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Unit: -IV
Lecture no.23

Heat conduction through composite walls

8 Heat Conduction through Composite Walls

(a) Series and Parallel Resistances

- The total resistance of resistances in series is

$$R_{tot} = \sum_i R_i$$

- The total resistance of resistances in parallel is

$$R_{tot} = \left(\sum_i R_i^{-1} \right)^{-1}$$

(b) Series Rectangular Composite

Problem: Consider a rectangular wall composed of three distinct materials. The left third ($x = x_0$ to $x = x_1$) is some arbitrary material 1, the middle ($x = x_1$ to $x = x_2$) is some arbitrary material 2, and the right third ($x = x_2$ to $x = x_3$) is some arbitrary material 3. Each has a unique k value. Find the effective thermal conductivity if $b_1 = x_1 - x_0$, $b_2 = x_2 - x_1$, and $b_3 = x_3 - x_2$.

1. The heat flux must be continuous. For instance, at $x = x_0$, $q_x = q_0$. At the x_1 interface, $q_0 = q_1$. At the x_2 interface, $q_1 = q_2$. At the x_3 interface, $q_2 = q_3$. Therefore, $q_x = q_1 = q_2 = q_3 = q_0$. The heat flux is a constant at each interface.

2. Since this is true, at region 1, 2, and 3,

$$q_0 = -k_{01} \frac{dT}{dx}$$

$$q_0 = -k_{12} \frac{dT}{dx}$$

$$q_0 = -k_{23} \frac{dT}{dx}$$

3. Integrating yields at region 1, 2, and 3,

$$T = -\frac{q_0}{k_{01}} x + C_1$$

$$T = -\frac{q_0}{k_{12}} x + C_2$$

$$T = -\frac{q_0}{k_{23}} x + C_3$$

4. At $x = x_0$, $T = T_0$, and at $x = x_1$, $T = T_1$ such that

$$T_0 - T_1 = \frac{q_0}{k_{01}}(x_1 - x_0)$$

5. Repeating this for the other regions yields

$$T_1 - T_2 = \frac{q_0}{k_{12}}(x_2 - x_1)$$

$$T_2 - T_3 = \frac{q_0}{k_{23}}(x_3 - x_2)$$

6. Adding the temperature from each region yields

$$T_0 - T_3 = q_0\left(\frac{b_1}{k_{01}} + \frac{b_2}{k_{12}} + \frac{b_3}{k_{23}}\right)$$

7. The heat flow can then be written as

$$Q = q_0 A = \frac{T_0 - T_3}{\left(\frac{b_1}{Ak_{01}} + \frac{b_2}{Ak_{12}} + \frac{b_3}{Ak_{23}}\right)} = \frac{\Delta T}{R_{eff}}$$

where

$$R_{eff} = \frac{b_1}{k_{01}A} + \frac{b_2}{k_{12}A} + \frac{b_3}{k_{23}A} = R_1 + R_2 + R_3$$

8. It is clear that the above system can be modeled with a circuit analog where there are three resistances in series!

9. Bringing the area term to the numerator,

$$Q = \frac{A\,(T_0 - T_3)}{\frac{b_1}{k_{01}} + \frac{b_2}{k_{12}} + \frac{b_3}{k_{23}}}$$

10. If we introduce k_{eff} as the effective heat transfer coefficient,

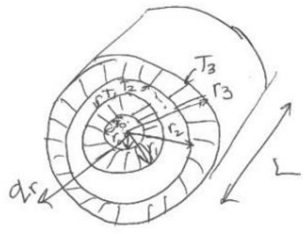
$$Q = \frac{k_{eff}A\,(T_0 - T_3)}{b_1 + b_2 + b_3}$$

11. Therefore,

$$k_{eff} = \left[\frac{1}{b_1 + b_2 + b_3}\left(\frac{b_1}{k_{01}} + \frac{b_2}{k_{12}} + \frac{b_3}{k_{23}}\right)\right]^{-1}$$

(c) Cylindrical Composite

Problem: There are three cylindrical shells surrounding one another. The inner temperature of the first surface from the center is T_1 and is at r_1. The inner temperature of the second surface (outer temperature of the first surface) is T_2 and is at r_2. The inner temperature of the third surface (outer temperature of the second surface) is T_3 and is at r_3. The cylinder has a length L. The heat flux is solely radial. Model the effective thermal resistance. A rough sketch is shown below to help visualize the scenario:

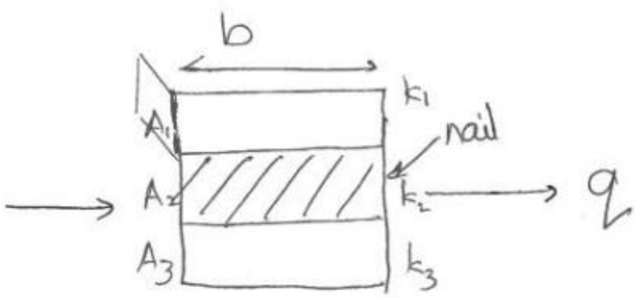


Solution: Using the definition of R_{th} from earlier in cylindrical coordinates and realizing that this is a system in series,

$$R_{eff} = R_1 + R_2 + R_3 = \frac{\ln(r_1/r_0)}{2\pi k_{01} L} + \frac{\ln(r_2/r_1)}{2\pi k_{12} L} + \frac{\ln(r_3/r_2)}{2\pi k_{23} L}$$

(d) Parallel Rectangular Composite

Problem: Consider the following system. Find R_{eff} in terms of k_{eff}.



Solution:

- Since there are now three components in parallel,

$$R_{eff} = \left(R_1^{-1} + R_2^{-1} + R_3^{-1}\right)^{-1} = b\left(\frac{1}{k_{01}A_1} + \frac{1}{k_{12}A_2} + \frac{1}{k_{23}A_3}\right)$$

- Introducing a k_{eff} yields

$$R_{eff} = \frac{b}{k_{eff}\left(A_1 + A_2 + A_3\right)}$$

- To figure out what k_{eff} is equal to, substitute back in for R_{eff}:

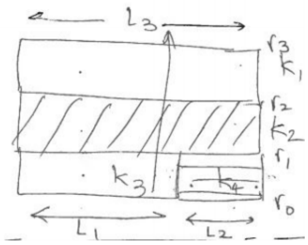
$$\frac{k_{eff}\left(A_1 + A_2 + A_3\right)}{b} = \frac{k_{01}A_1}{b} + \frac{k_2 A_2}{b} + \frac{k_3 A_3}{b}$$

- Therefore,

$$k_{eff} = \frac{\left(k_{01}A_1 + k_{12}A_2 + k_{23}A_3\right)}{A_1 + A_2 + A_3}$$

(e) Series and Parallel Cylindrical Composite

Problem: Consider the following system. Find R_{eff} in terms of k_{eff}. The system below is a rectangular cross-section of a cylinder. Therefore, r_0 is the innermost radius, and r_3 is the outermost radius.



Solution:

- Elements 1 and 2 are in series. Elements 3 and 4 are in parallel. The equivalent of elements 3 and 4 are in series with elements 1 and 2. Therefore,

$$R_{eq-34} = \left(R_3^{-1} + R_4^{-1}\right)^{-1}$$

  and

$$R_{eff} = R_1 + R_2 + R_{eq-34}$$

- Using the cylindrical definitions defined earlier,

$$R_{eff} = \frac{\ln(r_1/r_0)}{2\pi k_1 L_3} + \frac{\ln(r_2/r_1)}{2\pi k_2 L_3} + \frac{\ln(r_3/r_2)}{2\pi k_{eq-34} L_3}$$

- Introducing k_{eff} yields

$$R_{eff} = \frac{\ln(r_3/r_0)}{2\pi k_{eff} L_3}$$

- Using the general definition of $Q = \dfrac{\Delta T}{R_{eff}}$,

$$Q = \frac{\Delta T}{\left(\dfrac{\ln(r_3/r_0)}{2\pi k_{eff} L_3}\right)}$$

- To find out what k_{eff} is in this equation, substitute back in for R_{eff}

$$\frac{\ln(r_3/r_0)}{2\pi k_{eff} L_3} = \frac{\ln(r_3/r_2)}{2\pi k_1 L_3} + \frac{\ln(r_2/r_1)}{2\pi k_2 L_3} + \frac{\ln(r_1/r_0)}{2\pi k_{eq-34} L_3}$$

- This simplifies to

$$k_{eff} = \frac{\ln(r_3/r_0)}{\dfrac{\ln(r_3/r_2)}{k_1} + \dfrac{\ln(r_2/r_1)}{k_2} + \dfrac{\ln(r_1/r_0)}{k_{eq-34}}}$$

- However, we have not yet defined k_{eq-34} yet. To find k_{eq-34}, substitute back in for R_{eq-34}. As such,

$$\frac{2\pi k_{eq-34} L_3}{\ln(r_1/r_0)} = \frac{2\pi k_3 L_1}{\ln(r_1/r_0)} + \frac{2\pi k_4 L_2}{\ln(r_1/r_0)}$$

$$k_{eq-34} = \frac{k_3 L_1 + k_4 L_2}{L_1 + L_2} = \frac{k_3 L_1 + k_4 L_2}{L_3}$$

9 Newton's Law of Cooling

(a) Definitions

- Newton's law of cooling states the following where T_s is solid surface temperature and T_∞ is the bulk liquid temperature

$$q = h\left(T_s - T_\infty\right)$$

  - The constant h is the heat transfer coefficient

- The biot number (dimensionless) is defined as

$$\mathrm{Bi} = \frac{bh}{k} = \frac{\text{heat transfer by fluid}}{\text{heat transfer by solid}}$$

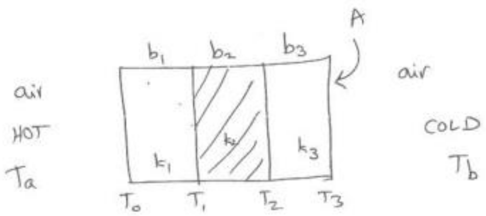
(b) Liquid Bound on One Side

Problem: Consider a bulk liquid at temperature T_∞ bounded on one side by a solid wall at temperature T_s. Derive R_{th}.

1. We know that $Q = \dfrac{\Delta T}{R_{th}}$
2. Using Newton's law of cooling $Q = qA = h\left(T_s - T_\infty\right)A$
3. Therefore, $R_{th} = \dfrac{1}{Ah}$ to make $Q = \dfrac{\Delta T}{R_{th}}$

(c) Solid Bound by Two Different Temperature Fluids - Rectangular

Problem: Solve the Series Rectangular Composite problem except for now there is a fluid bounding both sides of the rectangular composite (depicted below):



1. Using the same procedure as in the Series Rectangular Composite question, the following is true

$$T_0 - T_1 = \frac{q_0}{k_{01}} b_1$$

$$T_1 - T_2 = \frac{q_0}{k_{12}} b_2$$

$$T_2 - T_3 = \frac{q_0}{k_{23}} b_3$$

2. Now, we must consider the heat transfer at the solid-liquid interfaces as well. As such,

$$q_0 = h_a\left(T_a - T_0\right)$$

$$q_0 = h_b\left(T_3 - T_b\right)$$

3. The sum of these equations yields

$$T_a - T_b = q_0\left(\frac{b_1}{k_{01}} + \frac{b_2}{k_{12}} + \frac{b_3}{k_{23}} + \frac{1}{h_a} + \frac{1}{h_b}\right)$$

4. Therefore,

$$Q = q_0 A = \frac{T_a - T_b}{\dfrac{1}{A}\left(\dfrac{b_1}{k_{01}} + \dfrac{b_2}{k_{12}} + \dfrac{b_3}{k_{23}} + \dfrac{1}{h_a} + \dfrac{1}{h_b}\right)}$$

5. Since $Q = \dfrac{\Delta T}{R_{eff}}$, for the solid region, $R_{th} = \dfrac{b}{kA}$ as before, and the fluid region is $\dfrac{1}{hA}$

(d) General Equation

- To generalize the previous examples, a solid rectangular system bounded by fluid can be described by

$$Q = \frac{T_{hot} - T_{cold}}{\frac{1}{A}\left(\frac{1}{h_0} + \sum_{j=1}^{n} \frac{x_j - x_{j-1}}{k_{j-1,j}} + \frac{1}{h_n}\right)}$$

- A solid cylindrical system bounded by fluid can be described by

$$Q = \frac{T_{hot} - T_{cold}}{\frac{1}{2\pi L}\left(\frac{1}{r_0 h_0} + \sum_{j=1}^{n} \frac{\ln\left(r_j/r_{j-1}\right)}{k_{j-1,j}} + \frac{1}{r_n h_n}\right)}$$

- If there is no surrounding fluid to be considered, the terms with h can be dropped

9 The Equations of Change for Nonisothermal Systems

(a) The Energy Equation

- The general form of the energy equation states that

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho\hat{U}\right) = -\nabla\left[\left(\frac{1}{2}\rho v^2 + \rho\hat{U}\right)\vec{v}\right] - \nabla\cdot\vec{q} - \nabla\cdot P\vec{v} - \nabla\left(\boldsymbol{\tau} : \vec{v}\right) + \rho\left(\vec{v}\cdot\vec{g}\right)$$

- The equation of change of temperature states that

$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t} + \vec{v}\cdot\nabla T\right) = -\nabla\cdot\vec{q} - \left(\frac{\partial \ln\rho}{\partial \ln T}\right)_p\left(\frac{\partial T}{\partial t} + \vec{v}\cdot\nabla T\right) - \boldsymbol{\tau} : \nabla\vec{v}$$

- The following relationships also hold

$$\hat{H} = \hat{U} + \frac{P}{\rho}$$

$$d\hat{H} = \hat{C}_p dT$$