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Equation of Motion and Euler's Equation

1. Equation of Motion (Conservation of Momentum)

3.2 EQUATION OF MOTION (Conservation of Momentum)

Consider a differential control volume, $dx\,dy\,dz$, in the form of a cube as shown in Fig. (3.2.1).

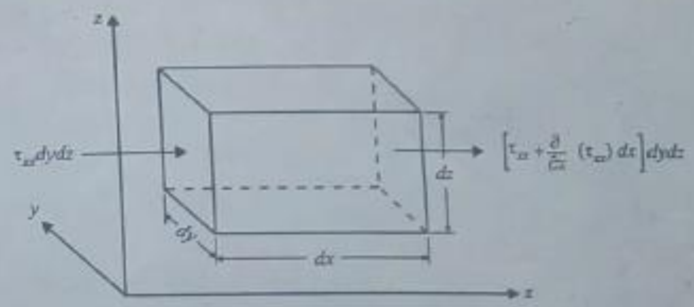

Fig. 3.2.1 Control volume $dx.dy.dz$ for momentum balance

Momentum enters and leaves the control volume by two mechanisms advection and molecular transfer. These enter through three faces and leave through three opposite faces.

The gross rate at which x-component of flow momentum enters into the control volume is

$$\rho v_x v_x\, dydz + \rho v_y v_x\, dxdz + \rho v_z v_x\, dydx$$

The gross rate at which x-component of flow momentum leaves the control volume is

$$\left[\rho v_x v_x + \frac{\partial}{\partial x}(\rho v_x v_x)\,dx\right]dydz + \left[\rho v_y v_x + \frac{\partial}{\partial y}(\rho v_y v_x)\,dy\right]dxdz + \left[\rho v_z v_x + \frac{\partial}{\partial z}(\rho v_z v_x)\,dz\right]dxdy$$

The net rate at which x- component of flow momentum enters the control volume, therefore, is

$$-\left[\frac{\partial}{\partial x}(\rho v_x v_x) + \frac{\partial}{\partial y}(\rho v_y v_x) + \frac{\partial}{\partial z}(\rho v_z v_x)\right]dxdydz$$

The gross rate at which x-component of momentum enters the control volume by **molecular transport** is

$$\tau_{xx}\, dydz + \tau_{yx}\, dxdz + \tau_{zx}\, dxdy$$

The gross rate at which x-component of momentum leaves the control volume by **molecular transport** is

$$\left[\tau_{xx} + \frac{\partial}{\partial x}(\tau_{xx})\,dx\right]dydz + \left[\tau_{yx} + \frac{\partial}{\partial y}(\tau_{yx})\,dy\right]dxdz + \left[\tau_{zx} + \frac{\partial}{\partial z}(\tau_{zx})\,dz\right]dxdy$$

The net rate at which x-component of momentum enters the control volume by molecular transport, therefore, is

$$-\left[\frac{\partial}{\partial x}(\tau_{xx})+\frac{\partial}{\partial y}(\tau_{yx})+\frac{\partial}{\partial z}(\tau_{zx})\right]dxdydz$$

If pressure and gravitational forces are the only forces acting on the control volume, then the resultant of these two forces in the x-direction is

$$\left\{p-\left(p+\frac{\partial p}{\partial x}dx\right)\right\}dydz+\rho g_x\,dxdydz$$

where g_x is the x-component of acceleration due to gravity.

The rate at which x-component of momentum is accumulated within the control volume is

$$\frac{\partial}{\partial t}(\rho v_x)\,dxdydz$$

Substituting these terms in the general statement of conservation of momentum, Eq. (1.1.4), and dividing both sides by control volume, we obtain,

$$\frac{\partial}{\partial t}(\rho v_x)=-\left\{\frac{\partial}{\partial x}(\rho v_x^2)+\frac{\partial}{\partial y}(\rho v_y v_x)+\frac{\partial}{\partial z}(\rho v_z v_x)\right\}-\left(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)-\frac{\partial p}{\partial x}+\rho g_x \quad ...(3.2.1)$$

Equation (3.2.1) is the x-component of the equation of motion.

Similarly, we obtain the y- and z- components of the equation of motion which, respectively, are

$$\frac{\partial}{\partial t}(\rho v_y)=-\left\{\frac{\partial}{\partial x}(\rho v_x v_y)+\frac{\partial}{\partial y}(\rho v_y^2)+\frac{\partial}{\partial z}(\rho v_z v_y)\right\}-\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\right)-\frac{\partial p}{\partial y}+\rho g_y \quad ...(3.2.2)$$

$$\frac{\partial}{\partial t}(\rho v_z)=-\left\{\frac{\partial}{\partial x}(\rho v_x v_z)+\frac{\partial}{\partial y}(\rho v_y v_z)+\frac{\partial}{\partial z}(\rho v_z^2)\right\}-\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}\right)-\frac{\partial p}{\partial z}+\rho g_z \quad ...(3.2.3)$$

The quantities $\rho v_x, \rho v_y, \rho v_z$ are the components of the mass velocity vector $\rho\vec{v}$, g_x, g_y, g_z are the components of the gravitational acceleration $\vec{g}$ and $\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}$ are the components of the ∇p, known as the gradient of p, often written as grad p. The terms $\rho v_x v_x$, $\rho v_x v_y$, $\rho v_x v_z$, etc. are the nine components of momentum flux due to advection, $\rho \vec{v}\,\vec{v}$. Similarly, τ_{xx}, τ_{xy}, τ_{xz}, etc. are the nine components of the stress tensor, τ. Eqs. (3.2.1), (3.2.2), (3.2.3) may be combined to give a single vector equation :

$$\frac{\partial}{\partial t}(\rho \vec{v}) = -[\nabla . \rho \vec{v}\,\vec{v}] - \nabla p - [\nabla . \tau] + \rho \vec{g} \qquad ...(3.2.4)$$

Equation (3.2.1) may be rearranged after performing indicated differentiation in the following form :

$$\rho\left[\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right] + v_x\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z)\right]$$
$$= -\left(\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx}\right) - \frac{\partial p}{\partial x} + \rho g_x \qquad ...(3.2.5)$$

Eq. (3.2.5) may be simplified by using the continuity equation (3.1.1) to give

$$\rho\frac{Dv_x}{\partial t} = -\frac{\partial p}{\partial x} - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{zx}}{\partial z}\right) + \rho g_x \qquad ...(3.2.6)$$

Similarly, Eqs. (3.2.2) and (3.2.3) may be rearranged to give, respectively,

$$\rho\frac{Dv_y}{\partial t} = -\frac{\partial p}{\partial y} - \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + \rho g_y \qquad ...(3.2.7)$$

$$\rho\frac{Dv_z}{\partial t} = -\frac{\partial p}{\partial z} - \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) + \rho g_z \qquad ...(3.2.8)$$

When these three components of the equation of motion are added vertically, we obtain,

$$\rho\frac{D\vec{v}}{Dt} = -\nabla p - [\nabla . \tau] + \rho \vec{g} \qquad ...(3.2.9)$$

The term on the left is mass times acceleration, where acceleration is expressed by the substantial derivative of velocity to account for the motion of system. The first, second and third terms on the right represent the forces, due respectively, to pressure, fluid friction and gravitational acceleration on the control volume. The equation of motion as written in the form of Eq. (3.2.9) states that a small volume element moving with the fluid is accelerated because of the forces acting upon it. In other words Eq. (3.2.9) is a statement of Newton's second law of motion i.e., mass times acceleration equals sum of the forces.

The next and most vital step involves the relationship between the stresses and the motion of the fluid. The stresses in many fluids are related linearly to the derivatives of the velocities. Most fluids are isotropic *i.e.,* these fluids have no preferred direction in space. The stresses do not explicitly depend on the position coordinate and the fluid velocities. Also the pairs of stresses with subscript which differ only in their order are equal. This follows by taking moments about any axis and using the condition of equilibrium of the differential control volume. Thus, for example, for the z-axis, we obtain

$$\tau_{xy}\, dy\, dz\, dx = \tau_{yx}\, dx\, dz\, dy$$

Hence $\tau_{xy} = \tau_{yx}$ and similarly,

$$\tau_{xz} = \tau_{zx} \quad \text{and} \quad \tau_{yz} = \tau_{zy}$$

For Newtonian fluids, the relationship between stresses and velocity gradients are :

$$\tau_{xx} = -2\mu \frac{\partial v_x}{\partial x} + \frac{2}{3}\mu(\nabla \cdot \vec{v}) \qquad \text{...(3.2.10)}$$

$$\tau_{yy} = -2\mu \frac{\partial v_y}{\partial y} + \frac{2}{3}\mu(\nabla \cdot \vec{v}) \qquad \text{...(3.2.11)}$$

$$\tau_{zz} = -2\mu \frac{\partial v_z}{\partial z} + \frac{2}{3}\mu(\nabla \cdot \vec{v}) \qquad \text{...(3.2.12)}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \qquad \text{...(3.2.13)}$$

$$\tau_{yz} = \tau_{zy} = -\mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) \qquad \text{...(3.2.14)}$$

$$\tau_{zx} = \tau_{xz} = -\mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right) \qquad \text{...(3.2.15)}$$

These equations constitute a more general statement of Newton's law of viscosity than that given in Eq. (1.4.3) and apply to flow situations with fluid flowing in all directions.

Substitutions of Eqs. (3.2.10) through (3.2.15) into Eq. (3.2.6), (3.2.7) and (3.2.8) give the general equation of motion for a Newtonion fluid with varying density and viscosity.

x-component :

$$\rho \frac{Dv_x}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[2\mu \frac{\partial v_x}{\partial x} - \frac{2}{3}\mu(\nabla \cdot \vec{v})\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right)\right] + \rho g_x \qquad \text{...(3.2.16)}$$

y-component :

$$\rho \frac{Dv_y}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[2\mu \frac{\partial v_y}{\partial y} - \frac{2}{3}\mu(\nabla \cdot \vec{v})\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right)\right] + \rho g_y \quad ...(3.2.17)$$

z-component :

$$\rho \frac{Dv_z}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right)\right] + \frac{\partial}{\partial z}\left[2\mu \frac{\partial v_z}{\partial z} - \frac{2}{3}\mu(\nabla \cdot \vec{v})\right] + \rho g_z \quad ...(3.2.18)$$

These equations are called the **Navier-Stokes Equations**. These equations can be simplified for the case of incompressible flow with constant viscosity because the continuity equation for incompressible fluid viz. $(\nabla \cdot \vec{v}) = 0$.

Thus for constant ρ and constant μ, the Navier-Stokes Equations become

x :

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \rho g_x \quad ...(3.2.19)$$

y :

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) + \rho g_y \quad ...(3.2.20)$$

z :

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_z \quad ...(3.2.21)$$

Adding these three components vertically, the Navier-Stokes equations for fluid of constant density and viscosity become

$$\rho \frac{D\vec{v}}{dt} = -\nabla p + \mu \nabla^2 \vec{v} + \rho g \quad ...(3.2.22)$$

2. Euler's Equation

Eular's Equation

For the case of frictionless flow $\mu = 0$ and thus $[\nabla \cdot \tau] = 0$, the equation of motion (3.2.9) or (3.2.22) reduces to

$$\rho \frac{D\vec{v}}{Dt} = -\nabla p + \rho \vec{g} \qquad \text{...(3.2.23)}$$

Equation (3.2.23) is called **Euler's equation** which applies to the motion of a fluid for which forces due to viscous friction are negligible compared with pressure and gravity forces. The classical science of hydrodynamics has evolved from the solution of Euler's equation of motion. The mathematical solutions of classical hydrodynamics are, however, not always in agreement with experimental results *e.g.*, for the drag of a solid body completely immersed in a body of uniformly moving fluid, classical hydrodynamics predicts a zero drag which is not in agreement with the experimental results. This type of contradiction is usually referred to as **d' Alembert's paradox.**

Note that in any equation of motion expressed in terms of substantial derivative, $\frac{Dv_x}{Dt}$, the term $\frac{\partial v_x}{\partial t}$ is the time rate of change of v_x at a point, and is the local acceleration while the remaining terms involve the velocity change from point to point *i.e.*, the convective acceleration. The sum of these terms is the total acceleration. Corresponding expressions hold for other two components of velocity. **The substantial derivative, therefore, accounts for the fact that a fluid element moving through a flow field, can experience a velocity change both due to the flow field changing with time and the fluid element changing its position in the flow field.**

The Navier-Stokes equations are nonlinear due to $v_x \frac{dv_x}{dx}$ etc. terms and exact solution can be obtained only in limited cases.

We shall discuss later how the Navier-Stokes equations are reduced to the simple forms for the cases considered previously in Chapter 2.