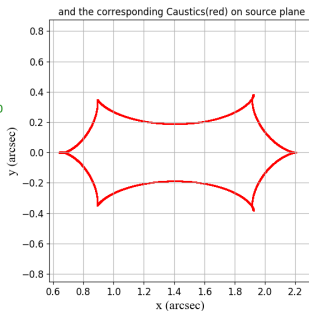
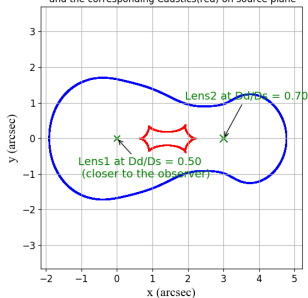


## Question 4

case 2) these two galaxies are at different redshifts

►  $\vec{\rho} = (3, 0)$ , and  $D_1 = 1, D_2 = 1.4, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc:  $(0,0,0,0), (3,0,0,0)$  and  $\sigma_v: 200, 200$  in  $\text{km/s}$   
and the corresponding Caustics(red) on source plane

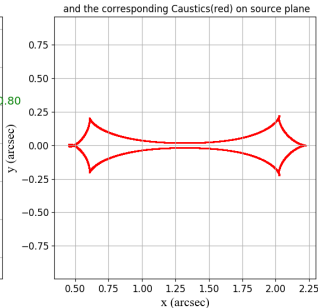
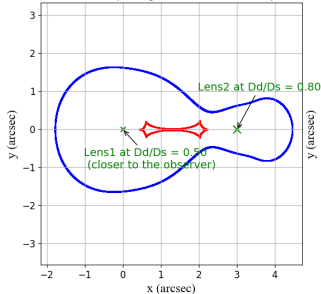


## Question 4

case 2) these two galaxies are at different redshifts

►  $\vec{\rho} = (3, 0)$ , and  $D_1 = 1, D_2 = 1.6, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc:  $(0.0, 0.0), (3.0, 0.0)$  and  $\sigma_v: 200, 200$  in  $km/s$   
and the corresponding Caustics(red) on source plane

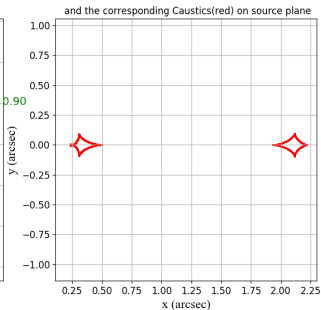
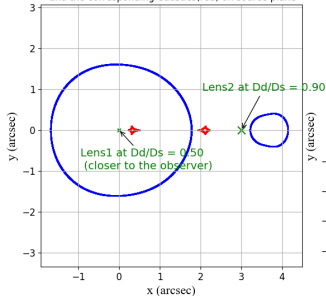


## Question 4

case 2) these two galaxies are at different redshifts

- $\vec{\rho} = (3, 0)$ , and  $D_1 = 1, D_2 = 1.8, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc:  $(0.0, 0.0), (3.0, 0.0)$  and  $\sigma_v: 200, 200$  in  $km/s$   
and the corresponding Caustics(red) on source plane

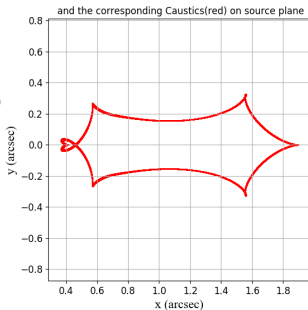
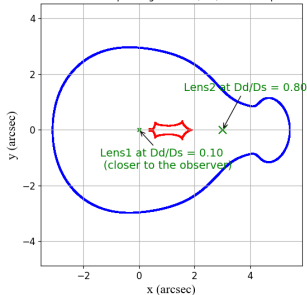


## Question 4

case 2) these two galaxies are at different redshifts

- $\vec{\rho} = (3, 0)$ , and  $D_1 = 0.2, D_2 = 1.6, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc:  $(0.0, 0.0), (3.0, 0.0)$  and  $\sigma_v: 200, 200$  in  $km/s$   
and the corresponding Caustics(red) on source plane

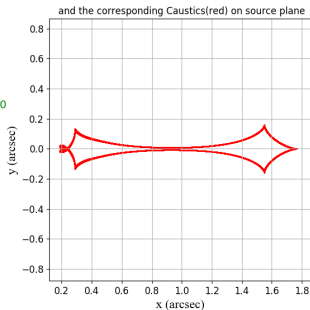
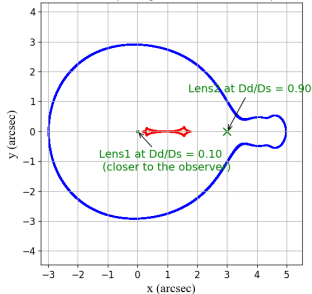


## Question 4

case 2) these two galaxies are at different redshifts

- $\vec{\rho} = (3, 0)$ , and  $D_1 = 0.2, D_2 = 1.8, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc:  $(0.0, 0.0), (3.0, 0.0)$  and  $\sigma_v: 200, 200$  in  $km/s$   
and the corresponding Caustics(red) on source plane

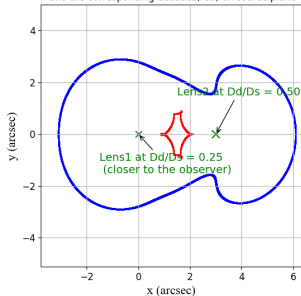


## Question 4

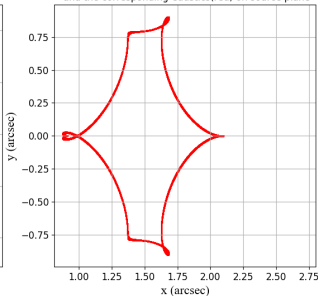
case 2) these two galaxies are at different redshifts

►  $\vec{\rho} = (3, 0)$ , and  $D_1 = 0.5, D_2 = 1, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc:  $(0.0, 0.0), (3.0, 0.0)$  and  $\sigma_v: 200, 200$  in  $km/s$   
and the corresponding Caustics(red) on source plane



and the corresponding Caustics(red) on source plane



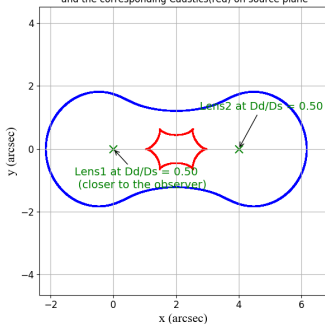


## Question 4

case 2) these two galaxies are at different redshifts

►  $\vec{\rho} = (4, 0)$ , and  $D_1 = 1, D_2 = 1, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc:  $(0.0, 0.0), (4.0, 0.0)$  and  $\sigma_v$ : 200, 200 in km/s  
and the corresponding Caustics(red) on source plane

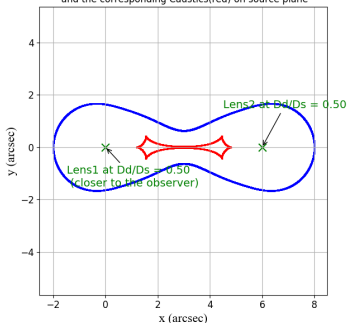


## Question 4

case 2) these two galaxies are at different redshifts

- $\vec{\rho} = (6, 0)$ , and  $D_1 = 1, D_2 = 1, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc:  $(0.0, 0.0), (6.0, 0.0)$  and  $\sigma_v: 200, 200$  in  $km/s$   
and the corresponding Caustics(red) on source plane

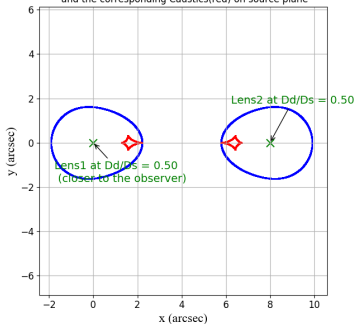


## Question 4

case 2) these two galaxies are at different redshifts

►  $\vec{\rho} = (8, 0)$ , and  $D_1 = 1, D_2 = 1, D_s = 2$

The Critical Curves(blue) of two SIS lens(green) located at different redshift  
loc:  $(0.0, 0.0), (8.0, 0.0)$  and  $\sigma_v$ : 200, 200 in km/s  
and the corresponding Caustics(red) on source plane



# Gravitational Lensing Applications

GravLens

Renkun Kuang

History

Gravitational  
Lensing Theories

Related Questions

Applications

- ▶ Cosmic telescopes: distant, faint objects observation
- ▶ 2-d mass distribution of lenses, dark matter
- ▶ Hubble constant, cosmological constant, density parameter
- ▶ .....

# The End, Thanks!

# References

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- ▶ <http://web.mit.edu/6.055/old/S2009/notes/bending-of-light.pdf>
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- ▶ <https://web.stanford.edu/~oas/SI/SRGR/notes/SchwarzschildSolution.pdf>
- ▶ .....

# Appendix-1, Newtonian prediction

$$\begin{aligned}\phi(b, z) &= -\frac{GM}{(b^2 + z^2)^{1/2}} \\ \alpha &= \frac{v_b}{c} = \frac{1}{c} \int \frac{d\Phi}{db} dt = \frac{1}{c^2} \int \frac{d\Phi}{db} dl \\ &\approx \frac{1}{c^2} \int \frac{d\Phi}{db} dz \\ &= \frac{GMb}{c^2} \int \frac{dz}{(b^2 + z^2)^{3/2}} \\ &= \frac{GMb}{c^2} \left[ \frac{z}{b^2 \sqrt{b^2 + z^2}} \right]_{-\infty}^{+\infty} \\ &= \frac{2GM}{c^2 b}\end{aligned}$$

$b$ : impact factor

Gravitational potential simplify the calculation by integrating not along the deflected ray but along  $z$  axis

