UIT-RGPV (Autonomous) Bhopal
Department of Petrochemical Engineering

Subject In-charge: Prof. M.S. Chouhan
Semester: VII
Subject code – PC 702 **Subject: Transport Phenomena**
Unit: -I
Lecture no. 6 (date:19.08.2020)

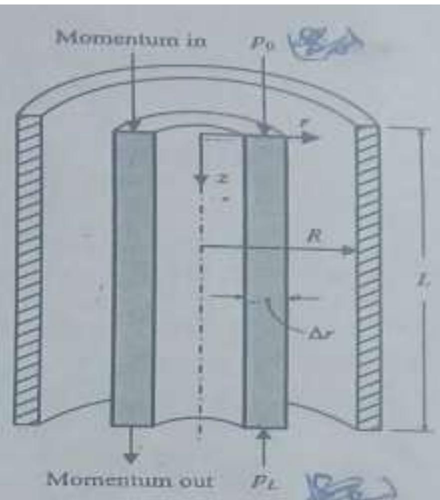
Laminar flow a power -law fluid in a circular tube

1. Fully development Laminar flow a power -law fluid in a circular tube

Consider steady laminar flow of a power law fluid of constant density ρ in a very long tube of length L and inner radius R. Since the tube is long end effects may be neglected.

We select as control volume, a cylindrical fluid shell of thickness Δr, at a distance r and length L. The various contributions to the momentum balance are :

Fig. 2.8.1 Cylindrical fluid shell for momentum balance.



rate of z-momentum in across the cylindrical surface at r	$(2\pi rL\tau_{rz})\|_r$
rate of z-momentum out across the cylindrical surface at $r+\Delta r$	$(2\pi rL\tau_{rz})\|_{r+\Delta r}$
rate of z-momentum in across the annular surface at $z=0$.	$(2\pi r\Delta r v_z)(\rho v_z)\|_{z=0}$
rate of z-momentum out across the annular surface at $z=L$	$(2\pi r\Delta r v_z)(\rho v_z)\|_{z=L}$
gravity force acting on the cylindrical shell	$(2\pi r\Delta rL)\rho g$
pressure force acting on the annular force at $z=0$	$(2\pi r\Delta r)\,p_0$
pressure force acting on the annular surface at $z=L$	$-(2\pi r\Delta r)\,p_L$

Since

(rate of momentum in) – (rate of momentum out) + (sum of forces) = 0

$$(2\pi rL\tau_{rz})\big|_r - (2\pi rL\,\tau_{rz})\big|_{r+\Delta r} + (2\pi r\,\Delta r\rho v_z^2)\big|_{z=0} - (2\pi r\Delta r\rho v_z^2)\big|_{z=L}$$
$$+\,2\pi r\Delta rL\,\rho g + 2\pi r\Delta r\,(p_0 - p_L) = 0 \qquad ...(2.8.1)$$

v_z is the same at $z = 0$ and $z = L$, because the fluid is incompressible and hence the third and fourth terms in Eq. (2.8.1) cancel one another. Dividing by $2\pi L\Delta r$ and taking limit as $\Delta r \to 0$, we obtain

$$\lim_{\Delta r \to 0} \frac{(r\,\tau_{rz})\big|_{r+\Delta r} - (r\,\tau_{rz})\big|_r}{\Delta r} = r\left(\frac{p_0 - p_L}{L} + \rho g\right) \qquad ...(2.8.2)$$

Note that the pressure and gravity forces are acting in the same direction.

If $\quad P = p - \rho g z,$ then

$P_0 = p_0 \quad$ and $\quad P_L = p_L - \rho g L$

and Eq. (2.8.2) may be written as

$$\frac{d}{dr}(r\tau_{rz}) = \left(\frac{P_0 - P_L}{L}\right) r \qquad ...(2.8.3)$$

Integrating Eq. (2.8.3) and dividing by r,

$$\tau_{rz} = \left(\frac{P_0 - P_L}{2L}\right) r + \frac{C_1}{r} \qquad ...(2.8.4)$$

We know from physical considerations that the velocity must be finite at $r = 0$ and hence the momentum flux (*i.e.* shear stress) must be finite at $r = 0$. This is possible only when $C_1 = 0$.

Hence the momentum flux distribution is

$$\boxed{\tau_{rz} = \left(\frac{P_0 - P_L}{2L}\right) r} \qquad ...(2.8.5)$$

C_1 (0)

For a Power-law fluid

$$\tau_{rz} = -K\left|\frac{dv_z}{dr}\right|^{n-1}\left(\frac{dv_z}{dr}\right) \qquad ...(2.8.6)$$

Since v_z decreases as r increases, $\dfrac{dv_z}{dr}$ is –ve.

$$\therefore \qquad \tau_{rz} = K\left|-\frac{dv_z}{dr}\right|^{n-1}\left(-\frac{dv_z}{dr}\right) = K\left(-\frac{dv_z}{dr}\right)^n \qquad ...(2.8.7)$$

Substituting Eq. (2.8.7) onto Eq. (2.8.5),

$$K\left(-\frac{dv_z}{dr}\right)^n = \left(\frac{P_0 - P_L}{2L}\right) r$$

or

$$\frac{dv_z}{dr} = -\left(\frac{P_0 - P_L}{2KL}\right)^{\frac{1}{n}} \cdot r^{\frac{1}{n}} \qquad ...(2.8.8)$$

Integrating Eq. (2.8.8),

$$v_z = -\left(\frac{P_0 - P_L}{2KL}\right)^{\frac{1}{n}} \frac{r^{\left(\frac{1}{n}+1\right)}}{\frac{1}{n}+1} + C_2$$

where C_2 is a constant of integration.

Since $v_z = 0$ at $r = R$,

$$C_2 = \left(\frac{P_0 - P_L}{2KL}\right)^{\frac{1}{n}} \frac{n}{n+1} R^{\frac{n+1}{n}}$$

Hence the velocity profile is

$$v_z = \left(\frac{P_0 - P_L}{2KL}\right)^{\frac{1}{n}} \frac{n}{n+1} R^{\frac{n+1}{n}} \left[1 - \left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right] \qquad \text{...(2.8.9)}$$

From Eq. (2.8.9), we can derive the following quantities :

(*a*) **Average velocity : $v_{z\,av}$**

$$V_{z,\,av} = \frac{\int_0^{2\pi}\int_0^R v_z\, r dr d\theta}{\int_0^{2\pi}\int_0^R r dr d\theta} = \frac{Q}{\pi R^2} \qquad \text{...(2.8.9a)}$$

where Q = Volume flow rate

$$= \int_0^{2\pi}\left[\int_0^R v_z \,.\, r dr\right] d\theta$$

$$= 2\pi \int_0^R \left[\left(\frac{n}{n+1}\right)\left(\frac{P_0 - P_L}{2KL}\right)^{\frac{1}{n}} R^{\frac{n+1}{n}} \left[r - r\left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right] dr\right]$$

$$= 2\pi\left(\frac{n}{n+1}\right)\left(\frac{P_0 - P_L}{2KL}\right)^{\frac{1}{n}} R^{\frac{n+1}{n}} \left[\frac{R^2}{2} - \frac{R^{\frac{2n+1}{n}+1}}{\frac{2n+1}{n}+1}\right]$$

or

$$Q = \pi\left(\frac{n}{3n+1}\right)\left(\frac{P_0 - P_L}{2KL}\right)^{\frac{1}{n}} R^{\frac{3n+1}{n}} \qquad \text{...(2.8.10)}$$

Therefore, the average velocity, $v_{z,\,av}$ is

$$v_{z,\,av} = \frac{Q}{\pi R^2} = \frac{n}{3n+1}\left(\frac{P_0 - P_L}{2KL}\right)^{\frac{1}{n}} R^{\frac{n+1}{n}} \qquad \text{...(2.8.11)}$$

Maximum velocity, $v_{z,\max}$, occurs at $r = 0$

$$v_{z,\max} = \frac{n}{n+1}\left(\frac{P_0 - P_L}{2KL}\right)^{\frac{1}{n}} R^{\frac{n+1}{n}} \quad ...(2.8.12)$$

For the case of a Newtonian fluid, $n = 1$ and $K = \mu$, and for this case, the velocity profile v_z, is

$$v_z = \frac{(P_0 - P_L) R^2}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2\right] \quad ...(2.8.13)$$

The average velocity, $v_{z,av}$, is

$$v_{z,av} = \frac{(P_0 - P_L) R^2}{8\mu L} \quad ...(2.8.14)$$

The maximum velocity, $v_{z,\max}$, is

$$v_{z,\max} = \frac{(P_0 - P_L) R^2}{4\mu L} \quad ...(2.8.15)$$

The ratio of average velocity, $v_{z,av}$ to maximum velocity, $v_{z,\max}$, is

$$\frac{v_{z,av}}{v_{z,\max}} = \frac{1}{2} \quad ...(2.8.16)$$

The volume flow rate Q is

$$Q = \frac{\pi (P_0 - P_L) R^4}{8\mu L} \quad ...(2.8.17)$$

Eq. (2.8.17) is called the Hagen-Poiseuille Law which gives the relationship between the volume flow rate and the forces associated with the pressure drop and the gravitational acceleration. These are the two forces which are responsible for causing the flow against the resistance of simple fluids characterized by viscosity, μ.

The z-component of the force of the fluid acting on the wetted surface of the tube, F_z, is the momentum flux integrated over the wetted surface area of the tube :

$$F_z = 2\pi R L \tau_{rz}\big|_R \quad ...(2.8.18)$$

For a Newtonian fluid, Eq. (2.8.18) becomes

$$F_z = 2\pi R L\left(-\mu \frac{dv_z}{dr}\right)\bigg|_R = \pi R^2 (P_0 - P_L)$$

or,
$$F_z = \pi R^2 (P_0 - P_L) = \pi R^2 \{p_0 - (p_L - \rho g L)\}$$

or,
$$F_z = \pi R^2 (p_0 - p_L) + \pi R^2 L \rho g \quad ...(2.8.19$$

Eq. (2.8.19) states that the net force acting downwards on the fluid cylinder by virtue of the pressure difference and gravitational acceleration is counter-balanced by the viscous force, F_Z, which tends to resist the motion of the fluid.

2. Flow Through an Annulus

An incompressible fluid is in steady laminar flow in the annular region of two coaxial cylinders. The inner radius of the outer cylinder is R while the outer radius of the inner cylinder is aR, as shown in Fig. 2.11.1.

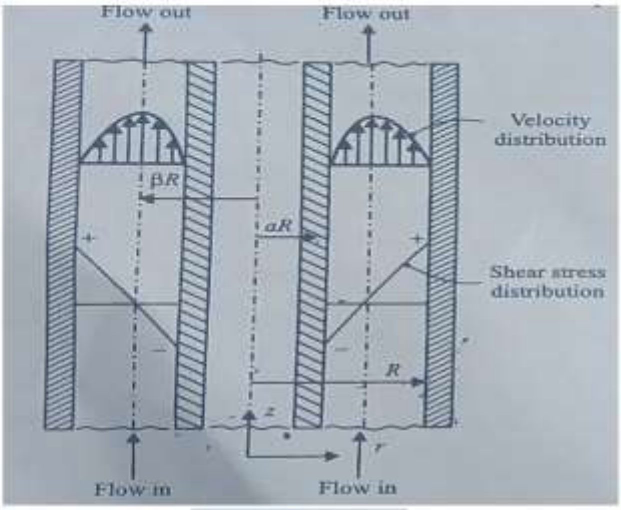


Fig. 2.11.1
Flow through an annulus

Considering a shell of thickness Δr in the annular region at a distance r from the axis of the cylinders and making a moment x balance, we obtain

$$\frac{d}{dr}(r\tau_{rz}) = \left(\frac{P_0 - P_L}{L}\right)r \quad \text{...(2.11.1)}$$

where $P = p + \rho g z$. Here z is the same as h used for inclined surfaces to designate the distance upward from any chosen reference datum. Note that pressure and gravity forces are acting in the opposite directions.

Integrating Eq. (2.11.1) and dividing by r we obtain,

$$\tau_{rz} = \left(\frac{P_0 - P_L}{2L}\right)r + \frac{C_1}{r}$$

where C_1 is a constant of integration. If maximum velocity, $v_{z,\max}$, occurs at $r = \beta R$ at which $\tau_{rz}|_{r=\beta R} = 0$, then $C_1 = -\left(\frac{P_0 - P_L}{2L}\right)(\beta R)^2$,

where β is an unknown constant,

and $$\tau_{rz} = \frac{(P_0 - P_L)R}{2L}\left[\left(\frac{r}{R}\right) - \beta^2\left(\frac{R}{r}\right)\right] \quad \text{...(2.11.2)}$$

Assuming Newtonian fluid, substituting

$$\tau_{rz} = -\mu\frac{dv_z}{dr}$$

into Eq. (2.11.2) and integrating, we obtain,

$$v_z = -\frac{(P_0 - P_L) R^2}{4\mu L}\left[\left(\frac{r}{R}\right)^2 - 2\beta^2 \ln\left(\frac{r}{R}\right) + C_2\right] \quad ...(2.11.5)$$

where C_2 is another constant. The constants β and C_2 may be evaluated from the boundary conditions :

$v_z = 0$ at $r = aR$

$v_z = 0$ at $r = R$.

Using these boundary conditions, we obtain two simultaneous equations

$$0 = -\frac{(P_0 - P_L) R^2}{4\mu L}(a^2 - 2\beta^2 \ln a + C_2)$$

$$0 = -\frac{(P_0 - P_L) R^2}{4\mu L}(1 + C_2)$$

Solving, $\quad C_2 = -1 \quad$ and $\quad 2\beta^2 = \dfrac{1 - a^2}{\ln\left(\frac{1}{a}\right)}$

Thus, the momentum flux distribution and the velocity distribution steady, laminar flow of an incompressible Newtonian fluid in an annulus (ring-shaped region) are, respectively :

$$\tau_{rz} = \frac{(P_0 - P_L) R}{2L}\left[\left(\frac{r}{R}\right) - \left\{\frac{1 - a^2}{2\ln(1/a)}\right\}\left(\frac{R}{r}\right)\right] \quad ...(2.1$$

$$v_z = \frac{(P_0 - P_L) R^2}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2 + \left\{\frac{1 - a^2}{\ln(1/a)}\right\}\ln\left(\frac{r}{R}\right)\right] \quad ...(2.1$$

Note that when $a \to 0$, these two results reduce to the corresponding expressions for flow of an incompressible Newtonian fluid in circular pipe.

We can now derive the following expressions :

(a) The maximum velocity :

$$v_{z,\,max} = v_z\big|_{r=\beta R} = \frac{(P_0 - P_L) R^2}{4\mu L}\left[1 - \left\{\frac{1 - a^2}{2\ln\left(\frac{1}{a}\right)}\right\}\left\{1 - \ln\frac{1 - a^2}{2\ln\left(\frac{1}{a}\right)}\right\}\right] \quad ...(2.11.6)$$

(b) The average velocity :

$$v_{z,\,av} = \frac{\int_0^{2\pi}\int_{aR}^{R} v_z\, r\, dr d\theta}{\int_0^{2\pi}\int_{aR}^{R} r\, dr d\theta} = \frac{(P_0 - P_L) R^2}{8\mu L}\left\{\frac{1 - a^4}{1 - a^2} - \frac{1 - a^2}{\ln\left(\frac{1}{a}\right)}\right\} \quad ...(2.11.7)$$

(*c*) **The volume flow rate :**

$$Q = (\pi R^2 - a^2 R^2 \pi)\, v_{z,\,av}$$

$$= \pi R^2 (1 - a^2)\, v_{z,\,av} = \frac{\pi (P_0 - P_L) R^4}{8\mu L} \left\{ (1 - a^4) - \frac{(1 - a^2)^2}{\ln\left(\frac{1}{a}\right)} \right\}$$

...(2.11.8)

(*d*) **The force exerted by the fluid on the solid surfaces :**

$$F_z = -\tau_{rz}\big|_{r=aR}\, 2\pi a R L + \tau_{rz}\big|_{r=aR}\, 2\pi R L.$$

$$= \pi R^2 (1 - a^2)(P_0 - P_L) \qquad \text{...(2.11.9)}$$

Note that momentum flows from the point of maximum momentum concentration towards the outer surface of the inner cylinder *i.e.*, in the negative radial direction and towards the inner surface of the outer cylinder in the positive radial direction.

Equations (2.11.6) to Eq. (2.11.9) reduce to corresponding results for flow through circular pipes when $a \to 0$. These equations also reduce to corresponding results for flow in plane slits when $a \to 1$.

The above solution is valid for laminar flow for which Reynolds number, defined as

$$Re = \frac{2R(1 - a)\, v_{z,\,av}\, \rho}{\mu} < 2000.$$