UIT-RGPV (Autonomous) Bhopal
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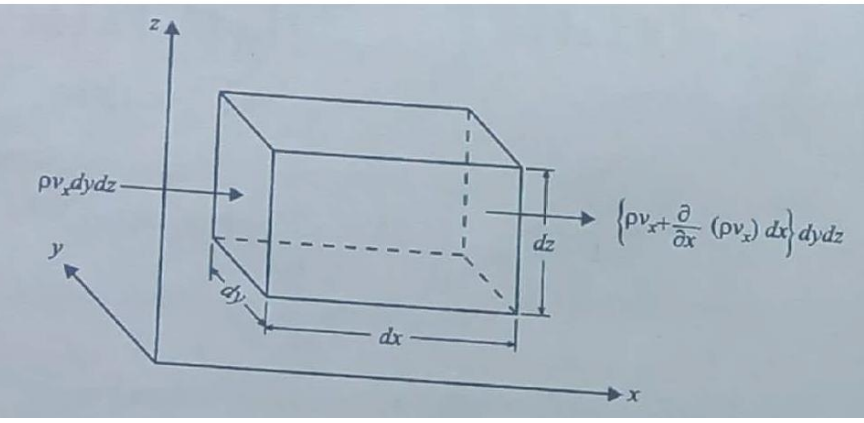
The continuity Equation

1. The continuity Equation (Conservation of Mass)

We now apply the principle of conservation of mass to a differential control volume $dx\,dy\,dz$ through which fluid flow takes place. This volume is an imaginary volume fixed in position and offering no resistance of any kind to the flow. We choose a Cartesian coordinate system (x, y, z) and the three velocity components in x, y and z directions are v_x, v_y and v_z respectively. The three velocity components and the density of the fluid, ρ are functions of position and time.

Fig. 3.1.1 Control volume, $dx.dy.dz$, fixed in a position through which a fluid is flowing.



The flow of mass per unit time and per unit area through a surface is the product of density and the velocity normal to the surface. Thus the x-component of the mass flux entering the surface at x is ρv_x. This flux changes from point to point. The x-component of the mass flux leaving the surface at $x + dx$ is $\rho v_x + \frac{\partial}{\partial x}(\rho v_x)\,dx$. Similar expressions may be written for other two pairs of faces. The gross rate of mass inflow into the control volume is

$$\rho v_x\,dy\,dz + \rho v_y\,dx\,dz + \rho v_z\,dx\,dy$$

The gross rate of mass outflow is

$$\left\{\rho v_x + \frac{\partial}{\partial x}(\rho v_x)\,dx\right\} dydz + \left\{\rho v_y + \frac{\partial}{\partial y}(\rho v_y)\,dy\right\} dxdz + \left\{\rho v_z + \frac{\partial}{\partial z}(\rho v_z)\,dz\right\} dxdy$$

The net rate of mass inflow, found by subtracting the outflow from the inflow for three directions, is

$$\left[\rho v_x - \left\{\rho v_x + \frac{\partial}{\partial x}(\rho v_x)\,dx\right\}\right]dydz + \left[\rho v_y - \left\{\rho v_y + \frac{\partial}{\partial y}(\rho v_y)\,dy\right\}\right]dxdz$$

$$+\left[\rho v_z - \left\{\rho v_z + \frac{\partial}{\partial z}(\rho v_z)\,dz\right\}\right]dxdy$$

$$= -\left[\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z)\right]dxdydz$$

and this must equal the rate of mass increase in the control volume, $\frac{\partial \rho}{\partial t}\, dx\, dy\, dz$.

Equating the expression for net rate of mass inflow in three directions with rate of mass increase in the control volume and then dividing by the control volume, *dx dy dz*, we obtain

$$\frac{\partial \rho}{\partial t} = -\left\{\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z)\right\} \quad \text{...(3.}$$

Eq. (3.1.1), called the continuity equation, is a general expression of the over mass conservation requirement and it must be satisfied at every point in the flow field. The continuity equation (Eq. 3.1.1) describes the rate of change density at a fixed point resulting from changes in the mass flux *i.e.*, the mass velocity vector $\rho\vec{v}$. Note that mass enters and leaves the control volume exclusively through gross fluid motion. Transport due to such motion is o referred to as advection. In case of a binary mixture ρ is the total mass density $(\rho_A + \rho_B)$ and v_x, v_y, v_z are the $x-$, $y-$, and $z-$component of the m average velocity, respectively, of the mixture. Equation (3.1.1) applies to a pure fluid or to a fluid mixture as a whole at a given point but it does apply to individual species in the mixture. Eq. (3.1.1) may be written in vector form as

$$\frac{\partial \rho}{\partial t} = -(\nabla \cdot \rho\vec{v}) \quad \text{...(3.}$$

where $(\nabla \cdot \rho\vec{v})$ is the divergence of $\rho\vec{v}$, often written as $div\rho\vec{v}$. The divergence of mass flux vector or mass velocity vector $\rho\vec{v}$ signifies the net rate of m efflux per unit volume. Eq. (3.1.2), then, states that the rate of change of to density within a small volume element fixed in space is equal to the net rate of mass influx to the element divided by its volume.

Equation (3.1.1) may be written in another form by performing the indicated differentiation and collecting all derivatives of density ρ on one side as follows :

$$\frac{\partial \rho}{\partial t}+v_x \frac{\partial \rho}{\partial x}+v_y \frac{\partial \rho}{\partial y}+v_z \frac{\partial \rho}{\partial z}=-\rho\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right) \quad ...(3.1.3)$$

The left side of Eq. (3.1.3) is called the substantial derivative or material derivative $\frac{D\rho}{Dt}$ of total density ρ and it represents the change in ρ with time that occurs when an observer moves with the velocity of the stream. The first term on the left represents the change that occurs at a fixed point and the sum of the three other terms on the left side represents the change due to the motion of the fluid.

A balance equation written in terms of $\frac{D}{Dt}$ is said to be in the Lagrangian form, to distinguish it form an equation in Eulerian form, which describes changes that occur at a fixed point in space.

The continuity equation (3.1.3) in Lagrangian form then becomes :

$$\frac{D\rho}{Dt}=-\rho\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right) \quad ...(3.1.4)$$

or in vector form :

$$\frac{D\rho}{Dt}=-\rho(\nabla . \vec{v}) \quad ...(3.1.5)$$

The continuity equation (3.1.4) or (3.1.5) describes the rate of change of density as seen by an observer floating along with the fluid.

For an incompressible fluid, the mass density ρ is constant and all derivatives of density are zero. Hence for an incompressible fluid,

$$\frac{\partial v_x}{dx}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}=0 \quad ...(3.1.6)$$

Or in vector form,

$$\nabla . \vec{v}=0 \quad ...(3.1.7)$$

The assumption of constant density results in considerable simplification with negligible error. Equation (3.1.6) is valid whether the velocity is time dependent or not. A flow for which all velocities and properties at a given location are independent of time is called steady. It follows that for a steady flow all partial derivatives with respect to time are equal to zero. For Eq. (3.1.6) or (3.1.7) to be valid, it is necessary only that ρ remain constant for a fluid element as it moves along a streamline i.e., that $\frac{D\rho}{Dt}=0$.