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Semester: VII
Subject code – PC 702 **Subject: Transport Phenomena**
Unit: -III

Lecture no.16

Problems of turbulent and Internal momentum analysis for turbulent boundary layer

EXAMPLE 4.7.3 *Thickness of laminar sublayer in turbulent pipe flow.*

Consider turbulent flow through a pipe of diameter D. The volumetric flow rate is Q and fluid density and viscosity are ρ and μ respectively. Assuming 1/7th power law of velocity distribution, find an expression for the thickness of laminar sublayer.

Solution.

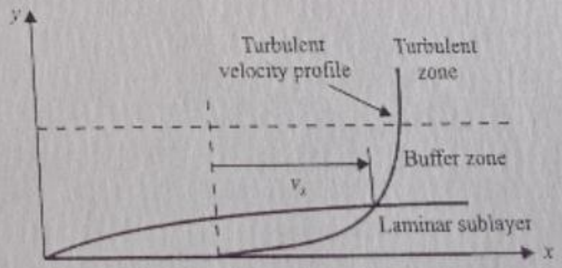

Fig. E4.7.3.1 Viscous sublayer in turbulent flow.

Following the nomenclature shown in Fig. E4.7.3.1 we may write following relations for laminar sublayer in turbulent flow field.

From $1/7^{th}$ power-law velocity field :

$$v_s / v_\infty = (\delta_s / R)^{1/7} \qquad \text{...(E4.7.3.1)}$$

Linear wall shear stress relation for laminar sublayer :

$$\tau_w = \mu(v_s / \delta_s) \qquad \text{...(E4.7.3.2)}$$

Blasius friction factor relation :

$$f = 0.3164\, Re^{-1/4} \qquad \text{...(E4.7.3.3)}$$

For pipe flow, wall shear stress is defined as :

$$\tau_w = (-dp / dx)(R / 2) \qquad \text{...(E4.7.3.4)}$$

Pressure gradient is expressed as :

$$-dp / dx = f \rho v_{av}^2 / 2D \qquad \text{...(E4.7.3.5)}$$

Velocity relation for turbulent pipe flow :

$$v_{av} / v_\infty = 0.816 \qquad \text{...(E4.7.3.6)}$$

Combining above relations laminar sublayer thickness may be expressed as,

$$\delta_s = 75.896\, v_\infty^{-2.04} D^{0.125} (\mu / \rho)^{0.875} \qquad \text{...(E4.7.3.7)}$$

1. **Internal momentum analysis for turbulent boundary layer**

We assume a small control volume in the boundary layer over a flat plate as shown in Fig. 4.8.1.

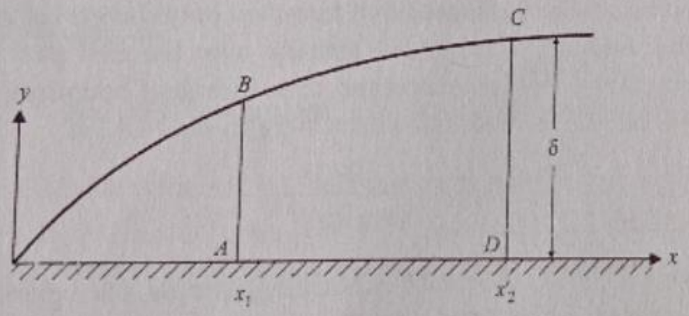

Fig. 4.8.1 Control volume for integral momentum analysis in boundary layer.

A steady state mass and momentum balance over the control volume gives,

$$\frac{\tau_0}{\rho} = \frac{d}{dx}\int_0^{\delta} v_x\,(v_\infty - v_x)\,dy \qquad \text{...(4.8.1)}$$

where, τ_0 is the shear stress at the solid surface ($y = 0$) and boundary layer thicknesses δ and τ_0 depend on x. Eq. (4.8.1) is valid for laminar and turbulent boundary layer analysis. Laminar boundary layer analysis has been discussed in details in sections 3.4 through 3.14. In this section we present momentum balance for turbulent boundary layer.

Blasius used an empirical velocity profile for fully developed turbulent flow in pipe. The equation is,

$$\frac{v_x}{v_{x_{\max}}} = \left(\frac{y}{r_0}\right)^{1/7} \qquad \text{...(4.8.1)}$$

This is known as Blasius $\frac{1}{7}$th power law and it is valid upto $Re = 10^5$. We apply a similar form of velocity profile for flow over flat plate in turbulent region.

SECTION 1

$$\frac{v_x}{v_\infty} = \left(\frac{y}{\delta}\right)^{1/7} \qquad \text{...(4.8.2)}$$

Substituting Eq. (4.8.2) in Eq. (4.8.1), we get,

$$\frac{\tau_0}{\rho} = \frac{d}{dx}\int_0^{\delta} v_\infty^2\left[\left(\frac{y}{\delta}\right)^{1/7} - \left(\frac{y}{\delta}\right)^{2/7}\right]dy \qquad \text{...(4.8.3)}$$

Defining a dimensionless velocity as

$$v_x^* = \frac{v_x}{v_\infty} \qquad \text{...(4.8.4)}$$

and a dimensionless y-coordinate,

$$y^* = \frac{y}{\delta} \qquad \text{...(4.8.5)}$$

Eq. (4.8.3) may be rewritten as,

$$\frac{\tau_0}{\rho v_\infty^2} = \frac{d}{dx}\delta\int_0^{1}\left[v_x^*\,(1 - v_x^*)\right]dy^* \qquad \text{...(4.8.6)}$$

The integral on the right-hand-side is evaluated as,

$$\int_0^1 [v_x^* (1 - v_x^*)]\, dy^* = \frac{7}{72} \qquad ...(4.8.7)$$

So we get a relation of the following form,

$$\frac{\tau_0}{\rho v_\infty^2} = \frac{7}{72}\frac{d\delta}{dx}$$

or

$$\frac{d\delta}{dx} = \frac{72}{7}\frac{\tau_0}{\rho v_\infty^2} \qquad ...(4.8.8)$$

Now we know that, for both laminar and turbulent flow,

$$\tau_0 = \mu \left.\frac{dv_x}{dy}\right|_{y=0} \qquad ...(4.8.9)$$

But if we substitute Eq. (4.8.2) in Eq. (4.8.9), the derivative, hence τ_0 goes to zero at the wall. So we have to make use of Blasius friction factor for turbulent pipe flow which is consistent with 1/7th power velocity distribution. The said friction factor expression is,

$$f = 0.079\left(\frac{D\, v_{av}\rho}{\mu}\right)^{-1/4} \qquad ...(4.8.10)$$

When Eq. (4.8.10) is applied for turbulent boundary layer on a flat plate, the following changes are made.

$$D = 2\delta, \qquad v_{av} = 0.817,\ v_{max} = 0.817\ v_\infty$$

With this change,

$$f = 0.079\left[\frac{(2\delta)(0.817\ v_\infty)\rho}{\mu}\right]^{-1/4}$$

or

$$f = 0.0698\left(\frac{\delta v_\infty \rho}{\mu}\right)^{-1/4} \qquad ...(4.8.11)$$

We know that,

$$f = \frac{\tau_0}{\frac{1}{2}\rho v_{av}^2}$$

For turbulent flow using the relation between v_{av} and v_∞, we write,

$$f = \frac{2\tau_0}{\rho\,(0.817\ v_\infty)^2} \qquad ...(4.8.12)$$

Combining Eqs. (4.8.11) and (4.8.12), we obtain,

$$0.0698\left(\frac{\delta v_\infty \rho}{\mu}\right)^{-1/4} = \frac{2\tau_0}{\rho(0.817\, v_\infty)^2}$$

or

$$\frac{\tau_0}{\rho v_\infty^2} = 0.0233\left(\frac{\delta v_\infty \rho}{\mu}\right)^{-1/4} \qquad \text{...(4.8.13)}$$

From equations (4.8.8) and (4.8.13), we get,

$$\frac{d\delta}{dx} = \frac{72}{7}\left[0.0233\left(\frac{\delta v_\infty \rho}{\mu}\right)^{-1/4}\right]$$

or

$$\frac{d\delta}{dx} = 0.2396\left(\frac{\delta v_\infty \rho}{\mu}\right)^{-1/4} \qquad \text{...(4.8.14)}$$

Integrating Eq. (4.8.14) between $\delta = 0$ to $\delta = \delta$ and $x = 0$ to $x = L$, we get

$$\int_0^\delta \delta^{1/4}\, d\delta = 0.2396\left(\frac{v_\infty \rho}{\mu}\right)^{-1/4} \int_0^L dx$$

or

$$\delta = 0.376\left(\frac{L v_\infty \rho}{\mu}\right)^{-1/5} L$$

or

$$\delta = 0.376 (Re_L)^{-1/5} L \qquad \text{...(4.8.15)}$$

It has been observed that Eq. (4.8.15) is valid when $5\times 10^5 < Re_L < 10^7$. This restriction is due to the fact that Blasius empirical relation for pipe flow, based on which Eq. (4.8.15) has been developed, is not valid beyond $Re = 10^5$. Based on Eq. (4.8.15) expression for drag coefficient is obtained as

$$C_D = 0.072 (Re_L)^{1/5} \qquad \text{...(4.8.16)}$$

For higher Reynold's number the drag coefficient on the flat plate is given by,

$$C_D = \frac{0.455}{(\log Re_L)^{2.58}} \qquad \text{...(4.8.17)}$$

The above expression for C_D is obtained by assuming logarithmic velocity profile instead of 1/7th power law.

So far we have assumed that turbulent boundary layer starts to develop from the start of the plate. But actually over the first part of the plate laminar boundary layer develops and then turbulent boundary layer starts to grow. Eq. (4.8.17) is modified accordingly in Eq. (4.8.18).

$$C_D = \frac{0.455}{(\log Re_L)^{2.58}} - \frac{\beta}{Re_L} \qquad \text{...(4.8.18)}$$

SECTION I

This is known as Prandtl-Schlichting formula. β is dependent on transition Re_L as given in the Table 4.8.1

Table 4.8.1

Transition Re_L	β
3×10^5	1050
4×10^5	1700
1×10^6	3300
3×10^6	8700