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Complex Dynamics of a Three Species Ecosystem

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Abstract

This paper investigates the dynamics of a model system by modifying key assumptions to explore alternative ecological interactions. The original model, outlined in the paper [11], initially assumes a competition interaction between species X and Y. However, we consider the implications of assuming a mutualism interaction instead. Additionally, we modify the assumption of a commensalism interaction between species X and Z to examine the consequences of an amensalism interaction. Furthermore, the model is expanded by assuming a logistic growth pattern for species Z. Through these modifications, we determine the new equilibrium points and conduct a comprehensive analysis of their characteristics. Numerical computations are performed to assess the stability or instability of the newly derived equilibrium points. The obtained results offer valuable insights into the dynamics and stability of the modified model, shedding light on the consequences of mutualism and amensalism interactions in the examined ecological system. This research contributes to a deeper understanding of the interrelationships between species interactions and population dynamics in ecological systems.

1 Introduction

In biology, there are many ecological systems that can be analyzed through mathematical modeling. Through mathematical modeling, one can determine the behavior of a complex ecological system. Some instances of mathematical modeling being used in ecological systems are the model proposed by Lotka and Volterra [5], Rosenzweig-MacArthur [3], and Hassell [7]. All of which involved two species. However, mathematical modeling is not limited to modeling two species in an ecological system. There have been mathematical models proposed by authors who consider a system of three species [9, 4, 10].

2 Original Model

In this section, we will go over the model that was featured in the paper written by Panja, Gayen, Kar, and Jana [11]. The authors wanted to build a model of three species X, Y, Z which contains three different types of species interactions. Namely nonlinear competition, predation, and commensalism.

2.1 Assumptions on species X

The authors have assumed that X grows logistically at a rate of r_x with a maximum capacity of K_x. The authors assumed that X and Y have a competition interaction with an interspecies competition coefficient α_{xy}. The authors have also assumed that X and Z are in a commensalism interaction where Z is a host of X with a commensal coefficient δ. From this, the authors have created the differential equation to represent the population of X:

$$\frac{\mathrm{d}X}{\mathrm{d}T} = r_x X - \frac{r_x}{K_x}X^2 - \frac{r_x\alpha_{xy}}{K_x}XY^2 + \delta XZ \tag{2.1}$$

2.2 Assumptions on species Y

The authors have assumed that Y grows logistically at a rate of r_y with a maximum capacity of K_y. The authors assumed that Y and X have a competition interaction with an interspecies competition coefficient α_{yx}. The authors have also assumed that Y and Z are in a predation interaction where Z preys on Y via the Holling type II functional response with a half saturation constant b. Z attacks Y at a rate of a. Furthermore, as a result of being the prey to Z, Y shows refuge behavior with a refuge rate of p. From this, the authors have created the differential equation to represent the population of Y:

$$\frac{\mathrm{d}Y}{\mathrm{d}T} = r_y Y - \frac{r_y}{K_y} Y^2 - \frac{r_y \alpha_{yx}}{K_y} X^2 Y - \frac{a(1-p)YZ}{b+(1-p)Y} \tag{2.2}$$

2.3 Assumptions on species Z

The authors assumed that Z grows as a result of being in a predation interaction with Y via the Holling type II functional response with a half saturation constant b and c as the conservation rate of Y. Furthermore, the authors assume that the population of Z decays at a rate of e. From this, the authors have created the differential equation to represent the population of Z:

$$\frac{\mathrm{d}Z}{\mathrm{d}T} = \frac{ac(1-p)YZ}{b+(1-p)Y} - eZ \tag{2.3}$$

2.4 Building the model

From the assumptions, the governing equations for this model are:

$$\frac{\mathrm{d}X}{\mathrm{d}T} = r_x X - \frac{r_x}{K_x} X^2 - \frac{r_x \alpha_{xy}}{K_x} XY^2 + \delta XZ \tag{2.4a}$$

$$\frac{\mathrm{d}Y}{\mathrm{d}T} = r_y Y - \frac{r_y}{K_y} Y^2 - \frac{r_y \alpha_{yx}}{K_y} X^2 Y - \frac{a(1-p)YZ}{b+(1-p)Y} \tag{2.4b}$$

$$\frac{\mathrm{d}Z}{\mathrm{d}T} = \frac{ac(1-p)YZ}{b+(1-p)Y} - eZ \tag{2.4c}$$

and with the following substitutions:

$$\begin{gathered}
T = \frac{t}{r_x},\ X = K_x x,\ Y = K_y y,\ Z = \frac{r_x K_y}{a} z \\
\alpha_{xy} = \frac{K_x}{K_y^2}\gamma_{12},\ \alpha_{yx} = \frac{K_y}{K_x^2}\gamma_{21},\ \delta = \frac{a}{K_y}\gamma \\
r = \frac{r_y}{r_x},\ b = K_y v_1,\ e = r_x v_2,\ c = \frac{r_x v_3}{a}
\end{gathered}$$

the authors have simplified and non-dimensionalized Model (2.4). This gives us the following model the authors have used throughout their paper:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x\left(1 - x - \gamma_{12} y^2\right) + \gamma xz \tag{2.5a}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ry\left(1 - y - \gamma_{21} x^2\right) - \frac{(1-p)yz}{v_1+(1-p)y} \tag{2.5b}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = z\left(\frac{v_3(1-p)y}{v_1+(1-p)y} - v_2\right) \tag{2.5c}$$

3 Revised/Modified Model

In Section 2, we discussed the model the authors used in their paper and the assumptions that were used to construct it. In this section, we will modify some of the initial assumptions that the authors have made when creating their model to obtain a new model that we will analyze throughout this paper.

3.1 Modification 1: The interaction between species X and Y

The original model assumes that species X and Y are in a competition interaction. Instead of considering an interaction that negatively impacts both species, lets consider an interaction that benefits both species. Thus, we will assume that species X and Y are in a mutualism interaction.

3.2 Modification 2: The interaction between species X and Z

The original model assumes that species X and Z are in a commensalism interaction where species X is benefiting from species Z. Instead of considering an interaction where species X gets positively impacted, lets consider an interaction where species X gets negatively impacted while being in an interaction with species Z. Thus, we will assume that species X and Z are in an amensalism interaction.

3.3 Modification 3: The growth rate of species Z

The original model assumes that species Z grows solely due to being in a predation interaction with species Y. Here, we will assume that species Z not only grows due to being in a predation interaction with species Y, but also grows logistically with a rate of r_z with a capacity of K_z.

3.4 Building the modified model

With these modified assumptions, the authors' original model become:

$$\frac{\mathrm{d}X}{\mathrm{d}T} = r_x X - \frac{r_x}{K_x} X^2 + \frac{r_x \alpha_{xy}}{K_x} XY^2 - \delta XZ \tag{3.1a}$$

$$\frac{\mathrm{d}Y}{\mathrm{d}T} = r_y Y - \frac{r_y}{K_y} Y^2 + \frac{r_y \alpha_{yx}}{K_y} X^2 Y - \frac{a\,(1-p)\,YZ}{b + (1-p)\,Y} \tag{3.1b}$$

$$\frac{\mathrm{d}Z}{\mathrm{d}T} = r_z Z - \frac{r_z}{K_z} Z^2 + \frac{ac\,(1-p)\,YZ}{b + (1-p)\,Y} - eZ \tag{3.1c}$$

and with the following substitutions:

$$X = K_x x,\ Y = K_y y,\ Z = \frac{r_x K_y}{a} z,\ T = \frac{1}{r_x} t$$

$$r_1 = \frac{r_y}{r_x},\ r_2 = \frac{r_z}{r_x},\ v_1 = \frac{b}{K_y},\ v_2 = \frac{e}{r_x},\ v_3 = \frac{ac}{r_x}$$

$$\gamma_{12} = \frac{\alpha_{xy} K_y^2}{K_x},\ \gamma_{21} = \frac{\alpha_{yx} K_x^2}{K_y},\ \gamma_{13} = \frac{\delta K_y}{a},\ \gamma_{31} = \frac{r_x K_y}{a K_z}$$

we can simplify and non-dimensionalize Model (3.1). This gives us the following model we will work on throughout this paper:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x\left(1 - x + \gamma_{12} y^2\right) - \gamma_{13} xz \tag{3.2a}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = r_1 y\left(1 - y + \gamma_{21} x^2\right) - \frac{(1-p)\,yz}{v_1 + (1-p)\,y} \tag{3.2b}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = r_2 z\,(1 - \gamma_{31} z) + z\left(\frac{v_3\,(1-p)\,y}{v_1 + (1-p)\,y} - v_2\right) \tag{3.2c}$$

4 Identifying Equilibria

In Section 3, we have modified some of the assumptions the authors have made and created Model (3.2) based on the new/modified assumptions. In this section, we will find all the equilibria present in our model. To find all the equilibria (x^*, y^*, z^*) of this system, we will set all the equations in Model (3.1) equal to 0 [6].

$$0 = x^*\left(1 - x^* + \gamma_{12}\,(y^*)^2\right) - \gamma_{13} x^* z^* \tag{4.1a}$$

$$0 = r_1 y^*\left(1 - y^* + \gamma_{21}\,(x^*)^2\right) - \frac{(1-p)\,y^* z^*}{v_1 + (1-p)\,y^*} \tag{4.1b}$$

$$0 = r_2 z^*\,(1 - \gamma_{31} z^*) + z^*\left(\frac{v_3\,(1-p)\,y^*}{v_1 + (1-p)\,y^*} - v_2\right) \tag{4.1c}$$

4.1 Case 1: The trivial equilibrium

The trivial equilibrium is where $(x^*.y^*.z^*) = (0,0,0)$. To verify that the trivial equilibrium exists in Model (3.2), we will plug in $x^* = 0$, $y^* = 0$, and $z^* = 0$ into the model and must conclude that each equation reduces to $0 = 0$. Plugging $x^* = 0$, $y^* = 0$, and $z^* = 0$ into Model (3.2) yields:

$$
\begin{aligned}
0 &= (0)\left(1 - (0) + \gamma_{12}(0)^2\right) - \gamma_{13}(0)(0) \\
0 &= r_1(0)\left(1 - (0) + \gamma_{21}(0)^2\right) - \frac{(1-p)\,(0)(0)}{v_1 + (1-p)\,(0)} \\
0 &= r_2(0)\,(1 - \gamma_{31}(0)) + (0)\left(\frac{v_3\,(1-p)\,(0)}{v_1 + (1-p)\,(0)} - v_2\right)
\end{aligned}
$$

By inspection, we can see that each equation reduces to $0 = 0$. Thus, we can say that the trivial equilibrium exists:

$$\boxed{E_0 = (0,0,0)}$$

4.2 Case 2: The x-axial equilibrium

The x-axial equilibrium is an equilibrium where x^* is non-zero and the other components of the equilibrium are 0. Since we are talking about population densities, it does not make sense to consider values of x^* where $x^* < 0$. Thus, the conditions to impose when finding the x-axial equilibrium are $x^* > 0$ and all other components are 0. To find the x-axial equilibrium of Model (3.2), we will plug in $y^* = 0$ and $z^* = 0$:

$$
\begin{aligned}
0 &= x^*\left(1 - x^* + \gamma_{12}(0)^2\right) - \gamma_{13}x^*(0) \\
0 &= r_1(0)\left(1 - (0) + \gamma_{21}\left(x^*\right)^2\right) - \frac{(1-p)\,(0)(0)}{v_1 + (1-p)\,(0)} \\
0 &= r_2(0)\,(1 - \gamma_{31}(0)) + (0)\left(\frac{v_3\,(1-p)\,(0)}{v_1 + (1-p)\,(0)} - v_2\right)
\end{aligned}
$$

which reduces to:

$$0 = x^*\,(1 - x^*) \tag{4.2}$$

Equation (4.2) yields solutions $x^* = \{0,1\}$. However, due the condition $x^* > 0$ we imposed, the only valid solution to consider is $x^* = 1$. With this value of x^*, we have concluded that the x-axial equilibrium exists:

$$\boxed{E_x = (1,0,0)}$$

4.3 Case 3: The y-axial equilibrium

The y-axial equilibrium is an equilibrium where y^* is non-zero and the other components of the equilibrium are 0. Since we are talking about population densities, it does not make sense to consider values of y^* where $y^* < 0$. Thus, the conditions to impose when finding the y-axial equilibrium are $y^* > 0$ and all other components are 0. To find the y-axial equilibrium of Model (3.2), we will plug in $x^* = 0$ and $z^* = 0$:

$$
\begin{aligned}
0 &= (0)\left(1 - (0) + \gamma_{12}\left(y^*\right)^2\right) - \gamma_{13}(0)(0) \\
0 &= r_1 y^*\left(1 - y^* + \gamma_{21}(0)^2 0\right) - \frac{(1-p)\,y^*(0)}{v_1 + (1-p)\,y^*} \\
0 &= r_2(0)\,(1 - \gamma_{31}(0)) + (0)\left(\frac{v_3\,(1-p)\,y^*}{v_1 + (1-p)\,y^*} - v_2\right)
\end{aligned}
$$

which reduces to:

$$r_1 y^*\,(1 - y^*) = 0 \tag{4.3}$$

Since all parameters are positive, Equation (4.3) yields solutions $y^* = \{0,1\}$. However, due the condition $y^* > 0$ we imposed, the only valid solution to consider is $y^* = 1$. With this value of y^*, we have concluded that the y-axial equilibrium exists:

$$\boxed{E_y = (0,1,0)}$$

4.4 Case 4: The z-axial equilibrium

The z-axial equilibrium is an equilibrium where z^* is non-zero and the other components of the equilibrium are 0. Since we are talking about population densities, it does not make sense to consider values of z^* where $z^* < 0$. Thus, the conditions to impose when finding the z-axial equilibrium are $z^* > 0$ and all other components are 0. To find the z-axial equilibrium of Model (3.2), we will plug in $x^* = 0$ and $y^* = 0$:

$$\begin{aligned}
0 &= (0)\left(1-(0)+\gamma_{12}(0)^2\right) - \gamma_{13}(0)z^* \\
0 &= r_1(0)\left(1-(0)+\gamma_{21}(0)^2\right) - \frac{(1-p)\,(0)z^*}{v_1+(1-p)\,(0)} \\
0 &= r_2 z^*\left(1-\gamma_{31}z^*\right) + z^*\left(\frac{v_3\,(1-p)\,(0)}{v_1+(1-p)\,(0)} - v_2\right)
\end{aligned}$$

which reduces to:

$$r_2 z^*\left(1-\gamma_{31}z^*\right) - v_2 z^* = 0 \tag{4.4}$$

Since all parameters are positive, Equation (4.3) yields solutions

$$z^* = \left\{0, \frac{r_2 - v_2}{\gamma_{31} r_2}\right\}$$

However, due the condition $z^* > 0$ we imposed, the only valid solution to consider is:

$$z^* = \frac{r_2 - v_2}{\gamma_{31} r_2}$$

With this value of z^*, we have concluded that the z-axial equilibrium exists:

$$\boxed{E_z = \left(0, 0, \frac{r_2 - v_2}{\gamma_{31} r_2}\right)}$$

under the condition:

$$r_2 > v_2$$

4.5 Case 5: The xy-boundary equilibrium

The xy-boundary equilibrium is an equilibrium where x^* and y^* are non-zero and all other components of the equilibrium are 0. Since we are talking about population densities, it does not make sense to consider values of x^* and y^* where $x^* < 0$ and $y^* < 0$. Thus, the conditions to impose when finding the xy-boundary equilibrium are $x^* > 0$ and $y^* > 0$ and all other components are 0. To find the xy-boundary equilibrium of Model (3.2), we will plug in $z^* = 0$:

$$\begin{aligned}
0 &= x^*\left(1-x^*+\gamma_{12}\left(y^*\right)^2\right) - \gamma_{13}x^*(0) \\
0 &= r_1 y^*\left(1-y^*+\gamma_{21}\left(x^*\right)^2\right) - \frac{(1-p)\,y^*(0)}{v_1+(1-p)\,y^*} \\
0 &= r_2(0)\left(1-\gamma_{31}(0)\right) + (0)\left(\frac{v_3\,(1-p)\,y^*}{v_1+(1-p)\,y^*} - v_2\right)
\end{aligned}$$

which reduces to:

$$0 = x^*\left(1-x^*+\gamma_{12}\left(y^*\right)^2\right) \tag{4.5a}$$

$$0 = r_1 y^*\left(1-y^*+\gamma_{21}\left(x^*\right)^2\right) \tag{4.5b}$$

With the conditions $x^* > 0$ and $y^* > 0$ we imposed, the System (4.5) reduces to:

$$0 = 1 - x^* + \gamma_{12}\left(y^*\right)^2 \tag{4.6a}$$

$$0 = 1 - y^* + \gamma_{21}\left(x^*\right)^2 \tag{4.6b}$$

We can solve for x^* in Equation (4.6a):

$$x^* = 1 + \gamma_{12} (y^*)^2 \tag{4.7}$$

and plug Equation (4.7) into Equation (4.6b) to get:

$$\gamma_{12}^2 \gamma_{21} (y^*)^4 + 2\gamma_{12}\gamma_{21} (y^*)^2 - y^* + (\gamma_{21} + 1) = 0 \tag{4.8}$$

There isn't a nice closed-form solution to Equation (4.8) so we will show that there exist a y^* such that Equation (4.8) is satisfied. To do this, we will use the intermediate value theorem [8]. Let

$$f(y^*) = \gamma_{12}^2 \gamma_{21} (y^*)^4 + 2\gamma_{12}\gamma_{21} (y^*)^2 - y^* + (\gamma_{21} + 1)$$

Here, we can see that $f(0) = \gamma_{21} + 1$. Since all parameters are positive, this means that $f(0) > 0$. Let $f(\beta) < 0$ for some value of β. $f(\beta) < 0$ implies:

$$\gamma_{12} < \frac{1}{\beta^2} \left(\sqrt{\frac{\beta - 1}{\gamma_{21}}} - 1 \right)$$

Therefore we can say that the xy-boundary equilibrium $E_{xy} = (\hat{x}, \hat{y}, 0)$ exists where

$$\hat{x} = 1 + \gamma_{12} (\hat{y})^2$$

and y^* is a positive solution to the equation:

$$\gamma_{12}^2 \gamma_{21} (\hat{y})^4 + 2\gamma_{12}\gamma_{21} (\hat{y})^2 - \hat{y} + \gamma_{21} + 1 = 0$$

if the following condition is satisfied for some value of $\beta > 0$:

$$\gamma_{12} < \frac{1}{\beta^2} \left(\sqrt{\frac{\beta - 1}{\gamma_{21}}} - 1 \right)$$

4.6 Case 6: The xz-boundary equilibrium

The xz-boundary equilibrium is an equilibrium where x^* and z^* are non-zero and all other components of the equilibrium are 0. Since we are talking about population densities, it does not make sense to consider values of x^* and z^* where $x^* < 0$ and $z^* < 0$. Thus, the conditions to impose when finding the xz-boundary equilibrium are $x^* > 0$ and $z^* > 0$ and all other components are 0. To find the xz-boundary equilibrium of Model (3.2), we will plug in $y^* = 0$:

$$\begin{aligned}
0 &= x^* \left(1 - x^* + \gamma_{12}(0)^2\right) - \gamma_{13} x^* z^* \\
0 &= r_1(0) \left(1 - (0) + \gamma_{21} (x^*)^2\right) - \frac{(1-p)(0) z^*}{v_1 + (1-p)(0)} \\
0 &= r_2 z^* (1 - \gamma_{31} z^*) + z^* \left(\frac{v_3 (1-p)(0)}{v_1 + (1-p)(0)} - v_2 \right)
\end{aligned}$$

which reduces to:

$$0 = x^* (1 - x^*) - \gamma_{13} x^* z^* \tag{4.9a}$$
$$0 = r_2 z^* (1 - \gamma_{31} z^*) - v_2 z^* \tag{4.9b}$$

With the conditions $x^* > 0$ and $z^* > 0$ we imposed, the System (4.9) reduces to:

$$0 = 1 - x^* - \gamma_{13} z^* \tag{4.10a}$$
$$0 = r_2 (1 - \gamma_{31} z^*) - v_2 \tag{4.10b}$$

We can solve for z^* in Equation (4.10b):

$$z^* = \frac{r_2 - v_2}{\gamma_{31} r_2} \tag{4.11}$$

To ensure that $z^* > 0$, we need to impose the condition $r_2 > v_2$. Plugging Equation (4.11) into Equation (4.10a), we get:

$$1 - x^* - \gamma_{13}\left(\frac{r_2 - v_2}{\gamma_{31} r_2}\right) = 0 \tag{4.12}$$

Solving for x^* in Equation (4.12), we get:

$$x^* = 1 - \frac{\gamma_{13}\left(r_2 - v_2\right)}{\gamma_{31} r_2} \tag{4.13}$$

To ensure that $x^* > 0$, we need to impose the condition

$$\frac{\gamma_{31} r_2}{\gamma_{13}} > 0$$

With Equation (4.13), $y^* = 0$, and Equation (4.11), we can conclude that the xz-boundary equilibrium exists:

$$\boxed{E_{xz} = \left(1 - \frac{\gamma_{13}\left(r_2 - v_2\right)}{\gamma_{31} r_2}, 0, \frac{r_2 - v_2}{\gamma_{31} r_2}\right)}$$

under the conditions:

$$r_2 > v_2, \quad \frac{\gamma_{31} r_2}{\gamma_{13}} > 0$$

4.7 Case 7: The yz-boundary equilibrium

The yz-boundary equilibrium is an equilibrium where y^* and z^* are non-zero and all other components of the equilibrium are 0. Since we are talking about population densities, it does not make sense to consider values of y^* and z^* where $y^* < 0$ and $z^* < 0$. Thus, the conditions to impose when finding the yz-boundary equilibrium are $y^* > 0$ and $z^* > 0$ and all other components are 0. To find the yz-boundary equilibrium of Model (3.2), we will plug in $x^* = 0$:

$$\begin{aligned}
0 &= (0)\left(1 - (0) + \gamma_{12}\left(y^*\right)^2\right) - \gamma_{13}(0) z^* \\
0 &= r_1 y^*\left(1 - y^* + \gamma_{21}(0)^2\right) - \frac{(1-p)\, y^* z^*}{v_1 + (1-p)\, y^*} \\
0 &= r_2 z^*\left(1 - \gamma_{31} z^*\right) + z^*\left(\frac{v_3\,(1-p)\, y^*}{v_1 + (1-p)\, y^*} - v_2\right)
\end{aligned}$$

which reduces to:

$$0 = r_1 y^*\left(1 - y^*\right) - \frac{(1-p)\, y^* z^*}{v_1 + (1-p)\, y^*} \tag{4.14a}$$

$$0 = r_2 z^*\left(1 - \gamma_{31} z^*\right) + z^*\left(\frac{v_3\,(1-p)\, y^*}{v_1 + (1-p)\, y^*} - v_2\right) \tag{4.14b}$$

With the conditions $y^* > 0$ and $z^* > 0$ we imposed, the System (4.14) reduces to:

$$0 = r_1\left(1 - y^*\right) - \frac{(1-p)\, z^*}{v_1 + (1-p)\, y^*} \tag{4.15a}$$

$$0 = r_2\left(1 - \gamma_{31} z^*\right) + \left(\frac{v_3\,(1-p)\, y^*}{v_1 + (1-p)\, y^*} - v_2\right) \tag{4.15b}$$

We can solve for z^* in Equation (4.15b):

$$z^* = \frac{r_2 - v_2}{\gamma_{31} r_2} + \frac{v_3\,(1-p)\, y^*}{\gamma_{31} r_2\left(v_1 + (1-p)\, y^*\right)} \tag{4.16}$$

To ensure that $z^* > 0$, we need to impose the condition $r_2 > v_2$. To solve for y^*, we will plug in Equation (4.16) into Equation (4.15a) to get the equation:

$$\frac{Y_3\left(y^*\right)^3 + Y_2\left(y^*\right)^2 + Y_1 y^* + Y_0}{\gamma_{31} r_2\left(v_1 + (1-p)\, y^*\right)^2} = 0 \tag{4.17}$$

where

$$
\begin{aligned}
Y_3 &= -\gamma_{31} r_1 r_2 (1-p)^2 \\
Y_2 &= \gamma_{31} r_1 r_2 ((1-p) - 2v_1)(1-p) \\
Y_1 &= \gamma_{31} r_1 r_2 v_1 (2(1-p) - v_1) + (v_2 - v_3 - r_2)(1-p)^2 \\
Y_0 &= \gamma_{31} r_1 r_2 v_1^2 + v_1 (v_2 - r_2)(1-p)
\end{aligned}
$$

Note that we can eliminate one value of y^* form Equation (4.17):

$$
(v_1 + (1-p) y^*)^2 \neq 0 \implies y^* \neq -\frac{v_1}{1-p}
$$

and simplify Equation (4.17) to:

$$
Y_3 (y^*)^3 + Y_2 (y^*)^2 + Y_1 y^* + Y_0 = 0 \tag{4.18}
$$

With a third degree polynomial with complex coefficients, it will be difficult to derive the closed-form solutions to Equation (4.18). However, we don't need to find the exact form of the y^* component in this equilibrium. It is sufficient to show that a positive solution to Equation (4.18) exists. This is because if $y^* = 0$, then it will lead to the z-axial equilibrium and if $y^* < 0$ or if y^* is complex, then we ignore it since it biologically does not make sense. Going through the coefficients of Equation (4.18), we can immediately see that $Y_3 < 0$. For Y_2, we can place a condition to determine when its positive or negative. In particular, we can say that

$$
\begin{cases}
Y_2 < 0 & \text{if} \quad 1-p < 2v_1 \\
Y_2 > 0 & \text{if} \quad 1-p > 2v_1
\end{cases}
$$

For Y_1, we have:

$$
\begin{cases}
Y_1 < 0 & \text{if} \quad \dfrac{\gamma_{31} r_1 r_2 v_1 (2(1-p) - v_1)}{(r_2 - v_2 + v_3)(1-p)^2} < 1 \\
Y_1 > 0 & \text{if} \quad \dfrac{\gamma_{31} r_1 r_2 v_1 (2(1-p) - v_1)}{(r_2 - v_2 + v_3)(1-p)^2} > 1
\end{cases}
$$

and for Y_0, we have:

$$
\begin{cases}
Y_0 < 0 & \text{if} \quad \dfrac{\gamma_{31} r_1 r_2 v_1}{(r_2 - v_2)(1-p)} < 1 \\
Y_0 > 0 & \text{if} \quad \dfrac{\gamma_{31} r_1 r_2 v_1}{(r_2 - v_2)(1-p)} > 1
\end{cases}
$$

Note that we had to impose a condition on z^*, which was $r_2 > v_2$. With this condition, we can rewrite one our conditions on Y_0:

$$
\begin{cases}
Y_0 < 0 & \text{if} \quad \dfrac{\gamma_{31} r_1 r_2 v_1}{1-p} < r_2 - v_2 \\
Y_0 > 0 & \text{if} \quad \dfrac{\gamma_{31} r_1 r_2 v_1}{1-p} > 0
\end{cases}
$$

and for Y_1, the conditions can be rewritten as:

$$
\begin{cases}
Y_1 < 0 & \text{if} \quad \dfrac{\gamma_{31} r_1 r_2 v_1 (2(1-p) - v_1)}{(1-p)^2} < r_2 - v_2 + v_3 \\
Y_1 > 0 & \text{if} \quad 1-p > \dfrac{v_1}{2}
\end{cases}
$$

By Descartes' rule of signs [2], we can say that Equation (4.18) has at least one positive solution if $Y_3 < 0$ and $Y_2, Y_1, Y_0 > 0$. This means that we must impose the conditions:

$$
1-p > 2v_1, \quad 1-p > \frac{v_1}{2}, \quad \frac{\gamma_{31} r_1 r_2 v_1}{1-p} > 0
$$

Since all parameters are positive, the third condition is always fulfilled. Also, we know that $2v_1 > v_1/2$ so the second condition is redundant. Therefore, we can say that the yz-boundary equilibrium $E_{yz} = (0, \bar{y}, \bar{z})$ exists where:

$$
\bar{z} = \frac{r_2 - v_2}{\gamma_{31} r_2} + \frac{v_3 (1-p) \bar{y}}{\gamma_{31} r_2 (v_1 + (1-p) \bar{y})}
$$

with a condition that $r_2 > v_2$ and $\bar{y}$ is a positive root to the equation:

$$Y_3(\bar{y})^3 + Y_2(\bar{y})^2 + Y_1\bar{y} + Y_0 = 0$$

where

$$\begin{aligned}
Y_3 &= -\gamma_{31}r_1r_2(1-p)^2 \\
Y_2 &= \gamma_{31}r_1r_2((1-p)-2v_1)(1-p) \\
Y_1 &= \gamma_{31}r_1r_2v_1(2(1-p)-v_1) + (v_2-v_3-r_2)(1-p)^2 \\
Y_0 &= \gamma_{31}r_1r_2v_1^2 + v_1(v_2-r_2)(1-p)
\end{aligned}$$

with a condition that $1-p > 2v_1$.

4.8 Case 8: The interior equilibrium

The interior equilibrium is an equilibrium where all the components of the equilibrium are non-zero. Since we are talking about population densities, having non-zero components that are also negative would not make sense. Thus, the conditions to impose when finding the interior equilibrium are when the components are positive. To find the interior equilibrium of Model (3.2), we know that $x^*, y^*, z^* \neq 0$. So we can reduce the model to:

$$0 = 1 - x^* + \gamma_{12}(y^*)^2 - \gamma_{13}z^* \tag{4.19a}$$

$$0 = r_1\left(1 - y^* + \gamma_{21}(x^*)^2\right) - \frac{(1-p)z^*}{v_1 + (1-p)y^*} \tag{4.19b}$$

$$0 = r_2(1-\gamma_{31}z^*) + \left(\frac{v_3(1-p)y^*}{v_1+(1-p)y^*} - v_2\right) \tag{4.19c}$$

We can solve for x^* in Equation (4.19a):

$$x^* = 1 + \gamma_{12}(y^*)^2 - \gamma_{13}z^* \tag{4.20}$$

To ensure that $x^* > 0$, we will need to impose the condition:

$$z^* < \frac{1+\gamma_{12}(y^*)^2}{\gamma_{13}}$$

We can also solve for z^* in Equation (4.19c):

$$z^* = \frac{r_2 - v_2}{\gamma_{31}r_2} + \frac{v_3(1-p)y^*}{\gamma_{31}r_2(v_1+(1-p)y^*)} \tag{4.21}$$

To ensure that $z^* > 0$, we will need to impose the condition $r_2 > v_2$. With Equation (4.20) and Equation (4.21), we can plug them into Equation (4.19b) to obtain the following Equation:

$$\frac{1}{\gamma_{13}^2 r_2^2 (v_1+(1-p)y^*)^2}\sum_{i=0}^{6} Y_i (y^*)^i = 0 \tag{4.22}$$

where

$$\begin{aligned}
Y_6 &= \gamma_{12}^2\gamma_{21}\gamma_{31}^2 r_1 r_2^2 (1-p)^2 \\
Y_5 &= 2\gamma_{12}^2\gamma_{21}\gamma_{31}^2 r_1 r_2^2 v_1 (1-p) \\
Y_4 &= \gamma_{12}\gamma_{21}\gamma_{31}r_1r_2\left(2\gamma_{13}(v_2-r_2-v_3)(1-p)^2 + \gamma_{31}r_2\left(\gamma_{12}v_1^2 + 2(1-p)^2\right)\right) \\
Y_3 &= \gamma_{31}r_1r_2(2\gamma_{12}\gamma_{13}\gamma_{21}v_1(2(v_2-r_2)-v_3) + \gamma_{31}r_2(4\gamma_{12}\gamma_{21}v_1 - (1-p)))(1-p) \\
Y_2 &= r_1\Big(2\gamma_{12}\gamma_{13}\gamma_{21}\gamma_{31}r_2v_1^2(v_2-r_2) + \gamma_{13}^2\gamma_{21}(r_2-v_2+v_3)^2(1-p)^2 + 2\gamma_{13}\gamma_{21}\gamma_{31}r_2(v_2-r_2-v_3)(1-p)^2 \\
&\quad + \gamma_{21}\gamma_{31}^2r_2^2\left(2\gamma_{12}v_1^2 + (1-p)^2\right) + \gamma_{31}^2r_2^2((1-p)-2v_1)(1-p)\Big) \\
Y_1 &= 2\gamma_{13}^2\gamma_{21}r_1v_1(r_2-v_2)(r_2-v_2+v_3)(1-p) + 2\gamma_{13}\gamma_{21}\gamma_{31}r_1r_2v_1(2(v_2-r_2)-v_3)(1-p) \\
&\quad + \gamma_{31}^2r_1r_2^2v_1(2(\gamma_{21}+1)(1-p)-v_1) + \gamma_{31}r_2(v_2-r_2-v_3)(1-p)^2 \\
Y_0 &= v_1\left(\gamma_{13}^2\gamma_{21}r_1v_1(r_2-v_2)^2 + \gamma_{31}^2r_1r_2^2v_1(\gamma_{21}+1) + \gamma_{31}r_2(2\gamma_{13}\gamma_{21}r_1v_1 + (1-p))(v_2-r_2)\right)
\end{aligned}$$

Note that we can eliminate one value of y^* form Equation (4.22):

$$(v_1 + (1-p)\, y^*)^2 \neq 0 \implies y^* \neq -\frac{v_1}{1-p}$$

and simplify Equation (4.22) to:

$$\sum_{i=0}^{6} Y_i \,(y^*)^i = 0 \tag{4.23}$$

What we are left with is an equation in the form of a polynomial of degree 6. The solutions to Equation (4.23) cannot be analytically solved. However, we don't need to find the exact form of the y^* component in this equilibrium. It is sufficient to show that a positive solution to Equation (4.23) exists. To do this, we will use Descartes' rule of signs. From the coefficients, we can conclude that $Y_5, Y_6 > 0$ since all parameters are positive. Then, to ensure that Equation (4.23) has at least one positive solution, we will need an odd number of sign changes after the fifth degree term. For simplicity, we will make all the other coefficients negative. Thus, we can say that the interior equilibrium $E_{xyz} = (x^*, y^*, z^*)$ exists where:

$$x^* = 1 + \gamma_{12}\,(y^*)^2 - \gamma_{13} z^*$$

and

$$z^* = \frac{r_2 - v_2}{\gamma_{31} r_2} + \frac{v_3\,(1-p)\,y^*}{\gamma_{31} r_2\,(v_1 + (1-p)\,y^*)}; \quad r_2 > v_2$$

and y^* is a positive root to the equation:

$$\sum_{i=0}^{6} Y_i \,(y^*)^i = 0$$

where $Y_0 < 0$, $Y_1 < 0$, $Y_2 < 0$, $Y_3 < 0$, $Y_4 < 0$ and

$$\begin{aligned}
Y_6 &= \gamma_{12}^2 \gamma_{21} \gamma_{31}^2 r_1 r_2^2 \,(1-p)^2 \\
Y_5 &= 2\gamma_{12}^2 \gamma_{21} \gamma_{31}^2 r_1 r_2^2 v_1 \,(1-p) \\
Y_4 &= \gamma_{12}\gamma_{21}\gamma_{31} r_1 r_2 \left(2\gamma_{13}\,(v_2 - r_2 - v_3)\,(1-p)^2 + \gamma_{31} r_2 \left(\gamma_{12} v_1^2 + 2\,(1-p)^2\right)\right) \\
Y_3 &= \gamma_{31} r_1 r_2 \,(2\gamma_{12}\gamma_{13}\gamma_{21} v_1 \,(2\,(v_2 - r_2) - v_3) + \gamma_{31} r_2 \,(4\gamma_{12}\gamma_{21} v_1 - (1-p)))\,(1-p) \\
Y_2 &= r_1 \Big(2\gamma_{12}\gamma_{13}\gamma_{21}\gamma_{31} r_2 v_1^2 \,(v_2 - r_2) + \gamma_{13}^2 \gamma_{21}\,(r_2 - v_2 + v_3)^2\,(1-p)^2 + 2\gamma_{13}\gamma_{21}\gamma_{31} r_2\,(v_2 - r_2 - v_3)\,(1-p)^2 \\
&\quad + \gamma_{21}\gamma_{31}^2 r_2^2 \left(2\gamma_{12} v_1^2 + (1-p)^2\right) + \gamma_{31}^2 r_2^2\,((1-p) - 2v_1)\,(1-p)\Big) \\
Y_1 &= 2\gamma_{13}^2 \gamma_{21} r_1 v_1 \,(r_2 - v_2)\,(r_2 - v_2 + v_3)\,(1-p) + 2\gamma_{13}\gamma_{21}\gamma_{31} r_1 r_2 v_1 \,(2\,(v_2 - r_2) - v_3)\,(1-p) \\
&\quad + \gamma_{31}^2 r_1 r_2^2 v_1 \,(2\,(\gamma_{21} + 1)\,(1-p) - v_1) + \gamma_{31} r_2 \,(v_2 - r_2 - v_3)\,(1-p)^2 \\
Y_0 &= v_1 \left(\gamma_{13}^2 \gamma_{21} r_1 v_1 \,(r_2 - v_2)^2 + \gamma_{31}^2 r_1 r_2^2 v_1 \,(\gamma_{21} + 1) + \gamma_{31} r_2 \,(2\gamma_{13}\gamma_{21} r_1 v_1 + (1-p))\,(v_2 - r_2)\right)
\end{aligned}$$

5 Stability Analysis via Linearization

In Section 4, we have computed all possible equilibria in Model (3.2). In this section, we will use mathematical analysis to analyze the stability of each equilibria and determine the conditions for stability. In order to find the stability of each equilibrium point with the linearization method [6], we will need the Jacobian matrix of Model (3.1), which is:

$$\mathbf{J} = \begin{bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ 0 & j_{32} & j_{33} \end{bmatrix} \tag{5.1}$$

where

$$
\begin{aligned}
j_{11} &= 1 - 2x + \gamma_{12} y^2 - \gamma_{13} z \\
j_{12} &= 2\gamma_{12} x y \\
j_{13} &= -\gamma_{13} x \\
j_{21} &= 2\gamma_{21} r_1 x y \\
j_{22} &= -\frac{v_1 (1-p) z}{(v_1 + (1-p) y)^2} + r_1 \left(1 - 2y + \gamma_{21} x^2\right) \\
j_{23} &= -\frac{(1-p) y}{v_1 + (1-p) y} \\
j_{32} &= \frac{v_1 v_3 (1-p) z}{(v_1 + (1-p) y)^2} \\
j_{33} &= r_2 (1 - 2\gamma_{31} z) + \frac{v_3 (1-p) y}{v_1 + (1-p) y} - v_2
\end{aligned}
$$

5.1 Analyzing the trivial equilibrium

Plugging the trivial equilibrium into Matrix (5.1) yields:

$$
\mathbf{J}(E_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r_1 & 0 \\ 0 & 0 & r_2 - v_2 \end{bmatrix} \tag{5.2}
$$

The characteristic equation for this Jacobian matrix is:

$$
-(\lambda - 1)(\lambda - r_1)(\lambda - (r_2 - v_2)) = 0 \tag{5.3}
$$

Solving for the eigenvalues in Equation (5.3), we get:

$$
\lambda = \{1,\ r_1,\ r_2 - v_2\}
$$

Since we have a positive eigenvalue $\lambda = 1$, we can conclude that the trivial equilibrium is unstable.

5.2 Analyzing the x-axial equilibrium

Plugging the x-axial equilibrium into Matrix (5.1) yields:

$$
\mathbf{J}(E_x) = \begin{bmatrix} -1 & 0 & -\gamma_{13} \\ 0 & r_1(\gamma_{12} + 1) & 0 \\ 0 & 0 & r_2 - v_2 \end{bmatrix} \tag{5.4}
$$

The characteristic equation for this Jacobian matrix is:

$$
-(\lambda + 1)(\lambda - r_1(\gamma_{12} + 1))(\lambda - (r_2 - v_2)) = 0 \tag{5.5}
$$

Solving for the eigenvalues in Equation (5.5), we get:

$$
\lambda = \{-1,\ r_2 - v_2,\ r_1(\gamma_{12} + 1)\}
$$

Since the eigenvalue $\lambda = r_1(\gamma_{12} + 1)$ is always positive, we can conclude that the x-axial equilibrium is unstable.

5.3 Analyzing the y-axial equilibrium

Plugging the y-axial equilibrium into Matrix (5.1) yields:

$$
\mathbf{J}(E_y) = \begin{bmatrix} 1 + \gamma_{12} & 0 & 0 \\ 0 & -r_1 & j_{23} \\ 0 & 0 & j_{33} \end{bmatrix} \tag{5.6}
$$

where

$$j_{23} = -\frac{1-p}{v_1+1-p}$$
$$j_{33} = -v_3 j_{23} + r_2 - v_2$$

The characteristic equation for this Jacobian matrix is:

$$-\left(\lambda - (1+\gamma_{12})\right)\left(\lambda + r_1\right)\left(\lambda - j_{33}\right) = 0 \tag{5.7}$$

Solving for the eigenvalues in Equation (5.7), we get:

$$\lambda = \{1+\gamma_{12},\ -r_1,\ -v_3 j_{23} + r_2 - v_2\}$$

Since the eigenvalue $\lambda = 1 + \gamma_{12}$ is always positive, we can conclude that the y-axial equilibrium is unstable.

5.4 Analyzing the z-axial equilibrium

Plugging the z-axial equilibrium into Matrix (5.1) yields:

$$\mathbf{J}\left(E_z\right) = \begin{bmatrix} j_{11} & 0 & 0 \\ 0 & j_{22} & 0 \\ 0 & j_{32} & v_2 - r_2 \end{bmatrix} \tag{5.8}$$

where

$$j_{11} = \frac{\gamma_{13}\left(v_2 - r_2\right) + \gamma_{31} r_2}{\gamma_{31} r_2}$$
$$j_{22} = r_1 + \frac{(1-p)\left(v_2 - r_2\right)}{\gamma_{31} r_2 v_1}$$
$$j_{32} = \frac{v_3\left(r_2 - v_2\right)(1-p)}{\gamma_{31} r_2 v_1}$$

The characteristic equation for this Jacobian matrix is:

$$-\left(\lambda - j_{11}\right)\left(\lambda - j_{22}\right)\left(\lambda - j_{33}\right) = 0 \tag{5.9}$$

Solving for the eigenvalues in Equation (5.9), we get:

$$\lambda = \{j_{11},\ j_{22},\ j_{33}\}$$

The first eigenvalue is negative when:

$$\frac{\gamma_{31} r_2}{\gamma_{13}} < r_2 - v_2$$

The second eigenvalue is negative when:

$$\frac{\gamma_{31} r_1 r_2 v_1}{1-p} < r_2 - v_2$$

The third eigenvalue is negative when $v_2 < r_2$. Therefore we can say that the z-axial equilibrium is stable if:

$$\frac{\gamma_{31} r_2}{\gamma_{13}} < r_2 - v_2,\ \frac{\gamma_{31} r_1 r_2 v_1}{1-p} < r_2 - v_2,\ 0 < r_2 - v_2$$

5.5 Analyzing the xy-boundary equilibrium

Plugging the xy-boundary equilibrium into Matrix (5.1) yields:

$$\mathbf{J}\left(E_{xy}\right) = \begin{bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ 0 & 0 & j_{33} \end{bmatrix} \tag{5.10}$$

where

$$
\begin{aligned}
j_{11} &= 1 - 2\hat{x} + \gamma_{12}\left(\hat{y}\right)^2 \\
j_{12} &= 2\gamma_{12}\hat{x}\hat{y} \\
j_{13} &= -\gamma_{13}\hat{x} \\
j_{21} &= 2\gamma_{21}r_1\hat{x}\hat{y} \\
j_{22} &= r_1\left(1 - 2\hat{y} + \gamma_{21}\left(\hat{x}\right)^2\right) \\
j_{23} &= -\frac{(1-p)\,\hat{y}}{v_1 + (1-p)\,\hat{y}} \\
j_{33} &= r_2 + \frac{v_3\,(1-p)\,\hat{y}}{v_1 + (1-p)\,\hat{y}} - v_2
\end{aligned}
$$

The characteristic equation for this Jacobian matrix is:

$$
-\left(j_{33} - \lambda\right)\left(\lambda^2 - \left(j_{11} + j_{22}\right)\lambda + j_{11}j_{22} - j_{12}j_{21}\right) = 0 \tag{5.11}
$$

Solving for the eigenvalues in Equation (5.11), we get:

$$
\lambda = \left\{ j_{33},\ \frac{\left(j_{11} + j_{22}\right) \pm \sqrt{\left(j_{11} - j_{22}\right)^2 + 4j_{12}j_{21}}}{2} \right\}
$$

The first eigenvalue is negative when:

$$
\frac{v_3\,(1-p)\,\hat{y}}{v_1 + (1-p)\,\hat{y}} < v_2 - r_2
$$

The other two eigenvalues are negative when:

$$
4\gamma_{12}\gamma_{21}\left(\hat{x}\right)^2\left(\hat{y}\right)^2 < \left(1 - 2\hat{x} + \gamma_{12}\left(\hat{y}\right)^2\right)\left(1 - 2\hat{y} + \gamma_{21}\left(\hat{x}\right)^2\right)
$$

Therefore we can say that the xy-boundary equilibrium is stable if:

$$
\frac{v_3\,(1-p)\,\hat{y}}{v_1 + (1-p)\,\hat{y}} < v_2 - r_2
$$

and

$$
4\gamma_{12}\gamma_{21}\left(\hat{x}\right)^2\left(\hat{y}\right)^2 < \left(1 - 2\hat{x} + \gamma_{12}\left(\hat{y}\right)^2\right)\left(1 - 2\hat{y} + \gamma_{21}\left(\hat{x}\right)^2\right)
$$

5.6 Analyzing the xz-boundary equilibrium

Plugging the xz-boundary equilibrium into Matrix (5.1) yields:

$$
\mathbf{J}\left(E_{xz}\right) = \begin{bmatrix} j_{11} & 0 & j_{13} \\ 0 & j_{22} & 0 \\ 0 & j_{32} & j_{33} \end{bmatrix} \tag{5.12}
$$

where

$$
\begin{aligned}
j_{11} &= \frac{\gamma_{13}\left(r_2 - v_2\right) - \gamma_{31}r_2}{\gamma_{31}r_2} \\
j_{13} &= \frac{\gamma_{13}\left(\gamma_{13}\left(r_2 - v_2\right) - \gamma_{31}r_2\right)}{\gamma_{31}r_2} \\
j_{22} &= \frac{\gamma_{21}r_1\left(\gamma_{13}\left(r_2 - v_2\right) - \gamma_{31}r_2\right)^2}{\gamma_{31}^2r_2^2} + r_1 - \frac{\left(r_2 - v_2\right)(1-p)}{\gamma_{31}r_2v_1} \\
j_{32} &= \frac{v_3\left(r_2 - v_2\right)(1-p)}{\gamma_{31}r_2v_1} \\
j_{33} &= v_2 - r_2
\end{aligned}
$$

The characteristic equation for this Jacobian matrix is:

$$-\left(j_{11}-\lambda\right)\left(j_{22}-\lambda\right)\left(j_{33}-\lambda\right)=0 \tag{5.13}$$

Solving for the eigenvalues in Equation (5.13), we get:

$$\lambda=\{j_{11},\ j_{22},\ j_{33}\}$$

The first eigenvalue is negative when:

$$r_2-v_2<\frac{\gamma_{31}r_2}{\gamma_{13}}$$

The second eigenvalue is negative when:

$$\frac{r_1v_1\left(\gamma_{21}\left(\gamma_{13}\left(r_2-v_2\right)-\gamma_{31}r_2\right)^2+\gamma_{31}^2r_2^2\right)}{\gamma_{31}r_2\left(1-p\right)}<r_2-v_2$$

The third eigenvalue is negative when:

$$r_2-v_2>0$$

Therefore we can say that the xz-boundary equilibrium is stable if $0<r_2-v_2$, $r_2-v_2<\dfrac{\gamma_{31}r_2}{\gamma_{13}}$, and:

$$\frac{r_1v_1\left(\gamma_{21}\left(\gamma_{13}\left(r_2-v_2\right)-\gamma_{31}r_2\right)^2+\gamma_{31}^2r_2^2\right)}{\gamma_{31}r_2\left(1-p\right)}<r_2-v_2$$

5.7 Analyzing the yz-boundary equilibrium

Plugging the yz-boundary equilibrium into Matrix (5.1) yields:

$$\mathbf{J}=\begin{bmatrix} j_{11} & 0 & 0 \\ 0 & j_{22} & j_{23} \\ 0 & j_{32} & j_{33} \end{bmatrix} \tag{5.14}$$

where

$$\begin{aligned}
j_{11}&=1+\gamma_{12}\left(\bar{y}\right)^2-\gamma_{13}\bar{z}\\
j_{22}&=r_1\left(1-2\bar{y}\right)-\frac{v_1\left(1-p\right)\bar{z}}{\left(v_1+\left(1-p\right)\bar{y}\right)^2}\\
j_{23}&=-\frac{\left(1-p\right)\bar{y}}{v_1+\left(1-p\right)\bar{y}}\\
j_{32}&=\frac{v_1v_3\left(1-p\right)\bar{z}}{\left(v_1+\left(1-p\right)\bar{y}\right)^2}\\
j_{33}&=r_2\left(1-2\gamma_{31}\bar{z}\right)+\frac{v_3\left(1-p\right)\bar{y}}{v_1+\left(1-p\right)\bar{y}}-v_2
\end{aligned}$$

The characteristic equation for this Jacobian matrix is:

$$-\left(j_{11}-\lambda\right)\left(\lambda^2-\left(j_{22}+j_{33}\right)\lambda+j_{22}j_{33}-j_{23}j_{32}\right)=0 \tag{5.15}$$

Solving for the eigenvalues in Equation (5.15), we get:

$$\lambda=\left\{j_{11},\ \frac{\left(j_{22}+j_{33}\right)\pm\sqrt{\left(j_{22}-j_{33}\right)^2+4j_{23}j_{32}}}{2}\right\}$$

The first eigenvalue is negative when:

$$\frac{1+\gamma_{12}\bar{y}}{\gamma_{13}}<\bar{z}$$

The other two eigenvalues are negative when:

$$j_{23}j_{32} < j_{22}j_{33}$$

Therefore we can say that the xy-boundary equilibrium is stable if:

$$\frac{1+\gamma_{12}\bar{y}}{\gamma_{13}} < \bar{z}, \quad j_{23}j_{32} < j_{22}j_{33}$$

where

$$\begin{aligned}
j_{22} &= -\frac{v_1(1-p)\bar{z}}{(v_1+(1-p)\bar{y})^2} + r_1(1-2\bar{y}) \\
j_{23} &= -\frac{(1-p)\bar{y}}{v_1+(1-p)\bar{y}} \\
j_{32} &= \frac{v_1 v_3 (1-p)\bar{z}}{(v_1+(1-p)\bar{y})^2} \\
j_{33} &= r_2(1-2\gamma_{31}\bar{z}) + \frac{v_3(1-p)\bar{y}}{v_1+(1-p)\bar{y}} - v_2
\end{aligned}$$

5.8 Analyzing the interior equilibrium

Plugging the interior equilibrium into Matrix (5.1) yields:

$$\mathbf{J} = \begin{bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ 0 & j_{32} & j_{33} \end{bmatrix} \tag{5.16}$$

where

$$\begin{aligned}
j_{11} &= 1-2x^* + \gamma_{12}(y^*)^2 - \gamma_{13}z^* \\
j_{12} &= 2\gamma_{12}x^*y^* \\
j_{13} &= -\gamma_{13}x^* \\
j_{21} &= 2\gamma_{21}r_1x^*y^* \\
j_{22} &= r_1\left(1-2y^*+\gamma_{21}(x^*)^2\right) - \frac{v_1(1-p)z^*}{(v_1+(1-p)y^*)^2} \\
j_{23} &= -\frac{(1-p)y^*}{v_1+(1-p)y^*} \\
j_{32} &= \frac{v_1v_3(1-p)z^*}{(v_1+(1-p)y^*)^2} \\
j_{33} &= r_2(1-2\gamma_{31}z^*) + \frac{v_3(1-p)y^*}{v_1+(1-p)y^*} - v_2
\end{aligned}$$

The characteristic equation for this Jacobian matrix is:

$$\lambda^3 + J_2\lambda^2 + J_1\lambda + J_0 = 0 \tag{5.17}$$

where

$$\begin{aligned}
J_2 &= -(j_{11}+j_{22}+j_{33}) \\
J_1 &= j_{11}(j_{22}+j_{33}) + j_{22}j_{33} - (j_{12}j_{21}+j_{23}j_{32}) \\
J_0 &= -j_{13}j_{21}j_{32} + j_{12}j_{21}j_{33} + j_{11}j_{23}j_{32} - j_{11}j_{22}j_{33}
\end{aligned}$$

By the Routh–Hurwitz stability criterion [1], we can say that the interior equilibrium is stable if:

$$J_2 > 0, \quad J_1 > 0, \quad J_0 > 0, \quad J_2J_1 > J_0$$

where

$$
\begin{aligned}
J_2 &= -\left(j_{11}+j_{22}+j_{33}\right) \\
J_1 &= j_{11}\left(j_{22}+j_{33}\right)+j_{22}j_{33}-\left(j_{12}j_{21}+j_{23}j_{32}\right) \\
J_0 &= -j_{13}j_{21}j_{32}+j_{12}j_{21}j_{33}+j_{11}j_{23}j_{32}-j_{11}j_{22}j_{33} \\
j_{11} &= 1-2x^*+\gamma_{12}\left(y^*\right)^2-\gamma_{13}z^* \\
j_{12} &= 2\gamma_{12}x^*y^* \\
j_{13} &= -\gamma_{13}x^* \\
j_{21} &= 2\gamma_{21}r_1x^*y^* \\
j_{22} &= r_1\left(1-2y^*+\gamma_{21}\left(x^*\right)^2\right)-\frac{v_1\left(1-p\right)z^*}{\left(v_1+\left(1-p\right)y^*\right)^2} \\
j_{23} &= -\frac{\left(1-p\right)y^*}{v_1+\left(1-p\right)y^*} \\
j_{32} &= \frac{v_1v_3\left(1-p\right)z^*}{\left(v_1+\left(1-p\right)y^*\right)^2} \\
j_{33} &= r_2\left(1-2\gamma_{31}z^*\right)+\frac{v_3\left(1-p\right)y^*}{v_1+\left(1-p\right)y^*}-v_2
\end{aligned}
$$

6 Numerical Simulations

In Section 5, we used mathematical analysis to analyze the stability of each equilibria and determine the conditions for stability. In this section, we will support and verify the stable equilibria determined in Section 5 through numerical simulations and compare the interior equilibrium of both models under same/similar parameters.

6.1 The z-axial equilibrium

In Section 4.4 and Section 5.4, we determined that the z-axial equilibrium

$$E_z = \left(0, 0, \frac{r_2-v_2}{\gamma_{31}r_2}\right)$$

exists if the condition $r_2 - v_2 > 0$ is satisfied and is stable if the following conditions are satisfied:

$$r_2-v_2>0, \quad r_2-v_2>\frac{\gamma_{31}r_2}{\gamma_{13}}, \quad r_2-v_2>\frac{\gamma_{31}r_1r_2v_1}{1-p}$$

To satisfy the conditions above, lets consider the following set of parameters:

$$\begin{cases} r_1 = 0.404 \\ r_2 = 0.903 \\ p = 0.182 \end{cases}, \quad \begin{cases} \gamma_{12} = 0.639 \\ \gamma_{21} = 0.283 \\ \gamma_{13} = 0.301 \\ \gamma_{31} = 0.110 \end{cases}, \quad \begin{cases} v_1 = 0.645 \\ v_2 = 0.175 \\ v_3 = 0.145 \end{cases} \tag{6.1}$$

Under this set of parameter values, the z-axial equilibrium is $E_z = (0, 0, 1.7343)$. This is further supported by Figure 1a, which is the result of numerically solving Model (3.2).

6.2 The xy-boundary equilibrium

In Section 4.5 and Section 5.5, we determined that the yz-boundary equilibrium $E_{yz} = (\hat{x}, \hat{y}, 0)$ exists where

$$\hat{x} = 1+\gamma_{12}\left(\hat{y}\right)^2$$

and y^* is a positive solution to the equation:

$$\gamma_{12}^2\gamma_{21}\left(\hat{y}\right)^4+2\gamma_{12}\gamma_{21}\left(\hat{y}\right)^2-\hat{y}+\gamma_{21}+1=0$$

if the following condition is satisfied for some value of $\beta > 0$:

$$\gamma_{12} < \frac{1}{\beta^2}\left(\sqrt{\frac{\beta - 1}{\gamma_{21}}} - 1\right)$$

and the equilibrium is stable if:

$$\frac{v_3\left(1-p\right)\hat{y}}{v_1 + \left(1-p\right)\hat{y}} < v_2 - r_2$$

and

$$4\gamma_{12}\gamma_{21}\left(\hat{x}\right)^2\left(\hat{y}\right)^2 < \left(1 - 2\hat{x} + \gamma_{12}\left(\hat{y}\right)^2\right)\left(1 - 2\hat{y} + \gamma_{21}\left(\hat{x}\right)^2\right)$$

To satisfy the conditions above, let $\beta = 2$ and consider the following set of parameters:

$$\begin{cases} r_1 = 0.978 \\ r_2 = 0.613 \\ p = 0.326 \end{cases}, \quad \begin{cases} \gamma_{12} = 0.245 \\ \gamma_{21} = 0.015 \\ \gamma_{13} = 0.920 \\ \gamma_{31} = 0.696 \end{cases}, \quad \begin{cases} v_1 = 0.523 \\ v_2 = 0.951 \\ v_3 = 0.570 \end{cases} \tag{6.2}$$

Under this set of parameter values, the xy-boundary equilibrium is $E_{xy} = (1.257, 1.024, 0)$. This is further supported by Figure 1b, which is the result of numerically solving Model (3.2).

6.3 The xz-boundary equilibrium

In Section 4.6 and Section 5.6, we determined that the xz-boundary equilibrium

$$E_{xz} = \left(1 - \frac{\gamma_{13}\left(r_2 - v_2\right)}{\gamma_{31}r_2}, 0, \frac{r_2 - v_2}{\gamma_{31}r_2}\right)$$

exists if the following conditions are satisfied:

$$r_2 - v_2 > 0, \quad \frac{\gamma_{31}r_2}{\gamma_{13}} > 0$$

and the equilibrium is stable if $0 < r_2 - v_2$, $r_2 - v_2 < \dfrac{\gamma_{31}r_2}{\gamma_{13}}$, and:

$$\frac{r_1v_1\left(\gamma_{21}\left(\gamma_{13}\left(r_2 - v_2\right) - \gamma_{31}r_2\right)^2 + \gamma_{31}^2r_2^2\right)}{\gamma_{31}r_2\left(1-p\right)} < r_2 - v_2$$

To satisfy the conditions above, lets consider the following set of parameters:

$$\begin{cases} r_1 = 0.102 \\ r_2 = 0.763 \\ p = 0.271 \end{cases}, \quad \begin{cases} \gamma_{12} = 0.182 \\ \gamma_{21} = 0.301 \\ \gamma_{13} = 0.109 \\ \gamma_{31} = 0.198 \end{cases}, \quad \begin{cases} v_1 = 0.983 \\ v_2 = 0.186 \\ v_3 = 0.113 \end{cases} \tag{6.3}$$

Under this set of parameter values, the xz-boundary equilibrium is $E_{xz} = (0.584, 0, 3.819)$. This is further supported by Figure 1c, which is the result of numerically solving Model (3.2).

6.4 The yz-boundary equilibrium

In Section 4.7 and Section 5.7, we determined that the yz-boundary equilibrium $E_{yz} = (0, \bar{y}, \bar{z})$ exists where:

$$\bar{z} = \frac{r_2 - v_2}{\gamma_{31}r_2} + \frac{v_3\left(1-p\right)\bar{y}}{\gamma_{31}r_2\left(v_1 + \left(1-p\right)\bar{y}\right)}$$

with a condition that $r_2 > v_2$ and $\bar{y}$ is a positive root to the equation:

$$Y_3\left(\bar{y}\right)^3 + Y_2\left(\bar{y}\right)^2 + Y_1\bar{y} + Y_0 = 0$$

where

$$
\begin{aligned}
Y_3 &= -\gamma_{31} r_1 r_2 (1-p)^2 \\
Y_2 &= \gamma_{31} r_1 r_2 \left((1-p) - 2v_1\right)(1-p) \\
Y_1 &= \gamma_{31} r_1 r_2 v_1 \left(2(1-p) - v_1\right) + (v_2 - v_3 - r_2)(1-p)^2 \\
Y_0 &= \gamma_{31} r_1 r_2 v_1^2 + v_1 (v_2 - r_2)(1-p)
\end{aligned}
$$

with a condition that $1-p > 2v_1$ and the equilibrium is stable if:

$$
\frac{1+\gamma_{12}\bar{y}}{\gamma_{13}} < \bar{z}, \quad j_{23}j_{32} < j_{22}j_{33}
$$

where

$$
\begin{aligned}
j_{22} &= -\frac{v_1(1-p)\bar{z}}{(v_1 + (1-p)\bar{y})^2} + r_1(1-2\bar{y}) \\
j_{23} &= -\frac{(1-p)\bar{y}}{v_1 + (1-p)\bar{y}} \\
j_{32} &= \frac{v_1 v_3 (1-p)\bar{z}}{(v_1 + (1-p)\bar{y})^2} \\
j_{33} &= r_2(1-2\gamma_{31}\bar{z}) + \frac{v_3(1-p)\bar{y}}{v_1 + (1-p)\bar{y}} - v_2
\end{aligned}
$$

To satisfy the conditions above, lets consider the following set of parameters:

$$
\begin{cases} r_1 = 0.978 \\ r_2 = 0.310 \\ p = 0.843 \end{cases}, \quad
\begin{cases} \gamma_{12} = 0.002 \\ \gamma_{21} = 0.407 \\ \gamma_{13} = 0.859 \\ \gamma_{31} = 0.446 \end{cases}, \quad
\begin{cases} v_1 = 0.872 \\ v_2 = 0.201 \\ v_3 = 0.959 \end{cases} \tag{6.4}
$$

Under this set of parameter values, the yz-boundary equilibrium is $E_{yz} = (0, 0.74, 1.603)$. This is further supported by Figure 1d, which is the result of numerically solving Model (3.2).

6.5 The interior equilibrium

In Section 4.8 and Section 5.8, we determined that the interior equilibrium $E_{xyz} = (x^*, y^*, z^*)$ exists where:

$$
x^* = 1 + \gamma_{12}(y^*)^2 - \gamma_{13} z^*
$$

and

$$
z^* = \frac{r_2 - v_2}{\gamma_{31} r_2} + \frac{v_3 (1-p) y^*}{\gamma_{31} r_2 (v_1 + (1-p) y^*)}; \quad r_2 > v_2
$$

and y^* is a positive root to the equation:

$$
\sum_{i=0}^{6} Y_i (y^*)^i = 0
$$

where

$$
\begin{aligned}
Y_6 &= \gamma_{12}^2\gamma_{21}\gamma_{31}^2 r_1 r_2^2 (1-p)^2 \\
Y_5 &= 2\gamma_{12}^2\gamma_{21}\gamma_{31}^2 r_1 r_2^2 v_1 (1-p) \\
Y_4 &= \gamma_{12}\gamma_{21}\gamma_{31} r_1 r_2 \left(2\gamma_{13}(v_2 - r_2 - v_3)(1-p)^2 + \gamma_{31} r_2 \left(\gamma_{12} v_1^2 + 2(1-p)^2\right)\right) \\
Y_3 &= \gamma_{31} r_1 r_2 \left(2\gamma_{12}\gamma_{13}\gamma_{21} v_1 (2(v_2 - r_2) - v_3) + \gamma_{31} r_2 (4\gamma_{12}\gamma_{21} v_1 - (1-p))\right)(1-p) \\
Y_2 &= r_1 \left(2\gamma_{12}\gamma_{13}\gamma_{21}\gamma_{31} r_2 v_1^2 (v_2 - r_2) + \gamma_{13}^2\gamma_{21}(r_2 - v_2 + v_3)^2 (1-p)^2 + 2\gamma_{13}\gamma_{21}\gamma_{31} r_2 (v_2 - r_2 - v_3)(1-p)^2\right. \\
&\quad \left. + \gamma_{21}\gamma_{31}^2 r_2^2 \left(2\gamma_{12} v_1^2 + (1-p)^2\right) + \gamma_{31}^2 r_2^2 ((1-p) - 2v_1)(1-p)\right) \\
Y_1 &= 2\gamma_{13}^2\gamma_{21} r_1 v_1 (r_2 - v_2)(r_2 - v_2 + v_3)(1-p) + 2\gamma_{13}\gamma_{21}\gamma_{31} r_1 r_2 v_1 (2(v_2 - r_2) - v_3)(1-p) \\
&\quad + \gamma_{31}^2 r_1 r_2^2 v_1 (2(\gamma_{21}+1)(1-p) - v_1) + \gamma_{31} r_2 (v_2 - r_2 - v_3)(1-p)^2 \\
Y_0 &= v_1 \left(\gamma_{13}^2\gamma_{21} r_1 v_1 (r_2 - v_2)^2 + \gamma_{31}^2 r_1 r_2^2 v_1 (\gamma_{21}+1) + \gamma_{31} r_2 (2\gamma_{13}\gamma_{21} r_1 v_1 + (1-p))(v_2 - r_2)\right)
\end{aligned}
$$

and the equilibrium is stable if:

$$J_2 > 0, \quad J_1 > 0, \quad J_0 > 0, \quad J_2 J_1 > J_0$$

where

$$
\begin{aligned}
J_2 &= -(j_{11} + j_{22} + j_{33}) \\
J_1 &= j_{11}(j_{22} + j_{33}) + j_{22} j_{33} - (j_{12} j_{21} + j_{23} j_{32}) \\
J_0 &= -j_{13} j_{21} j_{32} + j_{12} j_{21} j_{33} + j_{11} j_{23} j_{32} - j_{11} j_{22} j_{33} \\
j_{11} &= 1 - 2x^* + \gamma_{12}(y^*)^2 - \gamma_{13} z^* \\
j_{12} &= 2\gamma_{12} x^* y^* \\
j_{13} &= -\gamma_{13} x^* \\
j_{21} &= 2\gamma_{21} r_1 x^* y^* \\
j_{22} &= r_1 \left(1 - 2y^* + \gamma_{21}(x^*)^2\right) - \frac{v_1 (1-p) z^*}{(v_1 + (1-p) y^*)^2} \\
j_{23} &= -\frac{(1-p) y^*}{v_1 + (1-p) y^*} \\
j_{32} &= \frac{v_1 v_3 (1-p) z^*}{(v_1 + (1-p) y^*)^2} \\
j_{33} &= r_2 (1 - 2\gamma_{31} z^*) + \frac{v_3 (1-p) y^*}{v_1 + (1-p) y^*} - v_2
\end{aligned}
$$

To ensure that the interior equilibrium exist and is stable, lets consider the following set of parameters:

$$
\begin{cases} r_1 = 0.635 \\ r_2 = 0.742 \\ p = 0.853 \end{cases}, \quad
\begin{cases} \gamma_{12} = 0.142 \\ \gamma_{21} = 0.002 \\ \gamma_{13} = 0.148 \\ \gamma_{31} = 0.215 \end{cases}, \quad
\begin{cases} v_1 = 0.090 \\ v_2 = 0.891 \\ v_3 = 0.980 \end{cases} \tag{6.5}
$$

Under this set of parameter values, the interior equilibrium is $E_{xyz} = (0.941, 0.174, 0.426)$. This is further supported by Figure 2a, Figure 2b, Figure 2c, Figure 2d, and Figure 2e where Figure 2a shows the time evolution of each species, Figure 2b shows the phase portrait, and Figure 2c, Figure 2d, and Figure 2e are phase planes when numerically solving Model (3.2).

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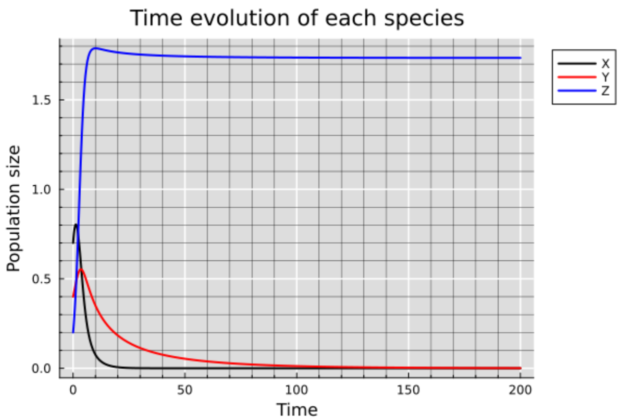
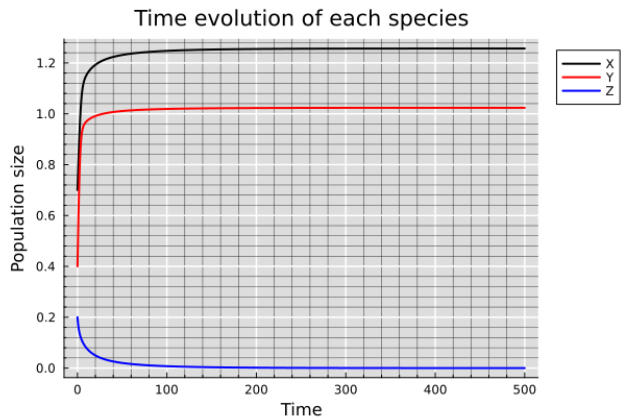
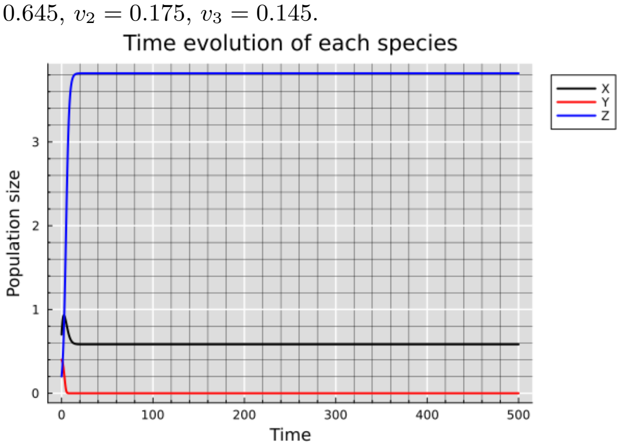
A Figures



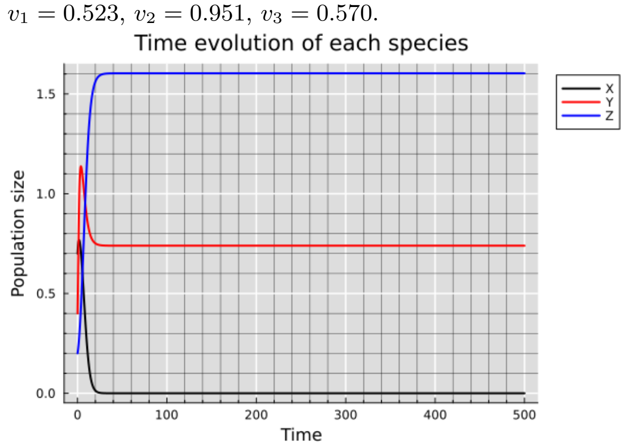
(a) z-axial equilibria; $r_1 = 0.404$, $r_2 = 0.903$, $p = 0.182$, $\gamma_{12} = 0.639$, $\gamma_{21} = 0.283$, $\gamma_{13} = 0.301$, $\gamma_{31} = 0.110$, $v_1 = 0.645$, $v_2 = 0.175$, $v_3 = 0.145$.



(b) xy-boundary equilibria; $r_1 = 0.978$, $r_2 = 0.613$, $p = 0.326$, $\gamma_{12} = 0.245$, $\gamma_{21} = 0.015$, $\gamma_{13} = 0.920$, $\gamma_{31} = 0.696$, $v_1 = 0.523$, $v_2 = 0.951$, $v_3 = 0.570$.

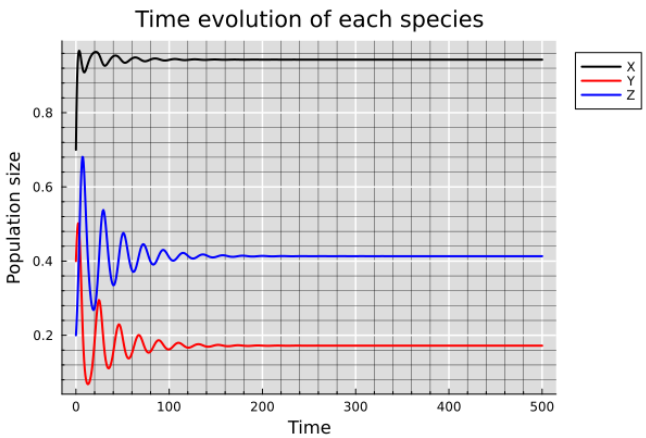


(c) xz-boundary equilibria; $r_1 = 0.102$, $r_2 = 0.763$, $p = 0.271$, $\gamma_{12} = 0.182$, $\gamma_{21} = 0.301$, $\gamma_{13} = 0.109$, $\gamma_{31} = 0.198$, $v_1 = 0.983$, $v_2 = 0.186$, $v_3 = 0.113$.

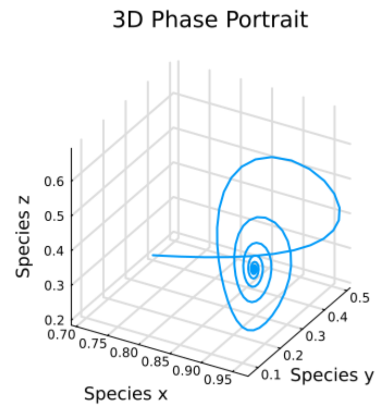


(d) yz-boundary equilibria; $r_1 = 0.978$, $r_2 = 0.310$, $p = 0.843$, $\gamma_{12} = 0.002$, $\gamma_{21} = 0.407$, $\gamma_{13} = 0.859$, $\gamma_{31} = 0.446$, $v_1 = 0.872$, $v_2 = 0.201$, $v_3 = 0.959$.

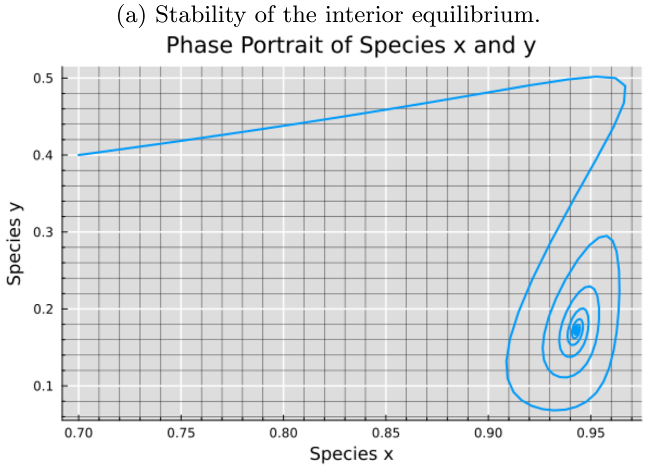
Figure 1: Showing the stability of equilibrium points with different set of parameters.



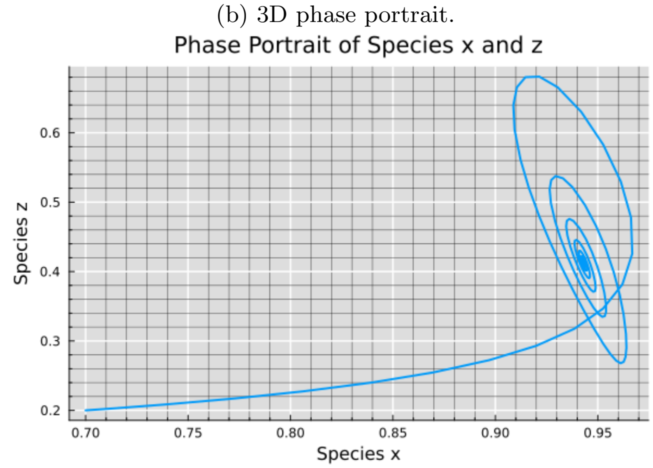
(a) Stability of the interior equilibrium.



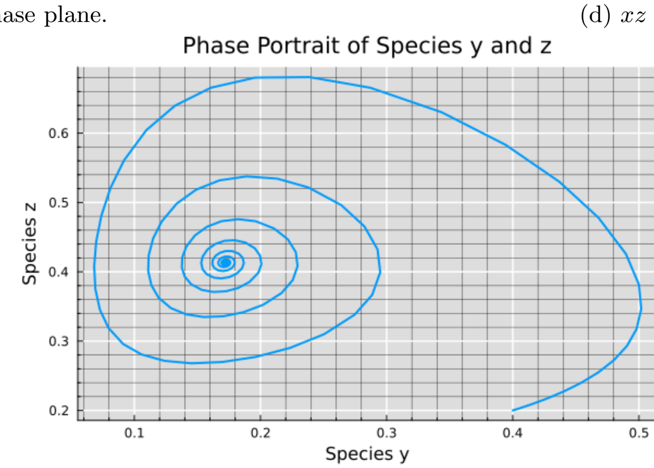
(b) 3D phase portrait.



(c) xy phase plane.



(d) xz phase plane.



(e) yz phase plane.

Figure 2: Different types of plots to show the behavior of Model (3.2) where $r_1 = 0.635$, $r_2 = 0.742$, $p = 0.853$, $\gamma_{12} = 0.142$, $\gamma_{21} = 0.002$, $\gamma_{13} = 0.148$, $\gamma_{31} = 0.215$, $v_1 = 0.090$, $v_2 = 0.891$, $v_3 = 0.980$.