

LENGTH-BASED CONJUGACY SEARCH IN THE BRAID GROUP

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ABSTRACT. Several key agreement protocols are based on the following *Generalized Conjugacy Search Problem*: Find, given elements

$$b_1, \dots, b_n \text{ and } xb_1x^{-1}, \dots, xb_nx^{-1}$$

in a nonabelian group G , the conjugator x . In the case of subgroups of the braid group B_N , Hughes and Tannenbaum suggested a length-based approach to finding x . Since the introduction of this approach, its effectiveness and successfulness were debated.

We introduce several effective realizations of this approach. In particular, a length function is defined on B_N which possesses significantly better properties than the natural length associated to the Garside normal form. We give experimental results concerning the success probability of this approach, which suggest that an unfeasible computational power is required for this method to successfully solve the Generalized Conjugacy Search Problem when its parameters are as in existing protocols.

1. INTRODUCTION

Assume that G is a nonabelian group. The following problem has a long history and many applications (see [12]).

Problem 1.1 (Generalized Conjugacy Search Problem). *Given elements $b_1, \dots, b_n \in G$ and their conjugations by an unknown element $x \in G$,*

$$xb_1x^{-1}, xb_2x^{-1}, \dots, xb_nx^{-1},$$

find x (or any element $\tilde{x} \in G$ such that $\tilde{x}b_i\tilde{x}^{-1} = xb_ix^{-1}$ for $i = 1, \dots, n$.)

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In the sequel, we will not make any distinction between the actual conjugator x and any other conjugator \tilde{x} yielding the same results.

The *braid group* B_N is the group generated by the $N - 1$ *Artin generators* $\sigma_1, \dots, \sigma_{N-1}$, with the relations

$$\begin{aligned}\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i \text{ when } |i - j| > 1\end{aligned}$$

Information on the basic algorithms in the braid group is available in [3] and the references therein. We will focus on the case where G is the subgroup of B_N generated by given elements a_1, \dots, a_m . A solution of the generalized conjugacy problem in this case immediately implies the vulnerability of several cryptosystems introduced in [1, 15], and the methods of solution may be applicable to several other cryptosystems from [1, 18].

History, motivation, and related work. The length-based approach to the Conjugacy Problem was suggested by Hughes and Tannenbaum in [14], as a potential attack on the cryptosystems introduced in [1, 15]. Based on [14], Garrett [10] has doubted the security of these cryptosystems. But soon afterwards he published an errata withdrawing these doubts (see [12]). The reason was that no known realization of Hughes and Tannenbaum's scheme (i.e., definition of actual, effective length functions) was given before, and in particular, the success probability of this approach could not be estimated. The purpose of the current paper is to introduce and compare several such realizations, and provide actual success probabilities for specific parameters.

We stress that we are not interested here in the best possible solution of the generalized conjugacy problem, but rather in settling the debate concerning the applicability of the Hughes-Tannenbaum length-based approach to the problem.

Other approaches appear in [13, 17] and turn out more successful. However, the length-based approach has several advantages: First, one does not need to know the conjugated element in order to find the conjugator using this approach, and second, it essentially deals with *arbitrary equations*. The current paper gives the foundations of this approach, on which we build in [9], where an extension of this approach is suggested and good success rates are achieved for arbitrary equations.

Some of the citations of the present paper (see [2, 5, 11, 16, 19, 20, 21, 6]) refer to its preliminary draft [8], which contains much more details and examples. We have tried to make the present version concise.

Length-based attacks. Throughout this paper we make the following assumptions:

- (1) The conjugator x belongs to a given finitely generated subgroup of B_N , whose generators

$$\{a_1, \dots, a_m, a_1^{-1}, \dots, a_m^{-1}\}$$

are given,

- (2) x was generated as a product of a fixed, known number of generators $a_i^{\pm 1}$ chosen at random from the set of generators;
 (3) We are given elements

$$xb_1x^{-1}, \dots, xb_nx^{-1}$$

where each $b_i \in B_N$ is generated by some (nontrivial) random process, and we wish to find x .

We try to find the conjugator x by using the property that for an appropriate, efficiently computable *length function* ℓ defined on B_N , $\ell(a^{-1}ba)$ is usually greater than $\ell(b)$ for elements $a, b \in B_N$. Therefore, we try to reveal x by peeling off generator after generator from the given braid elements $xb_1x^{-1}, \dots, xb_nx^{-1}$: Assume that

$$x = g_1 \cdot g_2 \cdots g_k,$$

where each g_i is a *generator*. We fix some *linear order* \preccurlyeq on the set of all possible n -tuples of lengths, and choose a generator g for which the lengths vector

$$\langle \ell(g^{-1}xb_1x^{-1}g), \dots, \ell(g^{-1}xb_nx^{-1}g) \rangle$$

is minimal with respect to \preccurlyeq . With some nontrivial probability, g is equal to g_1 (or at least, x can be rewritten as a product of k or fewer generators such that g is the first generator in this product), so that $g^{-1}x = g_2 \cdots g_k$ is a product of fewer generators and we may continue this way, until we get all g_i 's forming x .

If one is capable of doing $O((2m)^t)$ computations, it is better to check all possibilities of $g_1 \cdots g_t$ by peeling off $g_1 \cdots g_t$ from x and choosing the t -tuple which yielded the minimal lengths vector. We will call this approach *look ahead of depth t* .

In order for any of the above to be meaningful, we must define the length function ℓ and the linear ordering \preccurlyeq . We will consider several candidates for these.

2. REALIZATIONS OF THE LENGTH FUNCTION

We assume that each generator a_i is obtained by taking a product of some fixed number of (randomly chosen) Artin generators, to whom we refer as the “length” of the generators. Unless otherwise stated, in all of our experiments the length of each element a_i is 10. By a *generator*

we mean either an element a_i , $i = 1, \dots, m$, or its inverse. We will (informally) write $|x| = n$ when we mean that x was generated by a product of n generators chosen at random (with uniform distribution) from the list of $2m$ generators $a_1, \dots, a_m, a_1^{-1}, \dots, a_m^{-1}$.

2.1. The length function ℓ . The *Garside normal form* of an element $w \in B_N$ is the unique presentation of w in the form $\Delta_N^{-r} \cdot p_1 \cdots p_k$, where $r \geq 0$ is minimal and p_1, \dots, p_k are permutation braids in left canonical form [3]. Using the Garside normal form, one can assign a “length” to each $w \in B_N$ efficiently [3].

Definition 2.1. The *Garside length* of an element $w \in B_N$, $\ell_G(w)$, is the number of Artin generators needed to write w in its Garside normal form. If the Garside normal form of w is $\Delta_N^{-r} \cdot p_1 \cdots p_k$, then

$$\ell_G(w) = r \cdot \binom{N}{2} + \sum_{i=1}^k |p_i|,$$

where $|p|$ denotes the length of the permutation p .¹

The problem with this function is that it is not close enough to being monotone with $|x|$: One has to multiply many generators before an increase in the length function is observed. The left part of Figure 1 shows, for a fixed word b , $\ell_G(xbx^{-1})$ as a function of $|x|$. Its right part shows the *average* of $\ell_G(xbx^{-1})$ computed over 1200 random words.

FIGURE 1. The growth of $\ell_G(w)$: Specific case (left) and average growth (right)

We wish to have a length function that is closer to being monotone. For each permutation braid p , $\tilde{p} := p^{-1}\Delta_N$ is a permutation braid.

¹The length of a permutation p is the number of order distortions in p , that is, pairs (i, j) such that $i < j$ and $p(i) > p(j)$.

Thus, if $w = \Delta_N^{-r} \cdot p_1 \cdots p_k$ and $r > 0$, we can replace $\Delta_N^{-1} p_1$ with \tilde{p}_1^{-1} to get $w = \Delta_N^{-(r-1)} \cdot \tilde{p}_1^{-1} p_2 \cdots p_k$. Now, Δ_N almost commutes with any permutation braid: For each permutation braid q there exists a permutation braid q' such that $|q'| = |q|$ and $q\Delta_N = \Delta_N q'$, that is, $\Delta_N^{-1} q^{-1} = (q')^{-1} \Delta_N^{-1}$. Consequently, $w = \Delta_N^{-(r-2)} \cdot (\tilde{p}_1')^{-1} \Delta_N^{-1} p_2 \cdots p_k$, and we can replace $\Delta_N^{-1} p_2$ with \tilde{p}_2^{-1} as before. We iterate this process as much as possible, to get a presentation

$$w = \begin{cases} \Delta_N^{-(r-k)} (\tilde{p}_1')^{-1} \cdots (\tilde{p}_k')^{-1} & k < r \\ (\tilde{p}_1')^{-1} \cdots (\tilde{p}_r')^{-1} \cdot p_{r+1} \cdots p_k & r \leq k \end{cases}$$

In each case, w has the form $a^{-1}b$ where a, b are positive braid words or the identity element, and we define the *reduced Garside length* to be the sum of the length of a and the length of b .² This is equivalent to the following.

Definition 2.2. Let $w = \Delta_N^{-r} \cdot p_1 \cdots p_k$ be the Garside normal form of w . The *Reduced Garside length* of w is defined by

$$\ell_{\text{RG}}(w) = \ell_{\text{G}}(w) - 2 \sum_{i=1}^{\min(r,k)} |p_i|$$

This function turns out much closer to monotone than ℓ_{G} – see Figure 2.

FIGURE 2. The growth of $\ell_{\text{RG}}(w)$: Specific case (left) and average growth (right)

²The length of a positive braid word is well defined to be the number of generators in its presentation.

2.2. Statistical comparison of the length functions. The purpose of the length function ℓ is to distinguish between the case $|X| = k - 1$ (after peeling off a correct generator) and $|X| = k + 1$ (after trying to peel off a wrong generator). Thus, a natural measure for the effectiveness of the length function is the distance in standard deviations between $\ell(X')$ and $\ell(X)$ when $|X'| = |X| + 2$.

We fixed a random set of 20 generators in B_{81} , and computed (an approximation of) $E(\ell(X') - \ell(X)) / \sqrt{V(\ell(X') - \ell(X))}$ as a function of $|X|$ for $|X| = 1, \dots, 100$. (Roughly speaking, when n independent samples are added, the effectiveness of the comparison is \sqrt{n} times this number.) We did that for both ℓ_G and ℓ_{RG} . The results appear in Figure 3, and show that the score for ℓ_{RG} is significantly higher. This phenomenon is typical – we have checked several random subgroups of the braid group and all of them exhibited the same behavior.

FIGURE 3. Distance between right and wrong in standard deviations.

More evidence for the superiority of ℓ_{RG} over ℓ_G will be given in the following sections.

3. REALIZATIONS OF THE LINEAR ORDERING \preceq

Recall that after peeling off a candidate for a generator and evaluating the resulting lengths, we need to compare the vectors of lengths according to some linear ordering \preceq , and choose a generator which gave a minimal vector with respect to \preceq . We tested two natural linear orderings.

The most natural approach is to take the average of the lengths in the vector. This is equivalent to the following.

Definition 3.1 (Average based linear ordering).

$$\langle \alpha_1, \dots, \alpha_n \rangle \preceq_{\text{Av}} \langle \beta_1, \dots, \beta_n \rangle \quad \text{if} \quad \sum_{i=1}^n \alpha_i \leq \sum_{i=1}^n \beta_i.$$

With this at hand, we have performed the following experiment. We fixed a subgroup of B_{81} generated by $m = 20$ generators. Then we chose at random 200 elements of the form xw_j which share the same leading prefix x , and for each generator $a_i^{\pm 1}$ we computed $\ell(a_i^{\pm 1}xw_j)$ for each j (and $\ell = \ell_G$ or ℓ_{RG}). For each of these two length functions ℓ , we have sorted the resulting length vectors according to \preceq_{Av} and checked the position of the “correct” generator, i.e., the generator which appeared leftmost in our computation of the word.³ We repeated the computations for 138 distinct X ’s, and for $|X| = 40$ and $|X| = 100$. For an ideal length function (and an ideal linear ordering \preceq), the correct generator would always be ranked first, and the results in Figure 4 show that ℓ_{RG} is closer to this ideal than ℓ_G : In the graphs, we show the *distribution* (lower part of the graph) and the *accumulated distribution* (upper part of the graph) of the position of the correct generator, for each of the length functions.

FIGURE 4. Position of correct generator ℓ_G and ℓ_{RG}

³In principle there could be more than one “correct” generator, but when the generators are long enough this is unlikely to happen often.

However, it turns out that even for the better length function ℓ_{RG} , the task of identifying the correct generator is not trivial. To demonstrate this, we selected at random one of the cases of x from the previous experiment, and computed over the given 200 samples the distribution of ℓ_{RG} for each generator. Figure 5 shows the distribution for the correct generator (in boldface) and of arbitrarily chosen 7 out of the remaining 40 generators (for an aesthetic reason we did not plot all 40).

FIGURE 5. Actual distribution.

While the correct distribution tends more to the left (i.e., to smaller values), there is a large overlap with the rest of the distributions. We must emphasize that while Figure 5 demonstrates the typical case, there exist cases where the distribution of the correct generator is not the leftmost. In these cases the current method is doomed to fail, no matter how many conjugations we are given for the same conjugator.

Finally, for the sake of comparison, we define one more natural linear ordering of the space of length vectors. We expect the correct generator to yield the shortest length more often than the other generators. This motivates the following definition.

Definition 3.2 (Majority based linear ordering). Consider the set of all obtained length vectors. For each $i = 1, \dots, n$, consider the i th coordinate of each vector and let μ_i denote the minimum of all these i th coordinate values. Then

$$\langle \alpha_1, \dots, \alpha_n \rangle \preceq_{\text{Maj}} \langle \beta_1, \dots, \beta_n \rangle \quad \text{if} \quad |\{i : \alpha_i = \mu_i\}| \geq |\{i : \beta_i = \mu_i\}|.$$

In the following section we compare the success probabilities of the length-based approach using the two length functions and two linear orderings defined in this section.

4. EXPERIMENTAL RESULTS FOR THE CONJUGACY PROBLEM

4.1. The probability of obtaining the correct generator. In this experiment we determine the probability that the correct generator is indeed the minimal with respect to the length function ℓ and linear ordering \preceq used. The choice of parameters in the experiments throughout the paper are usually motivated by the choices given in [1], which are believed there to make the generalized conjugacy problem difficult.

We made 200 experiments using the following parameters: $N = 81$, n and m (the number of a_i 's and b_i 's, respectively) are both 20, the elements a_i and b_i are products of 10 random Artin generators, and x is the product of 5, 10, 20, 40, 60, or 100 random generators $a_i^{\pm 1}$, respectively. We tested look ahead depth $t = 1, 2$. In each cell of Table 1, below the probability that the correct generator is first, we wrote the probability of its being second.

	5	10	20	40	60	100
$\ell_G, \preceq_{Av}, t = 1$	0.56 0.16	0.478 0.188	0.322 0.1	0.267 0.167	0.233 0.089	0.156 0.1
$\ell_G, \preceq_{Maj}, t = 1$	0.43 0.14	0.344 0.178	0.222 0.144	0.244 0.122	0.178 0.1	0.156 0.044
$\ell_{RG}, \preceq_{Av}, t = 1$	0.74 0.13	0.589 0.233	0.567 0.189	0.456 0.122	0.311 0.167	0.233 0.167
$\ell_{RG}, \preceq_{Maj}, t = 1$	0.71 0.15	0.578 0.267	0.578 0.133	0.433 0.089	0.289 0.167	0.211 0.167
$\ell_G, \preceq_{Av}, t = 2$	0.433 0.156	0.287 0.08	0.111 0.087	0.1 0.038	0.114 0.055	0.099 0.035
$\ell_G, \preceq_{Maj}, t = 2$	0.25 0.033	0.147 0.036	0.103 0.024	0.058 0.008	0.086 0.023	0.03 0.03
$\ell_{RG}, \preceq_{Av}, t = 2$	0.578 0.183	0.526 0.127	0.333 0.135	0.242 0.138	0.2 0.105	0.168 0.05
$\ell_{RG}, \preceq_{Maj}, t = 2$	0.511 0.139	0.482 0.139	0.31 0.127	0.242 0.104	0.186 0.091	0.149 0.089

TABLE 1. The probability that the correct generator is first or second

Table 1 shows that the Reduced Garside length function ℓ_{RG} is significantly better than the standard Garside length function ℓ_G . Also,

observe that using look ahead depth 2 is preferable to using look ahead depth 1 twice (to see this, square the probabilities for $t = 1$). Another natural approach to using look ahead $t > 1$ is to consider only the first generator (of the word with the least score) as correct, and ignore the rest of the generators. This means that in the algorithm for finding x , we peel off only one generator at a time despite the fact that we used look ahead $t > 1$. This gives better success rates than just taking $t = 1$, and our experiments indicate that this approach may be slightly better than that of taking the whole look ahead word, but we did not extensively check this conjecture since the differences were not significant. Some other variants of the usage of look ahead are mentioned in [8].

4.2. Nonsymmetric parameters. This experiment checks the effect on the probability of success when the lengths of the generators a_i and elements b_i (in terms of Artin generators) are not equal.

We tested the probability of success for $N = 81$, $n = m = 20$, look ahead depth $t = 2$, and $|x| = 30$.

	length of b_i	ℓ_{RG}					ℓ_{G}				
		a_i of length:					a_i of length:				
		5	10	15	20	25	5	10	15	20	25
\preccurlyeq_{Av}	5	44	82	124	134	156	32	51	81	102	115
	10	59	97	113	141	150	56	69	79	91	96
	15	56	91	123	136	141	31	53	75	93	105
	20	49	77	115	132	149	31	49	77	86	107
	25	56	84	102	127	141	42	59	60	91	100
$\preccurlyeq_{\text{Maj}}$	5	39	70	121	134	160	28	41	66	84	87
	10	57	97	114	140	156	49	49	58	83	82
	15	50	85	118	136	144	19	45	59	73	89
	20	48	80	116	133	149	39	41	52	73	86
	25	60	89	101	141	152	39	50	56	72	78

TABLE 2. Number of success out of 200 tries for different lengths

As expected, Table 2 shows that if the length of the elements a_i increases then so does the probability to find a correct generator (this is like making look ahead deeper without exponentially increasing the number of candidates for the prefix of x). On the other hand, the effect of the length of the elements b_i is not significant.

4.3. Increasing the number of given conjugates. Several experiments showed that increasing the number n of given elements $xb_i x^{-1}$

from few (about 10) to many (about 3000) did not significantly increase the probability that the correct generator appears first.

In an instance of the problem the length function ℓ and the (unknown) element x are fixed, and this defines for each generator g the distribution F_g of $\ell(g^{-1}xbx^{-1}g)$ over random words b of a fixed given length (in terms of Artin generators). For each g , we have a sample of the distribution F_g for each given equation. In most cases, the expectancy of F_g where g is the first letter in x is smaller than the other expectancies (see Section 3), and then enough samples will allow us to identify g . However in some cases the minimal expectancy is obtained for another generator. In these cases adding more samples cannot help, and so the probability to find the correct generator does not tend to 1 when we increase the number of samples.

Another important observation is that few samples (about 15) are needed in order to get very close to the expectancy of the distributions F_g . In light of the preceding paragraph, the outcome of the algorithm can be decided after a relatively small number of samples (i.e., given conjugates) are collected. In particular, the success probability does not significantly improve when n is large.

4.4. Finding x . The simplest way to try and obtain all generators of x and therefore x would be to use any of the above algorithms iteratively, at each step peeling off the first generator. In the following experiment, the probability to find *all of* x this way was tested. Here too, the lengths of the a_i 's and b_i 's were 10 Artin generators. We made 500 experiments, using a weaker variant of ℓ_{RG} as the length function, and with no look ahead ($t = 1$). We repeated this for B_4, \dots, B_{20} and x of lengths 2 to 18 generators $a_i^{\pm 1}$. The result is the number of successes out of 500 tries.

N	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
4	429	361	289	262	204	181	137	120	107	94	77	52	50	37	38	25	31
5	436	378	327	269	215	185	173	120	119	106	75	67	56		44		28
6	446		324	282	243		183	154	115	107	88	68	65	59	36	47	
7	453	400	330	287		208	176	142	126	97	74	69	50		35	39	33
8	440	396	275	230	198	149	137	116	103	63	57	51	39	34	37	25	23
9	463	404	334	276	208	180	148	121	86	70	73	41	44	29	29	17	15
10	461	383	328	274	221	165	156	113	83	71	60	46	42	30	26	10	17
14	460	377	295	244		140	108	79	54	41	33	19	14	14	8	9	8
17	453	365	293	221	167	118	89	56	56	33	16	17	10	4	2	4	
20	455	373	305	226	153		73	43	36	21	11		8		3	3	2

TABLE 3. Number of successes for finding x out of 500 tries

The results suggest that while we already obtain solutions (with non-trivial probability) for some nontrivial parameters, we must extend the

approach in order to consider harder parameters. A successful extension is discussed in [9]. In the sequel we discuss some other possible extensions.

5. POSSIBLE IMPROVEMENTS AND CONCLUSIONS

One approach is to create new conjugates by multiplying any number of existing ones (or their inverses). In fact, if \mathcal{B} is the group generated by b_1, \dots, b_n , then the group generated by $xb_1x^{-1}, \dots, xb_nx^{-1}$ is $x\mathcal{B}x^{-1}$. By Section 4.3, this does not help much.

The algorithm can be randomized by conjugating the given elements $xb_1x^{-1}, \dots, xb_nx^{-1}$ by a random (known) element $y \in \langle a_1, \dots, a_m \rangle$, so that running it several times increases the success probability. The problem with this approach is that the conjugator becomes longer and therefore the probability of success in each single case decreases.

Our experiments showed that the peeling off process often enters a loop, that is, a stage to which we return every several steps. This can sometimes be solved by conjugating with a random known element after we enter the loop. We also tried to change the length function or the linear ordering when we enter a loop. These approaches were successful for small parameters but did not result in a significant improvement for large parameters.

We did not try approaches of learning algorithms, neural networks, etc. A simple example is to try and learn the distribution of the lengths for the correct generator and define the linear ordering according to the likelihood test.

The purpose of this paper was to check the applicability of Hughes and Tannenbaum's length-based approach against the key agreement protocols introduced in [1, 15]. Our results suggest that this approach requires an unfeasible computational power in order to solve the generalized conjugacy search problem for the parameters used in these protocols. However, this method has natural extensions which can make it applicable: In [9] we suggest one particularly successful extension, and it turns out that it can solve these and other problems with standard computational power.

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