UIT-RGPV (Autonomous) Bhopal
Department of Petrochemical Engineering

Subject In-charge: Prof. M.S. Chouhan

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Subject code – PC 702 **Subject: Transport Phenomena**

Unit: -IV

Lecture no.21

Thermal Conductivity and the Mechanisms of Energy Transport

1 Fourier's Law of Heat Conduction

- The heat flux ($\vec{q}$) is defined as (Fourier's Law of Cooling)

$$\vec{q} = -k\nabla T$$

- The heat flux is related to the heat flow (Q) by

$$Q = \vec{q}A = -Ak\nabla T$$

- The thermal diffusivity is defined as the following where $\hat{C}_p$ is heat capacity at constant pressure with per-mass units,

$$\alpha = \frac{k}{\rho\hat{C}_p}$$

  – The units of α are length-squared per unit time

- The Prandtl number (unitless) is defined as

$$\mathrm{Pr} = \frac{\nu}{\alpha} = \frac{\hat{C}_p\mu}{k}$$

2 The Microscopic Energy Balance

(a) Equation

- The microscopic energy balance states that

$$\rho \hat{C}_p \left[\frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T \right] = -\nabla \cdot \vec{q} + S$$

- Using Fourier's Law of cooling yields the equivalent[14]

$$\rho \hat{C}_p \left[\frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T \right] = \nabla \left(k \nabla T \right) + S$$

  - Note that $\nabla (k \nabla T) = k \nabla^2 T$ when k is constant

- The $\frac{\partial T}{\partial t}$ is the accumulation term, the $(\vec{v} \cdot \nabla) T$ term is convective, the $\nabla (k \nabla T)$ is the conductive term, and S is the source term
- Dividing by $\rho \hat{C}_P$ and assuming k is constant yields

$$\left(\frac{\partial T}{\partial t} + (\vec{v} \cdot \nabla) T \right) = \alpha \nabla^2 T + \frac{S}{\rho \hat{C}_p}$$

where

$$\alpha \equiv \frac{k}{\rho \hat{C}_P}$$

- Recall the following thermodynamic definition as well:

$$\hat{C}_p = \left(\frac{\partial H}{\partial T} \right)_P$$

(b) Boundary Conditions

- The temperature may be specified at a surface
- The heat flux normal to a surface may be given (this is the same as saying the normal component of the temperature gradient)
- There must be continuity of temperature and heat flux normal to the surface at the interfaces
- At a solid-liquid interface, $q = h (T_0 - T_b)$ where T_b is the bulk temperature and T_0 is the solid surface temperature

3 Conduction through a Block

Prompt: There is a rectangular prism with heat flow solely in the $+x$ direction. The left face of the object is at a temperature T_1, and the right face of the object is at a temperature T_2. The length of the box is $x = B$. Assume that $k = a + bT$, where a and b are constants. Solve for the heat flux through the object.

Solution:

1. The Cartesian coordinate system should be used. Also, there is only conduction, and it is in the x direction such that $T(x)$
2. Using the microscopic energy balance, realizing it is steady state so $\frac{\partial T}{\partial t} = 0$, that $\vec{v} = 0$, and that $S = 0$ yields

$$0 = \nabla (k \nabla T)$$

3. Substituting in for k and rewriting the equivalents for ∇ yields

$$0 = \frac{d}{dx}\left((a + bT)\frac{dT}{dx}\right)$$

4. Integrating once yields

$$C_1 = (a + bT)\frac{dT}{dx}$$

5. Integrating again yields

$$C_1 x + C_2 = aT + \frac{bT^2}{2}$$

6. At the interfaces, there must be continuity of temperature, so at $x = 0$, $T = T_1$, and at $x = B$, $T = T_2$

$$C_2 = aT_1 + \frac{bT_1^2}{2}$$

$$C_1 = \frac{a(T_2 - T_1)}{B} + \frac{b(T_2^2 - T_1^2)}{2B}$$

7. From this, the temperature profile can be fully described. However, since the heat flux is desired, Fourier's Law will be used. Note that step 4 indicated that $C_1 = k\frac{dT}{dx}$. Therefore,

$$q_x = -k\frac{dT}{dx} = -C_1 = -\left[\frac{a(T_2 - T_1)}{B} + \frac{b(T_2^2 - T_1^2)}{2B}\right]$$

4 Shell Energy Balance

- As with fluids, the shell balance can be used in place of the microscopic balance for heat flow. The equation is:

Convection In - Convection Out + Conduction In - Conduction Out + Work On System - Work By System + Rate of Energy Production = 0

- Conduction is given as Aq where A is the projected area (analogous to the stress term in the shell momentum balance)
- The rate of energy production is given as SV where S is the rate of heat production per unit volume and V is volume

5 Heat Conduction with an Electrical Heat Source

(a) Shell Energy Balance

Problem: Find the temperature profile of a cylindrical wire with radius R, length L, an outside temperature of T_0, and a constant rate of heat production per unit volume of S_e.

Solution:

1. This problem is best done with cylindrical coordinates. Temperature is only a function of r, and the shell will become infinitesimally small in the radial direction
2. Setting up the shell energy balance with only conduction and a source yields

$$Aq\big|_{in} - Aq\big|_{out} + VS_e = 0$$

3. The conduction areas are the projection, which is the circumference times the length. The volume is simply the volume of the cylindrical shell

$$(2\pi r L q_r)\big|_r - (2\pi r L q_r)\big|_{r+\Delta r} + 2\pi r L \Delta r S_e = 0$$

4. Factoring out constants yields

$$2\pi L\left[(rq_r)\big|_r - (rq_r)\big|_{r+\Delta r}\right] + 2\pi r L \Delta r S_e = 0$$

5. Rearranging terms to make use of the definition of the derivative yields

$$2\pi L\left[(rq_r)\big|_{r+\Delta r}-(rq_r)\big|_{r}\right]-2\pi rL\Delta rS_e=0$$

6. Dividing by $2\pi L\Delta r$ and using the definition of the derivative yields

$$\frac{d\left(rq_r\right)}{dr}=rS_e$$

7. Integrating once yields

$$q_r=\frac{rS_e}{2}+\frac{C_1}{r}$$

8. There must be a value at $r=0$, but at $r=0$, the C_1 term blows up to infinity. Since this is non-physical, $C_1=0$ and the expression becomes

$$q_r=\frac{rS_e}{2}$$

9. Fourier's Law can now be used to introduce temperature such that

$$-k\frac{dT}{dr}=\frac{rS_e}{2}$$

10. Integrating once yields

$$T=-\frac{r^2S_e}{4k}+C_2$$

11. Since temperature must be continuous at the interface, at $r=R$, $T=T_0$. Therefore,

$$C_2=T_0+\frac{R^2S_e}{4k}$$

12. Rewriting the expression for temperature and simplifying yields

$$T-T_0=\frac{S_eR^2}{4k}\left[1-\left(\frac{r}{R}\right)^2\right]$$

(b) **Microscopic Energy Balance**

Problem: Repeat the previous example using the microscopic energy balance.
Solution:

1. Cylindrical coordinates are best used for this problem. Temperature is only a function of r, the system is at steady-state, the system is not moving, and k is assumed constant such that the microscopic energy balance becomes

$$0=k\nabla^2T+S_e$$

2. Substituting in for ∇^2 in cylindrical coordinates yields

$$k\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)=-S_e$$

3. Integrating once yields

$$r\frac{dT}{dr}=-\frac{S_er^2}{2k}+C_1$$

4. Integrating a second time yields

$$T=-\frac{S_er^2}{4k}+C_1\ln|r|+C_2$$

5. Since the C_1 term becomes unphysical at $r=0$, it must be true that $C_1=0$ such that

$$T=-\frac{S_er^2}{4k}+C_2$$

6. The temperature must be continuous at the interface, so at $r = R$, $T = T_0$ such that

$$C_2 = T_0 + \frac{S_e R^2}{4k}$$

7. This makes the final expression for the temperature

$$T - T_0 = \frac{SR^2}{4k} \left[1 - \left(\frac{r}{R} \right)^2 \right]$$

8. To get the heat flux, use Fourier's Law at Step 3 such that

$$-\frac{q_r r}{k} = -\frac{S_e r^2}{2k} + C_1$$

   (a) This is because $q_r = -k \dfrac{dT}{dr}$

9. Since C_1 was found to be zero in Step 5,

$$-\frac{q_r r}{k} = -\frac{S_e r^2}{2k}$$

10. Simplifying yields

$$q_r = \frac{S_e r}{2}$$