

MOMENTUM BALANCE

For steady-flow, the momentum balance is:

$$\left\{ \begin{array}{l} \text{rate-of} \\ \text{momentum-in} \\ \text{by-con vective} \\ \text{transport} \end{array} \right\} - \left\{ \begin{array}{l} \text{rate-of} \\ \text{momentum-out} \\ \text{by-con vective} \\ \text{transport} \end{array} \right\} + \left\{ \begin{array}{l} \text{rate-of} \\ \text{momentum-in} \\ \text{by-molecular} \\ \text{transport} \end{array} \right\} - \left\{ \begin{array}{l} \text{rate-of} \\ \text{momentum-out} \\ \text{by-molecular} \\ \text{transport} \end{array} \right\} + \{ \text{gravity} \} = 0$$

The expressions to be used have already been discussed and given.

The procedure goes as follows.

- Identify non-vanishing velocity components
- Write momentum balance
- Take limits for vanishingly small shell dimensions
- Integrate equation
- Insert Newton's law of viscosity
- Integrate equation again
- Use the velocity distribution to calculate macroscopic quantities

NON-NEWTONIAN FLUIDS

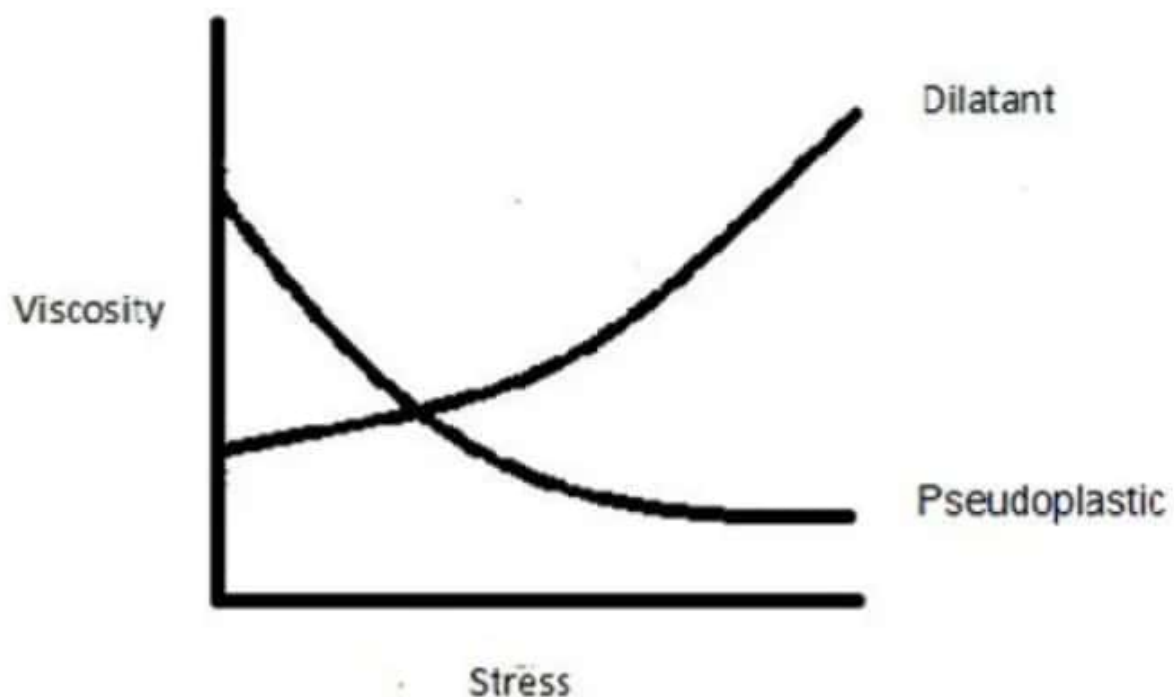
You can probably guess that non-Newtonian fluids are the opposite of Newtonian fluids. When shear is applied to non-Newtonian fluids, the viscosity of the fluid changes. The behavior of the fluid can be described one of four ways:

- **Dilatant** - Viscosity of the fluid increases when shear is applied. For example:
 - Quicksand
 - Cornflour and water
 - Silly putty
- **Pseudoplastic** - Pseudoplastic is the opposite of dilatant; the more shear applied, the less viscous it becomes. For

example:

- Ketchup

This chart shows how viscosity changes in respect to the amount of shear or stress applied to the fluid.



- **Rheopectic** - Rheopectic is very similar to dilatant in that when shear is applied, viscosity increases. The difference here, is that viscosity increase is time-dependent. For example:

- Gypsum paste
 - Cream
- **Thixotropic** - Fluids with thixotropic properties decrease in viscosity when shear is applied. This is a time dependent property as well. For example:
 - Paint
 - Cosmetics
 - Asphalt
 - Glue

EXAMPLE 2.2-1**Calculation of Film Velocity**

An oil has a kinematic viscosity of $2 \times 10^{-4} \text{ m}^2/\text{s}$ and a density of $0.8 \times 10^3 \text{ kg/m}^3$. If we want to have a falling film of thickness of 2.5 mm on a vertical wall, what should the mass rate of flow of the liquid be?

SOLUTION

According to Eq. 2.2-21, the mass rate of flow in kg/s is

$$w = \frac{\rho g \delta^3 W}{3\nu} = \frac{(0.8 \times 10^3)(9.80)(2.5 \times 10^{-3})^3 W}{3(2 \times 10^{-4})} = 0.204 W \quad (2.2-24)$$

To get the mass rate of flow one then needs to insert a value for the width of the wall in meters. This is the desired result provided that the flow is laminar and nonrippling. To determine the flow regime we calculate the Reynolds number, making use of Eqs. 2.2-21 and 24

$$\text{Re} = \frac{4\delta \langle v_z \rangle \rho}{\mu} = \frac{4w/W}{\nu\rho} = \frac{4(0.204)}{(2 \times 10^{-4})(0.8 \times 10^3)} = 5.1 \quad (2.2-25)$$

This Reynolds number is sufficiently low that rippling will not be pronounced, and therefore the expression for the mass rate of flow in Eq. 2.2-24 is reasonable.

THEORETICAL BACKGROUND

Forms of derivatives :

(1) Partial time derivative, $\frac{\partial c}{\partial t}$.

This means variation of c (the variable) with time with respect to a fixed position (x, y, z) in space.

(2) Total time derivative, $\frac{dc}{dt}$

$$\frac{dc}{dt} = \frac{\partial c}{\partial t} + \frac{\partial c}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial c}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial c}{\partial z} \cdot \frac{dz}{dt}$$

where, $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dz}{dt}$ are the variation of x , y and z respectively with respect to time.

(3) Substantial time derivative, $\frac{Dc}{Dt}$

It is a special kind of total time derivative also, called 'derivative following the motion'. The expression is

$$\frac{Dc}{Dt} = \frac{\partial c}{\partial t} + v_x \frac{\partial c}{\partial x} + v_y \frac{\partial c}{\partial y} + v_z \frac{\partial c}{\partial z}$$

where, v_x , v_y and v_z are components of local fluid velocity, v .

Equation of continuity :

This is based on conservation of mass. According to this,

Rate of change of density = Divergence of mass flux

Mathematically,

$$\frac{\partial \rho}{\partial t} = -(\nabla \cdot \rho v) \quad \dots(1)$$

or

$$\frac{D\rho}{Dt} = -\rho (\nabla \cdot v)$$

For steady state,

$$\frac{D\rho}{Dt} = 0 \quad \dots(2)$$

$$\begin{cases} \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = 0 \\ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{\nabla p}{\rho} = \mathbf{g} \\ \frac{\partial e}{\partial t} + \mathbf{u} \cdot \nabla e + \frac{p}{\rho} \nabla \cdot \mathbf{u} = 0 \end{cases}$$

The **Euler** equations can be applied to incompressible and to compressible flow – assuming the flow velocity is a solenoidal field, or using another appropriate energy **equation** respectively (the simplest form for **Euler** equations being the conservation of the specific entropy).

y-component :

$$\rho \frac{Dv_y}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x} \left[\mu \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right) \right] + \frac{\partial}{\partial y} \left[2\mu \frac{\partial v_y}{\partial y} - \frac{2}{3} \mu (\nabla \cdot \vec{v}) \right] + \frac{\partial}{\partial z} \left[\mu \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right) \right] + \rho g_y \quad \dots(3.2.17)$$

z-component :

$$\rho \frac{Dv_z}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x} \left[\mu \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right) \right] + \frac{\partial}{\partial y} \left[\mu \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right) \right] + \frac{\partial}{\partial z} \left[2\mu \frac{\partial v_z}{\partial z} - \frac{2}{3} \mu (\nabla \cdot \vec{v}) \right] + \rho g_z \quad \dots(3.2.18)$$

These equations are called the Navier-Stokes Equations. These equations can be simplified for the case of incompressible flow with constant viscosity because the continuity equation for incompressible fluid viz. $(\nabla \cdot \vec{v}) = 0$.

Thus for constant ρ and constant μ , the Navier-Stokes Equations become

x :

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \rho g_x \quad \dots(3.2.19)$$

y :

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) + \rho g_y \quad \dots(3.2.20)$$

z :

$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z \quad \dots(3.2.21)$$

Adding these three components vertically, the Navier-Stokes equations for fluid of constant density and viscosity become,

$$\rho \frac{D\vec{v}}{Dt} = -\nabla p + \mu \nabla^2 \vec{v} + \rho \vec{g} \quad \dots(3.2.22)$$

Euler's Equation

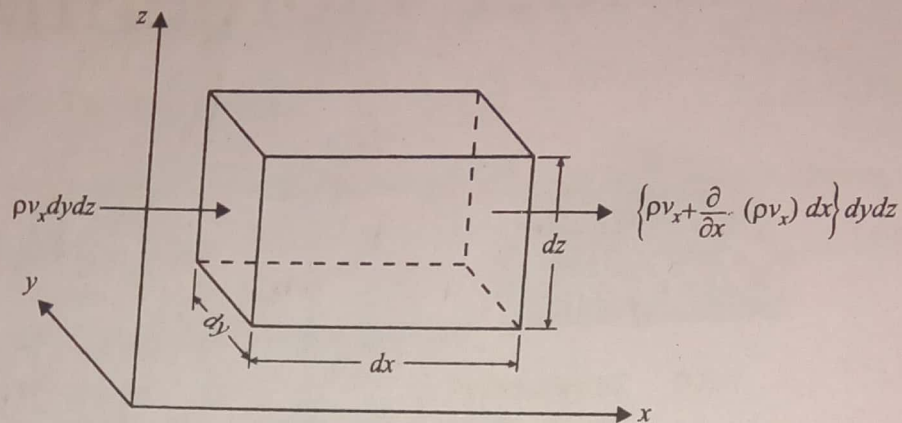
For the case of frictionless flow $\mu = 0$ and thus $[\nabla \cdot \tau] = 0$, the equation of motion (3.2.9) or (3.2.22) reduces to

$$\rho \frac{D\vec{v}}{Dt} = -\nabla p + \rho \vec{g} \quad \dots(3.2.23)$$

3.1 THE CONTINUITY EQUATION (Conservation of Mass)

We now apply the principle of conservation of mass to a differential control volume $dx\,dy\,dz$ through which fluid flow takes place. This volume is an imaginary volume fixed in position and offering no resistance of any kind to the flow. We choose a Cartesian coordinate system (x, y, z) and the three velocity components in x, y and z directions are v_x, v_y and v_z respectively. The three velocity components and the density of the fluid, ρ are functions of position and time.

Fig. 3.1.1
Control volume, dx, dy, dz , fixed in a position through which a fluid is flowing.



The flow of mass per unit time and per unit area through a surface is the product of density and the velocity normal to the surface. Thus the x -component of the mass flux entering the surface at x is ρv_x . This flux changes from point to point. The x -component of the mass flux leaving the surface at $x + dx$ is $\rho v_x + \frac{\partial}{\partial x}(\rho v_x) dx$. Similar expressions may be written for other two pairs of faces. The gross rate of mass inflow into the control volume is

$$\rho v_x dy dz + \rho v_y dx dz + \rho v_z dx dy$$

The gross rate of mass outflow is

$$\left\{ \rho v_x + \frac{\partial}{\partial x}(\rho v_x) dx \right\} dy dz + \left\{ \rho v_y + \frac{\partial}{\partial y}(\rho v_y) dy \right\} dx dz + \left\{ \rho v_z + \frac{\partial}{\partial z}(\rho v_z) dz \right\} dx dy$$

The net rate of mass inflow, found by subtracting the outflow from the inflow for three directions, is

$$\begin{aligned} & \left[\rho v_x - \left\{ \rho v_x + \frac{\partial}{\partial x} (\rho v_x) dx \right\} \right] dy dz + \left[\rho v_y - \left\{ \rho v_y + \frac{\partial}{\partial y} (\rho v_y) dy \right\} \right] dx dz \\ & + \left[\rho v_z - \left\{ \rho v_z + \frac{\partial}{\partial z} (\rho v_z) dz \right\} \right] dx dy \\ & = - \left[\frac{\partial}{\partial x} (\rho v_x) + \frac{\partial}{\partial y} (\rho v_y) + \frac{\partial}{\partial z} (\rho v_z) \right] dx dy dz \end{aligned}$$

and this must equal the rate of mass increase in the control volume, $\frac{\partial \rho}{\partial t} dx dy dz$.

Equating the expression for net rate of mass inflow in three directions with rate of mass increase in the control volume and then dividing by the control volume, $dx dy dz$, we obtain

$$\frac{\partial \rho}{\partial t} = - \left\{ \frac{\partial}{\partial x} (\rho v_x) + \frac{\partial}{\partial y} (\rho v_y) + \frac{\partial}{\partial z} (\rho v_z) \right\} \quad \dots(3.1.1)$$

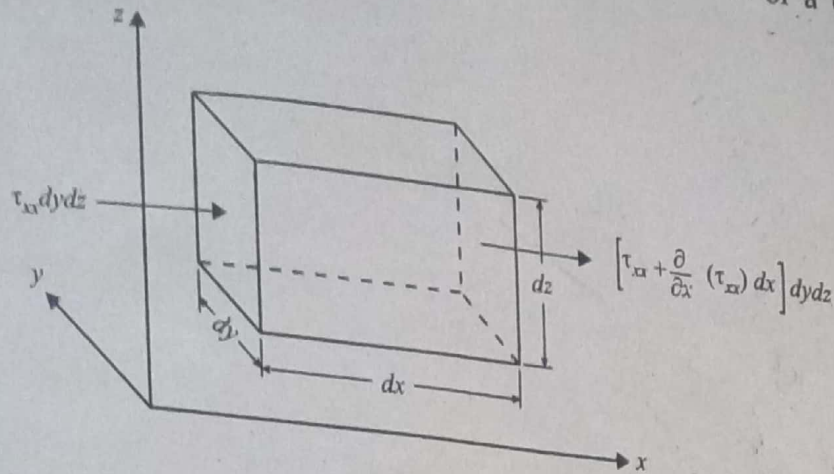
Eq. (3.1.1), called the continuity equation, is a general expression of the overall mass conservation requirement and it must be satisfied at every point in the

EQUATION OF MOTION (Conservation of Momentum)

Consider a differential control volume, $dx dy dz$, in the form of a cube as shown in Fig. (3.2.1).

Fig. 3.2.1

Control volume
 dx, dy, dz for
momentum
balance



Momentum enters and leaves the control volume by two mechanisms advection and molecular transfer. These enter through three faces and leave through three opposite faces.

The gross rate at which x -component of flow momentum enters into the control volume is

$$\rho v_x v_x dy dz + \rho v_y v_x dx dz + \rho v_z v_x dy dx$$

The gross rate at which x -component of flow momentum leaves the control volume is

$$\left[\rho v_x v_x + \frac{\partial}{\partial x}(\rho v_x v_x) dx \right] dy dz + \left[\rho v_y v_x + \frac{\partial}{\partial y}(\rho v_y v_x) dy \right] dx dz + \left[\rho v_z v_x + \frac{\partial}{\partial z}(\rho v_z v_x) dz \right] dx dy$$

The net rate at which x -component of flow momentum enters the control volume, therefore, is

$$-\left[\frac{\partial}{\partial x}(\rho v_x v_x) + \frac{\partial}{\partial y}(\rho v_y v_x) + \frac{\partial}{\partial z}(\rho v_z v_x) \right] dx dy dz$$

The gross rate at which x -component of momentum enters the control volume by molecular transport is

$$\tau_{xx} dy dz + \tau_{yx} dx dz + \tau_{zx} dx dy$$

The gross rate at which x -component of momentum leaves the control volume by molecular transport is

$$\left[\tau_{xx} + \frac{\partial}{\partial x}(\tau_{xx}) dx \right] dy dz + \left[\tau_{yx} + \frac{\partial}{\partial y}(\tau_{yx}) dy \right] dx dz + \left[\tau_{zx} + \frac{\partial}{\partial z}(\tau_{zx}) dz \right] dx dy$$

The net rate at which x -component of momentum enters the control volume by molecular transport, therefore, is

$$-\left[\frac{\partial}{\partial x} (\tau_{xx}) + \frac{\partial}{\partial y} (\tau_{yx}) + \frac{\partial}{\partial z} (\tau_{zx}) \right] dx dy dz$$

If pressure and gravitational forces are the only forces acting on the control volume, then the resultant of these two forces in the x -direction is

$$\left\{ p - \left(p + \frac{\partial p}{\partial x} dx \right) \right\} dy dz + \rho g_x dx dy dz$$

where g_x is the x -component of acceleration due to gravity.

The rate at which x -component of momentum is accumulated within the control volume is

$$\frac{\partial}{\partial t} (\rho v_x) dx dy dz$$

Substituting these terms in the general statement of conservation of momentum, Eq. (1.1.4), and dividing both sides by control volume, we obtain,

$$\begin{aligned} \frac{\partial}{\partial t} (\rho v_x) = - \left\{ \frac{\partial}{\partial x} (\rho v_x^2) + \frac{\partial}{\partial y} (\rho v_y v_x) + \frac{\partial}{\partial z} (\rho v_z v_x) \right\} \\ - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) - \frac{\partial p}{\partial x} + \rho g_x \end{aligned} \quad \dots(3.2.1)$$

Equation (3.2.1) is the x -component of the equation of motion.

Similarly, we obtain the y and z -components of the equation of motion which, respectively, are

$$\begin{aligned} \frac{\partial}{\partial t} (\rho v_y) = - \left\{ \frac{\partial}{\partial x} (\rho v_x v_y) + \frac{\partial}{\partial y} (\rho v_y^2) + \frac{\partial}{\partial z} (\rho v_z v_y) \right\} \\ - \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) - \frac{\partial p}{\partial y} + \rho g_y \end{aligned} \quad \dots(3.2.2)$$

$$\begin{aligned} \frac{\partial}{\partial t} (\rho v_z) = - \left\{ \frac{\partial}{\partial x} (\rho v_x v_z) + \frac{\partial}{\partial y} (\rho v_y v_z) + \frac{\partial}{\partial z} (\rho v_z^2) \right\} \\ - \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right) - \frac{\partial p}{\partial z} + \rho g_z \end{aligned} \quad \dots(3.2.3)$$

The quantities $\rho v_x, \rho v_y, \rho v_z$ are the components of the mass velocity vector $\rho \vec{v}$, g_x, g_y, g_z are the components of the gravitational acceleration \vec{g} and $\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}$ are

the components of the ∇p , known as the gradient of p , often written as $\text{grad } p$. The terms $\rho v_x v_x$, $\rho v_x v_y$, $\rho v_x v_z$, etc. are the nine components of momentum flux due to advection, $\rho \vec{v} \cdot \vec{v}$. Similarly, τ_{xx} , τ_{xy} , τ_{xz} , etc. are the nine components of the stress tensor, τ . Eqs. (3.2.1), (3.2.2), (3.2.3) may be combined to give a single vector equation :

$$\frac{\partial}{\partial t} (\rho \vec{v}) = -[\nabla \cdot \rho \vec{v} \vec{v}] - \nabla p - [\nabla \cdot \tau] + \rho \vec{g} \quad \dots(3.2.4)$$

Equation (3.2.1) may be rearranged after performing indicated differentiation in the following form :

$$\begin{aligned} \rho \left[\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right] + v_x \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho v_x) + \frac{\partial}{\partial y} (\rho v_y) + \frac{\partial}{\partial z} (\rho v_z) \right] \\ = - \left(\frac{\partial}{\partial x} \tau_{xx} + \frac{\partial}{\partial y} \tau_{yx} + \frac{\partial}{\partial z} \tau_{zx} \right) - \frac{\partial p}{\partial x} + \rho g_x \quad \dots(3.2.5) \end{aligned}$$

Eq. (3.2.5) may be simplified by using the continuity equation (3.1.1) to give

$$\rho \frac{Dv_x}{Dt} = - \frac{\partial p}{\partial x} - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) + \rho g_x \quad \dots(3.2.6)$$

Similarly, Eqs. (3.2.2) and (3.2.3) may be rearranged to give, respectively,

$$\rho \frac{Dv_y}{Dt} = - \frac{\partial p}{\partial y} - \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) + \rho g_y \quad \dots(3.2.7)$$

$$\rho \frac{Dv_z}{Dt} = - \frac{\partial p}{\partial z} - \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right) + \rho g_z \quad \dots(3.2.8)$$

When these three components of the equation of motion are added vertically, we obtain,

$$\rho \frac{D\vec{v}}{Dt} = - \nabla p - [\nabla \cdot \tau] + \rho \vec{g} \quad \dots(3.2.9)$$

The term on the left is mass times acceleration, where acceleration is expressed by the substantial derivative of velocity to account for the motion of system. The first, second and third terms on the right represent the forces, due respectively, to pressure, fluid friction and gravitational acceleration on the control volume. The equation of motion as written in the form of Eq.-(3.2.9) states that a small volume element moving with the fluid is accelerated because of the forces acting upon it. In other words Eq. (3.2.9) is a statement of Newton's second law of motion i.e., mass times acceleration equals sum of the forces.

The next and most vital step involves the relationship between the stresses and the motion of the fluid. The stresses in many fluids are related linearly to the derivatives of the velocities. Most fluids are isotropic i.e., these fluids have no preferred direction in space. The stresses do not explicitly depend on the position coordinate and the fluid velocities. Also the pairs of stresses with subscript which differ only in their order are equal. This follows by taking moments about any axis and using the condition of equilibrium of the differential control volume. Thus, for example, for the z-axis, we obtain

$$\tau_{xy} dy dz dx = \tau_{yx} dx dz dy$$

Hence $\tau_{xy} = \tau_{yx}$ and similarly,

$$\tau_{xz} = \tau_{zx} \text{ and } \tau_{yz} = \tau_{zy}$$

For Newtonian fluids, the relationship between stresses and velocity gradients are :

$$\tau_{xx} = -2\mu \frac{\partial v_x}{\partial x} + \frac{2}{3} \mu (\nabla \cdot \vec{v}) \quad \dots(3.2.10)$$

$$\tau_{yy} = -2\mu \frac{\partial v_y}{\partial y} + \frac{2}{3} \mu (\nabla \cdot \vec{v}) \quad \dots(3.2.11)$$

$$\tau_{zz} = -2\mu \frac{\partial v_z}{\partial z} + \frac{2}{3} \mu (\nabla \cdot \vec{v}) \quad \dots(3.2.12)$$

$$\tau_{xy} = \tau_{yx} = -\mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) \quad \dots(3.2.13)$$

$$\tau_{yz} = \tau_{zy} = -\mu \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right) \quad \dots(3.2.14)$$

$$\tau_{zx} = \tau_{xz} = -\mu \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right) \quad \dots(3.2.15)$$

These equations constitute a more general statement of Newton's law of viscosity than that given in Eq. (1.4.3) and apply to flow situations with fluid flowing in all directions.

Substitutions of Eqs. (3.2.10) through (3.2.15) into Eq. (3.2.6), (3.2.7) and (3.2.8) give the general equation of motion for a Newtonian fluid with varying density and viscosity.

x-component :

$$\rho \frac{Dv_x}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left[2\mu \frac{\partial v_x}{\partial x} - \frac{2}{3} \mu (\nabla \cdot \vec{v}) \right] + \frac{\partial}{\partial y} \left[\mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) \right] + \frac{\partial}{\partial z} \left[\mu \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right) \right] + \rho g_x \quad \dots(3.2.16)$$

y-component :

$$\rho \frac{Dv_y}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x} \left[\mu \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right) \right] + \frac{\partial}{\partial y} \left[2\mu \frac{\partial v_y}{\partial y} - \frac{2}{3} \mu (\nabla \cdot \vec{v}) \right] + \frac{\partial}{\partial z} \left[\mu \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right) \right] + \rho g_y \quad \dots(3.2.17)$$

z-component :

$$\rho \frac{Dv_z}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x} \left[\mu \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right) \right] + \frac{\partial}{\partial y} \left[\mu \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right) \right] + \frac{\partial}{\partial z} \left[2\mu \frac{\partial v_z}{\partial z} - \frac{2}{3} \mu (\nabla \cdot \vec{v}) \right] + \rho g_z \quad \dots(3.2.18)$$

These equations are called the **Navier-Stokes Equations**. These equations can be simplified for the case of incompressible flow with constant viscosity because the continuity equation for incompressible fluid viz. $(\nabla \cdot \vec{v}) = 0$.

Thus for constant ρ and constant μ , the Navier-Stokes Equations become

x :

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \rho g_x \quad \dots(3.2.19)$$

y :

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) + \rho g_y \quad \dots(3.2.20)$$

z :

$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z \quad \dots(3.2.21)$$

Adding these three components vertically, the Navier-Stokes equations for fluid of constant density and viscosity become,

$$\rho \frac{D\vec{v}}{dt} = -\nabla p + \mu \nabla^2 \vec{v} + \rho \vec{g} \quad \dots(3.2.22)$$

The Eddy Viscosity of Boussinesq

By analogy with Newton's law of viscosity, Eq. 1.1-1, one may write for a turbulent shear flow¹

$$\bar{\tau}_{xy}^{(t)} = -\mu^{(t)} \frac{d\bar{v}_x}{dy} \quad (5.4-1)$$

¹ C. S. Lin, R. W. Moulton, and G. L. Putnam, *Ind. Eng. Chem.*, **45**, 636-640 (1953); the numerical coefficient was determined from mass transfer experiments in circular tubes. The importance of the $y^{\frac{1}{2}}$ term in heat and mass transfer was recognized earlier by E. V. Murphree, *Ind. Eng. Chem.*, **24**, 726-736 (1932). Eger Vaughn Murphree (1898-1962) was captain of the University of Kentucky football team in 1920 and became President of the Standard Oil Development Company.

² J. Boussinesq, *Mém. prés. par div. savants à l'acad. sci. de Paris*, **23**, #1, 1-680 (1877), **24**, #2, 1-64 (1877). Joseph Valentin Boussinesq (1842-1929), university professor in Lille, wrote a two-volume treatise on heat, and is famous for the "Boussinesq approximation" and the idea of "eddy viscosity."

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in which $\mu^{(t)}$ is the *turbulent viscosity* (often called the *eddy viscosity*, and given the symbol ϵ). As one can see from Table 5.1-1, for at least one of the flows given there, the circular jet, one might expect Eq. 5.4-1 to be useful. Usually, however, $\mu^{(t)}$ is a strong function of position and the intensity of turbulence. In fact, for some systems² $\mu^{(t)}$ may even be negative in some regions. It must be emphasized that the viscosity μ is a property of the *fluid*, whereas the eddy viscosity $\mu^{(t)}$ is primarily a property of the *flow*.

For two kinds of turbulent flows (i.e., flows along surfaces and flows in jets and wakes), special expressions for $\mu^{(t)}$ are available:

$$(i) \text{ Wall turbulence: } \mu^{(t)} = \mu \left(\frac{yv_*}{14.5\nu} \right)^3 \quad 0 < \frac{yv_*}{\nu} < 5 \quad (5.4-2)$$

This expression, derivable from Eq. 5.3-13, is valid only very near the wall. It is of considerable importance in the theory of turbulent heat and mass transfer at fluid-solid interfaces.³

$$(ii) \text{ Free turbulence: } \mu^{(t)} = \rho \kappa_0 b (\bar{v}_{z,max} - \bar{v}_{z,min}) \quad (5.4-3)$$

in which κ_0 is a dimensionless coefficient to be determined experimentally, b is the width of the mixing zone at a downstream distance z , and the quantity in parentheses represents the maximum difference in the z -component of the time-smoothed velocities at that distance z . Prandtl⁴ found Eq. 5.4-3 to be a useful empiricism for jets and wakes.

The Mixing Length of Prandtl

By assuming that eddies move around in a fluid very much as molecules move around in a low-density gas (not a very good analogy) Prandtl⁵ developed an expression for momentum transfer in a turbulent fluid. The "mixing length" l plays roughly the same role as the mean free path in kinetic theory (see §1.4). This kind of reasoning led Prandtl to the following relation:

$$\bar{\tau}_{xy}^{(t)} = -\rho l^2 \left| \frac{d\bar{v}_x}{dy} \right| \frac{d\bar{v}_x}{dy} \quad (5.4-4)$$

If the mixing length were a universal constant, Eq. 5.4-4 would be very attractive, but in fact l has been found to be a function of position. Prandtl proposed the following expressions for l :

$$(i) \text{ Wall turbulence: } l = \kappa_1 y \quad (y = \text{distance from wall}) \quad (5.4-5)$$

$$(ii) \text{ Free turbulence: } l = \kappa_2 b \quad (b = \text{width of mixing zone}) \quad (5.4-6)$$

in which κ_1 and κ_2 are constants. A result similar to Eq. 5.4-4 was obtained by Taylor⁶ by his "vorticity transport theory" some years prior to Prandtl's proposal.

ity boundary layer, whereas for $Pr < 1$ the relative thicknesses are just reversed (keep in mind that for gases Pr is about $\frac{1}{2}$, whereas for ordinary liquids $Pr > 1$ and for liquid metals $Pr \ll 1$).

In §4.4 we showed that the boundary layer equation of motion could be integrated formally from $y = 0$ to $y = \infty$, if use is made of the equation of continuity. In a similar fashion the integration of Eqs. 12.4-1 to 3 can be performed to give

$$\begin{aligned} \text{Momentum} \quad \mu \frac{\partial v_x}{\partial y} \Big|_{y=0} &= \frac{d}{dx} \int_0^\infty \rho v_x (v_x - v_\infty) dy + \frac{dv_x}{dx} \int_0^\infty \rho (v_x - v_\infty) dy \\ &\quad + \int_0^\infty \rho g \beta (T - T_\infty) dy \end{aligned} \quad (12.4-4)$$

$$\text{Energy} \quad k \frac{\partial T}{\partial y} \Big|_{y=0} = \frac{d}{dx} \int_0^\infty \rho \hat{C}_p v_x (T_\infty - T) dy \quad (12.4-5)$$

Equations 12.4-4 and 5 are the **von Kármán momentum and energy balances**, valid for forced-convection and free-convection systems. The no-slip condition $v_y = 0$ at $y = 0$ has been used here, as in Eq. 4.4-4; nonzero velocities at $y = 0$ occur in mass transfer systems and will be considered in Chapter 20.

As mentioned in §4.4, there are two approaches for solving boundary layer problems: analytical or numerical solutions of Equations 12.4-1 to 3 are called "exact boundary layer solutions," whereas solutions obtained from Eqs. 12.4-4 and 5, with reasonable guesses for the velocity and temperature profiles, are called "approximate boundary layer solutions." Often considerable physical insight can be obtained by the second method, and with relatively little effort. Example 12.4-1 illustrates this method.

Extensive use has been made of the boundary layer equations to establish correlations of momentum- and heat-transfer rates, as we shall see in Chapter 14. Although in this section we do not treat free convection, in Chapter 14 many useful results are given along with the appropriate literature citations.

EXAMPLE 12.4-1

**Heat Transfer in
Laminar Forced
Convection along a
Heated Flat Plate
(von Kármán Integral
Method)**

Obtain the temperature profiles near a flat plate, along which a Newtonian fluid is flowing, as shown in Fig. 12.4-1. The wetted surface of the plate is maintained at temperature T_0 and the temperature of the approaching fluid is T_∞ .

SOLUTION

In order to use the **von Kármán balances** we first postulate reasonable forms for the velocity and temperature profiles. The following polynomial form gives 0 at the wall and 1 at the outer limit of the boundary layer, with a slope of zero at the outer limit:

$$\begin{cases} \frac{v_x}{v_\infty} = 2\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4 & y \leq \delta(x) \\ \frac{v_x}{v_\infty} = 1 & y \geq \delta(x) \end{cases} \quad (12.4-6, 7)$$

$$\begin{cases} \frac{T_0 - T}{T_0 - T_\infty} = 2\left(\frac{y}{\delta_T}\right) - 2\left(\frac{y}{\delta_T}\right)^3 + \left(\frac{y}{\delta_T}\right)^4 & y \leq \delta_T(x) \\ \frac{T_0 - T}{T_0 - T_\infty} = 1 & y \geq \delta_T(x) \end{cases} \quad (12.4-8, 9)$$

That is, we assume that the dimensionless velocity and temperature profiles have the same form within their respective boundary layers. We further *assume* that the boundary layer thicknesses $\delta(x)$ and $\delta_T(x)$ have a constant ratio, so that $\Delta = \delta_T(x)/\delta(x)$ is independent of x . Two possibilities have to be considered: $\Delta \leq 1$ and $\Delta \geq 1$. We consider here $\Delta \leq 1$ and relegate the other case to Problem 12D.8.

equation for a wide variety of boundary and initial conditions.²

§11.3 THE BOUSSINESQ EQUATION OF MOTION FOR FORCED AND FREE CONVECTION

The equation of motion given in Eq. 3.2-9 (or Eq. B of Table 3.5-1) is valid for both isothermal and nonisothermal flow. In nonisothermal flow, the fluid density and viscosity depend in general on temperature as well as on pressure. The variation in the density is particularly important because it gives rise to buoyant forces, and thus to free convection, as we have already seen in §10.9.

The buoyant force appears automatically when an equation of state is inserted into the equation of motion. For example, we can use the simplified equation of state introduced in Eq. 10.9-6 (this is called the *Boussinesq approximation*)¹

$$\rho(T) = \bar{\rho} - \bar{\rho}\bar{\beta}(T - \bar{T}) \quad (11.3-1)$$

in which $\bar{\beta}$ is $-(1/\rho)(\partial\rho/\partial T)_p$ evaluated at $T = \bar{T}$. This equation is obtained by writing the Taylor series for ρ as a function of T , considering the pressure p to be constant, and keeping only the first two terms of the series. When Eq. 11.3-1 is substituted into the $\rho\mathbf{g}$ term (but not into the $\rho(D\mathbf{v}/Dt)$ term) of Eq. B of Table 3.5-1, we get the *Boussinesq equation*:

$$\rho \frac{D\mathbf{v}}{Dt} = (-\nabla p + \bar{\rho}\mathbf{g}) - [\nabla \cdot \boldsymbol{\tau}] - \bar{\rho}\mathbf{g}\bar{\beta}(T - \bar{T}) \quad (11.3-2)$$

¹ The assumption of constant density is made here, instead of the less stringent assumption that $(\partial \ln \rho / \partial \ln T)_p = 0$, since Eq. 11.2-9 is customarily used along with Eq. 3.1-5 (equation of continuity for constant density) and Eq. 3.5-6 (equation of motion for constant density and viscosity). Note that the hypothetical equation of state $\rho = \text{constant}$ has to be supplemented by the statement that $(\partial p / \partial T)_\rho = \text{finite}$, in order to permit the evaluation of certain thermodynamic derivatives. For example, the relation

$$\dot{C}_p - \dot{C}_v = -\frac{1}{\bar{\rho}} \left(\frac{\partial \ln \rho}{\partial \ln T} \right)_p \left(\frac{\partial p}{\partial T} \right)_\rho \quad (11.2-9a)$$

leads to the result that $\dot{C}_p = \dot{C}_v$ for the "incompressible fluid" thus defined.

² J. B. Fourier, *Théorie analytique de la chaleur*, Œuvres de Fourier, Gauthier-Villars et Fils, Paris (1822).

³ H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, Oxford University Press, 2nd edition (1959).

⁴ J. Boussinesq, *Théorie Analytique de Chaleur*, Vol. 2, Gauthier-Villars, Paris (1903).

For the steady, fully developed, laminar flow in a circular tube of radius R we know that the **velocity distribution** and the average velocity are given by

$$\frac{v_z}{v_{z,\max}} = 1 - \left(\frac{r}{R}\right)^2 \quad \text{and} \quad \frac{\langle v_z \rangle}{v_{z,\max}} = \frac{1}{2} \quad (\text{Re} < 2100) \quad (5.1-1, 2)$$

and that the pressure drop and mass flow rate w are linearly related:

$$\mathcal{P}_0 - \mathcal{P}_L = \left(\frac{8\mu L}{\pi \rho R^4}\right)w \quad (\text{Re} < 2100) \quad (5.1-3)$$

For turbulent flow, on the other hand, the velocity is fluctuating with time chaotically at each point in the tube. We can measure a "time-smoothed velocity" at each point with, say, a Pitot tube. This type of instrument is not sensitive to rapid velocity fluctuations, but senses the velocity averaged over several seconds. The time-smoothed velocity (which is defined in the next section) will have a z -component represented by \bar{v}_z , and its shape and average value will be given very roughly by¹

$$\frac{\bar{v}_z}{v_{z,\max}} \approx \left(1 - \frac{r}{R}\right)^{1/7} \quad \text{and} \quad \frac{\langle \bar{v}_z \rangle}{v_{z,\max}} \approx \frac{4}{5} \quad (10^4 < \text{Re} < 10^5) \quad (5.1-4, 5)$$

This $\frac{1}{7}$ -power expression for the **velocity distribution** is too crude to give a realistic velocity derivative at the wall. The laminar and turbulent velocity profiles are compared in Fig. 5.1-1.

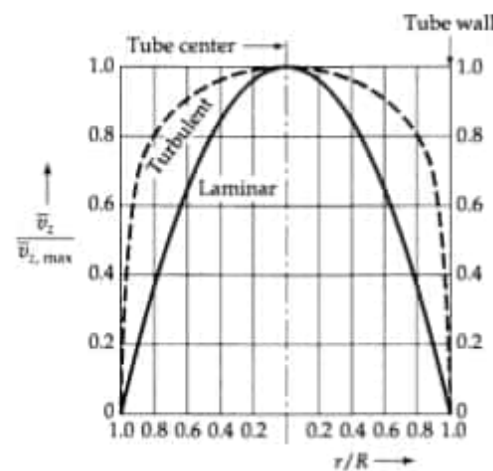


Fig. 5.1-1. Qualitative comparison of laminar and turbulent velocity profiles. For a more detailed description of the turbulent **velocity distribution** near the wall, see Fig. 5.5-3.

¹ H. Schlichting, *Boundary-Layer Theory*, McGraw-Hill, New York, 7th edition (1979), Chapter XX (tube flow), Chapters VII and XXI (flat plate flow), Chapters IX and XXIV (jet flows).

§5.1 Comparisons of Laminar and Turbulent Flows 155

Over the same range of Reynolds numbers the mass rate of flow and the pressure drop are no longer proportional but are related approximately by

$$\mathcal{P}_0 - \mathcal{P}_L \approx 0.198 \left(\frac{2}{\pi}\right)^{7/4} \left(\frac{\mu^{1/4} L}{\rho R^{19/4}}\right) w^{7/4} \quad (10^4 < \text{Re} < 10^5) \quad (5.1-6)$$

The stronger dependence of pressure drop on mass flow rate for turbulent flow results from the fact that more energy has to be supplied to maintain the violent eddy motion in the fluid.

The laminar-turbulent transition in circular pipes normally occurs at a *critical Reynolds number* of roughly 2100, although this number may be higher if extreme care is taken to eliminate vibrations in the system.² The transition from laminar flow to turbulent flow can be demonstrated by the simple experiment originally performed by Reynolds. One sets up a long transparent tube equipped with a device for injecting a small amount of dye into the stream along the tube axis. When the flow is laminar, the dye moves downstream as a straight, coherent filament. For turbulent flow, on the other hand, the dye spreads quickly over the entire cross section, similarly to the motion of particles in Fig. 2.0-1, because of the eddying motion (turbulent diffusion).

In §5.2 we introduced the notions of time-smoothed quantities and turbulent fluctuations. In this chapter we shall be primarily concerned with the temperature profiles. We introduce the time-smoothed temperature \bar{T} and temperature fluctuation T' , and write analogously to Eq. 5.2-1

$$T = \bar{T} + T' \quad (13.1-1)$$

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Clearly T' averages to zero so that $\bar{T'} = 0$, but quantities like $\overline{v_x' T'}$, $\overline{v_y' T'}$, and $\overline{v_z' T'}$ will not be zero because of the "correlation" between the velocity and temperature fluctuations at any point.

For a nonisothermal pure fluid we need three equations of change, and we want to discuss here their time-smoothed forms. The time-smoothed equations of continuity and motion for a fluid with constant density and viscosity were given in Eqs. 5.2-10 and 12, and need not be repeated here. For a fluid with constant μ , ρ , \hat{C}_p , and k , Eq. 11.2-5, when put in the $\partial/\partial t$ form by using Eq. 3.5-4, and with Newton's and Fourier's law included, becomes

$$\begin{aligned} \frac{\partial}{\partial t} \rho \hat{C}_p T = & - \left(\frac{\partial}{\partial x} \rho \hat{C}_p v_x T + \frac{\partial}{\partial y} \rho \hat{C}_p v_y T + \frac{\partial}{\partial z} \rho \hat{C}_p v_z T \right) + k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) \\ & + \mu \left[2 \left(\frac{\partial v_x}{\partial x} \right)^2 + \left(\frac{\partial v_x}{\partial y} \right)^2 + 2 \left(\frac{\partial v_x}{\partial y} \right) \left(\frac{\partial v_y}{\partial x} \right) + \dots \right] \end{aligned} \quad (13.1-2)$$

in which only a few sample terms in the viscous dissipation term $-(\tau \cdot \nabla \mathbf{v}) = \mu \Phi_v$ have been written (see Eq. B.7-1 for the complete expression).

In Eq. 13.1-2 we replace T by $T = \bar{T} + T'$, v_x by $\bar{v}_x + v'_x$, and so on. Then the equation is time-smoothed to give

$$\begin{aligned} \frac{\partial}{\partial t} \rho \hat{C}_p \bar{T} = & - \left(\frac{\partial}{\partial x} \rho \hat{C}_p \bar{v}_x \bar{T} + \frac{\partial}{\partial y} \rho \hat{C}_p \bar{v}_y \bar{T} + \frac{\partial}{\partial z} \rho \hat{C}_p \bar{v}_z \bar{T} \right) \\ & - \left(\frac{\partial}{\partial x} \rho \hat{C}_p \overline{v'_x T'} + \frac{\partial}{\partial y} \rho \hat{C}_p \overline{v'_y T'} + \frac{\partial}{\partial z} \rho \hat{C}_p \overline{v'_z T'} \right) \\ & + k \left(\frac{\partial^2 \bar{T}}{\partial x^2} + \frac{\partial^2 \bar{T}}{\partial y^2} + \frac{\partial^2 \bar{T}}{\partial z^2} \right) \\ & + \mu \left[2 \left(\frac{\partial \bar{v}_x}{\partial x} \right)^2 + \left(\frac{\partial \bar{v}_x}{\partial y} \right)^2 + 2 \left(\frac{\partial \bar{v}_x}{\partial y} \right) \left(\frac{\partial \bar{v}_y}{\partial x} \right) + \dots \right] \\ & + \mu \left[2 \left(\frac{\partial \overline{v'_x T'}}{\partial x} \right) \left(\frac{\partial \overline{v'_x T'}}{\partial x} \right) + \left(\frac{\partial \overline{v'_x T'}}{\partial y} \right) \left(\frac{\partial \overline{v'_x T'}}{\partial y} \right) + 2 \left(\frac{\partial \overline{v'_x T'}}{\partial y} \right) \left(\frac{\partial \overline{v'_y T'}}{\partial x} \right) + \dots \right] \end{aligned} \quad (13.1-3)$$

Comparison of this equation with the preceding one shows that the time-smoothed equation has the same form as the original equation, except for the appearance of the terms indicated by dashed underlines, which are concerned with the turbulent fluctuations. We are thus led to the definition of the turbulent heat flux $\bar{q}^{(t)}$ with components

$$\bar{q}_x^{(t)} = \rho \hat{C}_p \overline{v'_x T'} \quad \bar{q}_y^{(t)} = \rho \hat{C}_p \overline{v'_y T'} \quad \bar{q}_z^{(t)} = \rho \hat{C}_p \overline{v'_z T'} \quad (13.1-4)$$

and the turbulent energy dissipation function $\bar{\Phi}_v^{(t)}$:

$$\bar{\Phi}_v^{(t)} = \sum_{i=1}^3 \sum_{j=1}^3 \left(\left(\frac{\partial \overline{v'_i T'}}{\partial x_j} \right) \left(\frac{\partial \overline{v'_i T'}}{\partial x_j} \right) + \left(\frac{\partial \overline{v'_i T'}}{\partial x_j} \right) \left(\frac{\partial \overline{v'_j T'}}{\partial x_i} \right) \right) \quad (13.1-5)$$

The similarity between the components of $\bar{q}^{(t)}$ in Eq. 13.1-4 and those of $\bar{\tau}^{(t)}$ in Eq. 5.2-8 should be noted. In Eq. 13.1-5, v'_1 , v'_2 , and v'_3 are synonymous with v'_x , v'_y , and v'_z , and x_1 , x_2 , and x_3 have the same meaning as x , y , and z .

To summarize, we list all three time-smoothed equations of change for turbulent flows of pure fluids with constant μ , ρ , \hat{C}_p , and k in their D/Dt form (the first two were given in Eqs. 5.2-10 and 12):

$$\text{Continuity} \quad (\nabla \cdot \bar{\mathbf{v}}) = 0 \quad (13.1-6)$$

$$\text{Motion} \quad \rho \frac{D\bar{\mathbf{v}}}{Dt} = -\nabla \bar{p} - [\nabla \cdot (\bar{\tau}^{(v)} + \bar{\tau}^{(t)})] + \rho \mathbf{g} \quad (13.1-7)$$

$$\text{Energy} \quad \rho \hat{C}_p \frac{D\bar{T}}{Dt} = -(\nabla \cdot (\bar{\mathbf{q}}^{(v)} + \bar{\mathbf{q}}^{(t)})) + \mu (\bar{\Phi}_v^{(v)} + \bar{\Phi}_v^{(t)}) \quad (13.1-8)$$

We begin by considering a turbulent flow in a tube with a constant imposed pressure gradient. If at one point in the fluid we observe one component of the velocity as a function of time, we find that it is fluctuating in a chaotic fashion as shown in Fig. 5.2-1(a). The fluctuations are irregular deviations from a mean value. The actual velocity can be regarded as the sum of the mean value (designated by an overbar) and the **fluctuation** (designated by a prime). For example, for the z -component of the velocity we write

$$v_z = \bar{v}_z + v'_z \quad (5.2-1)$$

which is sometimes called the *Reynolds decomposition*. The mean value is obtained from $v_z(t)$ by making a time average over a large number of fluctuations

$$\bar{v}_z = \frac{1}{t_0} \int_{t-t_0}^{t+t_0} v_z(s) ds \quad (5.2-2)$$

§5.2 Time-Smoothed Equations of Change for Incompressible Fluids 157

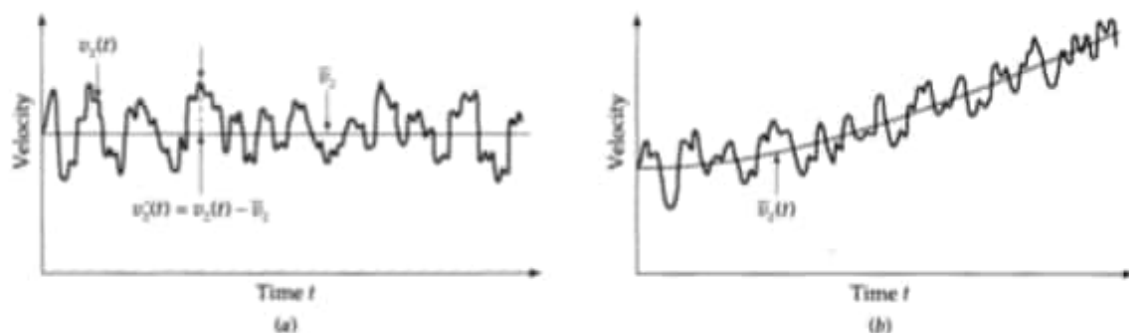


Fig. 5.2-1. Sketch showing the velocity component v_z as well as its time-smoothed value \bar{v}_z and its **fluctuation** v'_z in turbulent flow (a) for "steadily driven turbulent flow" in which \bar{v}_z does not depend on time, and (b) for a situation in which \bar{v}_z does depend on time.

the period t_0 being long enough to give a smooth averaged function. For the system at hand, the quantity \bar{v}_z , which we call the *time-smoothed velocity*, is independent of time, but of course depends on position. When the time-smoothed velocity does not depend on time, we speak of *steadily driven turbulent flow*. The same comments we have made for velocity can also be made for pressure.

Next we consider turbulent flow in a tube with a time-dependent pressure gradient. For such a flow one can define time-smoothed quantities as above, but one has to understand that the period t_0 must be small with respect to the changes in the pressure gradient, but still large with respect to the periods of fluctuations. For such a situation the time-smoothed velocity and the actual velocity are illustrated in Fig. 5.2-1(b).¹

According to the definition in Eq. 5.5-2, it is easy to verify that the following relations are true:

$$\overline{v'_z} = 0 \quad \overline{\bar{v}_z} = \bar{v}_z \quad \overline{v'_z v'_z} = 0 \quad \frac{\partial}{\partial x} \overline{v_z} = \frac{\partial}{\partial x} \bar{v}_z \quad \frac{\partial}{\partial t} \overline{v_z} = \frac{\partial}{\partial t} \bar{v}_z \quad (5.2-3)$$

The quantity $\overline{v'^2_z}$ will not, however, be zero, and in fact the ratio $\sqrt{\overline{v'^2_z}}/\bar{v}_z$ can be taken to be a measure of the magnitude of the turbulent fluctuations. This quantity, known as the *intensity of turbulence*, may have values from 1 to 10% in the main part of a turbulent stream and values of 25% or higher in the neighborhood of a solid wall. Hence, it must be emphasized that we are not necessarily dealing with tiny disturbances; sometimes the fluctuations are actually quite violent and large.

Quantities such as $\overline{v'_x v'_y}$ are also nonzero. The reason for this is that the local motions in the x and y directions are *correlated*. In other words, the fluctuations in the x direction are not independent of the fluctuations in the y direction. We shall see presently that these time-smoothed values of the products of fluctuating properties have an important role in turbulent momentum transfer. Later we shall find similar correlations arising in turbulent heat and mass transport.

We now combine the definition of f in Eq. 6.1-2 with the dimensional analysis of §3.7 to show what f must depend on in this kind of system. We consider a "test section" of inner radius R and length L , shown in Fig. 6.2-1, carrying a fluid of constant density and viscosity at a steady mass flow rate. The pressures \mathcal{P}_0 and \mathcal{P}_L at the ends of the test section are known.

² This friction factor definition is due to J. T. Fanning, *A Practical Treatise on Hydraulic and Water Supply Engineering*, Van Nostrand, New York, 1st edition (1877), 16th edition (1906); the name "Fanning" is used to avoid confusion with the "Moody friction factor," which is larger by a factor of 4 than the f used here [L. F. Moody, *Trans. ASME*, **66**, 671-684 (1944)].

If we use the "friction velocity" $v_* = \sqrt{\tau_0/\rho} = \sqrt{(\mathcal{P}_0 - \mathcal{P}_L)R/2L\rho}$, introduced in §5.3, then Eq. 6.1-4 assumes the form

$$f = 2(v_*/\langle v \rangle)^2 \quad (6.1-4a)$$

John Thomas Fanning (1837-1911) studied architectural and civil engineering, served as an officer in the Civil War, and after the war became prominent in hydraulic engineering. The 14th edition of his book *A Practical Treatise on Hydraulic and Water-Supply Engineering* appeared in 1899.

³ For the translational motion of a sphere in three dimensions, one can write approximately

$$F_s = (\pi R^2)(\frac{1}{2}\rho v_*^2)fn \quad (6.1-5a)$$

where n is a unit vector in the direction of \mathbf{v}_* . See Problem 6C.1.

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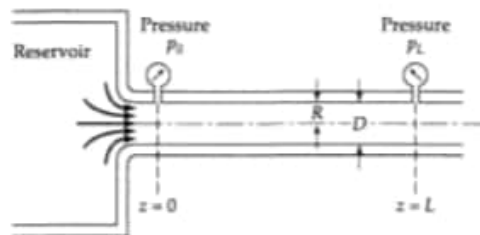


Fig. 6.2-1. Section of a circular pipe from $z = 0$ to $z = L$ for the discussion of dimensional analysis.

The system is either in steady laminar flow or steadily driven turbulent flow (i.e., turbulent flow with a steady total throughput). In either case the force in the z direction of the fluid on the inner wall of the test section is

$$F_z(t) = \int_0^L \int_0^{2\pi} \left(-\mu \frac{\partial v_z}{\partial r} \right) \bigg|_{r=R} R d\theta dz \quad (6.2-1)$$

In turbulent flow the force may be a function of time, not only because of the turbulent fluctuations, but also because of occasional ripping off of the boundary layer from the wall, which results in some distances with long time scales. In laminar flow it is understood that the force will be independent of time.

Equating Eqs. 6.2-1 and 6.1-2, we get the following expression for the friction factor:

$$f(t) = \frac{\int_0^L \int_0^{2\pi} \left(-\mu \frac{\partial v_z}{\partial r} \right) \bigg|_{r=R} R d\theta dz}{(2\pi RL)(\frac{1}{2}\rho \langle v \rangle^2)} \quad (6.2-2)$$

Next we introduce the dimensionless quantities from §3.7: $\tilde{r} = r/D$, $\tilde{z} = z/D$, $\tilde{v}_z = v_z/\langle v \rangle$, $\tilde{t} = \langle v \rangle t/D$, $\tilde{\mathcal{P}} = (\mathcal{P} - \mathcal{P}_0)/\rho \langle v \rangle^2$, and $Re = D \langle v \rangle \rho/\mu$. Then Eq. 6.2-2 may be rewritten as

$$f(\tilde{t}) = \frac{1}{\pi} \frac{D}{L} \frac{1}{Re} \int_0^{L/D} \int_0^{2\pi} \left(-\frac{\partial \tilde{v}_z}{\partial \tilde{r}} \right) \bigg|_{\tilde{r}=1/2} d\theta d\tilde{z} \quad (6.2-3)$$

This relation is valid for laminar or turbulent flow in circular tubes. We see that for flow systems in which the drag depends on viscous forces alone (i.e., no "form drag") the product of fRe is essentially a dimensionless velocity gradient averaged over the surface.

Recall now that, in principle, $\partial \tilde{v}_z / \partial \tilde{r}$ can be evaluated from Eqs. 3.7-8 and 9 along with the boundary conditions¹

$$\text{B.C. 1:} \quad \text{at } \tilde{r} = \frac{1}{2}, \quad \tilde{v} = 0 \quad \text{for } z > 0 \quad (6.2-4)$$

$$\text{B.C. 2:} \quad \text{at } \tilde{z} = 0, \quad \tilde{v} = \tilde{v}_z \quad (6.2-5)$$

$$\text{B.C. 3:} \quad \text{at } \tilde{r} = 0 \text{ and } \tilde{z} = 0, \quad \tilde{\mathcal{P}} = 0 \quad (6.2-6)$$

and appropriate initial conditions. The uniform inlet velocity profile in Eq. 6.2-5 is accurate except very near the wall, for a well-designed nozzle and upstream system. If Eqs. 3.7-8 and 9 could be solved with these boundary and initial conditions to get \tilde{v} and $\tilde{\Phi}$, the solutions would necessarily be of the form

$$\tilde{v} = \tilde{v}(\tilde{r}, \theta, \tilde{z}, \tilde{t}; \text{Re}) \quad (6.2-7)$$

$$\tilde{\Phi} = \tilde{\Phi}(\tilde{r}, \theta, \tilde{z}, \tilde{t}; \text{Re}) \quad (6.2-8)$$

That is, the functional dependence of \tilde{v} and $\tilde{\Phi}$ must, in general, include all the dimensionless variables and the one dimensionless group appearing in the differential equations. No additional dimensionless groups enter via the preceding boundary conditions. As a consequence, $\partial \tilde{v}_z / \partial \tilde{r}$ must likewise depend on $\tilde{r}, \theta, \tilde{z}, \tilde{t}$, and Re. When $\partial \tilde{v}_z / \partial \tilde{r}$ is evaluated at $\tilde{r} = \frac{1}{2}$ and then integrated over \tilde{z} and θ in Eq. 6.2-3, the result depends only on \tilde{t} , Re, and L/D (the latter appearing in the upper limit in the integration over \tilde{z}). Therefore we are led to the conclusion that $f(\tilde{t}) = f(\text{Re}, L/D, \tilde{t})$, which, when time averaged, becomes

$$f = f(\text{Re}, L/D) \quad (6.2-9)$$

when the time average is performed over an interval long enough to include any long-time turbulent disturbances. The measured friction factor then depends only on the Reynolds number and the length-to-diameter ratio.

The dependence of f on L/D arises from the development of the time-average velocity distribution from its flat entry shape toward more rounded profiles at downstream z values. This development occurs within an entrance region, of length $L_e \equiv 0.03D \text{ Re}$ for laminar flow or $L_e = 60D$ for turbulent flow, beyond which the shape of the velocity distribution is "fully developed." In the transportation of fluids, the entrance length is usually a small fraction of the total; then Eq. 6.2-9 reduces to the long-tube form

$$f = f(\text{Re}) \quad (6.2-10)$$

and f can be evaluated experimentally from Eq. 6.1-4, which was written for fully developed flow at the inlet and outlet.

Equations 6.2-9 and 10 are useful results, since they provide a guide for the systematic presentation of data on flow rate versus pressure difference for laminar and turbulent flow in circular tubes. For *long* tubes we need only a single curve of f plotted versus the single combination $D(\bar{v}_z)\rho/\mu$. Think how much simpler this is than plotting pressure drop versus the flow rate for separate values of D, L, ρ , and μ , which is what the uninitiated might do.

There is much experimental information for pressure drop versus flow rate in tubes, and hence f can be calculated from the experimental data by Eq. 6.1-4. Then f can be plotted versus Re for smooth tubes to obtain the *solid* curves shown in Fig. 6.2-2. These solid curves describe the laminar and turbulent behavior for fluids flowing in *long, smooth, circular* tubes.

Note that the *laminar* curve on the friction factor chart is merely a plot of the *Hagen-Poiseuille* equation in Eq. 2.3-21. This can be seen by substituting the expression for $(\Phi_0 - \Phi_L)$ from Eq. 2.3-21 into Eq. 6.1-4 and using the relation $w = \rho(\bar{v}_z)\pi R^2$; this gives

$$f = \frac{16}{\text{Re}} \begin{cases} \text{Re} < 2100 & \text{stable} \\ \text{Re} > 2100 & \text{usually unstable} \end{cases} \quad (6.2-11)$$

in which $\text{Re} = D(\bar{v}_z)\rho/\mu$; this is exactly the laminar line in Fig. 6.2-2.

Analogous *turbulent* curves have been constructed by using *experimental data*. Some analytical curve-fit expressions are also available. For example, Eq. 5.1-6 can be put into the form

$$f = \frac{0.0791}{\text{Re}^{1/4}} \quad 2.1 \times 10^3 < \text{Re} < 10^5 \quad (6.2-12)$$

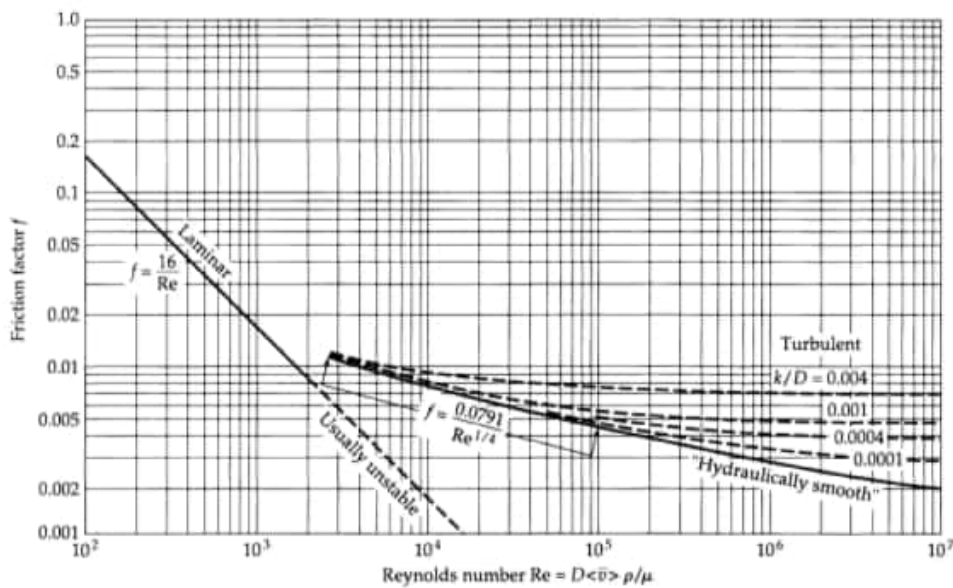


Fig. 6.2-2. Friction factor for tube flow (see definition of f in Eqs. 6.1-2 and 6.1-3. [Curves of L. F. Moody, *Trans. ASME*, **66**, 671-684 (1944) as presented in W. L. McCabe and J. C. Smith, *Unit Operations of Chemical Engineering*, McGraw-Hill, New York (1954).]

which is known as the *Blasius formula*.² Equation 5.5-1 (with 2.5 replaced by 2.45 and 1.75 by 2.00) is equivalent to

$$\frac{1}{\sqrt{f}} = 4.0 \log_{10} \text{Re} \sqrt{f} - 0.4 \quad 2.3 \times 10^3 < \text{Re} < 4 \times 10^6 \quad (6.2-13)$$

which is known as the *Prandtl formula*.³ Finally, corresponding to Eq. 5.5-2, we have

$$f = \frac{2}{\Psi^{2/(\alpha+1)}} \quad \text{where} \quad \Psi = \frac{e^{3/2}(\sqrt{3} + 5\alpha)}{2^{\alpha}(\alpha+1)(\alpha+2)} \quad (6.2-14)$$

and $\alpha = 3/(2 \ln \text{Re})$. This has been found to represent the experimental data well for $3.07 \times 10^3 < \text{Re} < 3.23 \times 10^6$. Equation 6.2-14 is called the *Barenblatt formula*.⁴

A further relation, which includes the dashed curves for rough pipes in Fig. 6.2-2, is the empirical *Haaland equation*⁵

$$\frac{1}{\sqrt{f}} = -3.6 \log_{10} \left[\frac{6.9}{\text{Re}} + \left(\frac{k/D}{3.7} \right)^{10/9} \right] \quad \begin{cases} 4 \times 10^4 < \text{Re} < 10^8 \\ 0 < k/D < 0.05 \end{cases} \quad (6.2-15)$$

² H. Blasius, *Forschungsarbeiten des Ver. Deutsch. Ing.*, no. 131 (1913).

³ L. Prandtl, *Essentials of Fluid Dynamics*, Hafner, New York (1952), p. 165.

⁴ G. I. Barenblatt, *Scaling, Self-Similarity, and Intermediate Asymptotics*, Cambridge University Press (1996), §10.2.

⁵ S. E. Haaland, *Trans. ASME, JFE*, **105**, 89-90 (1983). For other empiricisms see D. J. Zigrang and N. D. Sylvester, *AIChE Journal*, **28**, 514-515 (1982).

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This equation is stated⁵ to be accurate within 1.5%. As can be seen in Fig. 6.2-2, the frictional resistance to flow increases with the height, k , of the protuberances. Of course, k has to enter into the correlation in a dimensionless fashion and hence appears via the ratio k/D .

For turbulent flow in noncircular tubes it is common to use the following empiricism: First we define a "mean hydraulic radius" R_h as follows:

$$R_h = S/Z \quad (6.2-16)$$

in which S is the cross section of the conduit and Z is the wetted perimeter. Then we can use Eq. 6.1-4 and Fig. 6.2-2, with the diameter D of the circular pipe replaced by $4R_h$. That is, we calculate pressure differences by replacing Eq. 6.1-4 by

$$f = \left(\frac{R_h}{L} \right) \left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{\frac{1}{2}\rho \langle v_z \rangle^2} \right) \quad (6.2-17)$$

and getting f from Fig. 6.2-2 with a Reynolds number defined as

$$\text{Re}_h = \frac{4R_h \langle v_z \rangle \rho}{\mu} \quad (6.2-18)$$

This estimation method of Eqs. 6.2-16 to 18 should *not* be used for laminar flow.

§6.4 FRICTION FACTORS FOR PACKED COLUMNS

In the preceding two sections we have discussed the friction factor correlations for two simple flow systems of rather wide interest. Friction factor charts are available for a number of other systems, such as transverse flow past a cylinder, flow across tube

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banks, flow near baffles, and near flow around rotating disks. These and many more are summarized in various reference works.¹ One complex system of considerable interest in chemical engineering is the packed column, widely used for catalytic reactors and for separation processes.

There have been two main approaches for developing friction factor expressions for packed columns. In one method the packed column is visualized as a bundle of tangled tubes of weird cross section; the theory is then developed by applying the previous results for single straight tubes to the collection of crooked tubes. In the second method the packed column is regarded as a collection of submerged objects, and the pressure drop is obtained by summing up the resistances of the submerged particles.² The tube bundle theories have been somewhat more successful, and we discuss them here. Figure 6.4-1(a) depicts a packed column, and Fig. 6.4-1(b) illustrates the tube bundle model.

A variety of materials may be used for the packing in columns: spheres, cylinders, Berl saddles, and so on. It is assumed throughout the following discussion that the packing is statistically uniform, so that there is no "channeling" (in actual practice, channeling frequently occurs, and then the development given here does not apply). It is further assumed that the diameter of the packing particles is small in comparison to the diameter of the column in which the packing is contained, and that the column diameter is uniform.

We define the friction factor for the packed column analogously to Eq. 6.1-4:

$$f = \frac{1}{4} \left(\frac{D_p}{L} \right) \left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{\frac{1}{2} \rho v_0^2} \right) \quad (6.4-1)$$

in which L is the length of the packed column, D_p is the effective particle diameter (defined presently), and v_0 is the *superficial velocity*; this is the volume flow rate divided by the empty column cross section, $v_0 = w/\rho S$.

The pressure drop through a representative tube in the tube bundle model is given by Eq. 6.2-17

$$\mathcal{P}_0 - \mathcal{P}_L = \frac{1}{2} \rho (v)^2 \left(\frac{L}{R_h} \right) f_{\text{tube}} \quad (6.4-2)$$

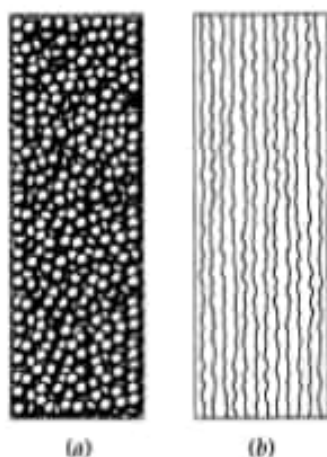


Fig. 6.4-1. (a) A cylindrical tube packed with spheres; (b) a "tube bundle" model for the packed column in (a).

in which the friction factor for a single tube, f_{tube} , is a function of the Reynolds number $Re_h = 4R_h\langle v \rangle\rho/\mu$. When this pressure difference is substituted into Eq. 6.4-1, we get

$$f = \frac{1}{4} \frac{D_p}{R_h} \frac{\langle v \rangle^2}{v_0^2} f_{\text{tube}} = \frac{1}{4\epsilon^2} \frac{D_p}{R_h} f_{\text{tube}} \quad (6.4-3)$$

In the second expression, we have introduced the *void fraction*, ϵ , the fraction of space in the column not occupied by the packing. Then $v_0 = \langle v \rangle\epsilon$, which results from the definition of the superficial velocity. We now need an expression for R_h .

The hydraulic radius can be expressed in terms of the void fraction ϵ and the wetted surface a per unit volume of bed as follows:

$$\begin{aligned} R_h &= \left(\frac{\text{cross section available for flow}}{\text{wetted perimeter}} \right) \\ &= \left(\frac{\text{volume available for flow}}{\text{total wetted surface}} \right) \\ &= \frac{\left(\frac{\text{volume of voids}}{\text{volume of bed}} \right)}{\left(\frac{\text{wetted surface}}{\text{volume of bed}} \right)} = \frac{\epsilon}{a} \end{aligned} \quad (6.4-4)$$

The quantity a is related to the "specific surface" a_v (total particle surface per volume of particles) by

$$a_v = \frac{a}{1 - \epsilon} \quad (6.4-5)$$

The quantity a_v is in turn used to define the mean particle diameter D_p as follows:

$$D_p = \frac{6}{a_v} \quad (6.4-6)$$

This definition is chosen because, for spheres of uniform diameter, D_p is exactly the diameter of a sphere. From the last three expressions we find that the hydraulic radius is $R_h = D_p\epsilon/6(1 - \epsilon)$. When this is substituted into Eq. 6.4-3, we get

$$f = \frac{3}{2} \left(\frac{1 - \epsilon}{\epsilon^3} \right) f_{\text{tube}} \quad (6.4-7)$$

We now adapt this result to laminar and turbulent flows by inserting appropriate expressions for f_{tube} .

(a) For *laminar flow* in tubes, $f_{\text{tube}} = 16/Re_h$. This is exact for circular tubes only. To account for the fact that the fluid is flowing through tubes that are noncircular and that its path is quite tortuous, it has been found that replacing 16 by 100/3 allows the tube bundle model to describe the packed-column data. When this modified expression for the tube friction factor is used, Eq. 6.4-7 becomes

$$f = \frac{(1 - \epsilon)^2}{\epsilon^3} \frac{75}{(D_p G_0 / \mu)} \quad (6.4-8)$$

in which $G_0 = \rho v_0$ is the mass flux through the system. When this expression for f is substituted into Eq. 6.4-1 we get

$$\frac{\mathcal{P}_0 - \mathcal{P}_L}{L} = 150 \left(\frac{\mu v_0}{D_p^2} \right) \frac{(1 - \epsilon)^2}{\epsilon^3} \quad (6.4-9)$$

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which is the *Blake-Kozeny equation*.³ Equations 6.4-8 and 9 are generally good for $(D_p G_0 / \mu(1 - \epsilon)) < 10$ and for void fractions less than $\epsilon = 0.5$.

(b) For *highly turbulent flow* a treatment similar to the above can be given. We begin again with the expression for the friction factor definition for flow in a circular tube. This time, however, we note that, for highly turbulent flow in tubes with any appreciable roughness, the friction factor is a function of the roughness only, and is independent of the Reynolds number. If we assume that the tubes in all packed columns have similar roughness characteristics, then the value of f_{tube} may be taken to be the same constant for all systems. Taking $f_{\text{tube}} = 7/12$ proves to be an acceptable choice. When this is inserted into Eq. 6.4-7, we get

$$f = \frac{7}{8} \left(\frac{1 - \epsilon}{\epsilon^3} \right) \quad (6.4-10)$$

When this is substituted into Eq. 6.4-1, we get

$$\frac{\mathcal{P}_0 - \mathcal{P}_L}{L} = \frac{7}{4} \left(\frac{\rho v_0^2}{D_p} \right) \frac{1 - \epsilon}{\epsilon^3} \quad (6.4-11)$$

When this is substituted into Eq. 6.4-1, we get

$$\frac{\mathcal{P}_0 - \mathcal{P}_L}{L} = \frac{7}{4} \left(\frac{\rho v_0^2}{D_p} \right) \frac{1 - \epsilon}{\epsilon^3} \quad (6.4-11)$$

which is the *Burke-Plummer*⁴ equation, valid for $(D_p G_0 / \mu(1 - \epsilon)) > 1000$. Note that the dependence on the void fraction is different from that for laminar flow.

(c) For the *transition region*, we may superpose the pressure drop expressions for (a) and (b) above to get

$$\frac{\mathcal{P}_0 - \mathcal{P}_L}{L} = 150 \left(\frac{\mu v_0}{D_p^2} \right) \frac{(1 - \epsilon)^2}{\epsilon^3} + \frac{7}{4} \left(\frac{\rho v_0^2}{D_p} \right) \frac{1 - \epsilon}{\epsilon^3} \quad (6.4-12)$$

For very small v_0 , this simplifies to the Blake-Kozeny equation, and for very large v_0 , to the Burke-Plummer equation. Such empirical superpositions of asymptotes often lead to satisfactory results. Equation 6.4-12 may be rearranged to form dimensionless groups:

$$\left(\frac{(\mathcal{P}_0 - \mathcal{P}_L)\rho}{G_0^2} \right) \left(\frac{D_p}{L} \right) \left(\frac{\epsilon^3}{1 - \epsilon} \right) = 150 \left(\frac{1 - \epsilon}{D_p G_0 / \mu} \right) + \frac{7}{4} \quad (6.4-13)$$

This is the *Ergun equation*,⁵ which is shown in Fig. 6.4-2 along with the Blake-Kozeny and Burke-Plummer equations and experimental data. It has been applied with success to gas flow through packed columns by using the density $\bar{\rho}$ of the gas at the arithmetic average of the end pressures. Note that G_0 is constant through the column, whereas v_0 changes through the column for a compressible fluid. For large pressure drops, however, it seems more appropriate to apply Eq. 6.4-12 locally by expressing the pressure gradient in differential form.

The Ergun equation is but one of many⁶ that have been proposed for describing packed columns. For example, the *Tallmadge equation*⁷

$$\left(\frac{(\mathcal{P}_0 - \mathcal{P}_L)\rho}{G_0^2} \right) \left(\frac{D_p}{L} \right) \left(\frac{\epsilon^3}{1 - \epsilon} \right) = 150 \left(\frac{1 - \epsilon}{D_p G_0 / \mu} \right) + 4.2 \left(\frac{1 - \epsilon}{D_p G_0 / \mu} \right)^{1/6} \quad (6.4-14)$$

is reported to give good agreement with experimental data over the range $0.1 < (D_p G_0 / \mu(1 - \epsilon)) < 10^5$.

³ F. C. Blake, *Trans. Amer. Inst. Chem. Engrs.*, **14**, 415-421 (1922); J. Kozeny, *Sitzungsber. Akad. Wiss. Wien, Abt. IIa*, **136**, 271-306 (1927).

⁴ S. P. Burke and W. B. Plummer, *Ind. Eng. Chem.*, **20**, 1196-1200 (1928).

⁵ S. Ergun, *Chem. Engr. Prog.*, **48**, 89-94 (1952).

⁶ I. F. Macdonald, M. S. El-Sayed, K. Mow, and F. A. Dullien, *Ind. Eng. Chem. Fundam.*, **18**, 199-208 (1979).

⁷ J. A. Tallmadge, *AIChE Journal*, **16**, 1092-1093 (1970).

192 Chapter 6 Interphase Transport in Isothermal Systems

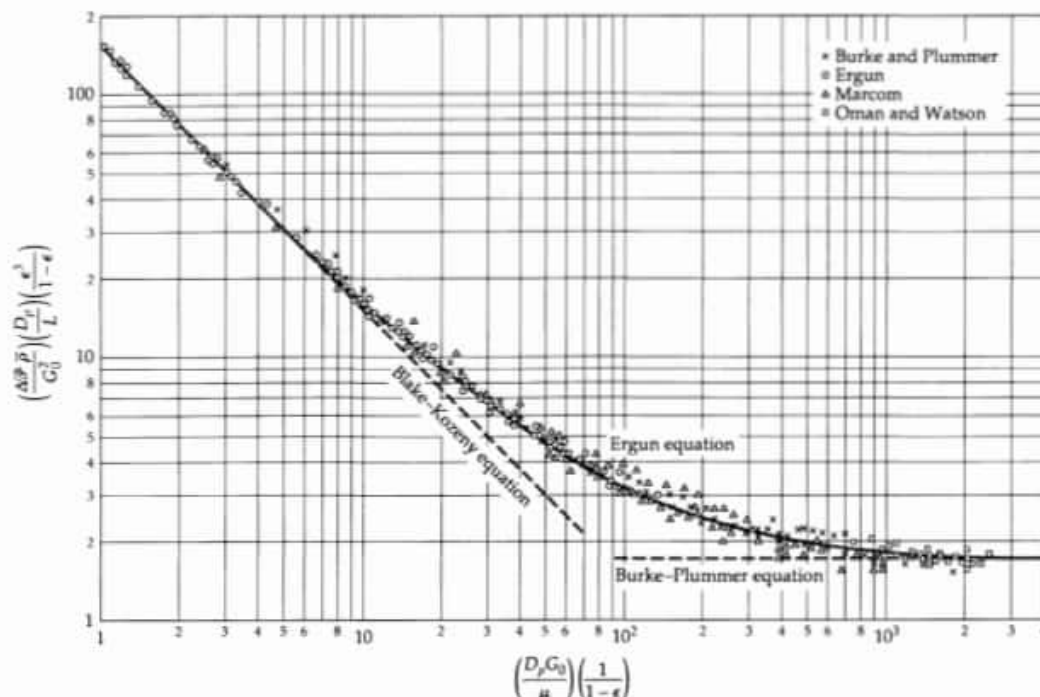


Fig. 6.4-2. The Ergun equation for flow in packed beds, and the two related asymptotes, the Blake-Kozeny equation and the Burke-Plummer equation [S. Ergun, *Chem. Eng. Prog.*, **48**, 89-94 (1952)].

As the outer cylinder rotates, each cylindrical shell of fluid "rubs" against an adjacent shell of fluid. This friction between adjacent layers of the fluid produces heat; that is, the mechanical energy is degraded into thermal energy. The volume heat source resulting from this "viscous dissipation," which can be designated by S_v , appears automatically in the shell balance when we use the combined energy flux vector \mathbf{e} defined at the end of Chapter 9, as we shall see presently.

If the slit width b is small with respect to the radius R of the outer cylinder, then the problem can be solved approximately by using the somewhat simplified system depicted in Fig. 10.4-2. That is, we ignore curvature effects and solve the problem in Cartesian coordinates. The velocity distribution is then $v_z = v_b(x/b)$, where $v_b = \Omega R$.

We now make an energy balance over a shell of thickness Δx , width W , and length L . Since the fluid is in motion, we use the combined energy flux vector \mathbf{e} as written in Eq. 9.8-6. The balance then reads

$$WLe_x|_x - WLe_x|_{x+\Delta x} = 0 \quad (10.4-1)$$

Dividing by $WL \Delta x$ and letting the shell thickness Δx go to zero then gives

$$\frac{de_x}{dx} = 0 \quad (10.4-2)$$

This equation may be integrated to give

$$e_x = C_1 \quad (10.4-3)$$

Since we do not know any boundary conditions for e_x , we cannot evaluate the integration constant at this point.

We now insert the expression for e_x from Eq. 9.8-6. Since the velocity component in the x direction is zero, the term $(\frac{1}{2}\rho v^2 + \rho U)\mathbf{v}$ can be discarded. The x -component of \mathbf{q} is $-k(dT/dx)$ according to Fourier's law. The x -component of $[\boldsymbol{\tau} \cdot \mathbf{v}]$ is, as shown in Eq. 9.8-1, $\tau_{xz}v_x + \tau_{xy}v_y + \tau_{zz}v_z$. Since the only nonzero component of the velocity is v_z and since $\tau_{xz} = -\mu(dv_z/dx)$ according to Newton's law of viscosity, the x -component of $[\boldsymbol{\tau} \cdot \mathbf{v}]$ is $-\mu v_z(dv_z/dx)$. We conclude, then, that Eq. 10.4-3 becomes

$$-k \frac{dT}{dx} - \mu v_z \frac{dv_z}{dx} = C_1 \quad (10.4-4)$$

When the linear velocity profile $v_z = v_b(x/b)$ is inserted, we get

$$-k \frac{dT}{dx} - \mu x \left(\frac{v_b}{b} \right)^2 = C_1 \quad (10.4-5)$$

in which $\mu(v_b/b)^2$ can be identified as the rate of viscous heat production per unit volume S_v .

When Eq. 10.4-5 is integrated we get

$$T = -\left(\frac{\mu}{k} \right) \left(\frac{v_b}{b} \right)^2 \frac{x^2}{2} - \frac{C_1}{k} x + C_2 \quad (10.4-6)$$

10 Chapter 10 Shell Energy Balances and Temperature Distributions in Solids and Laminar Flow

The two integration constants are determined from the boundary conditions

$$\text{B.C. 1:} \quad \text{at } x = 0, \quad T = T_0 \quad (10.4-7)$$

$$\text{B.C. 2:} \quad \text{at } x = b, \quad T = T_b \quad (10.4-8)$$

This yields finally, for $T_b \neq T_0$

$$\left(\frac{T - T_0}{T_b - T_0} \right) = \frac{1}{2} \text{Br} \frac{x}{b} \left(1 - \frac{x}{b} \right) + \frac{x}{b} \quad (10.4-9)$$

Here $\text{Br} = \mu v_b^2 / k(T_b - T_0)$ is the dimensionless *Brinkman number*,¹ which is a measure of the importance of the viscous dissipation term. If $T_b = T_0$, then Eq. 10.4-9 can be written as

$$\frac{T - T_0}{T_0} = \frac{1}{2} \frac{\mu v_b^2}{k T_0} \frac{x}{b} \left(1 - \frac{x}{b} \right) + \frac{x}{b} \quad (10.4-10)$$

and the maximum temperature is at $x/b = \frac{1}{2}$.

If the temperature rise is appreciable, the temperature dependence of the viscosity has to be taken into account. This is discussed in Problem 10C.1.

The viscous heating term $S_v = \mu(v_b/b)^2$ may be understood by the following arguments. For the system in Fig. 10.4-2, the rate at which work is done is the force acting on the upper plate times the velocity with which it moves, or $(-\mu dv_z/dx) \cdot W v_z$. The rate of en-

$$\cosh N$$

This result is reasonable only if the heat lost at the end and at the edges is negligible.

The "effectiveness" of the fin surface is defined³ by

$$\eta = \frac{\text{actual rate of heat loss from the fin}}{\text{rate of heat loss from an isothermal fin at } T_w} \quad (10.7-14)$$

For the problem being considered here η is then

$$\eta = \frac{\int_0^w \int_0^l h(T - T_s) dz dy}{\int_0^w \int_0^l h(T_w - T_s) dz dy} = \frac{\int_0^1 \Theta d\zeta}{\int_0^1 d\zeta} \quad (10.7-15)$$

or

$$\eta = \frac{1}{\cosh N} \left(-\frac{1}{N} \sinh N(1 - \zeta) \right) \Big|_0^1 = \frac{\tanh N}{N} \quad (10.7-16)$$

in which N is the dimensionless quantity defined in Eq. 10.7-8.

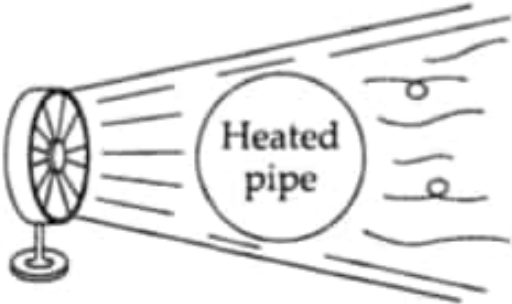
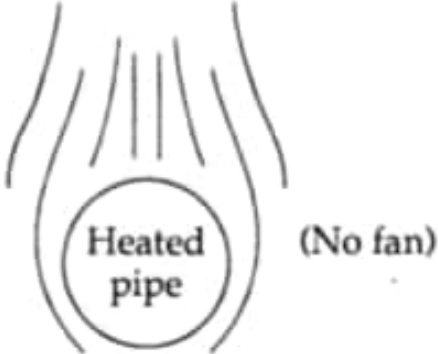
Forced Convection Heat Transfer	Free Convection Heat Transfer
 <p data-bbox="215 1025 703 1106">Heat swept to right by forced stream of air</p>	 <p data-bbox="813 1025 1289 1106">Heat transported upward by heated air that rises</p>
<p data-bbox="204 1153 644 1272">1. The flow patterns are determined primarily by some external force</p>	<p data-bbox="790 1153 1268 1272">1. The flow patterns are determined by the buoyant force on the heated fluid</p>
<p data-bbox="204 1328 718 1574">2 First, the velocity profiles are found; then they are used to find the temperature profiles (usual procedure for fluids with constant physical properties)</p>	<p data-bbox="790 1328 1236 1447">2. The velocity profiles and temperature profiles are interdependent</p>
<p data-bbox="204 1624 735 1742">3. The Nusselt number depends on the Reynolds and Prandtl numbers (see Chapter 14)</p>	<p data-bbox="790 1624 1321 1742">3. The Nusselt number depends on the Grashof and Prandtl numbers (see Chapter 14)</p>

Fig. 10.8-1. A comparison of forced and free convection in non-isothermal systems.

§10.2 HEAT CONDUCTION WITH AN ELECTRICAL HEAT SOURCE

The first system we consider is an electric wire of circular cross section with radius R and electrical conductivity k_e , $\text{ohm}^{-1} \text{cm}^{-1}$. Through this wire there is an electric current with current density I amp/ cm^2 . The transmission of an electric current is an irreversible process, and some electrical energy is converted into heat (thermal energy). The rate of heat production per unit volume is given by the expression

$$S_e = \frac{I^2}{k_e} \quad (10.2-1)$$

The quantity S_e is the heat source resulting from electrical dissipation. We assume here that the temperature rise in the wire is not so large that the temperature dependence of either the thermal or electrical conductivity need be considered. The surface of the wire is maintained at temperature T_0 . We now show how to find the radial temperature distribution within the wire.

For the energy balance we take the system to be a cylindrical shell of thickness Δr and length L (see Fig. 10.2-1). Since $\mathbf{v} = 0$ in this system, the only contributions to the energy balance are

$$\begin{array}{ll} \text{Rate of heat in} & \\ \text{across cylindrical} & (2\pi r L)q_r|_r = (2\pi r L q_r)|_r \\ \text{surface at } r & \end{array} \quad (10.2-2)$$

$$\begin{array}{ll} \text{Rate of heat out} & \\ \text{across cylindrical} & (2\pi(r + \Delta r)L)(q_r)|_{r+\Delta r} = (2\pi r L q_r)|_{r+\Delta r} \\ \text{surface at } r + \Delta r & \end{array} \quad (10.2-3)$$

$$\begin{array}{ll} \text{Rate of thermal} & \\ \text{energy production by} & (2\pi r \Delta r L)S_e \\ \text{electrical dissipation} & \end{array} \quad (10.2-4)$$

The notation q_r means "heat flux in the r direction," and $(\cdot \cdot \cdot)|_{r+\Delta r}$ means "evaluated at $r + \Delta r$." Note that we take "in" and "out" to be in the positive r direction.

We now substitute these quantities into the energy balance of Eq. 9.1-1. Division by $2\pi L \Delta r$ and taking the limit as Δr goes to zero gives

$$\lim_{\Delta r \rightarrow 0} \frac{(rq_r)|_{r+\Delta r} - (rq_r)|_r}{\Delta r} = S_e r \quad (10.2-5)$$

The expression on the left side is the first derivative of rq_r , with respect to r , so that Eq. 10.2-5 becomes

$$\frac{d}{dr}(rq_r) = S_e r \quad (10.2-6)$$

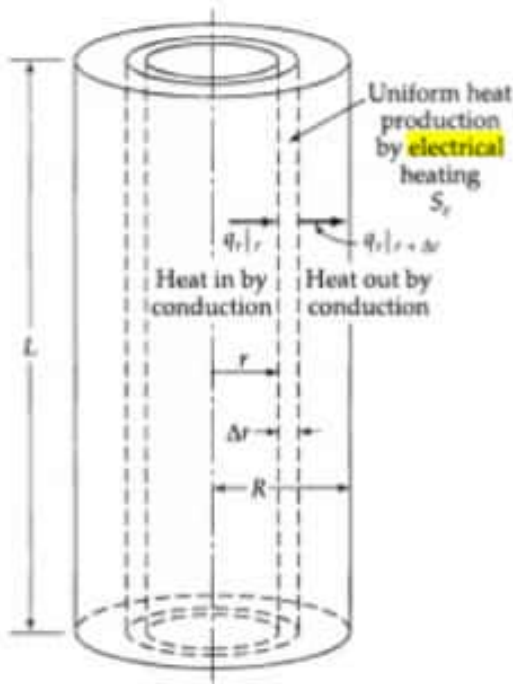


Fig. 10.2-1. An electrically heated wire, showing the cylindrical shell over which the energy balance is made.

This is a first-order differential equation for the energy flux, and it may be integrated to give

$$q_r = \frac{S_v r}{2} + \frac{C_1}{r} \quad (10.2-7)$$

The integration constant C_1 must be zero because of the boundary condition that

$$\text{B.C. 1:} \quad \text{at } r = 0, \quad q_r \text{ is not infinite} \quad (10.2-8)$$

Hence the final expression for the heat flux distribution is

$$q_r = \frac{S_v r}{2} \quad (10.2-9)$$

This states that the heat flux increases linearly with r .

We now substitute Fourier's law in the form $q_r = -k(dT/dr)$ (see Eq. B.2-4) into Eq. 10.2-9 to obtain

$$-k \frac{dT}{dr} = \frac{S_v r}{2} \quad (10.2-10)$$

When k is assumed to be constant, this first-order differential equation can be integrated to give

$$T = -\frac{S_v r^2}{4k} + C_2 \quad (10.2-11)$$

The integration constant is determined from

$$\text{B.C. 2:} \quad \text{at } r = R, \quad T = T_0 \quad (10.2-12)$$

Hence $C_2 = (S_v R^2/4k) + T_0$ and Eq. 10.2-11 becomes

$$T - T_0 = \frac{S_v R^2}{4k} \left[1 - \left(\frac{r}{R} \right)^2 \right] \quad (10.2-13)$$

Equation 10.2-13 gives the temperature rise as a parabolic function of the distance r from the wire axis.

Once the temperature and heat flux distributions are known, various information about the system may be obtained:

(i) *Maximum temperature rise (at $r = 0$)*

$$T_{\max} - T_0 = \frac{S_e R^2}{4k} \quad (10.2-14)$$

(ii) *Average temperature rise*

$$\langle T \rangle - T_0 = \frac{\int_0^{2\pi} \int_0^R (T(r) - T_0) r \, dr \, d\theta}{\int_0^{2\pi} \int_0^R r \, dr \, d\theta} = \frac{S_e R^2}{8k} \quad (10.2-15)$$

Thus the temperature rise, averaged over the cross section, is half the maximum temperature rise.

(iii) *Heat outflow at the surface (for a length L of wire)*

$$Q|_{r=R} = 2\pi RL \cdot q_r|_{r=R} = 2\pi RL \cdot \frac{S_e R}{2} = \pi R^2 L \cdot S_e \quad (10.2-16)$$

This result is not surprising, since, at steady state, all the heat produced by **electrical** dissipation in the volume $\pi R^2 L$ must leave through the surface $r = R$.

The reader, while going through this development, may well have had the feeling of *déjà vu*. There is, after all, a pronounced similarity between the heated wire problem and the viscous flow in a circular tube. Only the notation is different:

	Tube flow	Heated wire
First integration gives	$\tau_{rz}(r)$	$q_r(r)$
Second integration gives	$v_z(r)$	$T(r) - T_0$
Boundary condition at $r = 0$	$\tau_{rz} = \text{finite}$	$q_r = \text{finite}$
Boundary condition at $r = R$	$v_z = 0$	$T - T_0 = 0$
Transport property	μ	k
Source term	$(\mathcal{P}_0 - \mathcal{P}_L)/L$	S_e
Assumptions	$\mu = \text{constant}$	$k, k_z = \text{constant}$

That is, when the quantities are properly chosen, the differential equations *and* the boundary conditions for the two problems are identical, and the physical processes are said to be "analogous." Not all problems in momentum transfer have analogs in energy and mass transport. However, when such analogies can be found, they may be useful in taking over known results from one field and applying them in another. For example, the reader should have no trouble in finding a heat conduction analog for the viscous flow in a liquid film on an inclined plane.

There are many examples of heat conduction problems in the **electrical** industry.¹ The minimizing of temperature rises inside **electrical** machinery prolongs insulation life. One example is the use of internally liquid-cooled stator conductors in very large (500,000 kw) AC generators.

¹ M. Jakob, *Heat Transfer*, Vol. 1, Wiley, New York (1949), Chapter 10, pp. 167-199.

We consider a spherical nuclear fuel element as shown in Fig. 10.3-1. It consists of a sphere of fissionable material with radius $R^{(f)}$, surrounded by a spherical shell of aluminum "cladding" with outer radius $R^{(c)}$. Inside the fuel element, fission fragments are produced that have very high kinetic energies. Collisions between these fragments and the atoms of the fissionable material provide the major source of thermal energy in the reactor. Such a volume source of thermal energy resulting from nuclear fission we call S_v ($\text{cal}/\text{cm}^3 \cdot \text{s}$). This source will not be uniform throughout the sphere of fissionable material; it will be the smallest at the center of the sphere. For the purpose of this problem, we assume that the source can be approximated by a simple parabolic function

$$S_v = S_{v0} \left[1 + b \left(\frac{r}{R^{(f)}} \right)^2 \right] \quad (10.3-1)$$

Here S_{v0} is the volume rate of heat production at the center of the sphere, and b is a dimensionless positive constant.

We select as the system a spherical shell of thickness Δr within the sphere of fissionable material. Since the system is not in motion, the energy balance will consist only of heat conduction terms and a source term. The various contributions to the energy balance are:

Rate of heat in
by conduction
at r

$$q_r^{(f)}|_r \cdot 4\pi r^2 = (4\pi r^2 q_r^{(f)})|_r \quad (10.3-2)$$

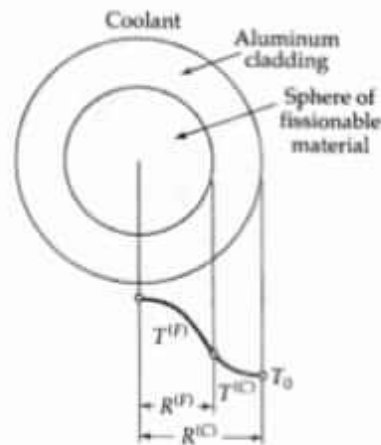


Fig. 10.3-1. A spherical nuclear fuel assembly, showing the temperature distribution within the system.

Rate of heat out
by conduction
at $r + \Delta r$

$$q_r^{(f)}|_{r+\Delta r} \cdot 4\pi(r + \Delta r)^2 = (4\pi r^2 q_r^{(f)})|_{r+\Delta r} \quad (10.3-3)$$

Rate of thermal
energy produced
by nuclear fission

$$S_v \cdot 4\pi r^2 \Delta r \quad (10.3-4)$$

Substitution of these terms into the energy balance of Eq. 10.1-1 gives, after dividing by $4\pi \Delta r$ and taking the limit as $\Delta r \rightarrow 0$

$$\lim_{\Delta r \rightarrow 0} \frac{(r^2 q_r^{(f)})|_{r+\Delta r} - (r^2 q_r^{(f)})|_r}{\Delta r} = S_v r^2 \quad (10.3-5)$$

Taking the limit and introducing the expression in Eq. 10.3-1 leads to

$$\frac{d}{dr} (r^2 q_r^{(f)}) = S_{v0} \left[1 + b \left(\frac{r}{R^{(f)}} \right)^2 \right] r^2 \quad (10.3-6)$$

The differential equation for the heat flux $q_r^{(c)}$ in the cladding is of the same form as Eq. 10.3-6, except that there is no significant source term:

$$\frac{d}{dr} (r^2 q_r^{(F)}) = S_{n0} \left[1 + b \left(\frac{r}{R^{(F)}} \right)^2 \right] r^2 \quad (10.3-6)$$

The differential equation for the heat flux $q_r^{(C)}$ in the cladding is of the same form as Eq. 10.3-6, except that there is no significant source term:

$$\frac{d}{dr} (r^2 q_r^{(C)}) = 0 \quad (10.3-7)$$

Integration of these two equations gives

$$q_r^{(F)} = S_{n0} \left(\frac{r}{3} + \frac{b}{R^{(F)2}} \frac{r^3}{5} \right) + \frac{C_1^{(F)}}{r^2} \quad (10.3-8)$$

$$q_r^{(C)} = + \frac{C_1^{(C)}}{r^2} \quad (10.3-9)$$

in which $C_1^{(F)}$ and $C_1^{(C)}$ are integration constants. These are evaluated by means of the boundary conditions:

$$\text{B.C. 1:} \quad \text{at } r = 0, \quad q_r^{(F)} \text{ is not infinite} \quad (10.3-10)$$

$$\text{B.C. 2:} \quad \text{at } r = R^{(F)}, \quad q_r^{(F)} = q_r^{(C)} \quad (10.3-11)$$

Evaluation of the constants then leads to

$$q_r^{(F)} = S_{n0} \left(\frac{r}{3} + \frac{b}{R^{(F)2}} \frac{r^3}{5} \right) \quad (10.3-12)$$

$$q_r^{(C)} = S_{n0} \left(\frac{1}{3} + \frac{b}{5} \right) \frac{R^{(F)3}}{r^2} \quad (10.3-13)$$

These are the heat flux distributions in the fissionable sphere and in the spherical-shell cladding.

Into these distributions we now substitute Fourier's law of heat conduction (Eq. B.2-7):

$$-k^{(F)} \frac{dT^{(F)}}{dr} = S_{n0} \left(\frac{r}{3} + \frac{b}{R^{(F)2}} \frac{r^3}{5} \right) \quad (10.3-14)$$

$$-k^{(C)} \frac{dT^{(C)}}{dr} = S_{n0} \left(\frac{1}{3} + \frac{b}{5} \right) \frac{R^{(F)3}}{r^2} \quad (10.3-15)$$

Chapter 10 Shell Energy Balances and Temperature Distributions in Solids and Laminar Flow

These equations may be integrated for constant $k^{(F)}$ and $k^{(C)}$ to give

$$T^{(F)} = -\frac{S_{n0}}{k^{(F)}} \left(\frac{r^2}{6} + \frac{b}{R^{(F)2}} \frac{r^4}{20} \right) + C_2^{(F)} \quad (10.3-16)$$

$$T^{(C)} = + \frac{S_{n0}}{k^{(C)}} \left(\frac{1}{3} + \frac{b}{5} \right) \frac{R^{(F)3}}{r} + C_2^{(C)} \quad (10.3-17)$$

The integration constants can be determined from the boundary conditions

$$\text{B.C. 3:} \quad \text{at } r = R^{(F)}, \quad T^{(F)} = T^{(C)} \quad (10.3-18)$$

$$\text{B.C. 4:} \quad \text{at } r = R^{(C)}, \quad T^{(C)} = T_0 \quad (10.3-19)$$

where T_0 is the known temperature at the outside of the cladding. The final expressions for the temperature profiles are

$$T^{(F)} = \frac{S_{n0} R^{(F)2}}{6k^{(F)}} \left\{ \left[1 - \left(\frac{r}{R^{(F)}} \right)^2 \right] + \frac{3}{10} b \left[1 - \left(\frac{r}{R^{(F)}} \right)^4 \right] \right\} + \frac{S_{n0} R^{(F)2}}{3k^{(C)}} \left(1 + \frac{3}{5} b \right) \left(1 - \frac{R^{(F)}}{R^{(C)}} \right) \quad (10.3-20)$$

where T_0 is the known temperature at the outside of the cladding. The final expressions for the temperature profiles are

$$T^{(F)} = \frac{S_{n0}R^{(F)2}}{6k^{(F)}} \left\{ \left[1 - \left(\frac{r}{R^{(F)}} \right)^2 \right] + \frac{3}{10} b \left[1 - \left(\frac{r}{R^{(F)}} \right)^4 \right] \right\} + \frac{S_{n0}R^{(F)2}}{3k^{(C)}} \left(1 + \frac{3}{5} b \right) \left(1 - \frac{R^{(F)}}{R^{(C)}} \right) \quad (10.3-20)$$

$$T^{(C)} = \frac{S_{n0}R^{(F)2}}{3k^{(C)}} \left(1 + \frac{3}{5} b \right) \left(\frac{R^{(F)}}{r} - \frac{R^{(F)}}{R^{(C)}} \right) \quad (10.3-21)$$

To find the maximum temperature in the sphere of fissionable material, all we have to do is set r equal to zero in Eq. 10.3-20. This is a quantity one might well want to know when making estimates of thermal deterioration.

This problem has illustrated two points: (i) how to handle a position-dependent source term, and (ii) the application of the continuity of temperature and normal heat flux at the boundary between two solid materials.

Effectiveness factor

We define the effectiveness factor as the average reaction rate with diffusion divided by the average reaction rate if the rate of reaction is evaluated at the bulk-stream (or boundary condition) values. This last quantity is the average reaction rate if diffusion is very fast, presenting no limitation to the mass transfer. The effectiveness factor is thus

$$\eta = \frac{\bar{q}^2 \int_0^1 R_1(z, T(z)) z^{a-1} dz}{\bar{q}^2 \int_0^1 R_1(z, T) z^{a-1} dz}$$

The parameter $a = 1, 2$, or 3 , respectively, for planar, cylindrical, or spherical geometry. We can integrate the differential equation

$$\frac{1}{z^{a-1}} \frac{d}{dz} \left[z^{a-1} \frac{dc}{dz} \right] = \bar{q}^2 R_1(z, T)$$

over the domain $r = 0$ to 1 to obtain

$$\int_0^1 \frac{1}{z^{a-1}} \frac{d}{dz} \left[z^{a-1} \frac{dc}{dz} \right] z^{a-1} dz = \bar{q}^2 \int_0^1 R_1(z, T) z^{a-1} dz = \frac{dc}{dz} (1)$$

Hence, we can rewrite the

§7.1 THE MACROSCOPIC MASS BALANCE

In the system shown in Fig. 7.0-1 the fluid enters the system at plane 1 with cross section S_1 and leaves at plane 2 with cross section S_2 . The average velocity is $\langle v_1 \rangle$ at the entry plane and $\langle v_2 \rangle$ at the exit plane. In this and the following sections, we introduce two assumptions that are not very restrictive: (i) at the planes 1 and 2 the time-smoothed veloc-

¹ R. B. Bird, *Chem. Eng. Sci.*, **6**, 123-131 (1957); *Chem. Eng. Educ.*, **27**(2), 102-109 (Spring 1993).

² J. C. Slattery and R. A. Gaggioli, *Chem. Eng. Sci.*, **17**, 893-895 (1962).

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ity is perpendicular to the relevant cross section, and (ii) at planes 1 and 2 the density and other physical properties are uniform over the cross section.

The law of conservation of mass for this system is then

$$\frac{d}{dt} m_{\text{tot}} = \rho_1 \langle v_1 \rangle S_1 - \rho_2 \langle v_2 \rangle S_2 \quad (7.1-1)$$

rate of
increase
of mass
rate of
mass in
at plane 1
rate of
mass out
at plane 2

Here $m_{\text{tot}} = \int \rho dV$ is the total mass of fluid contained in the system between planes 1 and 2. We now introduce the symbol $w = \rho \langle v \rangle S$ for the mass rate of flow, and the notation $\Delta w = w_2 - w_1$ (exit value minus entrance value). Then the *unsteady-state macroscopic mass balance* becomes

$$\frac{d}{dt} m_{\text{tot}} = -\Delta w \quad (7.1-2)$$

If the total mass of fluid does not change with time, then we get the *steady-state macroscopic mass balance*

$$\Delta w = 0 \quad (7.1-3)$$

which is just the statement that the rate of mass entering equals the rate of mass leaving.

For the macroscopic mass balance we use the term "steady state" to mean that the time derivative on the left side of Eq. 7.1-2 is zero. Within the system, because of the possibility for moving parts, flow instabilities, and turbulence, there may well be regions of unsteady flow.

scopic angular momentum balance can be obtained from it.

§3.5 THE EQUATIONS OF CHANGE IN TERMS OF THE SUBSTANTIAL DERIVATIVE

Before proceeding we point out that several different time derivatives may be encountered in transport phenomena. We illustrate these by a homely example—namely, the observation of the **concentration** of fish in the Mississippi River. Because fish swim around, the fish **concentration** will in general be a function of position (x, y, z) and time (t) .

The Partial Time Derivative $\partial/\partial t$

Suppose we stand on a bridge and observe the **concentration** of fish just below us as a function of time. We can then record the time rate of change of the fish **concentration** at a fixed location. The result is $(\partial c/\partial t)|_{x,y,z}$, the partial derivative of c with respect to t , at constant x , y , and z .

The Total Time Derivative d/dt

Now suppose that we jump into a motor boat and speed around on the river, sometimes going upstream, sometimes downstream, and sometimes across the current. All the time we are observing fish concentration. At any instant, the time rate of change of the observed fish **concentration** is

$$\frac{dc}{dt} = \left(\frac{\partial c}{\partial t}\right)_{x,y,z} + \frac{dx}{dt} \left(\frac{\partial c}{\partial x}\right)_{y,z,t} + \frac{dy}{dt} \left(\frac{\partial c}{\partial y}\right)_{x,z,t} + \frac{dz}{dt} \left(\frac{\partial c}{\partial z}\right)_{x,y,t} \quad (3.5-1)$$

in which dx/dt , dy/dt , and dz/dt are the components of the velocity of the boat.

The Substantial Time Derivative D/Dt

Next we climb into a canoe, and not feeling energetic, we just float along with the current, observing the fish concentration. In this situation the velocity of the observer is the same as the velocity \mathbf{v} of the stream, which has components v_x , v_y , and v_z . If at any instant we report the time rate of change of fish concentration, we are then giving

$$\frac{Dc}{Dt} = \frac{\partial c}{\partial t} + v_x \frac{\partial c}{\partial x} + v_y \frac{\partial c}{\partial y} + v_z \frac{\partial c}{\partial z} \quad \text{or} \quad \frac{Dc}{Dt} = \frac{\partial c}{\partial t} + (\mathbf{v} \cdot \nabla c) \quad (3.5-2)$$

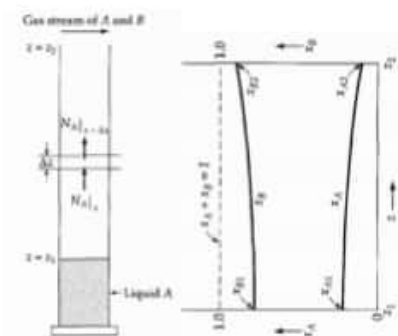
The special operator $D/Dt = \partial/\partial t + \mathbf{v} \cdot \nabla$ is called the *substantial derivative* (meaning that the time rate of change is reported as one moves with the "substance"). The terms *material derivative*, *hydrodynamic derivative*, and *derivative following the motion* are also used.

Now we need to know how to convert equations expressed in terms of $\partial/\partial t$ into equations written with D/Dt . For any scalar function $f(x, y, z, t)$ we can do the following manipulations:

$$\begin{aligned} & \frac{\partial}{\partial t}(\rho f) + \left(\frac{\partial}{\partial x} \rho v_x f\right) + \left(\frac{\partial}{\partial y} \rho v_y f\right) + \left(\frac{\partial}{\partial z} \rho v_z f\right) \\ &= \rho \left(\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y} + v_z \frac{\partial f}{\partial z}\right) + f \left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \rho v_x + \frac{\partial}{\partial y} \rho v_y + \frac{\partial}{\partial z} \rho v_z\right) \\ &= \rho \frac{Df}{Dt} \end{aligned} \quad (3.5-3)$$

6.2 Diffusion Through a Stagnant Gas Film

Problem: Consider the schematic shown below. Note that B is immiscible with A , so while B can be present in the system at steady state, there is no net flux of B down or out, just across such that $N_{B,z} = 0$. For the full description of the problem, see Section 18.2 of BSL.



1. We write the mass balance in the z direction as

$$N_{A,z} = -c\mathcal{D}_{AB} \frac{dx_A}{dz} + x_A N_{A,z}$$

2. Solving for $N_{A,z}$ yields

$$N_{A,z} = \frac{-c\mathcal{D}_{AB} \frac{dx_A}{dz}}{1 - x_A}$$

3. A steady-state mass balance can be written as

$$SN_{A,z}|_z - SN_{A,z}|_{z+\Delta z} = 0$$

where S is a cross-sectional area

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4. Dividing by $S\Delta z$ and letting $\Delta z \rightarrow 0$ yields

$$\frac{-dN_{A,z}}{dz} = 0$$

5. This can therefore be written as

$$\frac{d}{dz} \left(\frac{c\mathcal{D}_{AB} \frac{dx_A}{dz}}{1 - x_A} \right) = 0$$

- (a) For an ideal gas mixture, c is constant for a constant T and P . Also, for gases, \mathcal{D}_{AB} is usually independent of the composition such that

$$\frac{d}{dz} \left(\frac{1}{1 - x_A} \frac{dx_A}{dz} \right) = 0$$

which can be integrated to yield⁶

$$-\ln(1 - x_A) = C_1 z + C_2$$

6. Although this is not obvious, we can let $C_1 = -\ln K_1$ and $C_2 = -\ln K_2$ such that⁷

$$1 - x_A = K_1^z K_2$$

7. The boundary conditions are: $x_A(z_1) = x_{A1}$ and $x_A(z_2) = x_{A2}$

8. Applying the boundary conditions yields $1 - x_{A1} = K_1^{z_1} K_2$ and $1 - x_{A2} = K_1^{z_2} K_2$, which can be combined to yield

$$\frac{1 - x_{A2}}{1 - x_{A1}} = K_1^{z_2 - z_1}$$

- (a) A little algebraic manipulation yields

$$K_1 = \left(\frac{1 - x_{A2}}{1 - x_{A1}} \right)^{1/(z_2 - z_1)}$$

- (b) We need an expression for K_2 , so

$$1 - x_{A1} = \left(\frac{1 - x_{A2}}{1 - x_{A1}} \right)^{z_1/(z_2 - z_1)} K_2 \rightarrow K_2 = (1 - x_{A1}) \left(\frac{1 - x_{A2}}{1 - x_{A1}} \right)^{-z_1/(z_2 - z_1)}$$

9. Plugging in the results for K_1 and K_2 yields

$$1 - x_A = \left(\frac{1 - x_{A2}}{1 - x_{A1}} \right)^{z/(z_2 - z_1)} (1 - x_{A1}) \left(\frac{1 - x_{A2}}{1 - x_{A1}} \right)^{-z_1/(z_2 - z_1)}$$

which can be rearranged to

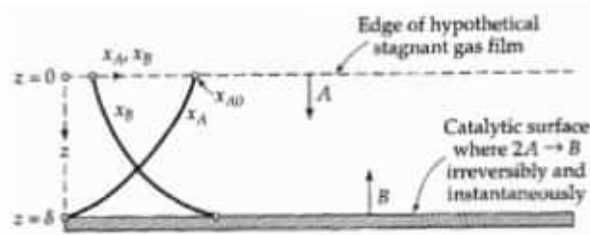
$$\frac{1 - x_A}{1 - x_{A1}} = \left(\frac{1 - x_{A2}}{1 - x_{A1}} \right)^{(z - z_1)/(z_2 - z_1)}$$

10. To obtain the profile for x_B , recognize that $x_A + x_B \equiv 1$

6.4 Diffusion with a Heterogeneous Chemical Reaction (e.g. Gas Reacting on Solid Catalyst)

6.4.1 Diffusion with an Instantaneous Heterogeneous Reaction

Problem: Consider the heterogeneous chemical reaction of $2A \rightarrow B$ shown in the diagram below. For the full problem statement, see Section 18.3 in BSL.



1. From the stoichiometry, we know that^a

$$N_{B,z} = -\frac{1}{2}N_{A,z}$$

2. We also know that

$$N_{A,z} = -c\mathcal{D}_{AB} + x_A \left(N_{A,z} - \frac{1}{2}N_{A,z} \right)$$

which simplifies to

$$N_{A,z} = -\frac{c\mathcal{D}_{AB}}{1 - \frac{1}{2}x_A} \frac{dx_A}{dz}$$

3. The shell mass balance states that

$$SN_{A,z}|_z - SN_{A,z}|_{z+\Delta z} = 0$$

which leads to

$$\frac{dN_{A,z}}{dz} = 0$$

4. This yields

$$\frac{d}{dz} \left(\frac{c\mathcal{D}_{AB}}{1 - \frac{1}{2}x_A} \frac{dx_A}{dz} \right) = 0$$

5. Integrating this yields

$$-2 \ln \left(1 - \frac{1}{2}x_A \right) = C_1 z + C_2$$

for constant $c\mathcal{D}_{AB}$

6. Substituting $C_1 = -2 \ln K_1$ and $C_2 = -2 \ln K_2$ yields

$$1 - \frac{x_A}{2} = K_1^z K_2$$

7. The boundary conditions are $x_A(0) = x_{A0}$ and $x_A(\delta) = 0$

^aFor a reaction $aA \rightarrow bB$, $N_{B,z} = -\frac{b}{a}N_{A,z}$

8. Applying the boundary conditions yields

$$1 - \frac{1}{2}x_A = \left(1 - \frac{1}{2}x_{A0} \right)^{1-z/\delta}$$

9. To get the molar flux, we need $\frac{dx_A}{dz}$. Since we want the molar flux at the film, and the film is at $z = 0$, we technically want $\frac{dx_A}{dz} \Big|_{z=0}$. The final result yields^b so^c

$$N_{A,z} = \frac{2c\mathcal{D}_{AB}}{\delta} \ln \left(\frac{1}{1 - \frac{1}{2}x_{A0}} \right)$$

6.4.2 Diffusion with a Slow Heterogeneous Reaction

Problem: Attempt the previous problem with a slow reaction (i.e. not instantaneous). Assume that the rate A disappears at the catalyst surface is given as $N_{A,z} = k_1'' c_A = k_1'' c x_A$, in which k_1'' is a rate constant for the pseudo-first-order surface reaction. For the full problem statement, see Example 18.3-1 in BSL.

1. The set-up is identical up until the boundary conditions at which point $x_A(\delta) = \frac{N_{A,z}}{k_1'' c}$ instead of $x_A(\delta) = 0$
2. Applying the boundary conditions yields

$$\left(1 - \frac{1}{2} x_A\right) = \left(1 - \frac{1}{2} \frac{N_{A,z}}{k_1'' c}\right)^{z/\delta} \left(1 - \frac{1}{2} x_{A0}\right)^{1-z/\delta}$$

3. Evaluating $\left.\frac{dx_A}{dz}\right|_{z=0}$ and solving for $N_{A,z}$ yields,

$$N_{A,z} = \frac{2c\mathcal{D}_{AB}}{\delta} \ln \left(\frac{1 - \frac{1}{2} \left(\frac{N_{A,z}}{k_1'' c} \right)}{1 - \frac{1}{2} x_{A0}} \right)$$

4. If k_1'' is large (note that this means the reaction is fast, but not so fast that it is instantaneous) then

$$N_{A,z} = \frac{2c\mathcal{D}_{AB}/\delta}{1 + \frac{\mathcal{D}_{AB}}{k_1'' \delta}} \ln \left(\frac{1}{1 - \frac{1}{2} x_{A0}} \right)$$

which can be obtained by a Taylor expansion on the logarithm term and keeping just the first term such that $\ln(1+p) \approx p$ for small p

5. The Damkohler Number of the second order can be defined as

$$\text{Da}^{\text{II}} = \frac{k_1'' \delta}{\mathcal{D}_{AB}}$$

- (a) In the limit of $\text{Da}^{\text{II}} \rightarrow \infty$, we obtain the expression for the instantaneous reaction
- (b) In words, the Damkohler number is the ratio of the chemical reaction rate compared to the diffusion rate (i.e. mass transfer)
- (c) A very fast reaction is governed by mass transfer, but a very slow reaction is governed by kinetics

*The following is a helpful identity: $\frac{d}{dx} (a^{bx}) = \ln(a) a^{bx}$