Chapter 1

Momentum Transfe[r] Overvie[w]

Outline

SECTION 1

1.0 INTRODUCTION

The momentum of a body is defined as the product of its mass and velocity. The velocity of a fluid at a point may be considered as its momentum per unit mass. The changes in the velocity of a fluid cause momentum transport just as the changes in temperature cause heat transport. The mathematical description of momentum transport is an important objective of fluid mechanics.

1.1 NEWTON'S SECOND LAW OF MOTION

According to this law, the force F acting on a body of mass m is proportional to the time rate of change of its momentum. If the mass remains constant, the force is proportional to the product of the mass and its acceleration. Thus,

$$F = K\frac{d\,(mV)}{dt} = K\,mf \qquad \text{...(1.1.1)}$$

where K is the proportionality constant,

and $f = \frac{dV}{dt}$ is the acceleration of the body.

The value of constant K depends on system of units used.

In the gravitational system of units, the units of mass and force are defined such that the body weight at sea level is numerically equal to the body mass. In English gravitational system of units, the unit of mass is taken as the pound mass (lb or lbm) and the unit of force, called the pound force (lb_f), is taken such that the weight in pound force of a body at sea level becomes numerically equal to its mass in pound mass. Since the acceleration due to gravity at sea level is $g = 32{\cdot}2\ \text{ft}/\text{s}^2$, the magnitude of constant K in Eq. (1.1.1) can be found by letting the weight in the pound force and the mass in pound mass have the same numerical value. When $F = W$ and $f = g = 32{\cdot}2\ \text{ft}/\text{s}^2$, Eq. (1.1.1) becomes

$$W = K\,mg$$

$$W\ \text{lb}_f = (Km\ \text{lb}_m)\,(32{\cdot}2\ \text{ft}^2/\text{s})$$

Letting $W = m$, we have,

$$K = \frac{1\ \text{lb}_f}{32{\cdot}2\ \text{lb}_m\ \ \text{ft}^2/\text{s}}$$

We commonly write

$$K = \frac{1}{g_c}$$

where g_c is a conversion factor, which is equal to $32{\cdot}2\ (\text{lbm}\,.\,\text{ft}/\text{s}^2)/\text{lb}_f$. Thus Newton's second law motion in terms of g_c is

$$F = \frac{1}{g_c}\,mf \qquad \text{...(1.1.2)}$$

Note that while g_c has the magnitude of g at sea level, its units are not same and it is not the acceleration due to gravity. g_c is simply a conversion factor required for the selection of units. Also note that while g_c is a pure constant, the acceleration due to gravity varies with distance from the earth.

In absolute system is units g_c equals to **1.0** and it is dimensionless. There are CGS system, International system (SI), and English system (pound mass, poundal, feet, second). In these systems the units in Eq. (1.1.1) are as follows :

System	*Force*	*Mass × Acceleration*
CGS	dyne (dyn)	$g \cdot cm/s^2$
SI	newton (N)	$kg \cdot m/s^2$
English	poundal	$lb \cdot ft/s^2$

Thus, in the absolute system of units,

$$1 \text{ dyn} = 1 \text{ g} \cdot \text{cm/s}, \; 1 \text{ N} = 10^5 \text{ dyn} = 1 \text{ kg} \cdot \text{m/s}^2, \; 1 \text{ poundal} = 1 \text{ lb}_m \cdot \text{ft/s}^2.$$

In elementary physics, Newton's second law of motion is applied to bodies or particles. But in fluid flow, a small volume element moving with the fluid is accelerated because of the forces acting upon the volume element. Thus Newton's second law as applied to fluid flow becomes :

$$\begin{pmatrix}\text{Rate of accumulation}\\ \text{of momentum in}\\ \text{volume element}\end{pmatrix} = \begin{pmatrix}\text{Net flow of}\\ \text{momentum into}\\ \text{volume element}\end{pmatrix} + \begin{pmatrix}\text{Sum of forces}\\ \text{acting on the}\\ \text{volume element}\end{pmatrix} \times g_c \quad ...(1.1.3)$$

For a gravitational system of units, the forces must be multiplied by g_c in order to convert to momentum units.

When absolute system of units is used,

$$\begin{pmatrix}\text{Rate of}\\ \text{momentum in}\end{pmatrix} - \begin{pmatrix}\text{Rate of}\\ \text{momentum out}\end{pmatrix} + \begin{pmatrix}\text{Sum of the}\\ \text{forces acting}\end{pmatrix} = \begin{pmatrix}\text{Rate of accumulation}\\ \text{of momentum}\end{pmatrix} \quad ...(1.1.4)$$

The rate of accumulation of momentum is zero for flow under steady state conditions.

1.2 MOMENTUM TRANSPORT BETWEEN PARALLEL PLATES : COUETTE FLOW

Consider a fluid, either gas or liquid contained between two parallel plates as shown in Fig. 1.2.1. The distance between the plates, L_y, is small compared

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the x-direction, from zero to some positive value. Since velocity of a fluid at a point is its momentum per unit mass, there is a corresponding increase in x-momentum. In other words, x-momentum is being transported in the y-direction from the lower plate to the fluid layer and then from one fluid layer to the next above. This is shown in Fig. 1.2.1(a).

In Fig. 1.2.1(b) the velocity profiles are plotted for different times. For $t=0$ there is a sharp drop at $y=0$ from $V_x = V$ to $V_x = 0$. At $t=t_1$, the velocity has increased near the lower plate but momentum has not yet reached (or penetrated or diffused) to the fluid near the upper plate. At $t=t_2$, the momentum has penetrated to a greater extent and is closer to the upper plate. Finally $t=\infty$ a steady state is reached in which the velocity V_x (y) is no longer dependent on time. The concept of mathematical infinite time in practice means a large time. For fluids with high viscosity it may require only a fraction of second to achieve 99 percent of the steady state condition.

The model discussed under flow between parallel plates is a classical problem in fluid dynamics referred to as plane Couette flow. This type of flow is found in lubrication systems and in Couette viscometers. A Couette viscometer consists of two concentric cylinders with liquid in the annulus and one or both of the cylinders capable of rotation. If the gap between the cylinders is small, the curvature can be neglected and the liquid behaves as if it were between two infinite flat parallel plates.

1.3 SHEAR STRESS AND MOMENTUM FLUX

In the discussion on flow between two parallel plates, we noted that the application of a force F_x imparted x-directed momentum to the fluid contained between the plates. F_x is a sheer force, a force that acts tangentially to the area over which it is applied. In this case F_x acts tangentially to the area of the plate, $A_y = L_x L_z$, where the subscript indicates that the area is perpendicular to the y-axis. If we multiply this area by a constant a, we would find that a force proportional to F_x is required to maintain the same velocity profile versus time in the fluid. Hence F_x is proportional to A_y in this case. The ratio

$$\frac{F_x}{A_y} = \sigma_{yx} \qquad ...(1.3.1)$$

is known as shear stress. A stress is any force per unit area. A normal force per unit area is called a normal stress. The first subscript on σ_{yx} refers to the area it acts (in this case the y-area or the area normal to the y-axis) and the second subscript refers to the direction in which the shear stress acts.

If M_x denotes the rate of transport of x-momentum, we can define a flux of x-momentum by using the general definition,

$$\text{Flux} = \text{rate of transport per unit area} \qquad ...(1.3.2)$$

1.4 NEWTON'S LAW OF VISCOSITY

Consider the flow between two parallel plates in Fig. 1.2.1. When the final steady motion has been attained, a constant force F_x is required to maintain the motion of the lower plate. If the flow is laminar, this force may be expressed as follows :

$$\sigma_{yx} = \frac{F_x}{A_y} = \frac{\mu}{g_c} \cdot \frac{V}{L_y} \qquad ...(1.4.1)$$

The force F_x and the shear stress σ_{yx} remain constant as long as the velocity V is unchanged. The constant of proportionality μ is called the viscosity of the fluid. Equation (1.4.1) is valid only for the special case of steady-state Couette flow.

To develop a more general relationship, consider any unsteady-state velocity profile of Fig. 1.2.1(*b*), showing a plot of V_x vs. y at constant t which is redrawn in Fig. 1.4.1.

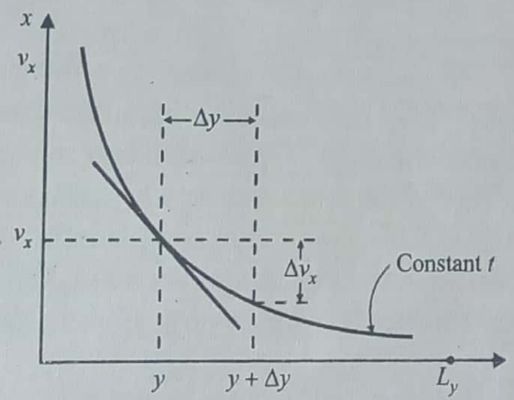


Fig. 1.4.1
Unsteady velocity profile for flow between parallel plates.

Consider a region of width Δy in which the change in velocity is Δv_x. Then in terms of difference operator,

$$\Delta v_x \equiv v_x\,(y + \Delta y, t) - v_x\,(y, t) \qquad ...(1.4.2)$$

Then $$g_c \sigma_{yx} = \mu \frac{\Delta v_x}{\Delta y}$$

where the average slope of the v_x versus y curve is $\frac{\Delta v_x}{\Delta y}$. When $\Delta y \to 0$, the true slope at y is given by the partial derivative $\frac{\partial v_x}{\partial y}$. Thus the Newton's law of viscosity for unsteady-state momentum transport in one dimension is,

$$g_c \sigma_{yx} = \tau_{yx} = -\mu \frac{\partial v_x}{\partial y} \qquad ...(1.4.3)$$

This states that the shear force per unit area is proportional to the negative of the local velocity gradient, and fluids, which obey this equation, are called

Newtonian fluids. It can be shown that for transport in more than one dimension, Newton's law of viscosity is

$$g_c \sigma_{yx} = \tau_{yx} = -\mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) \qquad \text{...(1.4.4)}$$

At steady-state v_x depends only on y and hence Newton's law of viscosity for one-dimensional transport under steady-state condition becomes,

$$g_c \sigma_{yx} = \tau_{yx} = -\mu \frac{dv_x}{dy} \qquad \text{...(1.4.5)}$$

All gases and most of the simple liquids obey Newton's law of viscosity. The diagram relating shear stress and shear rate for Newtonian fluid is a straight line of slope μ, which completely characterizes the fluid.

Note that Newton's law of viscosity, Eq. (1.4.5) may also be written as follows for constant density :

$$\tau_{yx} = -\nu \frac{d(v_x \rho)}{dy} \qquad \text{...(1.4.6)}$$

where τ_{yx} is the flux of x-directed momentum in the y-direction, $(\text{kg}\,.\,\text{m/s})/\text{s}\,.\,\text{m}^2$; ν is μ/ρ, the kinematic viscosity or the momentum diffusivity in m^2/s, y is the direction of transport or diffusion of momentum in m, ρ is the density in kg/m^3 and μ is the viscosity in $\text{kg/m}\,.\,\text{s}$. In this form Eq. (1.4.6) states that momentum diffuses from the region of higher momentum concentration to the region of lower momentum concentration, or momentum flows downhill in the direction of decreasing velocity.

1.5 NON-NEWTONIAN FLUIDS

Non-Newtonian fluids are those which do not obey Newton's law of viscosity. The "viscosity" of a non-Newtonian fluid is not a constant at a given temperature and pressure but depends on the other factors such as the rate of shear in the fluid, the time the fluid has been sheared, or its previous history. The subject of non-Newtonian flow is a subdivision of science of rheology, which deals with the deformation and flow of matter and includes the study of the mechanical properties of gases, liquids, plastics, and crystalline materials. Thus the science of rheology includes the mechanics of Newtonian fluids at one end of the spectrum of subject materials, and Hookeon elasticity at the other end. The region between these two ends concerns the deformation and flow of all sorts of sticky materials.

1.5.1 Time-independent non-Newtonian Fluids

Fluids of this type may be described by a rheological equation of the form

$$\dot{\gamma} = f(\tau) \qquad \text{...(1.5.1.1)}$$

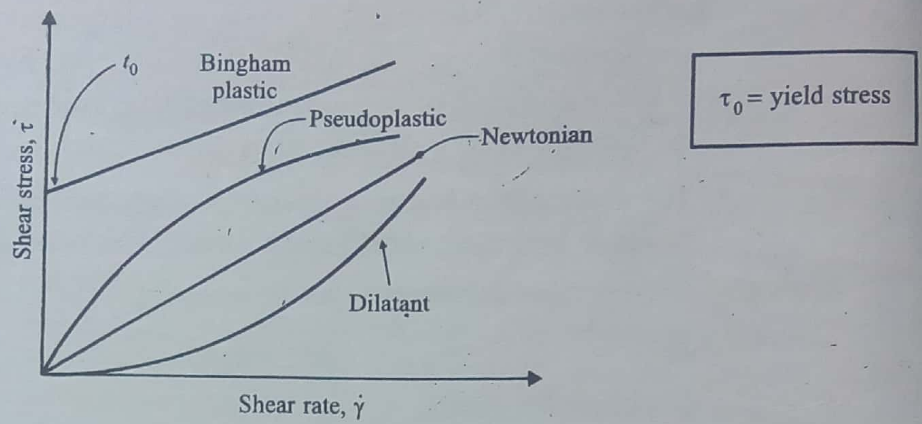
which states that the rate of shear, $\dot{\gamma}$, at any point in the fluid is a function of the shear stress at that point. Such fluids are termed non-Newtonian viscous fluids.

These fluids are subdivided into *three* distinct types :

1. Bingham plastics 2. Pseudoplastic fluids 3. Dilatant fluids

The flow curves of these three fluids along with Newtonian fluids are shown in Fig. 1.5.1.1.

Fig. 1.5.1.1
Flow curves for time independent non-Newtonian fluids.



The steady-state rheological behaviour of fluids of Fig. 1.5.1.1 can be expressed as

$$\tau_{yx} = -\eta \frac{dv_x}{dy} \quad ...(1.5.1.1)$$

where η may be expressed as a function of either shear rate, $\frac{dv_x}{dy}$, or shear stress τ_{yx}. When η decreases with increasing rate of shear, the behaviour is called pseudoplastic. When η increases with increasing rate of shear, the behaviour is called dilatant. If η is independent of the rate of shear, the behaviour is Newtonian in which case $\eta = \mu$.

The functional relationship between the shear stress and the rate of shear is called a constitutive equation. Numerous models have been proposed to expressed the steady-state relation between τ_{yx} and $\frac{dv_x}{dy}$. Each of these models contains empirical positive parameters, which are evaluated numerically to fit data on τ_{yx} versus $\frac{dv_x}{dy}$ at constant temperature and pressure.

1.5.2 The Bingham Plastic

This substance remains rigid when the shear stress, τ_{yx}, is of smaller magnitude than a critical value, τ_0, called the yield stress but flows like a

$$\tau_{yx} = m\left|-\frac{dv_x}{dy}\right|^{n-1}\left(-\frac{dv_x}{dy}\right)$$

or $$\tau_{yx} = m\left(-\frac{dv_x}{dy}\right)^n \quad ...(1.5.3.3)$$

The expression for τ_{yx} may then be substituted in the appropriate equation of the momentum balance for a specific problem to find the expression for velocity profile.

1.5.4 Time-dependent non-Newtonian Fluids

The apparent viscosity of complex fluids depends not only on the shear rate but also on the time of shear. These fluids are divided into *two* classes :

(*a*) thixotropic fluids (*b*) rheopectic fluids

If a thixotropic fluid is sheared at a constant rate after a period of rest, the structure of the material is progressively broken down and the apparent viscosity decreases with time. The rate of breakdown of structure during application of shear at a given rate depends on the number of linkages available for breaking and hence decrease with time. The simultaneous rate of reformation of structure increases with time as the number of possible new structural linkages increases. A state of dynamic equilibrium is eventually reached when the rate of reformation of structure equals the rate of breakdown. Thixotropy is, therefore, a reversible process and after resting, the structure of the material builds up gradually. A hysteresis loop for a thixotropic fluid is observed on the curve of shear-stress vs shear rate if the curve is plotted first for the rate of shear increasing at a constant rate and then for the rate of shear decreasing at a constant rate.

Rheopectic fluids are those in which gradual formation of structure is observed on application of shear. For example, a 42 per cent gypsum paste ($1 - 10\mu$) in water after shaking re-solidifies in 40 mins if at rest, but in 20 sec if the container is gently rolled in the palms of hands, indicating that small shearing motions facilitate structure build-up but large shearing (shaking) destroys it and there is a critical shear rate beyond which breakdown of structure occurs instead of reformation. Vanadium pentoxide and bentonite in dilute aqueous solutions show this type of behaviours.

1.5.5 Viscoelastic Fluids

An important distinction between fluids and solids is in the way these substances dispose of the work done upon them in shearing deformations. All the work done on a purely viscous fluid in shear is immediately dissipated as heat, whereas the work done on a perfectly elastic substance in shear is not dissipated but may be recovered at any time by allowing the elastic material to regain its original configurations.

There exist materials whose behaviour is partly fluid-like and partly solid-like ; the work of shearing deformation in these materials is not completely conserved, as in solids, nor is it completely dissipated as in fluids. These materials are called viscoelastic substances as they possess, both elastic and viscous properties.

SECTION I

Oldroyd Model

The simple constitutive equation for a viscoelastic fluid is

$$\tau + \lambda_1 \dot{\tau} = \mu_0 (\dot{\gamma} + \lambda_2 \ddot{\gamma}) \qquad \text{...(1.5.5.1)}$$

which relates shear stress τ with the shear rate, $\dot{\gamma}$ using three constants μ_0, λ_1 and λ_2 which can be determined in terms of the physical properties of the mixture. The constant μ_0 is identified as the viscosity at low rates of shear in the steady-state *i.e.*, when $\dot{\tau} = \ddot{\gamma} = 0$. The physical significance of constant λ_1 is that if the motion is suddenly stopped, the shear stress will decay as $\exp(-t/\lambda_1)$, λ_1 is called relaxation time. The physical significance of λ_2 is that if all stresses are removed the rate of strain decays as $\exp(-t/\lambda_2)$. λ_2 is called retardation time.

Dilute solutions of poly-methylmethacrylate in pyridine and some bitumens are described by this model.

For further details about non-Newtonian fluids and their engineering applications interested reader may refer to the book by Fredrickson (8).

1.6 SIGN CONVENTION FOR MOMENTUM AND FORCE BALANCES

The momentum flux, τ_{yx}, is taken as positive if the momentum physically diffuses in the positive y-direction and as negative if it diffuses in the – ve y-direction. Consider the case of Couette flow of Fig. 1.2.1, in which one of the plates is in motion while the other is stationary. In the first case, (Fig. 1.6.1), the lower plate moves with velocity $v_{x_1} = v_1$ and $v_{x_2} = 0$ and in the second case, the lower plate is stationary *i.e.*, $v_{x_1} = 0$ and the upper plate moves with velocity $v_{x_2} = v_2$.

In the first case, momentum flows in the positive y-direction and τ_{yx} is + ve but $\frac{dv_x}{dy}$ is – ve. However, in the second case, momentum is flowing in the – ve y-direction and τ_{yx} is – ve but $\frac{dv_x}{dy}$ is + ve. If the fluid is Newtonian, then in the first case,

$$\tau_{yx} = -\mu \frac{dv_x}{dy}$$

and in the second case,

$$-\tau_{yx} = \mu \frac{dv_x}{dy}$$

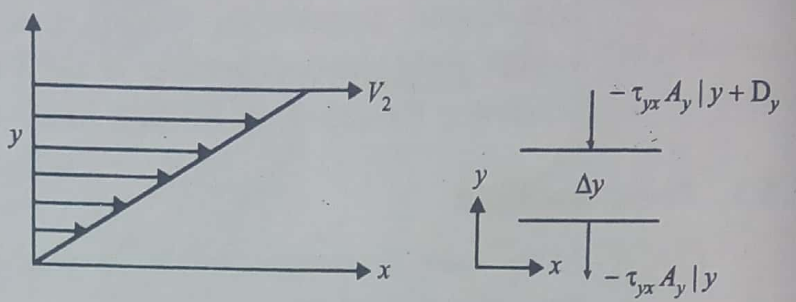
Thus – ve sign in the Newton's law of viscosity is always necessary.

If we consider a shell of thickness Δy at a distance y from the bottom plate, in the first case, the momentum is entering at y and leaving at $y + \Delta y$. Here τ_{yx} is +ve and the momentum balance at steady state becomes

$$\begin{pmatrix} \text{Rate at which momentum} \\ \text{enters the surface at } y \end{pmatrix} - \begin{pmatrix} \text{Rate at which} \\ \text{momentum leaves} \\ \text{the surface at } y + dy \end{pmatrix} = 0$$

or $$\tau_{yx} A_y \big|_y - \tau_{yx} A_y \big|_{y+dy} = 0 \qquad ...(1.6.1)$$

Fig. 1.6.1 Momentum transfer in the -ve y-direction



However, in the second case (Fig. 1.6.1), momentum is entering at $y + dy$ and leaving at y. Since γ_{yx} is – ve, the momentum balance becomes

$$(-\tau_{yx} A_y \big|_{y+\Delta y}) - (-\tau_{yx} A_y) | = 0 \qquad ...(1.6.2)$$

Rearranging we obtain the same momentum balance equation as that for the first case. The differential equation for flux is the same for both the cases, regardless of the sign of the momentum flux. The differential equation for flux can be derived by assuming the flux of momentum, τ_{yx} to be positive. The actual sign of the flux will be determined by the boundary conditions. For this specific problem, the sign of τ_{yx} will be determined by whether

$$v_{x_1} > v_{x_2} \quad \text{or} \quad v_{x_2} > v_{x_1}.$$

In the first case where lower plate is in motion and the upper plate is stationary, the shear-stress σ_{yx} is positive, and the force balance may be written as

$$\sigma_{yx} A_y \big|_y - \sigma_{yx} A_y \big|_{y+\Delta y} = 0 \qquad ...(1.6.3)$$

and for the second case, the shear-stress, σ_{yx} is negative, and the force balance becomes

$$-\sigma_{yx} A_y \big|_{y+\Delta y} - (-\sigma_{yx} A_y) \big|_y = 0 \qquad ...(1.6.4)$$

Since Eqs. (1.6.3) and (1.6.4) are identical, differential or shell force balances result in equations that apply to negative as well as to positive shear stresses.