UIT-RGPV (Autonomous) Bhopal
Department of Petrochemical Engineering

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Subject code – PC 702 **Subject: Transport Phenomena**
Unit: -I

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Calculation in Transport Phenomena

Calculations in Transport Phenomena

THEORETICAL BACKGROUND

Forms of derivatives :

(1) Partial time derivative, $\frac{\partial c}{\partial t}$.

This means variation of c (the variable) with time with respect to a fixed position (x, y, z) in space.

(2) Total time derivative, $\frac{dc}{dt}$

$$\frac{dc}{dt} = \frac{\partial c}{\partial t} + \frac{\partial c}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial c}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial c}{\partial z} \cdot \frac{dz}{dt}$$

where, $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dz}{dt}$ are the variation of x, y and z respectively with respect to time.

(3) Substantial time derivative, $\frac{Dc}{Dt}$

It is a special kind of total time derivative also, called 'derivative following the motion'. The expression is

$$\frac{Dc}{Dt} = \frac{\partial c}{\partial t} + v_x \frac{\partial c}{\partial x} + v_y \frac{\partial c}{\partial y} + v_z \frac{\partial c}{\partial z}$$

where, v_x, v_y and v_z are components of local fluid velocity, v.

Equation of continuity :

This is based on conservation of mass. According to this,

Rate of change of density = Divergence of mass flux

Mathematically,

$$\frac{\partial \rho}{\partial t} = -(\nabla \rho v) \quad \text{...(1)}$$

or

$$\frac{D\rho}{Dt} = -\rho (\nabla \cdot v)$$

For steady state, $$\frac{D\rho}{Dt} = 0 \quad \text{...(2)}$$

θ-component :

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v\,\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\cdot\frac{\partial v_\theta}{\partial \theta}+\frac{v_\varphi}{r\sin\theta}\cdot\frac{\partial v_\theta}{\partial \varphi}+\frac{v_r v_\theta}{r}-\frac{v_\varphi^2\cot\theta}{r}\right)$$

$$=-\frac{1}{r}\cdot\frac{\partial p}{\partial \theta}+\mu\left(\nabla^2 v_\theta+\frac{2}{r^2}\cdot\frac{\partial v_r}{\partial \theta}-\frac{v_\theta}{r^2\sin^2\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\cdot\frac{\partial v_\varphi}{\partial \varphi}\right)+\rho g_\theta \quad \text{...(15)}$$

φ-component :

$$\rho\left(\frac{\partial v_\varphi}{\partial t}+v_r\cdot\frac{\partial v_\varphi}{\partial r}+\frac{v_\theta}{r}\cdot\frac{\partial v_\varphi}{\partial \theta}+\frac{v_\varphi}{r\sin\theta}\cdot\frac{\partial v_\varphi}{\partial \varphi}+\frac{v_\varphi v_r}{r}+\frac{v_\theta v_\varphi}{r}\cot\theta\right)$$

$$=-\frac{1}{r\sin\theta}\cdot\frac{\partial p}{\partial \varphi}+\mu\left(\nabla^2 v_\varphi-\frac{v_\varphi}{r^2\sin^2\theta}+\frac{2}{r^2\sin^2\theta}\frac{\partial v_r}{\partial \varphi}+\frac{2\cos\theta}{r^2\sin^2\theta}\cdot\frac{\partial v_\theta}{\partial \varphi}\right)+\rho g_\varphi \quad \text{...(16)}$$

In equations (14) to (16),

$$\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial \varphi^2}\right) \quad \text{...(17)}$$

Shell Energy Balance :

For steady state, a statement of the law of conservation of energy gives

$$\begin{pmatrix}\text{rate of thermal}\\ \text{energy in}\end{pmatrix}-\begin{pmatrix}\text{rate of thermal}\\ \text{energy out}\end{pmatrix}+\begin{pmatrix}\text{rate of thermal}\\ \text{energy production}\end{pmatrix}=0 \quad \text{...(18)}$$

Thermal energy may enter or leave a system

(*i*) by virtue of heat conduction

(*ii*) by fluid motion (convective transport).

Thermal energy may be produced in various ways *viz.*

(1) by degradation of mechanical energy

(2) by degradation of electrical energy

(3) by conversion of chemical energy into heat.

Equation (18) can be applied for a system consisting of a thin slab or shell, the thickness of which may be allowed to approach zero.

Example 1. *With the help of momentum balance, derive expressions for the following in case of a steady laminar flow of incompressible fluid in a pipe*

(a) *Velocity profile* (b) *Average velocity*

(c) *Volumetric flow.*

the top plate is moving with a constant velocity 'U'. Derive expressions for velocity distribution and shear stress at the well.

Solution.

From momentum balance,

$$\frac{\partial \tau}{\partial y} = -\frac{\partial p}{\partial x}$$

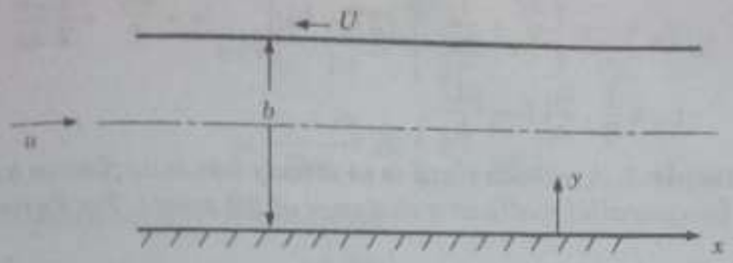

Fig. 9.2

Also, $$\tau = -\mu \frac{\partial u}{\partial y}$$

So, $$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x} \quad \therefore \quad \frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu} \cdot \frac{\partial p}{\partial x}$$

Integration w.r.t 'Y' gives, $$\frac{\partial u}{\partial y} = \frac{1}{\mu} \frac{\partial p}{\partial x} Y + C_1$$

Further integration w.r.t. 'Y', gives,

$$u = \frac{1}{\mu} \cdot \frac{\partial p}{\partial x} \cdot \frac{Y^2}{2} + C_1 Y + C_2$$

Boundary conditions :

(*i*) When $Y = 0, u = 0$

(*ii*) When $Y = b, u = U$

With the first boundary condition, $C_2 = 0$, and the second boundary condition yields

$$C_1 = \frac{U}{b} - \frac{1}{\mu} \frac{\partial p}{\partial x} \cdot \frac{b}{2}$$

Putting values of C_1 and C_2 in the velocity profile expression,

$$u = \frac{1}{\mu} \frac{\partial p}{\partial x} \cdot \frac{Y^2}{2} + \frac{U}{b} Y - \frac{1}{\mu} \frac{\partial p}{\partial x} \frac{y}{2} Y$$

$$= \frac{UY}{b} + \frac{1}{2\mu} \frac{\partial p}{\partial x} (Y^2 - by)$$

The expression for velocity distribution is

$$u = \frac{UY}{b} + \frac{1}{2\mu} \frac{\partial p}{\partial x} (Y^2 - by)$$

Shear stress at the wall :

$$\tau_w \Big|_{Y=0} = -\mu \left(\frac{\partial u}{\partial y} \right)_{Y=0}$$

$$\frac{\partial u}{\partial y} = \frac{U}{b} + \frac{1}{2\mu} \frac{\partial p}{\partial x} (2Y - b)$$

$$\tau_w = -\mu \left[\frac{U}{b} + \frac{1}{2\mu} \frac{\partial p}{\partial x} (2Y - b) \right]_{Y=0} = -\frac{\mu U}{b} + \frac{1}{2} \frac{\partial p}{\partial x} \cdot b$$

So, $$\tau_w = \frac{1}{2} \cdot \frac{\partial p}{\partial x} b - \frac{\mu U}{b}.$$

Example 3. *A viscous fluid is in steady laminar flow in a slit formed by two parallel walls at a distance of 2B apart. Find expressions for*

(a) *Velocity profile* (b) *Shear stress distribution*

(c) *Volumetric flow rate of fluid.*

Solution :

From shell momentum balance,

$$\frac{\partial \tau}{\partial x} = -\frac{\partial p}{\partial z}$$

Also, $$\tau = -\mu \left(\frac{\partial u}{\partial x} \right)$$

So, $$\mu \frac{\partial^2 u}{\partial x^2} = \frac{\partial p}{\partial z}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{1}{\mu} \frac{\partial p}{\partial z}$$

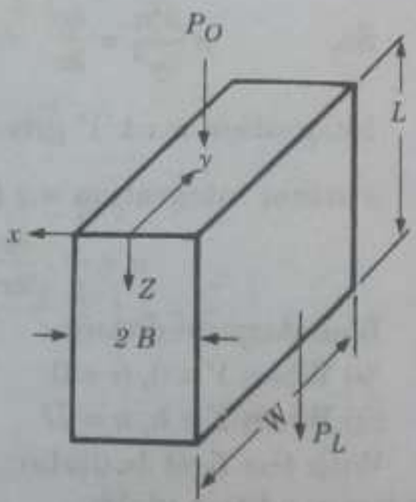

Fig. 9.3

Integration w.r.t. 'x' gives,

$$\frac{\partial u}{\partial x} = \frac{1}{\mu} \frac{\partial p}{\partial z} x + C_1$$

When $x = 0$, $u = u_{max}$ $\therefore$ $\frac{\partial u}{\partial x} = 0.$

This is satisfied when $C_1 = 0$.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1}{\mu} \cdot \frac{\partial p}{\partial z} x$$

Further integration w.r.t. 'x' gives,

$$u = \frac{1}{\mu} \cdot \frac{\partial p}{\partial z} \frac{x^2}{2} + C_2$$

When $x = B$, $u = 0$

This condition yields, $C_2 = -\frac{1}{\mu}\frac{\partial p}{\partial z}\cdot\frac{B^2}{2}$

Putting value of C_2 in velocity distribution equation,

$$u = \frac{1}{2\mu}\cdot\frac{\partial p}{\partial z}x^2 - \frac{1}{2\mu}\frac{\partial p}{\partial z}B^2$$

$$= \frac{1}{2\mu}\cdot\frac{\partial p}{\partial z}B^2\left(\frac{x^2}{B^2}-1\right) = -\frac{1}{2\mu}\cdot\frac{\partial p}{\partial z}B^2\left(1-\frac{x^2}{B^2}\right)$$

$$= \frac{P_0 - P_L}{2\mu L}B^2\left(1-\frac{x^2}{B^2}\right)$$

Shear stress distribution :

$$\tau = -\mu\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial u}{\partial x} = -\frac{1}{2\mu}\cdot\frac{\partial p}{\partial z}(-2x) = \frac{1}{2\mu}\cdot\frac{\partial p}{\partial z}2x = \frac{1}{\mu}\cdot\frac{\partial p}{\partial z}x$$

$$\tau = -\frac{\partial p}{\partial z}\cdot x = \frac{P_0 - P_L}{L}x$$

Volumetric flow rate of fluid :

$$\text{Average velocity} = u_{av} = \frac{\int_0^W\int_{-B}^{B} u\,dx\,dy}{\int_0^W\int_{-B}^{B} dx\,dy}$$

$$= \frac{W\int_{-B}^{B}\frac{1}{2\mu}\frac{\partial p}{\partial z}(x^2 - B^2)\,dx}{W\int_{-B}^{B}dx} = \frac{\frac{1}{2\mu}\cdot\frac{\partial p}{\partial z}\left(\frac{x^3}{3}-B^2x\right)_{-B}^{B}}{\left(x\right)_{-B}^{B}}$$

$$= \frac{P_0 - P_L}{3\mu L}B^2$$

Volumetric flow rate $= u_{av} \times$ **Area**

$$= \frac{P_0 - P_L}{3\mu L}B^2 \times 2BW = \frac{2}{3\mu}\frac{P_0 - P_L}{L}B^3W.$$