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Department of Petrochemical Engineering

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DIFFERENTIAL MOMENTUM BALANCE FOR FLOW THROUGH INCLINED SLIT FORMED BY PARALLEL PLATES

Consider the case of steady laminar flow of an incompressible fluid between inclined parallel plates inclined at an angle θ with the horizontal with upper plate moving with a constant speed, V, and imagine a differential control volume $dx\,.\,dy\,.1$ as shown in Fig. 2.6.1. Since the flow is fully developed ; the velocity v_x cannot vary with x and hence depends only on y, so that $v_x = v_x\,(y)$. Furthermore, there is no component of velocity in either y and z direction. The momentum flux entering the face of the control volume at x is equal in magnitude but opposite in sign to the momentum flux leaving through the face of the control volume at $x + dx$; there is not momentum flux through any of the remaining faces of the control volume.

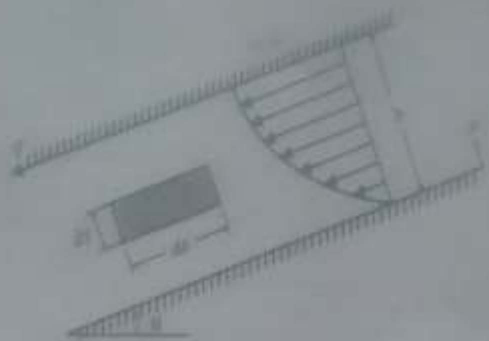

Fig. 2.6.1 Control volume for steady laminar flow between infinite parallel plates

The momentum balance for steady state flow gives

$$\tau_{yx}\,.dx\,.1 - \left(\tau_{yx}\,.dx\,.1 + \frac{\partial \tau_{yx}}{\partial y}\,dy dx\,.1\right) + \rho g dx dy\,.1 \sin\theta$$

$$+ p\,dy\,.1 - \left(p dy 1 + \frac{\partial p}{\partial x}\,dx dy\,.1\right) \qquad ...(2.6.1)$$

Simplifying and dividing through by the volume of the control element $dxdy\,.1$ gives

$$-\frac{\partial \tau_{yx}}{\partial y} = \frac{\partial p}{\partial x} - \rho g \sin\theta \qquad ...(2.6.2)$$

Since gravity acts vertically downward, h may be taken as a coordinate which is positive, vertically upward and

$$\sin\theta = -\frac{\partial h}{\partial x} \qquad \text{...(2.6.3)}$$

and Eq. (2.6.2) may be written as

$$-\frac{\partial \tau_{yx}}{\partial y} = \frac{\partial p}{\partial x} + \rho g \frac{\partial h}{\partial x}$$

or

$$-\frac{\partial \tau_{yx}}{\partial y} = \frac{\partial}{\partial x}(p + \rho g h) \qquad \text{...(2.6.4)}$$

Since τ_{yx} is a function of y only, $\frac{\partial \tau_{yx}}{\partial y} = \frac{d\tau_{yx}}{dy}$ and since $p + \rho g h$ does not change value in the y direction, $p + \rho g h$ is a function x only.

Hence,

$$\frac{\partial}{\partial x}(p + \rho g h) = \frac{d}{dx}(p + \rho g h)$$

Eq. (2.6.4) becomes

$$-\frac{d\tau_{yx}}{dy} = \frac{d}{dx}(p + \rho g h) \qquad \text{...(2.6.5)}$$

Integrating Eq. (2.6.5) w.r.t. y we obtain,

$$-\tau_{yx} = y\frac{d}{dx}(p + \rho g h) + C_1 \qquad \text{...(2.6.6)}$$

which indicates that shear stress varies linearly with y. For a Newtonian fluid

$$\tau_{yx} = -\mu\frac{dv_x}{dy} \qquad \text{...(2.6.7)}$$

then Eq. (2.6.6) becomes

$$\mu\frac{dv_x}{dy} = y\frac{d}{dx}(p + \rho g h) + C_1 \qquad \text{...(2.6.8)}$$

Integrating Eq. (2.6.8) w.r.t. y gives

$$v_x = \frac{y^2}{2\mu}\frac{d}{dx}(p + \rho g h) + \frac{C_1}{\mu}y + C_2 \qquad \text{...(2.6.9)}$$

Constants C_1 and C_2 may be evaluated by using boundary conditions :

At $y = 0$, $v_x = 0$

and at $y = b$, $v_x = V$

Thus,

$$v_x = \frac{Vy}{b} + \frac{b^2}{2\mu}\left[\left(\frac{y}{b}\right)^2 - \frac{y}{b}\right]\frac{d}{dx}(p + \rho g h) \qquad \text{...(2.6.10)}$$

Eq. (2.6.10) may be simplified when both plates are stationary in which case $V = 0$. When flow occurs in the channel formed by two horizontal stationary plates, the velocity profile is given by (since $\rho g h$ is constant),

$$v_x = \frac{b^2}{2\mu}\left[\left(\frac{y}{b}\right)^2 - \frac{y}{b}\right]\frac{dp}{dx} \qquad \text{...(2.6.11)}$$

Example

An oil having a density 900 kg / m³ and a viscosity of 0.105 kg / (m.s) flows in the channel formed by the two horizontal stationary plates spaced 0.014 m apart. If the average velocity is 1.5 m/s, determine :

(a) the velocity profile ; *(b) maximum velocity ;*

(c) shear stress at a distance of 0.005 m from one of the plates ;

(d) head loss in a distance of 15 m along the length of the plate.

Solution

Solution. $v_x = \frac{b^2}{2\mu}\left[\left(\frac{y}{b}\right)^2 - \frac{y}{b}\right]\frac{dp}{dx}$

$$= \frac{1}{2\mu}\left(-\frac{dp}{dx}\right)(by - y^2)$$

$$v_{x,\,av} = \frac{1}{2\mu}\left(-\frac{dp}{dx}\right)\frac{1}{b}\int_0^b (by - y^2)\,dy$$

$$= \frac{1}{2\mu}\frac{1}{b}\left(-\frac{dp}{dx}\right)\left[\frac{b}{2}y^2 - \frac{y^3}{3}\right]_0^b$$

$$v_{x,\,av} = \frac{b^2}{12\,\mu}\left(-\frac{dp}{dx}\right)$$

In this case $v_{x,\max}$ occurs at $\frac{b}{2}$ from the fixed plate. Then

$$v_{x\max} = \frac{1}{2\mu}\left(-\frac{dp}{dx}\right)\left\{b.\frac{b}{2}-\left(\frac{b}{2}\right)^2\right\} = \frac{b^2}{8\mu}\left(-\frac{dp}{dx}\right)$$

$$\frac{v_{x,av}}{v_{x,\max}} = \frac{\frac{b^2}{12\mu}\left(-\frac{dp}{dx}\right)}{\frac{b^2}{8\mu}\left(-\frac{dp}{dx}\right)} = \frac{2}{3}$$

Therefore, $\left(-\frac{dp}{dx}\right) = \frac{12\,\mu v_{x,av}}{b^2}$

$$= \frac{12\times 0.105\times 1.3}{(0.014)^2}$$

$$= 8357\,\frac{\text{N}}{\text{m}}$$

(a) The velocity profile is given by

$$v_x = \frac{1}{2\mu}\left(-\frac{dp}{dx}\right)(by - y^2)$$

$$= \frac{8357}{2\times 0.105}(0.014\,y - y^2)$$

$$v_x = 557\,y - 39796\,y^2 \text{ m/s}$$

(b) The maximum velocity, $v_{x,\max}$ is

$$v_{x,\max} = \frac{2}{3} v_{x,av} = \frac{2}{3} \times 1.3 \frac{\text{m}}{\text{s}}$$

$$= 0.867 \frac{\text{m}}{\text{s}}$$

(c) Shear stress at $y = 0.005$ m is

$$\tau_{yx} \Big|_{y=0.005} = -\mu \frac{dv_x}{dy} = -0.105[557 - 39796 \times 2y]_{y=0.005}$$

$$= -0.105(557 - 39796 \times 2 \times 0.005)$$

$$= -16.7 \frac{\text{N}}{\text{m}^2}$$

(d) Head loss H_f in a length L

$$H_f = \frac{1}{\rho g}\left(-\frac{dp}{dx}\right)$$

$$= \frac{12\mu\, v_{x,av}\, L}{\rho g b^2}$$

or

$$H_f = \frac{12 \times 0.105 \times 1.3 \times 15}{900 \times 9.866 \times (0.014)^2}$$

$$= 14.2 \text{ m.}$$