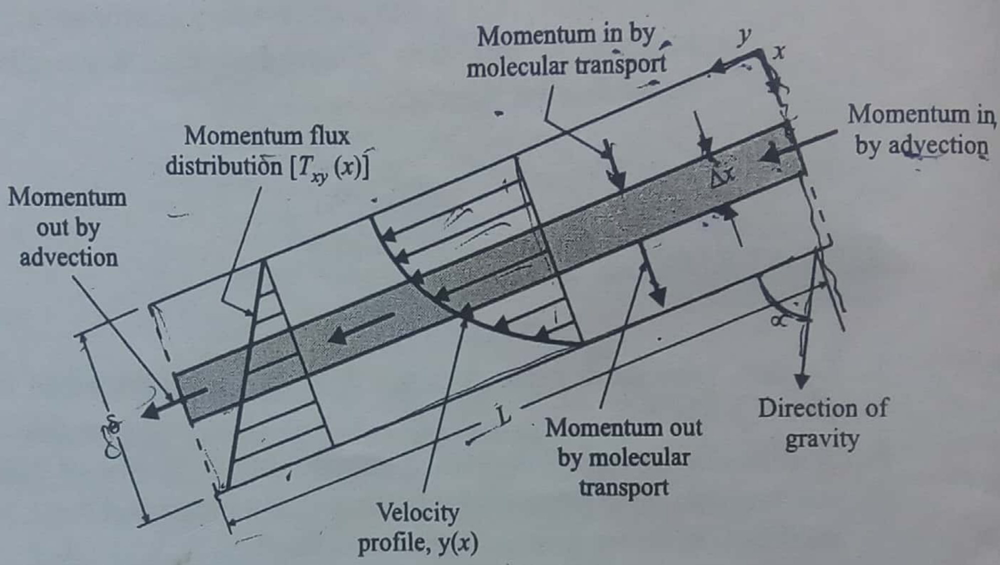
2.2 SHELL MOMENTUM BALANCE FOR FALLING FILM

Consider the flow of a fluid as a film under laminar condition along an inclined flat surface, as shown in Fig. 2.2.1. Falling films have been used to study various phenomena in gas absorption, evaporation and coatings on surfaces. We consider the viscosity and density of the fluid to be constant. We confine our attention to a region, called control volume, of thickness Δx, length L, sufficiently far from the entrance and exit regions so that the flow is not affected by the disturbances of these regions. The velocity $v_y(x)$ is, thus, independent of the position y.

Fig. 2.2.1 Laminar flow of a liquid down an inclined flat surface.



We set up a y-momentum balance over a shell of thickness Δx at a distance x from the free surface. The shell is bounded by planes $y=0$ and $y=L$ and extends a distance W in the Z-direction (which is perpendicular to the plane of the paper). The various terms of the momentum balance are ;

Rate of y-momentum in across the surface at x (by molecular transport)	$(LW)\,\tau_{xy}\big\|_x$
Rate of y-momentum out across the surface at $x+\Delta x$ (by molecular transport)	$(LW)\,(\tau_{xy})\big\|_{x+\Delta x}$
Rate of y-momentum in across the surface at $y=0$ (by advection)	$(W\Delta x v_y)\,(\rho v_y)\big\|_{y=0}$
Rate of y-momentum out across the surface at $y=L$	$(W\Delta x v_y)(\rho v_y)\big\|_{y=L}$
Gravity force acting on the fluid shell	$(LW\Delta x)(\rho g \cos\alpha)$

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Substituting these terms into the momentum balance Eq. (2.1.1), we get

$$\underset{1}{LW\tau_{xy}\Big|_x} \; \underset{2}{-LW\tau_{xy}\Big|_{x+\Delta x}} \; \underset{3}{+W\Delta x\rho v_y^2\Big|_{y=0}} \; \underset{4}{-W\Delta x\rho v_y^2\Big|_{y=L}} \; \underset{5}{+LW\Delta x\rho g\cos\alpha} = 0 \qquad \text{...(2.2.1)}$$

The third and the fourth terms cancel one another as v_y at $y=0$ is equal to v_y at $y=L$ for each value of x for constant density fluid. Dividing Eq. (2.2.1) by $LW\Delta x$ (volume of the element) and taking limit as Δx approaches zero, we get,

$$\lim_{\Delta x \to 0}\left(\frac{\tau_{xy}\big|_{x+\Delta x} - \tau_{xy}\big|_x}{\Delta x}\right) = \rho g\cos\alpha \qquad \text{...(2.2.2)}$$

or

$$\frac{d\tau_{xy}}{dx} = \rho g\cos\alpha \qquad \text{...(2.2.3)}$$

This differential equation describes momentum flux distribution. Integrating Eq. (2.2.3) we get,

$$\tau_{xy} = \rho g x\cos\alpha + C_1 \qquad \text{...(2.2.4)}$$

The constant of integration, C_1 is evaluated by using the boundary condition at the gas-liquid interface *i.e.*,

$$\text{at } x = 0, \quad \tau_{xy} = 0$$

Substituting this boundary condition in Eq. (2.2.4) gives $C_1 = 0$ and the momentum-flux distribution is

$$\tau_{xy} = \rho g x\cos\alpha \qquad \text{...(2.2.5)}$$

This indicates that the momentum-flux distribution is linear and the maximum value is at the solid surface, as shown in Fig. 2.2.1.

If the fluid is Newtonian, then the momentum flux is related to the velocity gradient according to Newton's law of viscosity :

$$\tau_{xy} = -\mu\frac{dv_y}{dx} \qquad \text{...(2.2.6)}$$

Combining Eqs. (2.2.5) and (2.2.6), we obtain the following differential equation for the velocity distribution.

$$\frac{dv_y}{dx} = -\left(\frac{\rho g\cos\alpha}{\mu}\right)x \qquad \text{...(2.2.7)}$$

Integrating Eq. (2.2.7), we obtain,

$$v_y = -\left(\frac{\rho g\cos\alpha}{2\mu}\right)x^2 + C_2 \qquad \text{...(2.2.8)}$$

Using the boundary condition that there is no slip at the solid-liquid interface *i.e.*, $v_y = 0$ at $x = \delta$, we evaluate the value of constant of integration C_2 :

$$C_2 = \frac{\rho g \cos\alpha}{2\mu}\delta^2$$

Therefore, the expression for the velocity distribution is

$$\boxed{v_y = \frac{\rho g \delta^2 \cos\alpha}{2\mu}\left[1 - \left(\frac{x}{\delta}\right)^2\right]} \quad \text{...(2.2.9)}$$

This means that the velocity profile is **parabolic** as shown in Fig. 2.2.1. The maximum velocity occurs at $x = 0$ and is given by,

$$\boxed{v_{y,\,max} = \frac{\rho g \delta^2 \cos\alpha}{2\mu}} \quad \text{...(2.2.10)}$$

The average velocity is given by

$$v_{y,\,av} = \frac{1}{A}\iint_A v_y\, dA = \frac{\int_0^W \int_0^\delta v_y\, dx dz}{\int_0^W \int_0^\delta dx dz}$$

$$= \frac{1}{W\delta}\int_0^W \int_0^\delta v_y\, dx dz = \frac{W}{W\delta}\int_0^\delta v_y\, dx$$

$$= \frac{1}{\delta}\int_0^\delta v_y\, dx$$

$$= \frac{\rho g \delta^2 \cos\alpha}{2\mu}\int_0^1 \left[1 - \left(\frac{x}{\delta}\right)^2 d\left(\frac{x}{\delta}\right)\right]$$

$$= \boxed{\frac{\rho g \delta^2 \cos\alpha}{3\mu}} \quad \text{...(2.2.11)}$$

Combining Eqs. (2.2.10) and (2.2.11), we obtain,

$$\boxed{\frac{v_{y,\,av}}{v_{y,\,max}} = \frac{2}{3}} \quad \text{...(2.2.12)}$$

The volumetric flow rate Q is obtained by multiplying the average velocity, $v_{y,\,av}$ with cross-sectional area (flow area), $W\delta$, or by integrating the velocity distribution :

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$$Q=\int_0^W\int_0^\delta v_y\,dx\,dy = W\,\delta v_{y,av}\,\rho$$

$$= \frac{\rho g W\delta^3}{3\mu}\cos\alpha \qquad ...(2.2.13)$$

The mass rate of flow per unit width of wall, Γ in $\frac{\text{Kg}}{s.m}$, is defined as

$$\Gamma = \rho\delta v_{y,av} \qquad ...(2.2.14)$$

and a Raynold's number for falling films is defined as

$$Re = \frac{4\delta v_{y,av}\rho}{\mu} = \frac{4\Gamma}{\mu} \qquad ...(2.2.15)$$

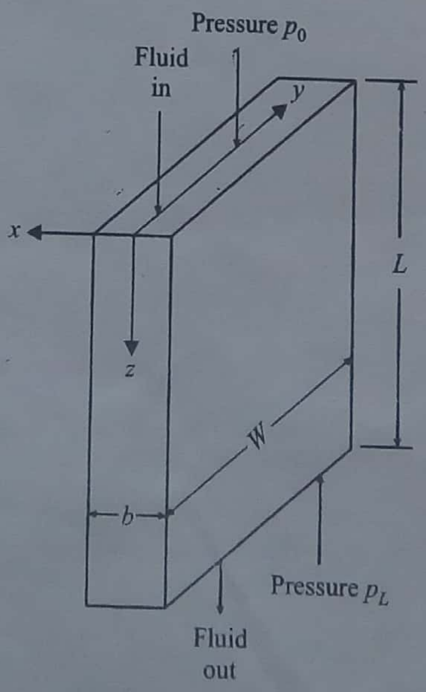
Laminar flow without rippling occurs for $R_e < 25$. Laminar flow with rippling occurs for $25 < Re < 1200$.

The falling film problem illustrates the role of gravity forces and the momentum balance in cartesian coordinates.

2.3 SHELL MOMENTUM BALANCE FOR LAMINAR FLOW IN A NARROW SLIT

Consider the flow of a fluid in a slit formed by two vertical parallel walls which are separated by a distance b. We assume that the fluid is Newtonian, incompressible and behaves as a continuum and flow is laminar, steady and fully developed. The origin of the coordinate system is chosen at the center of slit of width W and length L. We make the momentum balance over a shell of thickness Δx at a x from the center of the slit as shown in Fig. 2.3.1.

Fig. 2.3.1 Flow through a slit.



The various terms of the momentum balance are :

- The rate of z-momentum in across the surface at x

$$(LW\tau_{xz})|_x$$

- The rate of z-momentum out across the surface at $x+\Delta x$

$$(LW\tau_{xz})|_{x+\Delta x}$$

- The rate of z-momentum in across the surface at $z=0$

$$(W\Delta x\ v_z\,\rho v_z)|_{z=0}$$

- The rate of z-momentum out across the surface at $z=L$

$$(W\Delta x\, v_z\,\rho v_z)|_{z=L}$$

- Gravity force acting on the fluid shell

$$LW\Delta x\rho g$$

- Pressure force acting on the surface of the shell at $z=0$

$$W\Delta x\, p_0$$

- Pressure force acting on the surface of the shell at $z=L$

$$-W\Delta x\, p_L$$

Substituting these terms in the general statement of momentum balance fo steady state flow Eq. (2.1.1), we get,

$$(LW\tau_{xz})|_x-(LW\tau_{xz})|_{x+\Delta z}+(W\Delta x\ v_z\,\rho v_z)|_{z=0}-(W\Delta x\, v_z\,\rho v_z)|_{z=L}$$
$$+LW\Delta x\rho g+W\Delta x(p_0-p_L)=0 \quad \text{...(2.3.}$$

Since for fully developed flow, v_z is the same at $z=0$ as it is at $z=L$ for ea value of x, the third and fourth terms cancel one another. Dividing Eq. by $LW\Delta x$ and taking limit as Δx approaches zero, we obtain :

$$\lim_{\Delta x\to 0}\left(\frac{\tau_{xz}|_{x+\Delta x}-\tau_{xz}|_x}{\Delta x}\right)=\frac{p_0-p_L}{L}+\rho g$$

or

$$\frac{d}{dx}(\tau_{xz})=\frac{p_0-p_L}{L}+\rho g$$

If P represents the combined effect of static pressure and gravitational fo any z, then,

$$P=p_0-\rho gZ$$

so at $z=0$, $\quad P_0=p_0-\rho g.0=p_0$

and at $z=L$, $\quad P_L=p_L-\rho gL$

hence $\quad \dfrac{P_0-P_L}{L}=\dfrac{p_0-p_L}{L}+\rho g$ ...(

Combining Eq. (2.3.2) and (2.3.3) we get,

$$\frac{d}{dx}(\tau_{xz}) = \frac{P_0 - P_L}{L} \quad ...(2.3.4)$$

Eq. (2.3.4) may be integrated to give :

$$\tau_{xz} = \frac{P_0 - P_L}{L}x + C_1 \quad ...(2.3.5)$$

where C_1 is a constant of integration, which may be evaluated from boundary condition :

At $x = 0$, $\tau_{xz} = 0$. So $C_1 = 0$.

Hence the momentum flux distribution is

$$\tau_{xz} = \frac{P_0 - P_L}{L}x \quad ...(2.3.6)$$

For a Newtonian fluid,

$$\tau_{xz} = -\mu\frac{dv_z}{dx} \quad ...(2.3.7)$$

Substituting Eq. (2.3.7) into Eq. (2.3.6), we get

$$\frac{dv_z}{dx} = -\left(\frac{P_0 - P_L}{\mu L}\right)x \quad ...(2.3.8)$$

Integrating Eq. (2.3.8),

$$v_z = -\left(\frac{P_0 - P_L}{2L\mu}\right)x^2 + C_2 \quad ...(2.3.9)$$

The constant of integration C_2 may be evaluated from the second boundary condition : At $x = \frac{1}{2}b$ (*i.e.*, at the inner surface of the wall), $v_z = 0$. Then

$$0 = -\left(\frac{P_0 - P_L}{2\mu L}\right)\left(\frac{1}{2}b\right)^2 + C_2$$

or

$$C_2 = \left(\frac{P_0 - P_L}{8\mu L}\right)b^2$$

Hence the velocity distribution in the slit is

$$v_z = \left(\frac{P_0 - P_L}{2\mu L}\right)\left(\frac{b}{2}\right)^2\left[1 - \left(\frac{x}{\frac{b}{2}}\right)^2\right] \quad ...(2.3.10)$$

The maximum velocity occurs at $x = 0$ and has the value

$$v_{z,\max} = \frac{(P_0 - P_L)}{8\mu L}b^2 \quad ...(2.3.11)$$

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The average velocity, $v_{z,\,av}$, is obtained by summing up all the velocities over a cross-section and then dividing by the cross-sectional area

$$v_{z,\,av} = \frac{\int_0^W \int_0^{\frac{1}{2}b} v_z \, dx dy}{\int_0^W \int_0^{\frac{1}{2}b} dx dy} = \frac{\int_0^{\frac{1}{2}b} v_z \, dx}{\int_0^{\frac{1}{2}b} dx}$$

$$= \frac{\frac{P_0 - P_L}{2\mu L}\left(\frac{b}{2}\right)^2}{\frac{1}{2}b} \int_0^{\frac{1}{2}b} \left[1 - \left(\frac{x}{b/2}\right)^2\right] dx$$

$$= \frac{P_0 - P_L}{2\mu L} \cdot \left(\frac{b}{2}\right)\left[x - \frac{x^3}{3/4\, b^2}\right]_0^{\frac{b}{2}}$$

$$= \frac{2}{3} \cdot \frac{P_0 - P_L}{2\mu L}\left(\frac{b}{2}\right)^2 \qquad ...(2.3.12)$$

The ratio of the average velocity to the maximum velocity *i.e.*,

$$\frac{v_{z,\,av}}{v_{z,\,\max}} = \frac{2}{3}. \qquad ...(2.3.13)$$

The volumetric flow rate, Q, is the product of the cross-sectional area of the slit and the average velocity :

$$Q = v_{z,\,av} \times b \,.\, W \qquad [W \text{ is the width of the plate}]$$

$$= \frac{2}{3} \frac{P_0 - P_L}{2\mu\, L}\left(\frac{b}{2}\right)^2 . bW$$

or $$Q = \frac{2}{3}\left(\frac{P_0 - P_L}{8\mu L}\right) b^3 W \qquad ...(2.3.14)$$

This is the analog of the Hagen-Poiseuille law for the slit. This problem illustrates the role of pressure and gravity forces and the use of cartesian coordinates.

2.4 LAMINAR FLOW OF A POWER-LAW FLUID THROUGH A RECTANGULAR SLIT

Consider a shell of thickness Δx at a distance x from the center of the slit of thickness b, as shown in Fig. 2.3.1. The momentum flux distribution for steady laminar flow of a power-law fluid is given by,

$$\tau_{xz} = \left(\frac{P_0 - P_L}{L}\right) x \qquad ...(2.4.1)$$

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Now for a power-law fluid,

$$\tau_{xz} = -K\left|\frac{dv_z}{dx}\right|^{n-1} \cdot \frac{dv_z}{dx} \qquad \text{...(2.4.2)}$$

v_z decreases as x increases so that $\frac{dv_z}{dx}$ is –ve. Thus,

$$\tau_{xz} = K\left|-\frac{dv_z}{dx}\right|^{n-1}\left(-\frac{dv_z}{dx}\right) = K\left(-\frac{dv_z}{dx}\right)^n \qquad \text{...(2.4.2)}$$

Therefore,

$$K\left(-\frac{dv_z}{dx}\right)^n = \left(\frac{P_0 - P_L}{L}\right)x \qquad \text{...(2.4.3)}$$

or

$$-\frac{dv_z}{dx} = \left(\frac{P_0 - P_L}{KL}\right)^{\frac{1}{n}} . x^{\frac{1}{n}}$$

or

$$v_z = -\left(\frac{P_0 - P_L}{KL}\right)^{\frac{1}{n}} \int_0^x x^{\frac{1}{n}}\, dx + C_2$$

or

$$v_z = -\left(\frac{P_0 - P_L}{KL}\right)^{\frac{1}{n}} x^{\frac{n+1}{n}} \Big/ \left(\frac{n+1}{n}\right) + C_2 \qquad \text{...(2.4.4)}$$

where C_2 is a constant of integration which may be evaluated from the second boundary condition :

At $x = \frac{b}{2}$, $v_z = 0$

$$\therefore \qquad C_2 = \left(\frac{P_0 - P_L}{KL}\right)^{\frac{1}{n}} \frac{n}{n+1}\left(\frac{b}{2}\right)^{\frac{n+1}{n}} \qquad \text{...(2.4.5)}$$

Substituting C_2 from Eq. (2.4.5) into Eq. (2.4.4), we obtain

$$v_z = \frac{n}{n+1}\left(\frac{P_0 - P_L}{KL}\right)^{\frac{1}{n}} \cdot \left(\frac{b}{2}\right)^{\frac{n+1}{n}}\left[1 - \left(\frac{x}{b/2}\right)^{\frac{n+1}{n}}\right] \qquad \text{...(2.4.6)}$$

Eq. (2.4.6) gives the velocity distribution in the slit.

The average velocity, $v_{z,\,av}$ is given by

$$v_{z,\,av} = \frac{\int_0^W \int_0^{\frac{b}{2}} v_z\, dxdy}{\int_0^W \int_0^{\frac{b}{2}} dxdy} = \frac{W}{W \cdot \frac{b}{2}} \int_0^{\frac{b}{2}} v_z\, dx$$

or $$v_{z,\,av} = \frac{2}{b}\int_0^{\frac{b}{2}} v_z\, dx = \frac{2}{b}\left(\frac{n}{n+1}\right)\left(\frac{P_0 - P_L}{KL}\right)^{\frac{1}{n}} \int_0^{\frac{b}{2}} \left\{\left(\frac{b}{2}\right)^{\frac{n+1}{n}} - x^{\frac{n+1}{n}}\right\} dx$$

or $$v_{z,\,av} = \left(\frac{n}{2n+1}\right)\left(\frac{P_0 - P_L}{KL}\right)^{\frac{1}{n}}\left(\frac{b}{2}\right)^{\frac{n+1}{n}} \qquad \text{...(2.4.7)}$$

The volume rate of flow, Q, is given by

$$Q = bW\, v_{z,\,av}$$

or $$Q = 2\left(\frac{n}{2n+1}\right)\left(\frac{P_0 - P_L}{KL}\right)^{\frac{1}{n}}\left(\frac{b}{2}\right)^{\frac{2n+1}{n}} W \qquad \text{...(2.4.8)}$$

where W is the width of the channel.

The expressions for velocity distribution, v_z, the average velocity $v_{z,\,av}$ and the volume rate of flow, Q, for a power law fluid reduce to those for a Newtonian fluid when we substitute $n = 1$ and $K = \mu$.

2.5 LAMINAR FLOW OF A BINGHAM FLUID THROUGH A RECTANGULAR SLIT

Consider steady flow of a Bingham fluid in a rectangular slit and a shell of thickness Δx at a distance x from the center of the slit of thickness b as shown in Fig. 2.3.1.

The momentum flux distribution is given by :

$$\tau_{xz} = \left(\frac{P_0 - P_L}{L}\right)x \qquad \text{...(2.5.1)}$$

Momentum diffuses from the center of the slit towards the solid surface and hence τ_{xz} is positive, therefore,

$$\tau_{xz} = -\mu_0 \frac{dv_z}{dx} + \tau_0 \qquad \text{...(2.5.2)}$$

Substituting Eq. (2.5.2) into Eq. (2.5.1), we obtain,

$$\frac{dv_z}{dx} = -\left(\frac{P_0 - P_L}{\mu_0 L}\right)x + \frac{\tau_0}{\mu_0} \qquad \text{...(2.5.3)}$$

Integrating Eq. (2.5.3),

$$v_z = -\left(\frac{p_0 - p_L}{2\mu_0 L}\right)x^2 + \frac{\tau_0}{\mu_0}x + C_2$$

where C_2 is a constant of integration which is evaluated from the boundary condition that $v_z = 0$ at $x = \frac{b}{2}$. Thus

$$C_2 = \left(\frac{P_0 - P_L}{8\mu_0 L}\right) b^2 - \frac{\tau_0 b}{2\mu_0}$$

Then the velocity distribution becomes

$$v_z = \left(\frac{P_0 - P_L}{8\mu_0 L}\right) b^2 \left[1 - \left(\frac{x}{b/2}\right)^2\right] - \frac{\tau_0 b}{2\mu_0}\left(1 - \frac{x}{b/2}\right) \quad \text{...(2.5.4)}$$

The average velocity, $v_{z,av}$, is given by

$$v_{z,av} = \frac{2}{b}\int_0^{\frac{b}{2}} v_z \, dx$$

$$= \frac{2}{b}\left(\frac{P_0 - P_L}{8\mu_0 L}\right)\int_0^{\frac{b}{2}}\left(b^2 - \frac{4x^2}{b^2}\right)dx - \frac{\tau_0}{2\mu_0}\int_0^{\frac{b}{2}}\left(b - \frac{2x}{b}\right)dx$$

or

$$v_{z,av} = \left(\frac{P_0 - P_L}{3\mu_0 L}\right)\left(\frac{b}{2}\right)^2 - \frac{\tau_0}{2\mu_0}\left(\frac{b}{2}\right) \quad \text{...(2.5.5)}$$

The volume flow rate, Q, is given by

$$Q = bW\, v_{z,av} = \left\{\left(\frac{P_0 - P_L}{3\mu_0 L}\right)\left(\frac{b}{2}\right)^2 - \frac{\tau_0}{2\mu_0}\left(\frac{b}{2}\right)\right\} b.W \quad \text{...(2.5.6)}$$

The expression for v_z, $v_{z,av}$ and Q reduces to the corresponding expression for flow of a Newtonian fluid through a rectangular slit, when we substitute $\tau_0 = 0$.

2.6 DIFFERENTIAL MOMENTUM BALANCE FOR FLOW THROUGH INCLINED SLIT FORMED BY PARALLEL PLATES

Consider the case of steady laminar flow of an incompressible fluid between inclined parallel plates inclined at an angle θ with the horizontal with upper plate moving with a constant speed, V, and imagine a differential control volume $dx \cdot dy \cdot 1$ as shown in Fig. 2.6.1. Since the flow is fully developed ; the velocity v_x cannot vary with x and hence depends only on y, so that $v_x = v_x(y)$. Furthermore, there is no component of velocity in either y and z direction. The momentum flux entering the face of the control volume at x is equal in magnitude but opposite in sign to the momentum flux leaving through the face of the control volume at $x + dx$; there is not momentum flux through any of the remaining faces of the control volume.