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Semester: VII
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Unit: -III

Lecture no.14

Turbulent Flow: Velocity distribution in turbulent flow

1. Introduction

In the previous sections Laminar flow phenomena have been discussed in detail. But in practice most transport operations involve turbulent flow rather than laminar motion. The nature of flow, whether it is laminar or turbulent, is characterized by magnitude of Reynold's number.

The Reynold's number is a dimensionless quantity which is a ratio of inertial force to viscous force and defined by $\frac{LV\rho}{\mu}$. In laminar flow viscous force is predominant compared to inertial force, but in turbulent flow inertial force overpowers viscous force and momentum transport takes place with the movement of swarm of fluid elements, called eddy, of various sizes. Thus momentum transfer is much faster in turbulent motion compared to laminar motion. The turbulent flow phenomena is much more complex than laminar flow as movement of eddies is not predictable. Hence mathematical description of turbulent flow is more rigorous but less accurate in predicting turbulent flow field. As a result one has to depend more on experimental results rather than theoretical prediction of transport properties in turbulent flow.

2. Transition from Laminar to Turbulent

In turbulent flow eddies are in random motion. For convenience the velocity field in turbulent motion is described by the sum of an average velocity $\bar{u}$ and a fluctuating component u'. Thus, the three-dimensional velocity field is defined as,

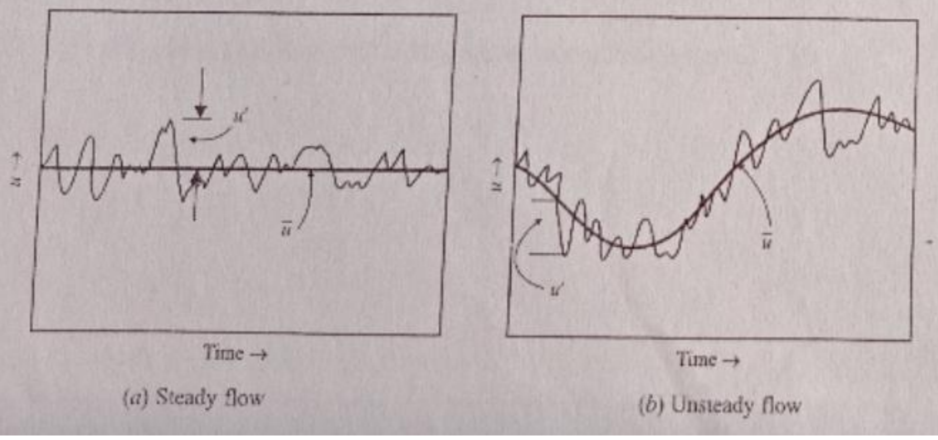
$$u = \bar{u} + u' \quad \text{...(4.1.1)}$$

$$v = \bar{v} + v' \quad \text{...(4.1.2)}$$

$$w = \bar{w} + w' \quad \text{...(4.1.3)}$$

Experimentally obtained form of point velocity fluctuation in turbulent flow may be represented as in Fig. 4.1.1.

Fig. 4.1.1 Steady and unsteady velocity field in turbulent flow.



(a) Steady flow (b) Unsteady flow

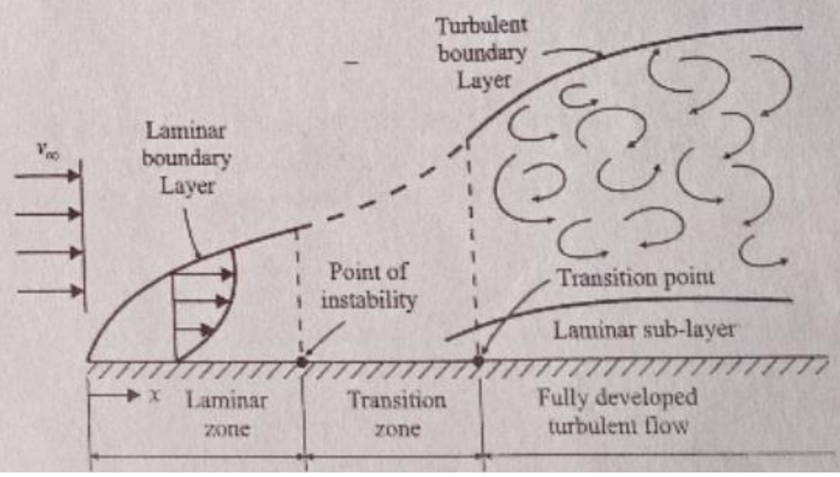
Transition from laminar to turbulent flow is most commonly quantified by the magnitude of Reynold's No. There is a value of Reynold's No. beyond which eddy motion becomes important and laminar flow becomes unstable. The Reynold's No. at which this hydrodynamic instability is incepted is called critical Reynold's No. Further increase of Reynold's No. causes amplification of disturbances and fully developed turbulent flow results. Table 4.1.1 shows the value of critical Reynold's No. for different flow geometry.

Table 4.1.1 Critical Reynold's No. for Laminar to turbulent flow transition

Geometry	*Reynold's No.*	*Critical Value*
Flat plate	$\frac{x v_\infty \rho}{\mu}$	$\sim 5 \times 10^5$
Cylinder in cross flow	$\frac{D v_\infty \rho}{\mu}$	$\sim 2 \times 10^5$
Sphere	$\frac{D v_\infty \rho}{\mu}$	$\sim 2 \times 10^5$
Pipe flow	$\frac{D \bar{v} \rho}{\mu}$	$\sim 2000 - 4000$
Non-circular duct	$\frac{D_e \bar{v} \rho}{\mu}$	$\sim 2000 - 4000$

Fig. 4.1.2 shows the laminar to turbulent flow transition over a flat plate (external flow).

Fig. 4.1.2 Development of turbulent flow over a flat plate.



In the fully developed turbulent boundary layer there exists a laminar sublayer over the solid boundary. The laminar sublayer controls transport of momentum as transport resistance is higher compared to turbulent zone above it. In the turbulent zone transport of momentum is fast and efficient due to random motion of eddies.

3. Transport equation in turbulent flow

In turbulent flow velocity fluctuations developed shear stress in the flow field. These shear stresses are much larger in magnitude compared to shear stresses developed in laminar flow. The basic continuity and momentum equations are valid for turbulent flow. But the velocity terms in those equations are replaced by

$$v_i = \bar{v}_i + v_i' \quad \text{...(4.3.1)}$$

and pressure is replaced by

$$P = \bar{P} + P' \quad \text{...(4.3.2)}$$

to recast the equations in terms of average velocities and fluctuating components of velocities. Substituting v_i in continuity and momentum equations for incompressible fluid with constant density and constant viscosity, continuity equation becomes,

$$\frac{\partial}{\partial x}(\bar{v}_x + v_x') + \frac{\partial}{\partial y}(\bar{v}_y + v_y') + \frac{\partial}{\partial z}(\bar{v}_z + v_z') = 0 \quad \text{...(4.3.3)}$$

and x-component momentum equation becomes,

$$\frac{\partial}{\partial t}[\rho(\bar{v}_x + v_x')] + \frac{\partial}{\partial x}[\rho(\bar{v}_x + v_x')(\bar{v}_x + v_x')] + \frac{\partial}{\partial y}[\rho(\bar{v}_x + v_x')(\bar{v}_y + v_y')] + \frac{\partial}{\partial z}[\rho(\bar{v}_x + v_x')(\bar{v}_z + v_z')]$$

$$= \mu\left[\frac{\partial^2}{\partial x^2}(\bar{v} + v_x') + \frac{\partial^2}{\partial y^2}(\bar{v}_x + v_x') + \frac{\partial^2}{\partial z^2}(\bar{v}_x + v_x')\right] - \frac{\partial(\bar{P} + P')}{\partial x} + \rho g_x \quad \text{...(4.3.4)}$$

Now, the following Reynold's averaging rules are applied in order to express field equations in terms of time-average velocities and their fluctuation components.

(*i*) The time average of v_i' is zero.

$$\bar{v}_i' = 0 \quad \text{...(4.3.5)}$$

(*ii*) Distribution law holds good for average quantities.

$$\bar{v}_i + \bar{v}_j = \overline{v_i + v_j} \quad \text{...(4.3.6)}$$

(*iii*) Average quantities act as constant while further averaged.

$$\bar{\bar{v}} = \bar{v} \quad ; \quad \overline{\bar{v}_i v_j} = \overline{v_i\, v_j} \quad \text{...(4.3.7)}$$

(*iv*) Derivatives follow averaging law :

$$\overline{\frac{\partial v_i}{\partial x}} = \frac{\partial \bar{v}_i}{\partial x} \quad \text{...(4.3.8)}$$

(v) Product of two fluctuating components and instantaneous quantities obey following algebraic equalities.

$$\overline{v_i v_j} = \overline{(\overline{v}_i + v_i')(\overline{v}_j + v_j')} = \overline{\overline{v}_i \overline{v}_j} + \overline{\overline{v}_i v_j'} + \overline{\overline{v}_j v_i'} + \overline{v_i' v_j'}$$
$$= \overline{v}_i \overline{v}_j + \overline{v_i' v_j'} \qquad \text{...(4.3.9)}$$

After utilizing Eq. (4.3.5) through (4.3.9) Eqs. (4.3.3) and (4.3.4) take the respective forms.

Reynold's averaged continuity equation

$$\frac{\partial \overline{v}_x}{\partial x} + \frac{\partial \overline{v}_y}{\partial y} + \frac{\partial \overline{v}_z}{\partial z} = 0 \qquad \text{...(4.3.10)}$$

Reynold's averaged *x*-component momentum equation

$$\frac{\partial}{\partial t}(\rho \overline{v}_x) + \frac{\partial}{\partial x}(\rho \overline{v}_x \overline{v}_x) + \frac{\partial}{\partial y}(\rho \overline{v}_x \overline{v}_y) + \frac{\partial}{\partial z}(\rho \overline{v}_x \overline{v}_z)$$
$$+ \left[\frac{\partial}{\partial x}(\rho \overline{v_x' v_x'}) + \frac{\partial}{\partial y}(\rho \overline{v_x' v_y'}) + \frac{\partial}{\partial z}(\rho \overline{v_x' v_z'}) \right]$$
$$= \mu \left[\frac{\partial^2}{\partial x^2}(\overline{v}_x) + \frac{\partial^2}{\partial y^2}(\overline{v}_x) + \frac{\partial^2}{\partial z^2}(\overline{v}_x) \right] - \frac{\partial \overline{P}}{\partial x} + \rho g_x \qquad \text{...(4.3.11)}$$

The above two equations (4.3.10) and (4.3.11) can be solved for $\overline{v}_x$, $\overline{v}_y$, and $\overline{v}_z$ if equations for v_x', v_y' and v_z' are available, otherwise the problem remains unclosed. For closure of the problem two approaches are commonly used : (*a*) eddy diffusivity model and (*b*) Prandtl's mixing length theory. Both have a common objective to replace Reynold's stress terms with expressions involving diffusivity like term and velocity gradient. These two approaches will be discussed in the following sections.

4. Reynold's Stresses

4.4 REYNOLD'S STRESSES

The product terms of average fluctuating components $\overline{v_i v_j}$ are called Reynold's stresses. Nine components of Reynold's stress terms is a second order tensor and represented as,

$$\begin{vmatrix} \rho(\overline{v_x' v_x'}) & \rho(\overline{v_y' v_x'}) & \rho(\overline{v_z' v_x'}) \\ \rho(\overline{v_x' v_y'}) & \rho(\overline{v_y' v_y'}) & \rho(\overline{v_z' v_y'}) \\ \rho(\overline{v_x' v_z'}) & \rho(\overline{v' v_z'}) & \rho(\overline{v_z' v_z'}) \end{vmatrix} \quad ...(4.4.1)$$

Each of the components of the array bears unit of stress, N/m^2. Reynold's stress terms are measure of extra energy needed to maintain turbulence, and for heat dissipation due to eddy-structure breakdown. It comprises of three normal components $\rho(\overline{v_i' v_i'})$ and six tangential components $\rho(\overline{v_i' v_j'})$. Commonly all nine terms are different in magnitude, but for **isotropic turbulence** all normal stresses are equal,

i.e., $\rho(\overline{v_x' v_x'}) = \rho(\overline{v_y' v_y'}) = \rho(v_z' v_z')$;

and all tangential components are zero,

i.e., $\rho\,\overline{v_i' v_j'} = 0.$

For **homogeneous turbulence** all stress components are of equal magnitude.