CHAPTER TWO

MATHEMATICAL FOUNDATION OF THE FINITE ELEMENT METHOD

From a mathematical point of view the finite element method is a special form of the well-known Galerkin and Rayleigh-Ritz methods for finding approximate solution of differential equations. In both methods the governing differential equation first is converted into an equivalent integral form. The Rayleigh-Ritz method employs calculus of variations to define an equivalent variational or energy functional. A function that minimizes this energy functional represents a solution of the governing differential equation. The Galerkin method uses a more direct approach. An approximate solution, with one or more unknown parameters, is chosen. In general, this assumed solution will not satisfy the differential equation. The integral form represents the *residual* obtained by integrating the error over the solution domain. Employing a criteria adopted to minimize the residual gives equations for finding the unknown parameters.

For most practical problems solutions of differential equations are required to satisfy not only the differential equation but also the specified boundary conditions at one or more points along the boundary of the solution domain. In both methods some of the boundary conditions must be satisfied explicitly by the assumed solutions while others are satisfied implicitly through the minimization process. The boundary conditions are thus divided into two categories, *essential* and *natural*. The essential boundary conditions are those that must explicitly be satisfied while the natural boundary conditions are incorporated into the integral formulation. In general, therefore, the approximate solutions will not satisfy the natural boundary conditions exactly.

The basic concepts will be explained with reference to the problem of axial deformation of bars. The derivation of the governing differential equation is considered in the next section. Approximate solutions using the classical form of Galerkin and Rayleigh-Ritz methods are presented. Finally the methods are cast into the form that is suitable for developing finite element equations.

2.1 AXIAL DEFORMATION OF BARS

2.1.1 Differential Equation for Axial Deformations

Consider a bar of any arbitrary cross section subjected to loads in the axial direction only, as shown in Figure 2.1. The area of cross section is denoted by A and it could vary over the length of the bar. The modulus of elasticity is denoted by E. The bar may be subjected to a distributed axial load $q(x, t)$ along its length. The load can vary over the bar length and may also be a function of time. The axial displacement is denoted by $u(x, t)$.

The governing differential equation can be written by considering equilibrium of a differential element as shown in Figure 2.2. Note the sign convention adopted in drawing the free-body diagram assumes that the tension in the bar is positive. The axial force at x is denoted by F. The axial force at $x + dx$ is $F + (\partial F/\partial x)\,dx$ based on the Taylor series expansion. The acceleration is indicated by $\ddot{u} \equiv \partial^2 u/\partial t^2$. From Newton's second law of motion a force, called the inertia force, is produced that is proportional to the mass of the element and acts in the direction opposite to the direction of motion. Denoting the mass density by ρ, the mass per unit length is $m = A\rho\,dx$ and thus the inertia force is $m\ddot{u} = A\rho\,dx\,\ddot{u}$. The only other force acting on the element is the applied axial load $q(x, t)$. Considering summation of forces in the x direction, we have

$$A(x)\rho\,dx\,\ddot{u}(x,t) + F = q\,dx + F + \frac{\partial F}{\partial x}dx \Longrightarrow A\rho\ddot{u} = q + \frac{\partial F}{\partial x}$$

The axial force is related to the axial stress σ_x as follows:

$$F = A\sigma_x$$

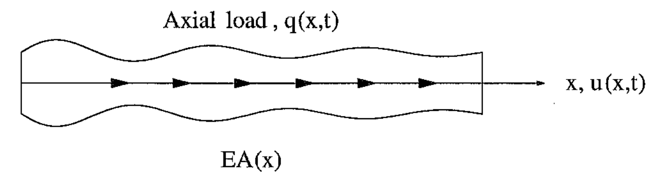

Figure 2.1. Axially loaded bar

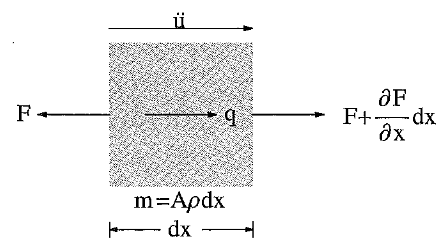

Figure 2.2. Forces acting on a differential element in an axially loaded bar

Assuming linear elastic material, axial stress is related to the axial strain ϵ_x through the modulus of elasticity. Thus

$$F = AE\epsilon_x$$

Assuming small displacements, the axial strain is related to the first derivative of the axial displacement. Thus the axial force is related to displacement as follows:

$$F = AE\frac{\partial u}{\partial x}$$

Substituting this into the equilibrium equation, the governing differential equation for axial deformation of bars is as follows:

$$\frac{\partial}{\partial x}\left(AE\frac{\partial u}{\partial x}\right) + q = A\rho\ddot{u}$$

This is a second-order partial differential equation. Since the equation involves second-order derivatives with respect to both x and t, we need two boundary conditions and two initial conditions for a proper solution. The initial conditions are specified displacement and velocity along the bar at time $t = 0$:

$$u(x, 0) = u_0; \qquad \dot{u}(x, 0) = v_0$$

where u_0 and v_0 are the specified values. The boundary conditions involve specification of displacement or its first derivative at the ends. Since the first derivative of displacement is related to the axial force, the derivative boundary condition is expressed in terms of the applied force. The possible boundary conditions at the right end of the bar $x = x_l$ are as follows:

$$u(x_l, t) = u_{x_l} \quad \text{or} \quad A(x_l)E(x_l)\frac{\partial u(x_l, t)}{\partial x} = P_{x_l}(t); \qquad x_l \rightarrow \text{right end of the bar}$$

where u_{x_l} is a specified displacement and P_{x_l} is a specified force. Both these quantities could be functions of time. Care must be exercised in assigning proper signs to force boundary condition terms. Applied forces are considered positive when they act in the positive coordinate direction. From the free-body diagram in Figure 2.2, it should be clear that, if a force is specified at the left end of a bar, then the appropriate force boundary condition must include a negative sign as follows:

$$-A(x_0)E(x_0)\frac{\partial u(x_0, t)}{\partial x} = P_{x_0}(t); \qquad x_0 \rightarrow \text{left end of the bar}$$

If the forces and displacements do not vary with time, we have a static analysis situation and the equilibrium equation is an ordinary second-order differential equation as follows:

$$\frac{d}{dx}\left(AE\frac{du}{dx}\right) + q(x) = 0; \qquad x_0 < x < x_l$$

with the boundary conditions of the form

$$u(x_0) = u_{x_0} \quad \text{or} \quad -A(x_0)E(x_0)\frac{du(x_0)}{dx} = P_{x_0}$$

$$u(x_l) = u_{x_l} \quad \text{or} \quad A(x_l)E(x_l)\frac{\partial u(x_l)}{\partial x} = P_{x_l}$$

An integration of the second-order differential equation, if possible, would yield two arbitrary constants, and we would need two conditions to evaluate these constants. Thus at least two boundary conditions must be specified for a unique solution. On physical grounds, it is easy to see that we must specify displacement at least at one point along the bar to prevent the whole bar from moving as a rigid body. The second boundary condition could be either of the displacement or the force type. Note that at the same point both a displacement and a force cannot be specified independently. The reason for this is that, if a displacement is specified, then the corresponding force represents a reaction at the point and in general is unknown.

2.1.2 Exact Solutions of Some Axial Deformation Problems

It is possible to directly integrate the differential equation to obtain exact solutions for the axial deformation problems in which the loading and the area of cross section are simple functions of x. Solutions for a uniform bar and a tapered bar subjected to linearly varying load are presented in this section. In later sections these exact solutions will be used to check the quality of the approximate solutions that are obtained using the Galerkin, the Rayleigh-Ritz, and the finite element methods.

Axial Deformation of a Uniform Bar Consider a uniform bar fixed at one end and subjected to a static point load at the other end, as shown in Figure 2.3. The bar is also subjected to a linearly varying axial load $q(x) = cx$, where c is a given constant. The problem is described in terms of the following boundary value problem:

$$EA\frac{d^2u}{dx^2} + cx = 0; \qquad 0 < x < L$$

$$u(0) = 0; \qquad EA\frac{du(L)}{dx} = P$$

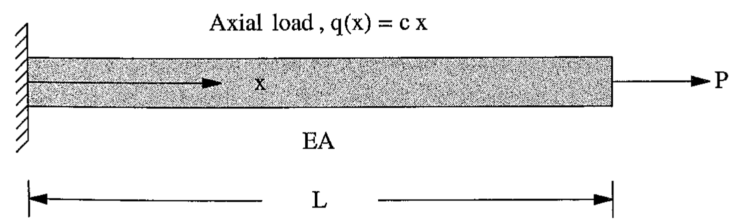

Figure 2.3. Uniform axially loaded bar

An exact solution of the problem can easily be obtained by integrating the differential equation twice and then using the boundary conditions to evaluate the resulting integration constants. Integrating both sides of the differential equation once, we get

$$EA\frac{du}{dx} + \frac{cx^2}{2} = C_1$$

where C_1 is an integration constant. Integrating once again, we get

$$EAu(x) + \frac{cx^3}{6} = C_1x + C_2$$

where C_2 is another integration constant. Rearranging terms,

$$u(x) = \frac{1}{EA}\left(C_1x + C_2 - \frac{cx^3}{6}\right)$$

Using the boundary conditions

$$u(0) = 0 \Longrightarrow C_2 = 0$$

and

$$EA\frac{du(L)}{dx} = P \Longrightarrow \tfrac{1}{6}(-3cL^2 + 6C_1) = P \Longrightarrow C_1 = \tfrac{1}{2}(2P + cL^2)$$

Thus the exact solution of the problem is

$$u(x) = \frac{x(6P + 3cL^2 - cx^2)}{6EA}$$

2.2 AXIAL DEFORMATION OF BARS USING GALERKIN METHOD

The governing differential equation for axial deformation of bars is the following second-order differential equation:

$$\frac{d}{dx}\left(AE\frac{du}{dx}\right) + q = 0; \qquad x_0 < x < x_l$$

where x_0 and x_l are the coordinates of the ends of the bar. The primary unknown is the axial displacement $u(x)$. Once the displacement is known, axial strain, stress, and force can be computed from the following relationships:

$$\epsilon_x = \frac{du}{dx}; \qquad \sigma_x = E\epsilon_x; \qquad F = A\sigma_x$$

At the ends either a displacement or an axial force can be specified. Thus the boundary conditions for the problem are of the following form:

$$u(x_0) = u_{x_0} \quad \text{or} \quad -A(x_0)E(x_0)\frac{du(x_0)}{dx} = P_{x_0}$$

$$u(x_l) = u_{x_l} \quad \text{or} \quad A(x_l)E(x_l)\frac{\partial u(x_l)}{\partial x} = P_{x_l}$$

where $u_{x0}, P_{x0}, \ldots$ are appropriate specified values. Mathematically speaking, for a second-order differential equation either u is specified or its first derivative du/dx is specified. Both u and du/dx cannot have specified values at the same point.

2.2.1 Weak Form for Axial Deformations

In the Galerkin method we assume a general form of the solution. This assumed solution must contain some unknown parameters whose values are determined so that the error between the assumed solution and the exact solution is as small as possible. The assumed solution can be of any form. As an example, we will consider a solution in the form of a polynomial:

$$\tilde{u}(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

where $a_0, a_1, \ldots$ are the unknown parameters. In general, this assumed solution will not satisfy the differential equation for all values of x. When this assumed solution is substituted into the differential equation, the error in satisfying the governing differential equation is

$$e(x) = \frac{d}{dx}\left(AE\frac{d\tilde{u}}{dx}\right) + q(x) \neq 0$$

The tilde (~) over u is used to emphasize that it is an assumed solution and not necessarily the same as the exact solution of the differential equation. In the following development we will have to perform several mathematical operations, such as differentiation and integration, on the assumed solution. Carrying the tilde symbol through all these manipulations becomes tedious. Since we are dealing primarily with the approximate solutions, for notational convenience, we will drop the tilde and use only u instead to indicate an approximate solution. An occasional reference to the exact solution will be indicated by using the word *exact* explicitly.

The total error, called the residual, for the entire solution domain can be obtained by integrating $e(x)$ over the domain. However, in a straight integration of e, the negative and positive errors at different points may cancel each other. To avoid error cancellation, $e(x)$ is multiplied by a suitable weighting function and then integrated over the solution domain. Since there are n unknown parameters, we need n weighting functions to establish weighted residual equations as follows:

$$\int_{x_0}^{x_l} e(x)w_i(x)\,dx = 0; \qquad i = 0, 1, \ldots, n$$

where $w_i(x)$ are suitable weighting functions. This form is known as the *weak form*. The differential equation is the *strong form* because it requires error term $e(x)$ to vanish at every x. The integral form makes the total error go to zero but does not necessarily satisfy the governing differential equation for all values of x.

In a method known as the *least-squared weighted residual method* the error term is squared to define the total squared error as follows:

$$e_T = \int_{x_0}^{x_l} e^2 \, dx$$

The necessary conditions for the minimum of the total squared error give n equations that can be solved for the unknown parameters:

$$\frac{\partial e_T}{\partial a_i} = 2 \int_{x_0}^{x_l} e \frac{\partial e}{\partial a_i} dx = 0; \qquad i = 0, 1, \ldots$$

Thus in the least-squares method the weighting functions are the partial derivatives of the error term $e(x)$.

Least-squares weighting functions:

$$w_i(x) = \frac{\partial e}{\partial a_i}; \qquad i = 0, 1, \ldots, n$$

A more popular method in the finite element applications is the *Galerkin method*. In this method, instead of taking partial derivatives of the error function, the weighting functions are defined as the partial derivatives of the assumed solution.

Galerkin weighting functions:

$$w_i(x) = \frac{\partial u}{\partial a_i}; \qquad i = 0, 1, \ldots, n$$

Thus the Galerkin weighted residual method defines the following n equations to solve for the unknown parameters:

$$\int_{x_0}^{x_l} e \frac{\partial u}{\partial a_i} dx = 0; \qquad i = 0, 1, \ldots$$

It turns out that for a large number of engineering applications the Galerkin method gives the same solution as another popular method, the Rayleigh-Ritz method, presented in a later section. Furthermore, since the least-squares method has no particular advantage over the Galerkin method for the kinds of problems discussed in this book, only the Galerkin method is presented in detail.

So far in developing the residual we have considered error in satisfying the differential equation alone. A solution must also satisfy boundary conditions. In order to be able to introduce the boundary conditions into the weighted residual, we use mathematical manipulations involving integration by parts.

The *integration-by-parts* formula is used to rewrite an integral of a product of a derivative of a function, say $f(x)$, and another function, say $g(x)$, as follows:

$$\int_{x_0}^{x_l} \left[\frac{d}{dx}(f(x))\right] g(x) \, dx = f(x_l)g(x_l) - f(x_0)g(x_0) - \int_{x_0}^{x_l} \left[\frac{d}{dx}(g(x))\right] f(x) \, dx$$

Note that the integrand must involve the product of a function and the derivative of another function. The application of the formula produces two terms that are evaluated at the ends of integration domain and another integral in which the derivative shifts from one function to the other.

For a second-order differential equation, we know that the boundary conditions specify either u or du/dx at the ends. Thus we are looking for a way of introducing these terms into the weighted residual. Since the integration-by-parts formula gives rise to boundary terms, it is precisely the tool that we need. Note that the highest derivative term in the residual $e(x)$ involves a second derivative on u. Using integration by parts on this term will result in two terms evaluated at the integration limits that are used to incorporate the boundary conditions into the residual.

For the axial deformation problem, the weighted residual is

$$\int_{x_0}^{x_l} e(x) w_i(x) \, dx = \int_{x_0}^{x_l} \left(\frac{d}{dx}\left(AE\frac{du}{dx}\right) + q(x)\right) w_i(x) \, dx = 0; \qquad i = 0, 1, \ldots$$

Note that, in general, A and E could be functions of x, and therefore, care must be taken when carrying out differentiation and integration. Also $q(x)$ and $w_i(x)$ are arbitrary functions of x at this stage and cannot be taken out of the integral. Writing the two terms in the weighted residual as separate integrals, we have

$$\int_{x_0}^{x_l} \frac{d}{dx}\left(AE\frac{du}{dx}\right) w_i(x) \, dx + \int_{x_0}^{x_l} q(x) w_i(x) \, dx = 0; \qquad i = 0, 1, \ldots$$

The first integral contains the second-order derivative on u and is written exactly in the form to which integration by parts is applicable with $f(x) = AE(du/dx)$ and $g(x) = w_i(x)$. Thus the application of integration by parts to the first integral gives

$$A(x_l)E(x_l)\frac{du(x_l)}{dx} w_i(x_l) - A(x_0)E(x_0)\frac{du(x_0)}{dx} w_i(x_0) - \int_{x_0}^{x_l} AE\frac{du}{dx}\frac{dw_i}{dx} dx + \int_{x_0}^{x_l} q(x) w_i(x) \, dx = 0$$

The first two terms in the weak form give us a way to incorporate the specified force or derivative boundary conditions into the weak form. If a force P_{x0} is applied at end x_0, then

$$-A(x_0)E(x_0)\frac{du(x_0)}{dx} = P_{x_0} \Longrightarrow \text{Second term in the weak form: } P_{x_0} w_i(x_0)$$

If a force P_{xl} is applied at end x_l, then

$$A(x_l)E(x_l)\frac{du(x_l)}{dx} = P_{x_l} \Longrightarrow \text{First term in the weak form: } P_{x_l} w_i(x_l)$$

Thus, when a force is specified, the boundary condition can be *naturally* incorporated into the weak form. Hence this type of boundary condition is called a *natural boundary condition* (NBC).

The situation is very different if the u is specified at one or both ends. The derivative du/dx at the corresponding point is unknown. (Physically $AE\, du/dx$ at the point represents the unknown reaction.) The specified displacement boundary condition therefore cannot be incorporated into the weak form directly. The assumed solutions must satisfy this type of boundary condition explicitly, and thus such a boundary condition is called the *essential boundary condition* (EBC). An assumed solution that satisfies the EBC is known as an *admissible solution*. Since the weighting functions are partial derivatives of the assumed solution, with an admissible assumed solution, all weighting functions corresponding to the location of an EBC are zero. Therefore, the boundary term in the weak form vanishes at the point where an EBC is specified. Thus we must deal with the boundary terms as follows:

Boundary Condition Term	Specified Value	Boundary Term in the Weak Form	Requirement on the Assumed Solution	Type
$-A(x_0)E(x_0)\dfrac{du(x_0)}{dx}$	P_{x_0}	$P_{x_0}w_i(x_0)$	None	NBC
$A(x_l)E(x_l)\dfrac{du(x_l)}{dx}$	P_{x_l}	$P_{x_l}w_i(x_l)$	None	NBC
$u(x_0)$	u_{x_0}	None	Must satisfy	EBC
$u(x_l)$	u_{x_l}	None	Must satisfy	EBC

For structural problems, the weak form can be interpreted as the well-known *principle of virtual displacements*. To see this, assume we have specified force boundary conditions at the ends. Then the weak form can be written as follows:

$$P_{x_l}w_i(x_l) + P_{x_0}w_i(x_0) + \int_{x_0}^{x_l}\left(-AE\frac{du}{dx}\frac{dw_i}{dx} + q(x)w_i(x)\right)dx = 0$$

Rearranging terms, we have

$$\int_{x_0}^{x_l} E\frac{du}{dx}\frac{dw_i}{dx}A\,dx = \int_{x_0}^{x_l} q(x)w_i(x)\,dx + P_{x_l}w_i(x_l) + P_{x_0}w_i(x_0)$$

If we interpret $w_i(x)$ as a virtual displacement, then the right-hand side is the virtual work done by the applied forces. The left-hand side is the total internal virtual work, since $E\,du/dx \equiv \sigma_x$ is the axial stress and dw_i/dx is axial virtual strain. Thus the weak form implies that when a bar is given a virtual displacement, then the external virtual work is equal to the total internal virtual work, which is a statement of the principle of virtual displacements. Since this principle is widely used in structural mechanics, it is one of the main reasons why the Galerkin weighted residual method is more popular in developing finite element equations. Also recall that the virtual displacements are required to satisfy the displacement (i.e., essential) boundary conditions.

From the final weak form we note that another advantage of the integration by parts is that the order of the derivative on the terms remaining inside the integral sign is reduced by 1. Thus for a second-order problem the weak form involves only first-order derivatives. It may not appear to be a big deal here, but it has important consequences in developing simple finite elements for practical problems. It will be seen in a later example that, when dealing with a fourth-order problem, integration by parts must be carried out twice to reduce the highest order derivative to 2.

2.2.2 Uniform Bar Subjected to Linearly Varying Axial Load

We now use the Galerkin method to find approximate solutions for a uniform bar (EA constant) fixed at one end and subjected to a static point load at the other end, as shown in Figure 2.3. The bar is also subjected to a linearly varying axial load $q(x) = cx$, where c is a given constant. The problem is described in terms of the following boundary value problem:

$$EA\frac{d^2u}{dx^2} + cx = 0; \qquad 0 < x < L$$

$$u(0) = 0; \qquad EA\frac{du(L)}{dx} = P$$

The following exact solution of the problem was obtained in Section 2.1:

$$u(x) = \frac{x(6P + 3cL^2 - cx^2)}{6EA}$$

With the solution domain from $(0, L)$, the EA constant, $q(x) = cx$, the essential boundary condition $u(0) = 0 \Rightarrow w(0) = 0$, and the natural boundary condition $EAu'(L) = P$, the weak form specific to this problem is as follows:

$$Pw_i(L) + \int_0^L\left(-AE\frac{du}{dx}\frac{dw_i}{dx} + cxw_i(x)\right)dx = 0$$

$$\text{EBC:} \quad u(0) = 0$$

This weak form can now be used to find a variety of approximate solutions to the problem.

Linear Solution The simplest possible solution that we can assume is a linear polynomial. An approximate solution of the problem is obtained using the following starting solution:

$$u(x) = a_0 + xa_1$$

To satisfy the essential boundary condition, we must have

$$u(0) = 0 \Longrightarrow a_0 + 0a_1 = 0 \Longrightarrow a_0 = 0$$

Thus the admissible assumed solution is

$$u(x) = xa_1 \qquad \text{giving} \qquad \frac{du}{dx} = a_1$$

Substituting into the weak form, we have

$$Pw_i(L) + \int_0^L \left(-AE(a_1)\frac{dw_i}{dx} + cxw_i\right) dx = 0$$

There is only one unknown parameter left in the solution and therefore we need only one equation to find it. The Galerkin weighting function is

$$w_1 = \frac{\partial u}{\partial a_1} = x; \qquad \frac{\partial w_1}{\partial x} = 1; \qquad w_1(0) = 0; \qquad w_1(L) = L$$

Substituting this into the weak form, we have

$$PL + \int_0^L (-AEa_1 + cx^2)\, dx = 0$$

Carrying out integration and simplifying, we get

$$\frac{cL^3}{3} + PL - EAa_1 L = 0$$

Solving this equation for a_1, we get

$$a_1 = -\frac{-cL^2 - 3P}{3EA}$$

Thus a linear approximate solution for the problem is as follows:

$$u(x) = a_1 x = \frac{(cL^2 + 3P)x}{3EA}$$

Quadratic Solution A better solution can be obtained if we start with a quadratic polynomial:

$$u(x) = a_2 x^2 + a_1 x + a_0$$

To satisfy the essential boundary condition, we must have

$$u(0) = 0 \Longrightarrow a_2 0 + a_1 0 + a_0 = 0 \Longrightarrow a_0 = 0$$

Thus the admissible assumed solution is

$$u(x) = a_2 x^2 + a_1 x; \qquad \frac{du}{dx} = 2a_2 x + a_1$$

Substituting into the weak form, we have

$$Pw_i(L) + \int_0^L \left(-AE(2a_2 x + a_1)\frac{dw_i}{dx} + cxw_i\right) dx = 0$$

There are two unknown parameters left in the solution and therefore we need two equations to find them. We get these equations by using the two Galerkin weighting functions

$$w_1 = \frac{\partial u}{\partial a_1} = x; \qquad \frac{\partial w_1}{\partial x} = 1; \qquad w_1(0) = 0; \qquad w_1(L) = L$$

$$w_2 = \frac{\partial u}{\partial a_2} = x^2; \qquad \frac{\partial w_2}{\partial x} = 2x; \qquad w_2(0) = 0; \qquad w_2(L) = L^2$$

Substituting w_1 into the weak form, we have

$$PL + \int_0^L (-AE(2a_2 x + a_1) + cx^2)\, dx = 0 \Longrightarrow \frac{cL^3}{3} - EAa_2 L^2 + PL - EAa_1 L = 0$$

Substituting w_2 into the weak form, we have

$$PL^2 + \int_0^L (-AE(2a_2 x + a_1)(2x) + cx^3)\, dx = 0 \Longrightarrow \frac{cL^4}{4} - \frac{4}{3}EAa_2 L^3 + PL^2 - EAa_1 L^2 = 0$$

Solving the two equations, we have

$$a_1 = -\frac{-7cL^2 - 12P}{12EA}; \qquad a_2 = -\frac{cL}{4EA}$$

Thus a quadratic approximate solution is as follows:

$$u(x) = a_2 x^2 + a_1 x = -\frac{cL}{4EA}x^2 - \frac{-7cL^2 - 12P}{12EA}x = \frac{(12P + cL(7L - 3x))x}{12EA}$$

Cubic Solution The exact solution of the problem is a cubic polynomial. As demonstrated below, the Galerkin method finds an exact solution if we start with a cubic polynomial:

$$u(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$$

To satisfy the essential boundary condition, we must have

$$u(0) = 0 \Longrightarrow a_3 0 + a_2 0 + a_1 0 + a_0 = 0 \Longrightarrow a_0 = 0$$

Thus the admissible assumed solution is

$$u(x) = a_3 x^3 + a_2 x^2 + a_1 x; \qquad \frac{du}{dx} = 3a_3 x^2 + 2a_2 x + a_1$$

Substituting into the weak form, we have

$$Pw_i(L) + \int_0^L \left(-AE(3a_3 x^2 + 2a_2 x + a_1)\frac{dw_i}{dx} + cxw_i\right) dx = 0$$

There are three unknown parameters left, which we find by using the three Galerkin weighting functions into the weak form, giving the following equations:

$$w_1 = \frac{\partial u}{\partial a_1} = x; \quad \frac{\partial w_1}{\partial x} = 1; \quad w_1(0) = 0; \quad w_1(L) = L$$
$$w_2 = \frac{\partial u}{\partial a_2} = x^2; \quad \frac{\partial w_2}{\partial x} = 2x; \quad w_2(0) = 0; \quad w_2(L) = L^2$$
$$w_3 = \frac{\partial u}{\partial a_3} = x^3; \quad \frac{\partial w_3}{\partial x} = 3x^2; \quad w_3(0) = 0; \quad w_3(L) = L^3$$

Substitute admissible solution and weights into the weak form and perform integrations to get the following:

	Weight	Equation
1	x	$\frac{1}{3}(c - 3EAa_3)L^3 - EAa_2L^2 + PL - EAa_1L = 0$
2	x^2	$\frac{1}{4}(c - 6EAa_3)L^4 - \frac{4}{3}EAa_2L^3 + PL^2 - EAa_1L^2 = 0$
3	x^3	$\frac{1}{5}(c - 9EAa_3)L^5 - \frac{3}{2}EAa_2L^4 + PL^3 - EAa_1L^3 = 0$

Solving these equations,

$$a_1 = -\frac{-cL^2 - 2P}{2EA}, \qquad a_2 = 0, \qquad a_3 = -\frac{c}{6EA}$$

Thus a cubic solution is as follows:

$$u(x) = a_3x^3 + a_2x^2 + a_1x = -\frac{c}{6EA}x^3 - \frac{-cL^2 - 2P}{2EA}x = \frac{x(3cL^2 - cx^2 + 6P)}{6EA}$$

Since this is the exact solution, trying any higher order polynomial will not make any difference.

Carefully note the distinction between the two types of boundary conditions. To make the assumed solution admissible, we required it to satisfy only the displacement boundary condition [$u(0) = 0$]. We never explicitly require the assumed solution to satisfy the force boundary condition. We can easily see that for this example the linear and quadratic solutions in fact do not satisfy this boundary condition. Only the cubic solution satisfies this condition exactly.

	$u(x)$	$\partial u/\partial x$	$\partial u(L)/\partial x$	$EA\,\partial u(L)/\partial x$
Linear	$\frac{(cL^2 + 3P)x}{3EA}$	$\frac{cL^2 + 3P}{3EA}$	$\frac{cL^2 + 3P}{3EA}$	$\frac{cL^2}{3} + P$
Quadratic	$\frac{(12P + cL(7L - 3x))x}{12EA}$	$\frac{12P + cL(7L - 3x)}{12EA} - \frac{cLx}{4EA}$	$\frac{4cL^2 + 12P}{12EA} - \frac{cL^2}{4EA}$	$\frac{cL^2}{12} + P$
Cubic	$\frac{x(3cL^2 - cx^2 + 6P)}{6EA}$	$\frac{3cL^2 - cx^2 + 6P}{6EA} - \frac{cx^2}{3EA}$	$\frac{2cL^2 + 6P}{6EA} - \frac{cL^2}{3EA}$	P

The logic in using the terminology *essential* and *natural* boundary conditions should now be clear. Essential boundary conditions are those that must explicitly be satisfied by the assumed solution while the natural boundary conditions are only implicitly satisfied through the weak form. A solution that satisfies the differential equation and all boundary conditions is obviously the exact solution.

2.2.3 Tapered Bar Subjected to Linearly Varying Axial Load

Consider now the tapered bar fixed at one end and subjected to a static point load at the other end, as shown in Figure 2.4. The bar is also subjected to a linearly varying axial load $q(x) = cx$, where c is a given constant. The problem is described in terms of the following boundary value problem:

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) + cx = 0; \quad 0 < x < L$$
$$A(x) = A_0 - \frac{A_0 - A_L}{L}x = \frac{(L + (-1 + r)x)A_0}{L}$$
$$u(0) = 0; \qquad ErA_0\frac{du(L)}{dx} = P$$

where A_0 is the area of cross section at $x = 0$, A_L is that at $x = L$, and $r = A_L/A_0$. The weak form specific to this problem is the same as that for the uniform bar in the previous section, except that A is now a function of x:

$$Pw_i(L) + \int_0^L \left(-AE\frac{du}{dx}\frac{dw_i}{dx} + cxw_i(x)\right)dx = 0$$

EBC: $\quad u(0) = 0$

Linear Solution Starting assumed solution: $u(x) = a_0 + xa_1$

The admissible solution must satisfy EBC:

EBC	Equation
$u(0) = 0$	$a_0 = 0$

Thus the admissible assumed solution is $u(x) = xa_1$.
Weighting function $\to \{x\}$
Substitute into the weak form and perform integrations to get:

Weight	Equation
x	$LP + \frac{cL^4/3 - (1/2)rEa_1A_0L^2 - (1/2)Ea_1A_0L^2}{L} = 0$

Solving this equation,

$$a_1 = \frac{-(cL^3/3) - PL}{-(1/2)LEA_0 - (1/2)LrEA_0}$$

Substituting into the admissible solution, we get the following solution of the problem:

$$u(x) = \frac{2cxL^2 + 6Px}{3rEA_0 + 3EA_0}$$

Quadratic Solution Starting assumed solution: $u(x) = a_2x^2 + a_1x + a_0$

The admissible solution must satisfy the EBC:

EBC	Equation
$u(0) = 0$	$a_0 = 0$

Thus the admissible assumed solution is $u(x) = a_2x^2 + a_1x$.
Weighting functions $\to \{x, x^2\}$
Substitute into the weak form and perform integrations to get:

Weight	Equation
x	$LP + \frac{(cL^4/3) - (2/3)rEa_2A_0L^3 - (1/3)Ea_2A_0L^3 - (1/2)rEa_1A_0L^2 - (1/2)Ea_1A_0L^2}{L} = 0$
x^2	$PL^2 + \frac{(cL^5/4) - rEa_2A_0L^4 - (1/3)Ea_2A_0L^4 - (2/3)rEa_1A_0L^3 - (1/3)Ea_1A_0L^3}{L} = 0$

Solving these equations,

$$a_1 = -\frac{-cL^2 - 6crL^2 - 12Pr}{2(r^2 + 4r + 1)EA_0}; \qquad a_2 = -\frac{-cL^2 + 7crL^2 - 12P + 12Pr}{4L(r^2 + 4r + 1)EA_0}$$

Substituting into the admissible solution, we get the following solution of the problem:

$$u(x) = \frac{x(c(2L(6r + 1) - 7rx + x)L^2 + 12P(2Lr - xr + x))}{4L(r^2 + 4r + 1)EA_0}$$

The exact solution of the problem was derived earlier. The linear and quadratic solutions are compared with the exact solution in Figure 2.6. The following numerical values are used:

$$c = 0; \quad P = 1; \quad A_0 = 1; \quad L = 1; \quad E = 1; \quad r = \tfrac{1}{2}$$

The quadratic solution is reasonably close to the exact solution. Of course the solution can be improved further by starting with a cubic or a higher-order polynomial.

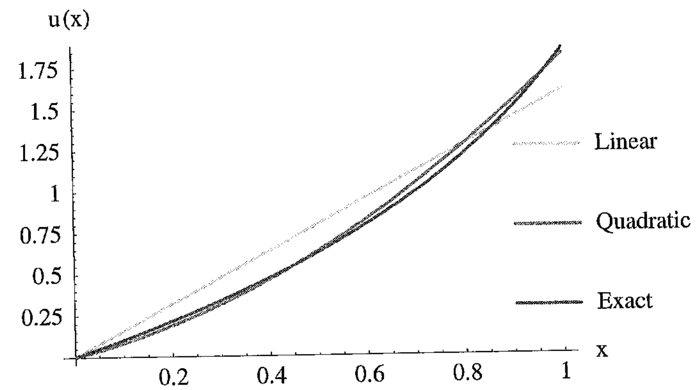

Figure 2.6. Solutions of tapered axially loaded bar subjected to end load

2.3 ONE-DIMENSIONAL BVP USING GALERKIN METHOD

So far only the axial deformation problem has been used to illustrate the Galerkin method. The method in fact is applicable to any differential equation. It is particularly well suited to boundary value problems (BVPs) in which a solution must satisfy differential equations and several boundary conditions over the domain. This section first summarizes the overall solution procedure and then presents several examples of finding approximate solutions of one-dimensional boundary value problems.

2.3.1 Overall Solution Procedure Using Galerkin Method

Given a boundary value problem, the overall procedure for obtaining an approximate solution using the Galerkin method is summarized here. Several examples in this section further clarify the details.

(i) Construct the weak form.

1. Move all terms in the differential equation to the left-hand side. With an assumed solution the left-hand side now represents the residual $e(x)$.
2. Define the total weighted residual. It consists of integral of the left-hand side of the differential equation multiplied by a weighting function w_i.
3. Use integration by parts to incorporate boundary conditions. For a second-order problem, all terms involving second-order derivatives are integrated by parts once. The highest order of remaining terms in the residual should therefore be 1. For a fourth-order problem integration by parts is used twice on all terms involving fourth-order derivatives and once on the terms involving third-order derivatives. The highest order of remaining terms in the residual should therefore be 2.
4. Identify essential and natural boundary conditions. For second-order problems boundary conditions involving first-order derivatives are natural because they

2.7 FINITE ELEMENT SOLUTION OF AXIAL DEFORMATION PROBLEMS

As mentioned earlier, using the assumed solution in the form of interpolation functions, it is possible to perform computations independently for each element and then simply add the contributions from different elements. Thus we can employ the standard six-step finite element procedure to obtain approximate solutions:

1. Development of element equations
2. Discretization of solution domain into a finite element mesh
3. Assembly of element equations
4. Introduction of boundary conditions
5. Solution for nodal unknowns
6. Computation of solution and related quantities over each element

Except for the derivation of element equations, all steps have been discussed in detail in Chapter 1. In this section the procedure for deriving element equations is illustrated considering a two-node element for axial deformation problem. Several numerical examples are then presented that use these equations to solve a variety of axial deformation problems.

2.7.1 Two-Node Uniform Bar Element for Axial Deformations

The simplest element for axial deformations of bars is a two-node line element, as shown in Figure 2.19. The element extends from x_1 to x_2 and has a length $L = x_2 - x_1$. The circles represent nodes. The unknown solutions at the nodes are indicated by u_1 and u_2. In addition to the distributed load $q(x)$, the element may be subjected to concentrated axial loads P_i applied at the ends of the element.

With these assumptions the governing differential equation over an element is

$$\frac{d}{dx}\left(AE\frac{du}{dx}\right) + q = 0; \qquad x_1 < x < x_2$$

Any possible concentrated loads at element ends form natural boundary conditions. With the sign convention explained earlier we have

$$-AE\frac{du(x_1)}{dx} = P_1; \qquad AE\frac{du(x_2)}{dx} = P_2$$

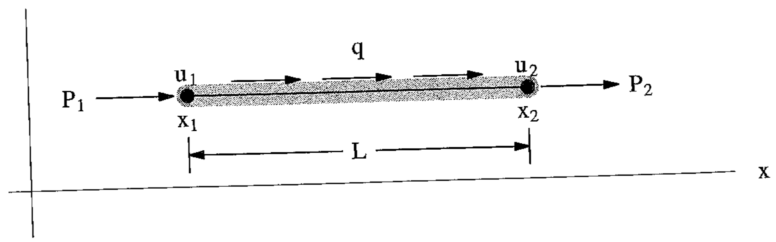

Figure 2.19. A simple two-node element for axial deformation

The primary unknown is the axial displacement u. Once the displacement is known, axial strain, stress, and force can be computed from the following relationships:

$$\epsilon_x = \frac{du}{dx}; \qquad \sigma_x = E\epsilon_x; \qquad F = A\sigma_x$$

Linear Assumed Solution The assumed solution is a linear interpolation between the nodal unknowns. Thus

$$u(x) = \begin{pmatrix} \frac{x - x_2}{x_1 - x_2} & \frac{x - x_1}{x_2 - x_1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} N_1 & N_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \equiv \boldsymbol{N}^T \boldsymbol{d}$$

$$N_1 = \frac{x - x_2}{x_1 - x_2} = -\frac{x - x_2}{L}; \qquad N_2 = \frac{x - x_1}{x_2 - x_1} = \frac{x - x_1}{L}$$

We will need $u'(x)$ in the later derivation. Differentiating with respect to x, we get

$$u'(x) = \frac{du(x)}{dx} = \begin{pmatrix} \frac{1}{x_1 - x_2} & \frac{1}{x_2 - x_1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} N_1' & N_2' \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \equiv \boldsymbol{B}^T \boldsymbol{d}$$

$$N_1' = \frac{1}{x_1 - x_2} = -\frac{1}{L}; \qquad N_2' = \frac{1}{x_2 - x_1} = \frac{1}{L}$$

Element Equations Using Galerkin Method The weak form can be written using standard steps of writing the weighted residual, integration by parts, and incorporating the natural boundary conditions. The weighting functions are the same as the interpolation functions N_i.

With $u(x)$ an assumed solution the residual is

$$e(x) = q + \frac{d}{dx}(AEu')$$

Multiplying by $N_i(x)$ and writing the integral over the given limits, the Galerkin weighted residual is

$$\int_{x_1}^{x_2} \left(qN_i + \frac{d}{dx}(AEu')N_i\right) dx = 0$$

Using integration by parts,

$$AEN_i(x_2)u'(x_2) - AEN_i(x_1)u'(x_1) + \int_{x_1}^{x_2} (qN_i - AEu'N_i')\, dx = 0$$

Given the NBC for the problem,

$$-P_1 - AEu'(x_1) = 0; \qquad AEu'(x_2) - P_2 = 0$$

Rearranging,

$$u'(x_1) \to -\frac{P_1}{AE}; \qquad u'(x_2) \to \frac{P_2}{AE}$$

Thus the boundary terms in the weak form reduce to

$$P_1 N_i(x_1) + P_2 N_i(x_2)$$

Assuming admissible solutions, the final weak form is as follows:

$$P_1 N_i(x_1) + P_2 N_i(x_2) + \int_{x_1}^{x_2} (qN_i - AEu'N_i')\,dx = 0$$

With the two interpolation functions, the two equations for the element are as follows:

$$\int_{x_1}^{x_2} (qN_1(x) - AEu'(x)N_1'(x))\,dx + P_1 N_1(x_1) + P_2 N_1(x_2) = 0$$

$$\int_{x_1}^{x_2} (qN_2(x) - AEu'(x)N_2'(x))\,dx + P_1 N_2(x_1) + P_2 N_2(x_2) = 0$$

Noting the property of the Lagrange interpolation functions that $N_1(x_1) = 1$, $N_1(x_2) = 0$, $N_2(x_1) = 0$, and $N_2(x_2) = 1$ and writing the two equations together using matrix notation, we have

$$\int_{x_1}^{x_2} \left(\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} q - \begin{pmatrix} N_1' \\ N_2' \end{pmatrix} AEu'(x) \right) dx + \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Substituting for $u'(x)$, we have

$$\int_{x_1}^{x_2} \left(\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} q - \begin{pmatrix} N_1' \\ N_2' \end{pmatrix} AE \begin{pmatrix} N_1' & N_2' \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) dx + \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\int_{x_1}^{x_2} \boldsymbol{N} q\,dx - \int_{x_1}^{x_2} \boldsymbol{BAEB}^T \boldsymbol{d}\,dx + \boldsymbol{r}_p = \boldsymbol{0}$$

where $\boldsymbol{r}_p$ is the 2×1 vector of element nodal loads $(P_1, P_2)^T$ and $\boldsymbol{0}$ is a 2×1 vector of zeros. Rearranging terms and taking nodal degrees of freedom out of the integral sign because they are not functions of x, we have

$$\int_{x_1}^{x_2} \boldsymbol{BAEB}^T\,dx\,\boldsymbol{d} = \int_{x_1}^{x_2} \boldsymbol{N} q\,dx + \boldsymbol{r}_p \Longrightarrow \boldsymbol{kd} = \boldsymbol{r}_q + \boldsymbol{r}_p$$

where

$$\boldsymbol{k} = \int_{x_1}^{x_2} \boldsymbol{BAEB}^T\,dx; \qquad \boldsymbol{r}_q = \int_{x_1}^{x_2} \boldsymbol{N} q\,dx; \qquad \boldsymbol{r}_p = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

Note the careful arrangement of terms when the two equations were written together. Since $AEu'(x)$ is a scalar, the second term inside the integral of the weak form can be written in two different ways as follows:

$$\int_{x_1}^{x_2} \begin{pmatrix} N_1' \\ N_2' \end{pmatrix} AEu'(x)\,dx \qquad \text{or} \qquad \int_{x_1}^{x_2} AEu'(x) \begin{pmatrix} N_1' \\ N_2' \end{pmatrix} dx$$

The first form was used above to get the equations in the convenient form presented. With the second form we have the following situation:

$$\int_{x_1}^{x_2} AEu'(x) \begin{pmatrix} N_1' \\ N_2' \end{pmatrix} dx = \int_{x_1}^{x_2} AE\boldsymbol{B}^T \boldsymbol{dB}\,dx$$

Notice that now $\boldsymbol{d}$ cannot be taken out of the integral sign (recall that with matrices $\boldsymbol{dB} \neq \boldsymbol{Bd}$) and thus the form is not suitable for writing element equations in a compact form as a system of linear equations. We must use the first form and take unknown nodal values out so that the remaining terms can be integrated to give element equations.

To write explicit equations, we must substitute derivatives of the interpolation functions and carry out integrations. For simplicity, it is assumed that EA and q are constant over an element. Thus we have

$$\boldsymbol{k} = \int_{x_1}^{x_2} \boldsymbol{BAEB}^T\,dx = \begin{pmatrix} \int_{x_1}^{x_2} AE \frac{1}{L^2}\,dx & -\int_{x_1}^{x_2} AE \frac{1}{L^2}\,dx \\ -\int_{x_1}^{x_2} AE \frac{1}{L^2}\,dx & \int_{x_1}^{x_2} AE \frac{1}{L^2}\,dx \end{pmatrix} = \frac{AE}{L} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\boldsymbol{r}_q = \int_{x_1}^{x_2} \boldsymbol{N} q\,dx = \begin{pmatrix} \int_{x_1}^{x_2} -\frac{x-x_2}{L} q\,dx \\ \int_{x_1}^{x_2} \frac{x-x_1}{L} q\,dx \end{pmatrix} = \frac{qL}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We now have the two-node element equations for the axial deformation problem:

$$\frac{AE}{L} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{qL}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

$$\boldsymbol{kd} = \boldsymbol{r}_q + \boldsymbol{r}_p$$

$$\boldsymbol{k} = \frac{AE}{L} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}; \qquad \boldsymbol{r}_q = \frac{qL}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \qquad \boldsymbol{r}_p = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

Element Equations Using Rayleigh-Ritz Method The Galerkin method was used to derive element equations in the previous section. The same equations can also be derived using the Rayleigh-Ritz method. The potential energy function for the element is as follows:

$$\Pi = U - W = \frac{1}{2} \int_{x_1}^{x_2} AE\,(u'(x))^2\,dx - \int_{x_1}^{x_2} qu\,dx - P_1 u_1 - P_2 u_2$$

The square of the first derivative of the assumed solution must be written carefully so that the nodal unknowns can be taken out of the integral sign. In order to achieve this goal we proceed as follows:

$$u'(x) = \boldsymbol{B}^T \boldsymbol{d}$$
$$(u'(x))^2 = (u'(x))^T u'(x) = (\boldsymbol{B}^T \boldsymbol{d})^T \boldsymbol{B}^T \boldsymbol{d} = \boldsymbol{d}^T \boldsymbol{B}\boldsymbol{B}^T \boldsymbol{d}$$

where we have used the rule for the transpose of a product of two matrices $(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$. The strain energy term can now be evaluated as follows:

$$U = \tfrac{1}{2}\int_{x_1}^{x_2} AE\boldsymbol{d}^T\boldsymbol{B}\boldsymbol{B}^T\boldsymbol{d}\,dx = \tfrac{1}{2}\boldsymbol{d}^T\int_{x_1}^{x_2} AE\boldsymbol{B}\boldsymbol{B}^T\,dx\,\boldsymbol{d} \equiv \tfrac{1}{2}\boldsymbol{d}^T\boldsymbol{k}\boldsymbol{d}$$

where

$$\boldsymbol{k} = \int_{x_1}^{x_2} AE\boldsymbol{B}\boldsymbol{B}^T\,dx = \begin{pmatrix} \int_{x_1}^{x_2} AE\frac{1}{L^2}\,dx & -\int_{x_1}^{x_2} AE\frac{1}{L^2}\,dx \\ -\int_{x_1}^{x_2} AE\frac{1}{L^2}\,dx & \int_{x_1}^{x_2} AE\frac{1}{L^2}\,dx \end{pmatrix} = \frac{AE}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

The work done by the distributed load can be evaluated as follows:

$$W_q = \int_{x_1}^{x_2} qu\,dx = \int_{x_1}^{x_2} q\boldsymbol{N}^T\boldsymbol{d}\,dx = \boldsymbol{r}_q^T\boldsymbol{d} \equiv \boldsymbol{d}^T\boldsymbol{r}_q$$

where

$$\boldsymbol{r}_q^T = \int_{x_1}^{x_2} q\boldsymbol{N}^T\,dx \Longrightarrow \boldsymbol{r}_q = \int_{x_1}^{x_2} \boldsymbol{N}q\,dx = \begin{pmatrix} \int_{x_1}^{x_2} -\frac{x-x_2}{L}q\,dx \\ \int_{x_1}^{x_2} \frac{x-x_1}{L}q\,dx \end{pmatrix} = \frac{qL}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The work done by the concentrated loads applied at element ends is

$$W_p = P_1u_1 - P_2u_2 = (P_1 \quad P_2)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \equiv \boldsymbol{r}_p^T\boldsymbol{d} \equiv \boldsymbol{d}^T\boldsymbol{r}_p$$

The potential energy can now be written as follows:

$$\Pi = U - W = \tfrac{1}{2}\boldsymbol{d}^T\boldsymbol{k}\boldsymbol{d} - \boldsymbol{r}_q^T\boldsymbol{d} - \boldsymbol{r}_p^T\boldsymbol{d} = \boldsymbol{d}^T\left(\tfrac{1}{2}\boldsymbol{k}\boldsymbol{d} - \boldsymbol{r}_q - \boldsymbol{r}_p\right)$$

The necessary conditions for the minimum of the potential energy give (see details below)

$$\frac{\partial\Pi}{\partial\boldsymbol{d}} = \left(\frac{1}{2}\boldsymbol{k}\boldsymbol{d} - \boldsymbol{r}_q - \boldsymbol{r}_p\right) + \frac{1}{2}\boldsymbol{k}\boldsymbol{d} = 0$$

Rearranging terms, we get the same element equations as those obtained by using the Galerkin method:

$$\boldsymbol{k}\boldsymbol{d} = \boldsymbol{r}_q + \boldsymbol{r}_p \Longrightarrow \frac{AE}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{qL}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

The detailed justification for the way the differentiation of the potential energy with respect to the nodal variables is carried out is as follows:

$$\Pi = \boldsymbol{d}^T\left(\tfrac{1}{2}\boldsymbol{k}\boldsymbol{d} - \boldsymbol{r}_q - \boldsymbol{r}_p\right) = (u_1 \quad u_2)\left(\frac{1}{2}\boldsymbol{k}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \boldsymbol{r}_q - \boldsymbol{r}_p\right)$$

Using the product rule of differentiation,

$$\frac{\partial\Pi}{\partial u_1} = (1 \quad 0)\left(\frac{1}{2}\boldsymbol{k}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \boldsymbol{r}_q - \boldsymbol{r}_p\right) + (u_1 \quad u_2)\left(\frac{1}{2}\boldsymbol{k}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = 0$$
$$\frac{\partial\Pi}{\partial u_2} = (0 \quad 1)\left(\frac{1}{2}\boldsymbol{k}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \boldsymbol{r}_q - \boldsymbol{r}_p\right) + (u_1 \quad u_2)\left(\frac{1}{2}\boldsymbol{k}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = 0$$

Combining the two equations together, we have

$$\begin{pmatrix} \frac{\partial\Pi}{\partial u_1} \\ \frac{\partial\Pi}{\partial u_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\left(\frac{1}{2}\boldsymbol{k}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \boldsymbol{r}_q - \boldsymbol{r}_p\right) + (u_1 \quad u_2)\left(\frac{1}{2}\boldsymbol{k}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\frac{\partial\Pi}{\partial\boldsymbol{d}} = \left(\frac{1}{2}\boldsymbol{k}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \boldsymbol{r}_q - \boldsymbol{r}_p\right) + (u_1 \quad u_2)\left(\frac{1}{2}\boldsymbol{k}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\frac{\partial\Pi}{\partial\boldsymbol{d}} = \left(\frac{1}{2}\boldsymbol{k}\boldsymbol{d} - \boldsymbol{r}_q - \boldsymbol{r}_p\right) + \boldsymbol{d}^T\left(\frac{1}{2}\boldsymbol{k}\right) = 0$$

Since $\boldsymbol{k}$ is a symmetric matrix, $\boldsymbol{d}^T\left(\tfrac{1}{2}\boldsymbol{k}\right) = \tfrac{1}{2}\boldsymbol{k}\boldsymbol{d}$, and we have

$$\frac{\partial\Pi}{\partial\boldsymbol{d}} = \left(\frac{1}{2}\boldsymbol{k}\boldsymbol{d} - \boldsymbol{r}_q - \boldsymbol{r}_p\right) + \frac{1}{2}\boldsymbol{k}\boldsymbol{d} = 0 \Longrightarrow \boldsymbol{k}\boldsymbol{d} = \boldsymbol{r}_q + \boldsymbol{r}_p$$

Thus, even though $\boldsymbol{d}$ is a vector, in the compact matrix form, the expression for the derivative of Π with respect to $\boldsymbol{d}$ appears as if $\boldsymbol{d}$ is a scalar variable. Based on this observation, in the remainder of the book, after writing the energy function, its derivatives with respect to the nodal variables will be written directly without going through a detailed justification every time.

2.7.2 Numerical Examples

Example 2.8 Column in a Multistory Building An interior column in a multistory building is subjected to axial loads from different floors, as shown in Figure 2.20. Determine axial displacements at the story levels. Using these displacements, compute the axial strain and stress distribution in the column. Use the following data:

Story height, h = 15 ft

Story loads: P_1 = 50,000 lb, $P_2 = P_3$ = 40,000 lb, P_4 = 35,000 lb

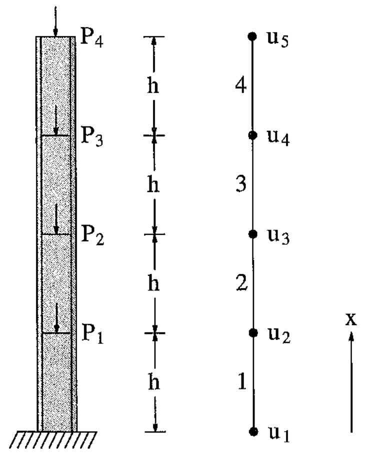

Figure 2.20. Column in a multistory building and a four-element model

Modulus of elasticity, $E = 29 \times 10^6$ lb/in^2
Area of cross section, $A = 21.8$ in^2

Since during the derivation of element equations it was assumed that the concentrated loads can only be applied at the element ends, we must divide the column into elements at the story levels. Thus the simplest possible finite element model is the four-element model shown in Figure 2.20.

The concentrated nodal loads will be added directly to the global equations after assembly. All elements have the same length and other properties. Thus the element equations are as follows:

$$\frac{EA}{h}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

Use pound-inches. The computed nodal displacements will be in inches and the stresses in pounds per square inch.

Nodal locations: {0, 180, 360, 540, 720}

Element 1:

$$\begin{pmatrix} 3.51222 \times 10^6 & -3.51222 \times 10^6 \\ -3.51222 \times 10^6 & 3.51222 \times 10^6 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Element 2:

$$\begin{pmatrix} 3.51222 \times 10^6 & -3.51222 \times 10^6 \\ -3.51222 \times 10^6 & 3.51222 \times 10^6 \end{pmatrix}\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Element 3:

$$\begin{pmatrix} 3.51222 \times 10^6 & -3.51222 \times 10^6 \\ -3.51222 \times 10^6 & 3.51222 \times 10^6 \end{pmatrix}\begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Element 4:

$$\begin{pmatrix} 3.51222 \times 10^6 & -3.51222 \times 10^6 \\ -3.51222 \times 10^6 & 3.51222 \times 10^6 \end{pmatrix}\begin{pmatrix} u_4 \\ u_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Global equations after assembly and incorporating the NBC (placing concentrated applied loads):

$$\begin{pmatrix} 3.51222 \times 10^6 & -3.51222 \times 10^6 & 0 & 0 & 0 \\ -3.51222 \times 10^6 & 7.02444 \times 10^6 & -3.51222 \times 10^6 & 0 & 0 \\ 0 & -3.51222 \times 10^6 & 7.02444 \times 10^6 & -3.51222 \times 10^6 & 0 \\ 0 & 0 & -3.51222 \times 10^6 & 7.02444 \times 10^6 & -3.51222 \times 10^6 \\ 0 & 0 & 0 & -3.51222 \times 10^6 & 3.51222 \times 10^6 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = \begin{pmatrix} 0 \\ -50000. \\ -40000. \\ -40000. \\ -35000. \end{pmatrix}$$

Essential boundary conditions:

dof	Value
u_1	0

Since the EBC is 0, we simply remove the first row and column to get the final system of equations as follows:

$$\begin{pmatrix} 7.02444 \times 10^6 & -3.51222 \times 10^6 & 0 & 0 \\ -3.51222 \times 10^6 & 7.02444 \times 10^6 & -3.51222 \times 10^6 & 0 \\ 0 & -3.51222 \times 10^6 & 7.02444 \times 10^6 & -3.51222 \times 10^6 \\ 0 & 0 & -3.51222 \times 10^6 & 3.51222 \times 10^6 \end{pmatrix}\begin{pmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = \begin{pmatrix} -50000. \\ -40000. \\ -40000. \\ -35000. \end{pmatrix}$$

Solution for nodal unknowns:

dof	x	Solution
u_1	0	0
u_2	180	−0.0469788
u_3	360	−0.0797216
u_4	540	−0.101076
u_5	720	−0.111041

Once the nodal displacements are known, the displacement over any element is obtained from the linear interpolation functions as follows:

$$u(x) = \begin{pmatrix} \frac{x - x_2}{x_1 - x_2} & \frac{x - x_1}{x_2 - x_1} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Element 1:

Nodes: $\{x_1 \to 0, x_2 \to 180\}$

Interpolation functions: $N^T = \left\{1 - \frac{x}{180}, \frac{x}{180}\right\}$

Nodal values: $d^T = \{0, -0.0469788\}$
Solution: $N^T d = -0.000260993x$

Element 2:
Nodes: $\{x_1 \to 180, x_2 \to 360\}$
Interpolation functions: $N^T = \{2 - \frac{x}{180}, \frac{x}{180} - 1\}$
Nodal values: $d^T = \{-0.0469788, -0.0797216\}$
Solution: $N^T d = -0.000181904x - 0.014236$

Element 3:
Nodes: $\{x_1 \to 360, x_2 \to 540\}$
Interpolation functions: $N^T = \{3 - \frac{x}{180}, \frac{x}{180} - 2\}$
Nodal values: $d^T = \{-0.0797216, -0.101076\}$
Solution: $N^T d = -0.000118633\,x - 0.0370136$

Element 4:
Nodes: $\{x_1 \to 540, x_2 \to 720\}$
Interpolation functions: $N^T = \{4 - \frac{x}{180}, \frac{x}{180} - 3\}$
Nodal values: $d^T = \{-0.101076, -0.111041\}$
Solution: $N^T d = -0.0000553622x - 0.07118$

Solution summary:

	Range	$u(x)$
1	$0 \le x \le 180$	$-0.000260993x$
2	$180 \le x \le 360$	$-0.000181904x - 0.014236$
3	$360 \le x \le 540$	$-0.000118633x - 0.0370136$
4	$540 \le x \le 720$	$-0.0000553622x - 0.07118$

From these displacements we get the following axial strains, stresses, and axial forces:

$$\epsilon_x = \frac{du}{dx}; \qquad \sigma_x = E\epsilon_x; \qquad F = A\sigma_x$$

	Range	ϵ	σ_x	F
1	$0 \le x \le 180$	−0.000260993	−7568.81	−165000.
2	$180 \le x \le 360$	−0.000181904	−5275.23	−115000.
3	$360 \le x \le 540$	−0.000118633	−3440.37	−75000.
4	$540 \le x \le 720$	−0.0000553622	−1605.5	−35000.

The negative sign indicates compression. The problem is statically determinate, and thus simple application of the static equilibrium condition indicates that the computed axial forces are exact.

Example 2.9 Tapered Bar Consider solution of the tapered axially loaded bar shown in Figure 2.21. Use a two-node uniform axial deformation element to model the bar and determine the axial stress and force distribution in the bar. Compute the support reactions

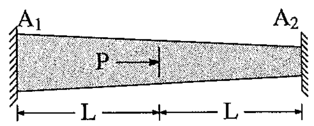

Figure 2.21. Tapered bar

from the axial force and see whether the overall equilibrium is satisfied. Comment on the quality of the finite element solution. Use the following numerical data:

$$E = 70\,\text{GPa}; \quad F = 20\,\text{kN}; \quad L = 300\,\text{mm}; \quad A_1 = 2400\,\text{mm}^2; \quad A_2 = 600\,\text{mm}^2$$

where A_1 and A_2 are the areas of cross section at the two ends of the bar. The area of cross section can be expressed as a linear function of x using the Lagrange interpolation formula as follows:

$$A(x) = \frac{x - 2L}{-2L}A_1 + \frac{x}{2L}A_2$$

Since the displacements are generally small, numerically it is convenient to use newton-millimeters. Then the computed nodal displacements are in millimeters and the stresses in megapascals.

A four-element model is as shown in Figure 2.22. Since the element equations derived earlier are based on the assumption of a uniform cross section, we must assign an average area to each element in the finite element model. Denoting the area of cross section at the left end of an element by A_l and that at the right end by A_r, the average area for each element is $(A_l + A_r)/2$. The concentrated nodal loads will be added directly to the global equations after assembly. Thus, with l the length of an element, the element equations are as follows:

$$\frac{(A_l + A_r)E}{2l}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u_l \\ u_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Nodal locations: {0, 150., 300., 450., 600.}
Areas at the nodes: {2400., 1950., 1500., 1050., 600.}
Average area for each element: {2175., 1725., 1275., 825.}

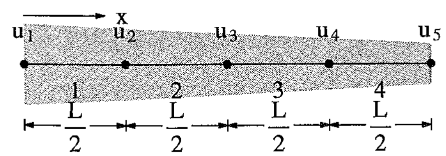

Figure 2.22. Four-element model for the tapered bar

Element 1:

$$\begin{pmatrix} 1.015\times 10^6 & -1.015\times 10^6 \\ -1.015\times 10^6 & 1.015\times 10^6 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Element 2:

$$\begin{pmatrix} 805000. & -805000. \\ -805000. & 805000. \end{pmatrix}\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Element 3:

$$\begin{pmatrix} 595000. & -595000. \\ -595000. & 595000. \end{pmatrix}\begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Element 4:

$$\begin{pmatrix} 385000. & -385000. \\ -385000. & 385000. \end{pmatrix}\begin{pmatrix} u_4 \\ u_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Global equations after incorporating the NBC:

$$\begin{pmatrix} 1.015\times 10^6 & -1.015\times 10^6 & 0 & 0 & 0 \\ -1.015\times 10^6 & 1.82\times 10^6 & -805000. & 0 & 0 \\ 0 & -805000. & 1.4\times 10^6 & -595000. & 0 \\ 0 & 0 & -595000. & 980000. & -385000. \\ 0 & 0 & 0 & -385000. & 385000. \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 20000 \\ 0 \\ 0 \end{pmatrix}$$

Essential boundary conditions:

dof	Value
u_1	0
u_5	0

Incorporating the EBC, the final system of equations is

$$\begin{pmatrix} 1.82\times 10^6 & -805000. & 0 \\ -805000. & 1.4\times 10^6 & -595000. \\ 0 & -595000. & 980000. \end{pmatrix}\begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0. \\ 20000. \\ 0. \end{pmatrix}$$

Solution for nodal unknowns:

dof	x	Solution
u_1	0	0
u_2	150.	0.0129577
u_3	300.	0.0292958
u_4	450.	0.0177867
u_5	600.	0

Solution over element 1:

Nodes: $\{x_1 \to 0, x_2 \to 150.\}$

Interpolation functions: $N^T = \{1. - 0.00666667x, 0.00666667x\}$

Nodal values: $d^T = \{0, 0.0129577\}$

Solution: $N^T d = 0.000086385x$

Solution over element 2:

Nodes: $\{x_1 \to 150., x_2 \to 300.\}$

Interpolation functions: $N^T = \{2. - 0.00666667x, 0.00666667x - 1.\}$

Nodal values: $d^T = \{0.0129577, 0.0292958\}$

Solution: $N^T d = 0.00010892x - 0.00338028$

Solution over element 3:

Nodes: $\{x_1 \to 300., x_2 \to 450.\}$

Interpolation functions: $N^T = \{3. - 0.00666667x, 0.00666667x - 2.\}$

Nodal values: $d^T = \{0.0292958, 0.0177867\}$

Solution: $N^T d = 0.0523139 - 0.000076727x$

Solution over element 4:

Nodes: $\{x_1 \to 450., x_2 \to 600.\}$

Interpolation functions: $N^T = \{4. - 0.00666667x, 0.00666667x - 3.\}$

Nodal values: $d^T = \{0.0177867, 0\}$

Solution: $N^T d = 0.0711469 - 0.000118578x$

Solution summary:

	Range	Solution
1	$0 \le x \le 150.$	$0.000086385x$
2	$150. \le x \le 300.$	$0.00010892x - 0.00338028$
3	$300. \le x \le 450.$	$0.0523139 - 0.000076727x$
4	$450. \le x \le 600.$	$0.0711469 - 0.000118578x$

From these displacements we get the following axial strains, stresses, and axial forces:

$$\epsilon_x = \frac{du}{dx}; \qquad \sigma_x = E\epsilon_x; \qquad F = A\sigma_x$$

	Range	ϵ	σ	F
1	$0 \le x \le 150.$	0.000086385	6.04695	13152.1
2	$150. \le x \le 300.$	0.00010892	7.62441	13152.1
3	$300. \le x \le 450.$	−0.000076727	−5.37089	−6847.89
4	$450. \le x \le 600.$	−0.000118578	−8.30047	−6847.89

The axial forces at the ends must balance the support reactions. With the sign convention for axial forces discussed earlier, the reaction at the left support is the negative of the axial

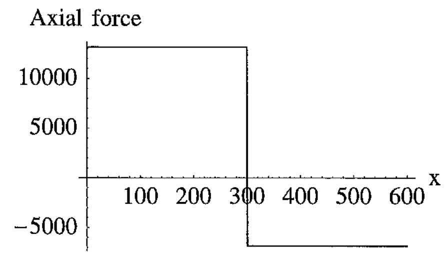

Figure 2.23. Four-element solution for the tapered bar—axial force distribution

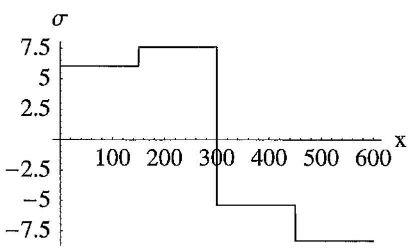

Figure 2.24. Four-element solution for the tapered bar—stress distribution

force at this point and that at the right support is equal to the axial force. Thus from the axial forces we get the support reactions as

$$\text{Reactions} = \{-13152.1, -6847.89\}$$

$$\text{Sum of reactions} = -20000.$$

The sum of reactions is equal and opposite to the applied load, and therefore the overall equilibrium is satisfied. Plots of the axial stress and axial force are shown in Figures 2.23 and 2.24. The axial force plot looks reasonable. In the stress plot we expect a discontinuity at the middle because of the concentrated applied force. However, the stress at nodes 2 and 4 should be continuous. A large discontinuity in the stress at these locations indicates that the solution is not very accurate.

The following solution is obtained using eight equal-length elements:

	Range	ϵ	σ	F
1	$0 \leq x \leq 75.$	0.0000824431	5.77101	13201.2
2	$75. \leq x \leq 150.$	0.0000914369	6.40058	13201.2
3	$150. \leq x \leq 225.$	0.000102633	7.18432	13201.2
4	$225. \leq x \leq 300.$	0.000116954	8.18679	13201.2
5	$300. \leq x \leq 375.$	−0.0000700005	−4.90004	−6798.8
6	$375. \leq x \leq 450.$	−0.000083549	−5.84843	−6798.8
7	$450. \leq x \leq 525.$	−0.000103601	−7.25206	−6798.8
8	$525. \leq x \leq 600.$	−0.000136317	−9.54218	−6798.8

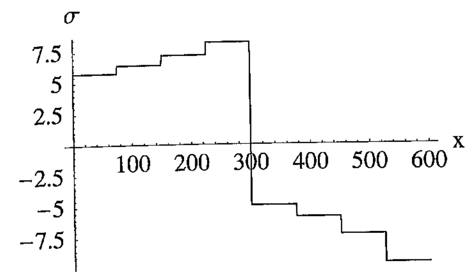

Figure 2.25. Eight-element solution for the tapered bar—stress distribution

The stresses are plotted in Figure 2.25. Because of the constant-area assumption over each element, we still have discontinuities in the stress. However, if we take the average of the stresses from the two elements at common nodes, the solution is very close to the exact solution:

```
avg = Map[Apply[Plus, #]/2&,
    {σ[[{1, 2}]], σ[[{2, 3}]], σ[[{3, 4}]], σ[[{5, 6}]],
    σ[[{6, 7}]], σ[[{7, 8}]]}]
```

{6.0858, 6.79245, 7.68556, −5.37424, −6.55024, −8.39712}

◆ *Mathematica*/MATLAB Implementation 2.4 on the Book Web Site: *Solution of axial deformation problems*

PROBLEMS

Exact Solution of Differential Equations

2.1 A uniform bar is subjected to uniform axial load along its length. The bar is fixed at the left end but there is a gap g between the support and the right end before the load is applied, as shown in the Figure 2.26. Assuming that the applied load is large enough to close the gap, write the governing differential equation with appropriate boundary conditions. Use direct integration to determine the exact solution of the

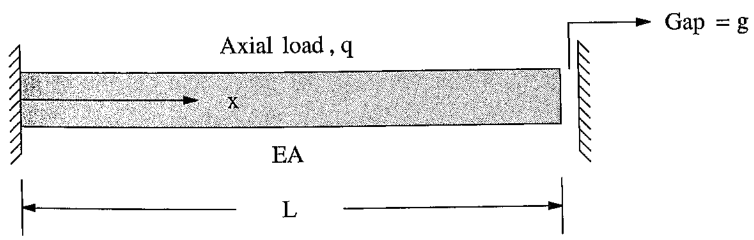

Figure 2.26. Axially loaded bar