*Subject In-charge: Prof. M.S. Chouhan*

**Semester: VII**

**Subject code – PC 702** **Subject: Transport Phenomena**

**Unit: -III**

## Lecture no.15

## Reynold's Flux components and eddy viscosity

### 1. Reynold's Flux components and eddy viscosity

As the Reynolds stress components bear the dimension of $N/m^2$, it is convenient to express them as a product of eddy viscosity and time-average velocity gradient which bears analogy with Newton's Law of viscosity. This model of eddy viscosity was proposed by Boussinesq (1877).

The eddy viscosity components in the $x$-direction are given by,

$$\rho(\overline{v'_x\, v'_x}) = -\mu^e_{xx}\frac{\partial \overline{v}_x}{\partial x} \quad ...(4.5.1)$$

$$\rho(\overline{v'_y\, v'_x}) = -\mu^e_{yx}\frac{\partial \overline{v}_x}{\partial y} \quad ...(4.5.2)$$

$$\rho(\overline{v'_z\, v'_x}) = -\mu^e_{zx}\frac{\partial \overline{v}_x}{\partial z} \quad ...(4.5.3)$$

Eddy viscosity $\mu^e$ is a strong function of flow and position hence is not a property of fluid like molecular viscosity $\mu$. The nine components of $\mu^e$ indicate that eddy viscosity in a tensor, and may be represent as follows :

$$\mu^e = \begin{bmatrix} \mu^e_{xx} & \mu^e_{yx} & \mu^e_{zx} \\ \mu^e_{xy} & \mu^e_{yy} & \mu^e_{zy} \\ \mu^e_{xz} & \mu^e_{yz} & \mu^e_{zz} \end{bmatrix} = \underbrace{\begin{bmatrix} \mu^e & 0 & 0 \\ 0 & \mu^e & 0 \\ 0 & 0 & \mu^e \end{bmatrix}}_{\text{isotropic turbulence}} + \underbrace{\begin{bmatrix} \mu^e & \mu^e & \mu^e \\ \mu^e & \mu^e & \mu^e \\ \mu^e & \mu^e & \mu^e \end{bmatrix}}_{\text{homogeneous turbulence}} \quad ...(4.5.4)$$

The turbulent shear stress may also be written in terms of eddy diffusivity $\nu^t$ and velocity gradient $\frac{d\overline{v}}{\partial y}$ as follows :

$$\rho(\overline{v'_y\, v'_x}) = \overline{\tau}^t_{yx} = -\mu^e_{yx}\frac{d\overline{v}_x}{dy} \quad ...(4.5.5)$$

or

$$\overline{\tau}^t_{yx} = -\rho\,\nu^e_{yx}\frac{d\overline{v}_x}{dy} \quad ...(4.5.6)$$

where, eddy diffusivity

$$= \nu^e = \frac{\mu^e}{\rho}\ (m^2/s) \quad ...(4.5.7)$$

Replacement of Reynold's stress terms with $\left(-\mu^e\frac{d\overline{v}}{dx}\right)$ or $\left(-\rho\nu^e\frac{d\overline{v}}{dx}\right)$ eliminates fluctuating components in Eq. (4.3.11), but a new unknown $\mu^e$ or $\nu^e$ is introduced. Eq. (4.3.11) now contains two similar terms

$$\mu^e\frac{d^2\overline{v}}{dx} \text{ and } \mu\frac{d^2 v}{dx}$$

one of which contains eddy viscosity and the other contains molecular viscosity. Eddy viscosity may be obtained from experimental observations or from semi-empirical approach. This is an important contribution of Boussinesq theory.

## 2. Prandtl's mixing length model

Prandtl's mixing length model is another approach to replace Reynold's stress terms with an expression containing a characteristic length scale of eddy and a velocity gradient term. The characteristic Length $L$ is called *Prandtl's mixing length*.

In the development of his mixing length model, Prandtl assumed random eddy motion as the basis ; just like random molecular motion is the basis of kinetic theory of gas. In the model, it is assumed that an eddy which has no preferential direction of motion moves through a distance L before it loses its identity by mixing or coalescing with neighbouring eddies.

According to Prandtl, a moving eddy travels a distance $L$ in $y$-direction, while the bulk flow is in $x$-direction. At the end of distance L it loses its identity by mixing with other eddies. The velocity fluctuation $v'_x$ is due to the movement of eddy. This mechanistic description is quite simplified, because an eddy being a lump of fluid loses its identity gradually during its travel through a distance $L$. As conceptually $L$ being the "Prandtl's mixing length", it is assumed that eddy character remains unchanged through this distance and mathematical formulation is done based on this assumption. In the formulation we assume,

| | |
|---|---|
| $\bar{v}_x\|_y$ | is the velocity of eddy at $y$ ; |
| $\bar{v}_x\|_{y+L}$ | is the velocity at some neighbouring point $L$ distance away in $y$-direction. |
| $v'_x\|_y$ | is the fluctuating component arising out of the difference in velocities |

$$\left(\bar{v}_x\big|_{y+L} - \bar{v}_x\big|_y\right)$$

So, we can write,

$$v'_x\big|_y = \bar{v}_x\big|_{y+L} - \bar{v}_x\big|_y \qquad \text{...(4.6.1)}$$



As $L$ is very small, Eq. (4.6.1) may be written as,

$$v'_x\big|_y = \bar{v}_x\big|_{y+L} - \bar{v}_x\big|_y = L\frac{d\bar{v}_x}{dy} \qquad \text{...(4.6.2)}$$

It follows in general from Eq. (4.6.2)

$$v'_x = L\frac{d\bar{v}_x}{dy} \qquad \text{...(4.6.3)}$$

Prandtl assumed isotropic turbulence. Which means, $v'_x \approx v'_y$ and time average quantity $\overline{v'_x v'_y}$ is,

$$\overline{v'_x v'_y} = -L^2 \left| \frac{d\bar{v}_x}{dy} \right| \frac{d\bar{v}_x}{dy} \qquad ...(4.6.4)$$

The term $\frac{d\bar{v}_x}{dy}$ takes care of the `sign' of the product term. Now it follows from Eq. (4.5.5) or Eq. (4.5.6) that,

$$\rho\,(\overline{v'_y v'_x}) = \bar{\tau}^t_{yx} = -L^2 \rho \left| \frac{d\bar{v}_x}{dy} \right| \frac{dv_x}{dy} \qquad ...(4.6.5)$$

Comparing Eq. (4.6.5) and Eq. (4.5.5) we get the definition of eddy viscosity as,

$$\mu^e_{yx} = L^2 \left| \frac{d\bar{v}_x}{dy} \right| \qquad ...(4.6.6)$$

...ary layer analysis.

**EXAMPLE 4.7.1** ***Relationship between average velocity and maximum velocity in turbulent pipe flow.***

*Assuming a general power low velocity profile derive an expression for average velocity for turbulent pipe flow. Hence find the ratio between $v_{av}$ and $v_{max}$ when $n=7$.*

**Solution.** Let us assume a general power law velocity profile.

$$\frac{v}{v_{max}} = \left(\frac{y}{R}\right)^{1/n}$$

where, $y$ is the distance from the wall.

$$\therefore \qquad y = R - r$$

The volumetric flow rate is given by,

$$Q = 2\pi R^2 v_{av} = \int_0^R 2\pi r v\, dr$$

$$= 2\pi v_{max} \int_0^R (R-y)\,(y/R)^{1/n}\,(-dy)$$

$$= \pi R^2 v_{max} \frac{2n^2}{(n+1)(2n+1)}$$

$$\therefore \qquad \frac{Q}{\pi R^2} = v_{av} = v_{max} \frac{2n^2}{(n+1)(2n+1)}$$

or

$$\frac{v_{av}}{v_{max}} = \frac{2n^2}{(n+1)(2n+1)}$$

For $n=7$,

$$\frac{v_{av}}{v_{max}} = 0.816$$

EXAMPLE 4.7.2 *Wall shear stress for turbulent pipe flow*

*Derive an expression for wall shear stress for turbulent flow through a smooth pipe.*

**Solution.** The wall shear stress is given by,

$$\tau_w = \frac{f}{2}\rho v_{av}^2$$

where $f$ is the Fanning friction factor. Blasius factor is $4f$. Assuming Blasius $1/7^{th}$ velocity law, shear stress is expressed as,

$$\tau_W = \frac{f_b}{8}\rho v_{av}^2 \qquad (f_b \text{ Blasius friction factor})$$

Blasius obtained $f_b = 0.3164\, Re^{-0.25}$

where, $Re = D v_{av}\, \rho / \mu$

Now using the value of $f_b$ in the expression of $\tau_W$, we get,

$$\tau_W = \frac{0.3164}{8} . Re^{-0.25}\, \rho v_{av}^2$$

$$= 0.03955\, \rho v_{av}^2 \left[\frac{\mu}{(2R)\, v_{av} \rho}\right]^{1/4}$$

$$= 0.03325\, \rho\, v_{av}^{7/4} \left(\frac{\mu}{R\rho}\right)^{1/4}$$

$$= 0.03325\, \rho \left(\frac{v_{av}}{v_{max}}\right)^{7/4} . (v_{max})^{7/4} . \left(\frac{\mu}{R\rho}\right)^{1/4}$$

$$= 0.03325\, \rho\, (0.816)^{7/4}\, v_{max}^2 \left(\frac{\mu}{R\rho\, v_{max}}\right)^{1/4}$$

$$\therefore \quad \tau_W = \mathbf{0.0225}\, \rho\, v_{max}^2 \left(\frac{\mu}{R\rho\, v_{max}}\right)^{1/4} \qquad (\text{Applicable for } Re < 10^5)$$

An expression for friction velocity now can be derived as,

$$v^{*2} = \frac{\tau_W}{\rho} = 0.0225\, v_{max}^2 \left(\frac{\mu}{R\rho v_{max}}\right)^{1/4}$$

where, $v^*$ is the friction velocity.

SECTION 1