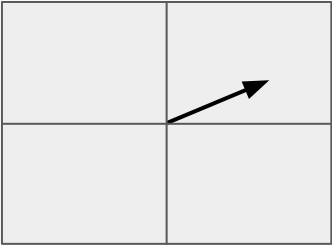
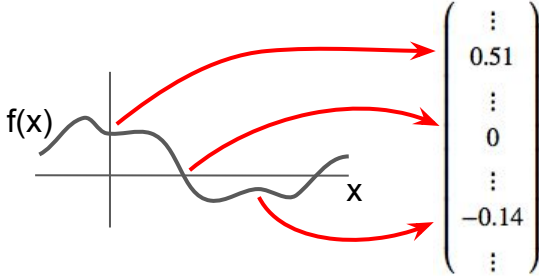
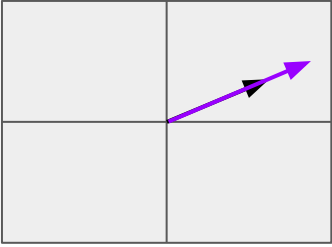
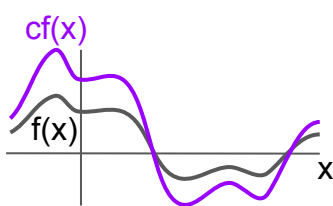
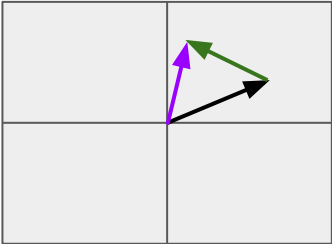
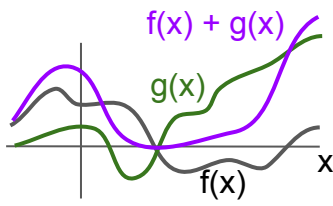
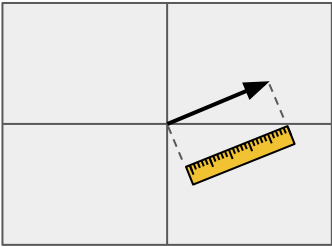
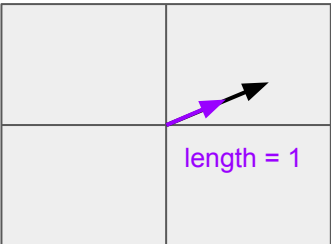
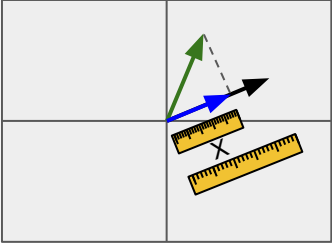
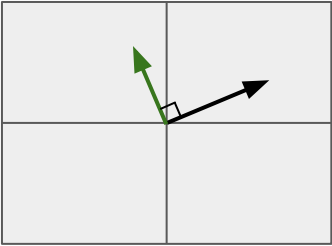
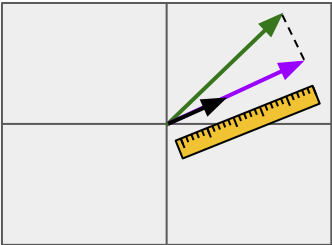
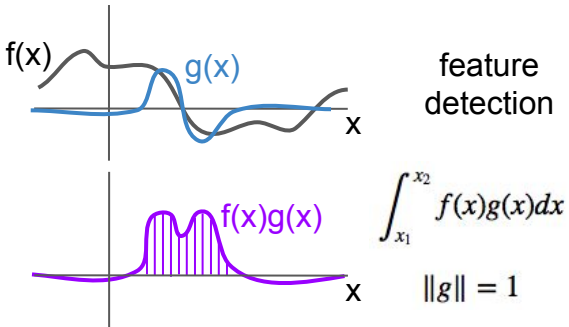
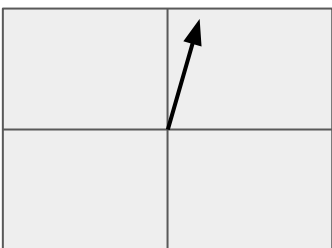


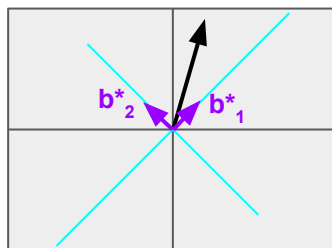
2-D	N-D	$\infty$ -D
<p>general description</p> 	<p>general description</p> <p>column representation</p> $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}$ <p>row representation</p> $\mathbf{v}^T = (v_1 \ v_2 \ \dots \ v_N)$	<p>general description</p> 
<p>scalar multiplication</p> 	<p>scalar multiplication</p> $c\mathbf{v} = c \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix} = \begin{pmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_N \end{pmatrix}$	<p>scalar multiplication</p> 
<p>addition</p> 	<p>addition</p> $\mathbf{v} + \mathbf{u} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} = \begin{pmatrix} v_1 + u_1 \\ v_2 + u_2 \\ \vdots \\ v_N + u_N \end{pmatrix}$	<p>addition</p> 
<p>norm</p> 	<p>norm</p> $\ \mathbf{v}\  = \sqrt{v_1^2 + v_2^2 + \dots + v_N^2}$	<p>norm</p> $\ f\  = \int_{x_1}^{x_2} f(x)^2 dx$
<p>unit vector</p> 	<p>unit vector</p> $\hat{\mathbf{v}} = \frac{\mathbf{v}}{\ \mathbf{v}\ }$	<p>unit vector</p> $f^{unit} = \frac{f}{\ f\ }$

2-D	N-D	$\infty$ -D
<p>dot (inner) product</p> 	<p>dot (inner) product</p> $\mathbf{v} \cdot \mathbf{u} = \langle \mathbf{v}, \mathbf{u} \rangle = \mathbf{v}^T \mathbf{u} = v_1 u_1 + v_2 u_2 + \dots + v_N u_N$ <p>takes in two vectors, returns scalar</p>	<p>dot (inner) product</p> $\langle f, g \rangle = \int_{x_1}^{x_2} f(x)g(x)dx$
<p>orthogonality</p> 	<p>orthogonality</p> <p>two vectors are <i>defined</i> to be orthogonal if</p> $\mathbf{v} \cdot \mathbf{u} = \langle \mathbf{v}, \mathbf{u} \rangle = \mathbf{v}^T \mathbf{u} = 0$	<p>orthogonality</p> <p>two functions are <i>defined</i> to be orthogonal if</p> $\langle f, g \rangle = \int_{x_1}^{x_2} f(x)g(x)dx = 0$
<p>projection (filtering)</p> 	<p>projection (filtering)</p> $\hat{\mathbf{v}} \cdot \mathbf{u} = \langle \hat{\mathbf{v}}, \mathbf{u} \rangle = \hat{\mathbf{v}}^T \mathbf{u}$	<p>projection (filtering)</p>  <p>feature detection</p> $\int_{x_1}^{x_2} f(x)g(x)dx$ <p><math>\ g\  = 1</math></p>

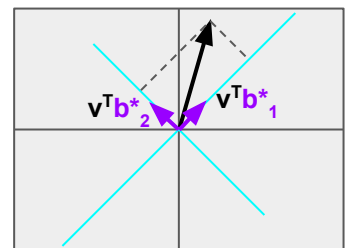
**Example:** rewrite the following vector in a coordinate system rotated 45 degrees CCW



$$\mathbf{v} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$



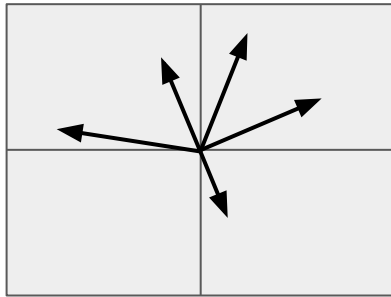
$$\mathbf{b}^*_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \mathbf{b}^*_2 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$



$$\mathbf{v}^* = \begin{pmatrix} \mathbf{v}^T \mathbf{b}^*_1 \\ \mathbf{v}^T \mathbf{b}^*_2 \end{pmatrix}$$

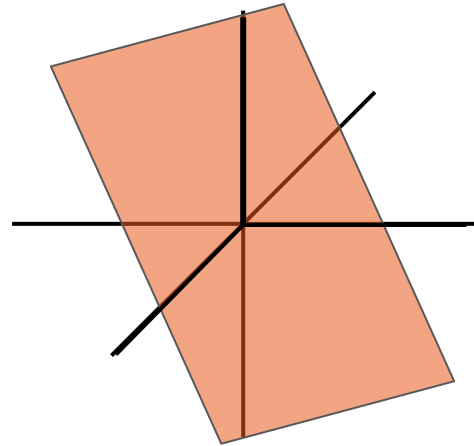
$$\mathbf{v}^* = \begin{pmatrix} 1/\sqrt{2} + 4/\sqrt{2} \\ -1/\sqrt{2} + 4/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 5/\sqrt{2} \\ 3/\sqrt{2} \end{pmatrix} \approx \begin{pmatrix} 3.54 \\ 2.12 \end{pmatrix}$$

vectors live in a *vector space*



2D vector space

one vector space can be a *subspace* of another vector space



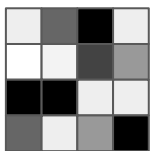
2D vector space as a subspace of a 3D vector space

## TIPS AND TRICKS

Always ask:

- Is the quantity I'm working with a scalar? vector? function?
  - If vector: what is dimensionality? what does dimensionality represent? what do indices represent? what do elements represent?
  - If function: what is domain of function? what does domain of function represent? what does function argument represent? what do function values represent?

representing an image as a vector



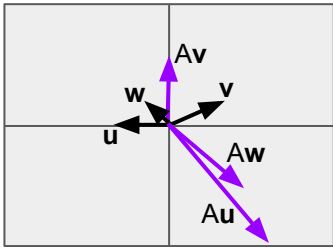
stack columns →



inner product between functions of multiple arguments

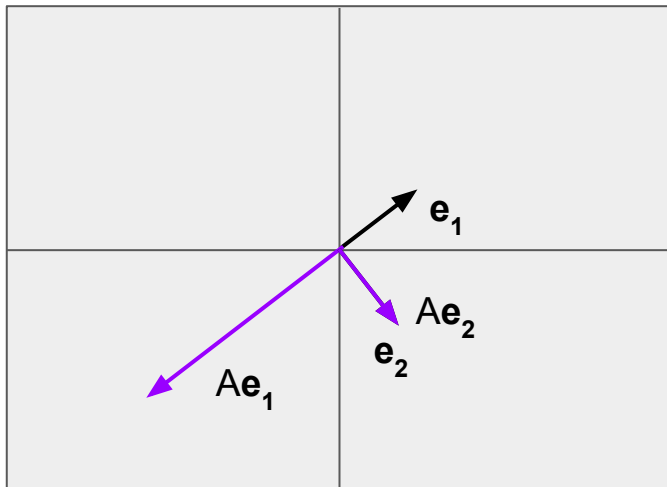
$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y)g(x, y)dx dy$$

$$\int_{t_1}^{t_2} \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y, t)g(x, y, t)dx dy dt$$

2 x 2	M x N
<p>general description</p> $A = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$	<p>general description</p> $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \ddots & a_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ a_{M1} & a_{N2} & \dots & a_{MN} \end{pmatrix}$ $= \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{a}_1^c & \mathbf{a}_2^c & \dots & \mathbf{a}_N^c \\ \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} \leftarrow (\mathbf{a}_1^r)^T \rightarrow \\ \leftarrow (\mathbf{a}_2^r)^T \rightarrow \\ \leftarrow \vdots \rightarrow \\ \leftarrow (\mathbf{a}_M^r)^T \rightarrow \end{pmatrix}$
<p>matrix times vector</p> $A\mathbf{v} = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} [-1 \times 2] + [2 \times 1] \\ [2 \times 2] + [-1 \times 1] \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$  <p>think of a matrix in terms of its actions on a vector space</p>	<p>matrix times vector</p> $A\mathbf{v} = \begin{pmatrix} (\mathbf{a}_1^r)^T \mathbf{v} \\ (\mathbf{a}_2^r)^T \mathbf{v} \\ \vdots \\ (\mathbf{a}_M^r)^T \mathbf{v} \end{pmatrix} = v_1 \mathbf{a}_1^c + v_2 \mathbf{a}_2^c + \dots + v_N \mathbf{a}_N^c$ <p>dot product of each row with <math>\mathbf{v}</math> or weighted sum of columns, weighted by elements of <math>\mathbf{v}</math></p>
<p>matrix times matrix</p> $AB = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} [-1 \times 2] + [2 \times 1], [-1 \times 3] + [2 \times -1] \\ [2 \times 2] + [-1 \times 1], [2 \times 3] + [-1 \times -1] \end{pmatrix} = \begin{pmatrix} 0 & -5 \\ 3 & 7 \end{pmatrix}$ <p><math>AB \neq BA</math></p>	<p>matrix times matrix</p> $AB = \begin{pmatrix} (\mathbf{a}_1^r)^T \mathbf{b}_1^c & (\mathbf{a}_1^r)^T \mathbf{b}_2^c & \dots & (\mathbf{a}_1^r)^T \mathbf{b}_P^c \\ (\mathbf{a}_2^r)^T \mathbf{b}_1^c & (\mathbf{a}_2^r)^T \mathbf{b}_2^c & \dots & (\mathbf{a}_2^r)^T \mathbf{b}_P^c \\ \vdots & \ddots & \ddots & \vdots \\ (\mathbf{a}_M^r)^T \mathbf{b}_1^c & (\mathbf{a}_M^r)^T \mathbf{b}_2^c & \dots & (\mathbf{a}_M^r)^T \mathbf{b}_P^c \end{pmatrix} = \begin{pmatrix} \uparrow & \uparrow & \dots & \uparrow \\ A\mathbf{b}_1^c & A\mathbf{b}_2^c & \dots & A\mathbf{b}_P^c \\ \downarrow & \downarrow & \dots & \downarrow \end{pmatrix}$ <p>each column of result is A multiplied by corresponding column of B</p>

$2 \times 2$ 

$$A = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$



$$\lambda_1 = -3, \lambda_2 = 1$$

eigenvectors of matrix  $A$  are vectors that change only length or sign, but not direction, when acted upon by  $A$

 $N \times N$ 

$$A\mathbf{e}_1 = \lambda_1\mathbf{e}_1$$

$$A\mathbf{e}_2 = \lambda_2\mathbf{e}_2$$

$$\vdots$$

$$A\mathbf{e}_N = \lambda_N\mathbf{e}_N$$

$$A = E\Lambda E^T$$

matrix whose  
columns  
are eigenvectors

diagonal matrix  
whose diagonal  
elements are  
eigenvalues

MATLAB Code:

```
[E, L] = eig(A)
```

$E$  is matrix whose columns are  
eigenvectors

$L$  is matrix whose diagonals are  
corresponding eigenvalues

Python Code:

```
E, L = np.linalg.eig(A)
```

$E$  is matrix whose columns are  
eigenvectors

$L$  is 1-D array whose elements  
are corresponding eigenvalues

fun facts

- eigenvalues and eigenvectors of matrix are *invariant* to change of basis.
  - eigenvalues will be identical
  - eigenvectors will be same vectors, just rewritten in new coordinates
- if  $A$  is symmetric ( $A = A^T$ ), then eigenvalues are real and eigenvectors are orthogonal.
- eigenvalues and eigenvectors often have special meaning:
  - e.g., in PCA related to directions of maximum variance
  - e.g., in dynamical systems related to system stability

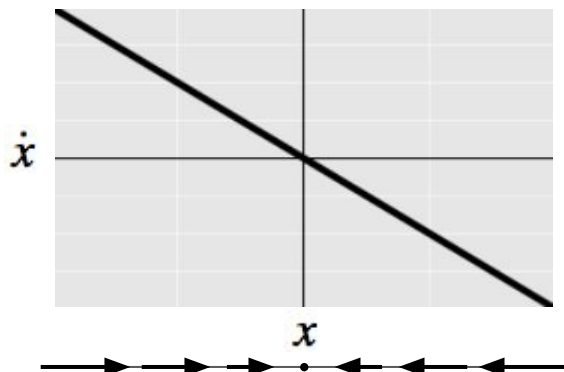
- “dynamical system” = system that changes in time
- represent with system of differential equations
- solving DEs is hard -- what can we learn about system without finding explicit solution?

1D	2D	ND
<p>description</p> $\dot{x} = f(x)$ <p>note: <math>\dot{x} \equiv \frac{dx}{dt}</math></p> <p>linear system</p> $\dot{x} = ax$	<p>description</p> $\dot{x}_1 = f_1(x_1, x_2)$ $\dot{x}_2 = f_2(x_1, x_2)$ $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ <p>linear system</p> $\dot{x}_1 = a_{11}x_1 + a_{12}x_2$ $\dot{x}_2 = a_{21}x_1 + a_{22}x_2$ $\dot{\mathbf{x}} = A\mathbf{x}$	<p>description</p> $\dot{x}_1 = f_1(x_1, \dots, x_N)$ $\vdots$ $\dot{x}_N = f_N(x_1, \dots, x_N)$ $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ <p>linear system</p> $\dot{x}_1 = a_{11}x_1 + \dots + a_{1N}x_N$ $\vdots$ $\dot{x}_N = a_{N1}x_1 + \dots + a_{NN}x_N$ $\dot{\mathbf{x}} = A\mathbf{x}$

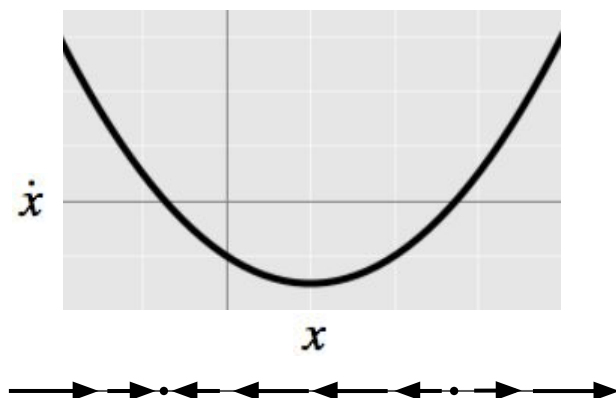
## 1D

graphical representation

linear

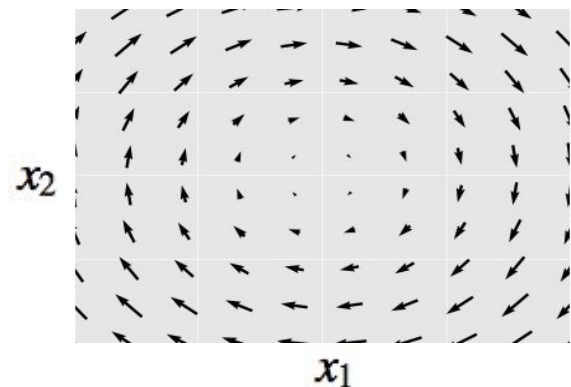
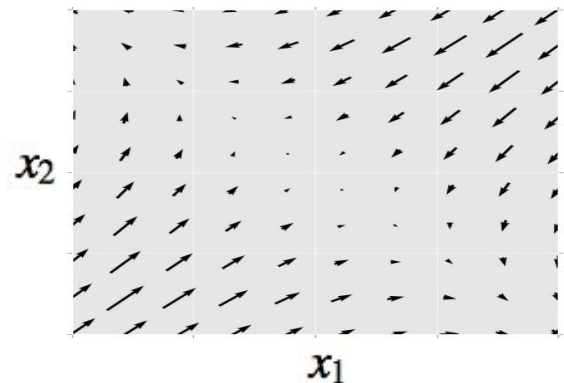


nonlinear

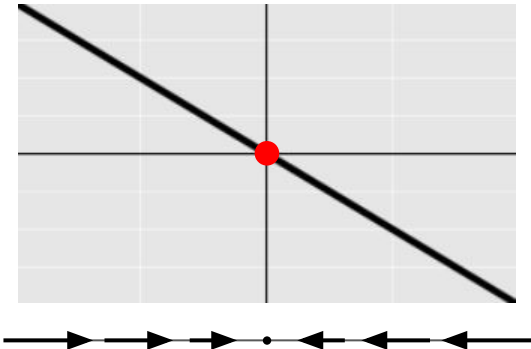
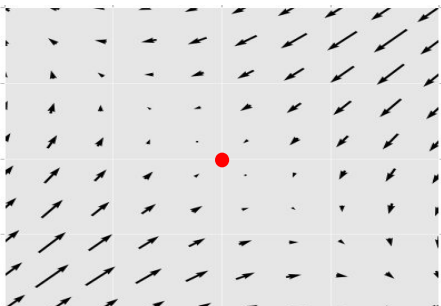
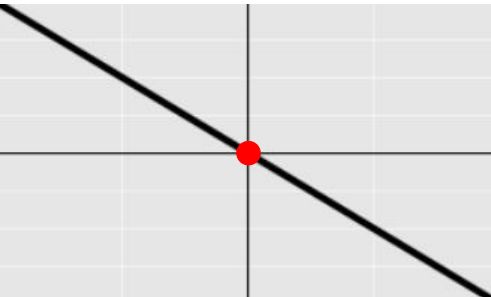
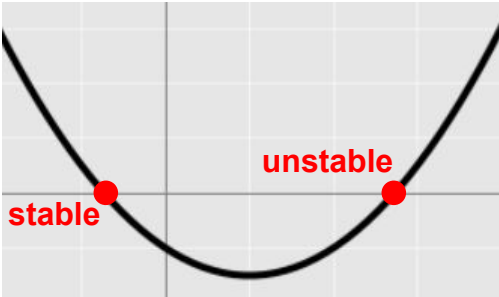
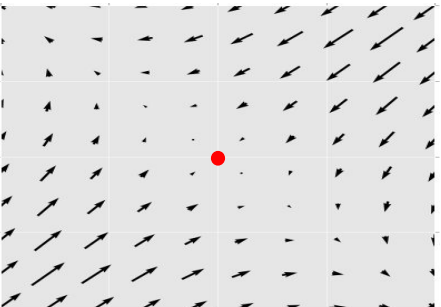
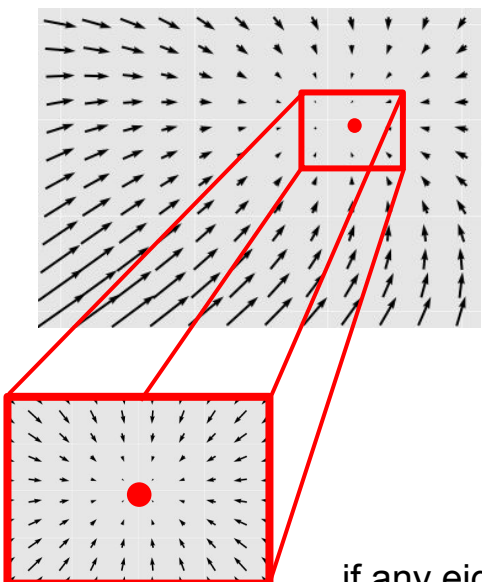


## 2D

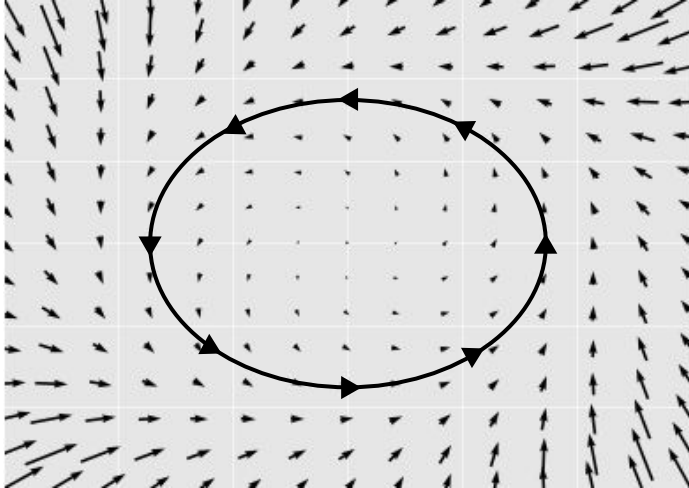
graphical representation (phase portrait)





1D	2D	ND
<p>fixed points</p> 	<p>fixed points</p> 	$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \mathbf{0}$
1D	ND	
<p>stability analysis</p> <p><i>linear</i></p>  <p><i>slope &gt; 0 : unstable</i> <i>slope &lt; 0 : stable</i></p> <p><i>nonlinear</i></p>  <p><math>\frac{df}{dx} &gt; 0 : \text{unstable}</math> <math>\frac{df}{dx} &lt; 0 : \text{stable}</math></p> <p>(evaluated at fixed point)</p>	<p>stability analysis</p> <p><i>linear</i></p>  <p><math>\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}</math></p> <p>if real part of any <i>eigenvalue</i> of <math>\mathbf{A}</math> is greater than 0, fixed point is unstable</p>	
	<p><i>nonlinear</i></p>  <p><math>\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})</math></p> $\mathbf{J} = \begin{bmatrix} \frac{\partial \mathbf{f}}{\partial x_1} & \cdots & \frac{\partial \mathbf{f}}{\partial x_n} \\ \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$ <p>if any eigenvalue of <math>\mathbf{J}</math> (evaluated at fixed point) is greater than 0, fixed point is unstable</p>	

limit cycles



strange attractors (chaos)

