Problem Set 2.

Exercise 1.)

a) Show that the linear projection coefficient solve the problem

$$\min_{a_0,a_1,a_2\dots} E\{[Y-(a_0+a_1X_1+a_2X_2+\dots)]^2\}$$

Solution:

$$\hat{Y} := a_0 + a_1X_1 + a_2X_2 + \dots$$

$$\min E\{[Y-\hat{Y}]^2\} \Leftrightarrow \min E\{[Y-\hat{E}(Y|X) + \hat{E}(Y|X) - \hat{Y}]^2\}$$

$$\Leftrightarrow [Y-\hat{E}(Y|X) + \hat{E}(Y|X) - \hat{Y}]^2 = [(Y-\hat{E}(Y|X)) - (\hat{Y}-\hat{E}(Y|X))]^2 =$$

$$= [Y-\hat{E}(Y|X)]^2 - 2\underbrace{(Y-\hat{E}(Y|X))}(\hat{Y}-\hat{E}(Y|X)) + (\hat{Y}-\hat{E}(Y|X))^2$$

by definition

$$E(Y-\hat{E}(Y|X)) = 0$$

linear projection

$$E\{(Y-\hat{E}(Y|X))(\hat{Y}-\hat{E}(Y|X))\} = E\{(Y-\hat{E}(Y|X))\}E\{\hat{Y}-\hat{E}(Y|X)\}$$

$$Y-\hat{E}(Y|X) \underset{(*)}{\perp} \underbrace{\hat{Y}-\hat{E}(Y|X)}_{\hat{\Gamma}}$$

$$\hat{E}(Y|X) = \alpha_0^* + \alpha_1^* X_1 + \dots$$

$$\hat{\Gamma} = \underbrace{\hat{Y}-\hat{E}(Y|X)}_{} = \underbrace{a_0-\alpha_0^*}_{\gamma_0} + \underbrace{(a_1-\alpha_1^*)}_{\gamma_1}X_1 + \underbrace{(a_2-\alpha_2^*)}_{\gamma_2}X_2 + \dots$$

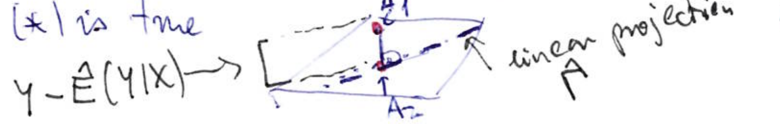
nothing else then a linear projection with different coefficients

Linear projection, iii) $E([Y_{t+h}-\hat{E}(Y_{t+h}|I_t)]I_{jt}) = 0 \quad \forall_j = 1,2,3\dots$

e.g. $E(\{\hat{Y}-\hat{E}(Y|X)\}\hat{\Gamma}) = 0$

it can be interpreted as an information set (linear combination of X_is) I_X contains the same

(*) is true



(**)

A_1 red ~~pro~~ dot is projected in linear way to a hyperplane (A_2). The distance between A_1 and A_2 ($= Y-\hat{E}(Y|X)$) is orthogonal to the ~~xxx~~ every line on hyperplane.

$$E\left[(Y-\hat{E}(Y|X)) - (\hat{Y}-\hat{E}(Y|X))\right]^2 = E\left[(Y-\hat{E}(Y|X))^2\right.$$

$$\left. -2(Y-\hat{E}(Y|X))[\hat{Y}-\hat{E}(Y|X)] + (\hat{Y}-\hat{E}(Y|X))^2\right] =$$

$$= E\left((Y-\hat{E}(Y|X))^2 + (\hat{Y}-\hat{E}(Y|X))^2\right) - 2E\{(Y-\hat{E}(Y|X))(\hat{Y}-\hat{E}(Y|X)$$

$$\| \ast\ast$$

$$E(Y-\hat{E}(Y|X))\, E(\hat{Y}-\hat{E}(Y|X)$$

$$= \underbrace{E(Y-\hat{E}(Y|X))^2 + E[\hat{Y}-\hat{E}(Y|X))^2]}$$

only this expression should be minimized.

$$\underbrace{Y-\hat{E}(Y|X)}_{(a)} \perp \underbrace{\hat{Y}-\hat{E}(Y|X)}_{(b)}$$

(a) and (b) are "ortogonal"

$$\min_{a_0,a_1,\ldots} E\left([Y-\hat{E}(Y|X)]^2 + [\hat{Y}-\hat{E}(Y|X)]^2\right) =$$

$$= \min_{a_0,a_1,\ldots} E\left([Y-(\alpha_0^* + \alpha_1^* X_1 + \ldots)]^2 + \{[a_0+a_1X_1+\ldots] - [\alpha_0^*+\alpha_1^* X_1+\ldots]\}^2\right)$$

only this part depends on a_i $i=0,\ldots$

$$\frac{\partial E(\cdot)}{\partial a_i} = 2\cdot E\left([a_0+a_1X_1+\ldots] - [\alpha_0^*+\alpha_1^*X_1+\ldots]\right)\cdot X_i = 0$$

chain rule

$$\Updownarrow \quad \text{if } X_i \neq 0$$

$$E[(a_0-\alpha_0^*) + (a_1-\alpha_1^*)X_1 + \ldots] = 0$$

$$\Updownarrow$$

$$a_0 = \alpha_0^* ;\ a_1 = \alpha_1^* ; \ldots$$

$\Rightarrow$ linear projection solves the problem.

Exercise 1.)

Part b)

Suppose that X is a scalar and write down the explicit expression for α_0^* and α_1^*?

exercise $\mathcal{L} := \min_{a_0, a_1} E\{[Y - (a_0 + a_1 X)]^2\}$

$$\frac{\partial \mathcal{L}}{\partial a_0} = 2E(Y - a_0 - a_1 X)\cdot(-1) = 0 \quad \cdot -\tfrac{1}{2}$$

$$E(Y - a_0 - a_1 X) = 0 \quad (*)$$

$$\frac{\partial \mathcal{L}}{\partial a_1} = 2E(Y - a_0 - a_1 X)(-X) = 0 \quad \cdot -\tfrac{1}{2}$$

$$E((Y - a_0 - a_1 X)X) = 0$$

$$E(YX - a_0 X - a_1 X^2) = 0 \quad (**)$$

$(*)$ $$E(Y - a_0 - a_1 X) = E(Y) - a_0 - a_1 E(X) = 0$$

$$\bar{Y} - a_0 - a_1 \bar{X} = 0$$

$$\Rightarrow a_0 = \bar{Y} - a_1 \bar{X} \quad (\square)$$

$(**)$ $$E(YX - a_0 X - a_1 X^2) = E(YX) - a_0 E(X) - a_1 E(X^2) = 0$$

$$Var(X) = E(X^2) - E^2(X) \rightarrow E(X^2) = Var(X) + (E(X))^2$$

$$E(YX) - a_0 EX - a_1 (Var(X) + (E(X))^2) = 0$$

$$E(YX) - a_0 EX - a_1 Var(X) - a_1 (E(X))^2 = 0$$

$$E(YX) - a_0 \bar{X} - a_1 Var(X) - a_1 \bar{X}^2 = 0$$

$$E(YX) - (\bar{Y} - a_1 \bar{X})\bar{X} - a_1 Var(X) - a_1 \bar{X}^2 = 0$$

$$E(YX) - \bar{Y}\bar{X} + \cancel{a_1 \bar{X}^2} - a_1 Var(X) \cancel{- a_1 \bar{X}^2} = 0$$

$$\underbrace{E(YX) - E(Y)E(X)} - a_1 Var(X) = 0$$

$$Cov(Y, X) - a_1 Var(X) = 0 \rightarrow \boxed{a_1 = \frac{Cov(Y, X)}{Var(X)}} \quad (\square\square)$$

(I) $a_0 = \overline{Y} - a_1 \overline{X} = \overline{Y} - \frac{Cov(Y,X)}{Var(X)} \overline{X}$

(II) $a_1 = \frac{Cov(Y,X)}{Var(X)}$

We know the linear projection results $a_0 = \alpha_0^*$ and $a_1 = \alpha_1^*$

(Showed in part a).)

$$\Rightarrow \alpha_0^* = a_0 = \overline{Y} - \frac{Cov(Y,X)}{Var(X)} \overline{X} = E(Y) - \frac{Cov(Y,X)}{Var(X)} E(X)$$

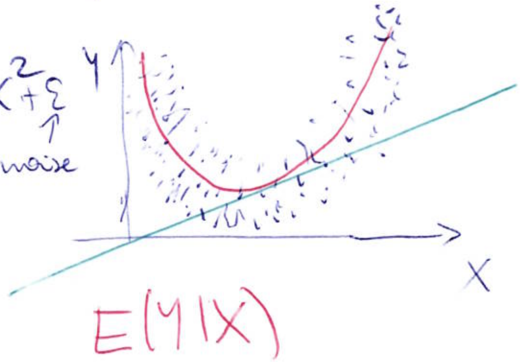
$$\alpha_1^* = a_1 = \frac{Cov(Y,X)}{Var(X)}$$

Remark: α_0^* and α_1^* are nothing else what we got in the case of linear regression (OLS). $\rightarrow$ OLS is a linear projection.

Exercise 1

c) Given example $\hat{E}(Y|X) \neq E(Y|X)$

1) $Y = X^2 + \varepsilon$ ← noise



$\hat{E}(Y|X) = \alpha_0^* + \alpha_1^* X$

$$E(Y|X) = E(X^2|X) = X^2$$

X^2 can be calculated from X

$Y = X^2$ measurable on X.

$$\hat{E}(Y|X) = \alpha_0 + \alpha_1 X$$

Exercise 2.)

$\Delta Y_t = AR(1)$

$$\Delta Y_t = c + \phi \Delta Y_{t-1} + \varepsilon_t \qquad |\phi| < 1$$

$$\Delta Y_t = \frac{c + \varepsilon_t}{1 - \phi L}$$

a)

$c = \text{drift} = \delta$ The answer is: the drift is only the constant c.

$$\hat{Y}_{t+h|t} = Y_t + \Delta Y_{t+1} + \Delta Y_{t+2} + \dots + \Delta Y_{t+h}$$

$$= Y_t + (c + \phi \Delta Y_t + \varepsilon_{t+1}) + (c + \phi c + \phi^2 \Delta Y_t + \phi \varepsilon_{t+1}$$

$$+ \varepsilon_{t+2} + \dots + \phi^h \Delta Y_t + \dots \quad (*1)$$

$$\hat{Y}_{t+h|t} = E(Y_{t+h} | I_t)$$

$E(\varepsilon_{t+j}) = 0$, so in the calculation of $\hat{Y}_{t+h|t}$

$\forall \varepsilon_{t+j}$, $j \geq 1$ do not matter

$$= Y_t + \sum_{i=1}^{h} \phi^i \Delta Y_t + \underbrace{\sum_{i=1}^{h} (h - i + 1) c \phi^{i-1}}_{} =$$

$$\sum_{i=1}^{h} \left(hc\phi^{i-1} - i\phi^{i-2} + c\phi^{i-1} \right)$$

$$\frac{hc(1-\phi^h)}{1-\phi} - c\sum^{h} i\phi^{i-1} + \frac{c(1-\phi^h)}{1-\phi}$$

$$= Y_t + \frac{(h+1) \cdot c(1-\phi^h)}{1-\phi} + \frac{\phi(1-\phi^h)}{1-\phi} \Delta Y_t - c\sum_{i=1}^{h} \phi^{i-1} \cdot i$$

b) $\hat{y}_{t+h|t} = y_t + \frac{(h+1)c(1-\phi^h)}{1-\phi} + \frac{\phi(1-\phi^h)\Delta y_t}{1-\phi}$ (7

$- c\sum_{i=1}^{h} \phi^{i-1} i$

c) $\lim_{h\to\infty} \left(\hat{y}_{t+h|t} - y_t - \delta h\right)$

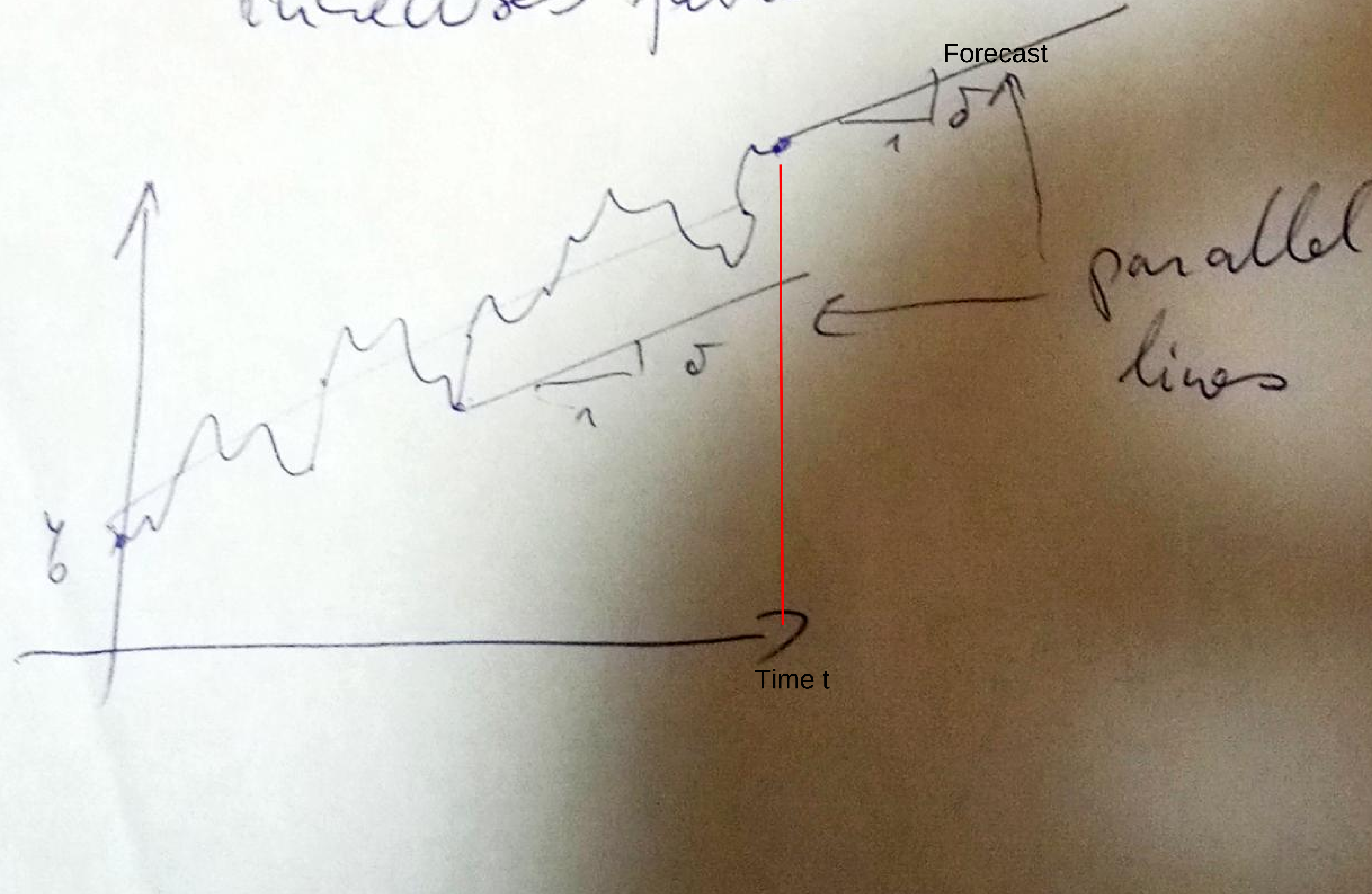
$\lim \big(\cancel{y_t} + \underbrace{\frac{\phi(1-\phi^h)}{1-\phi}}_{\to \frac{\phi}{1-\phi}} \Delta y_t + \underbrace{\frac{(h+1)c(1-\phi^h)}{1-\phi}}_{\to 0} -$

$\underbrace{- c\sum_{i=1}^{h} \phi^{i-1} i}_{\to 0} - \cancel{y_t} - \underset{c}{\overset{}{\delta}} h \longrightarrow -\infty$

$x\,\phi^x \longrightarrow 0$

increases faster than x

d)



Exercise 3.)

ARMA (1,1) : $Y_t = 0.2\, Y_{t-1} + \varepsilon_t + 0.1\, \varepsilon_{t-1}$; $Var(\varepsilon_t) = 1$ -

AR(2) : $Y_t = 0.32\, Y_{t-1} - 0.03\, Y_{t-2} + \eta_t$; $Var(\eta_t) \approx 1$.

-> general notation: ARMA(1,1) : $Y_t = \gamma_1 Y_{t-1} + \varepsilon_t + \gamma_2 \varepsilon_{t-1}$

AR(2) : $Y_t = \phi_1 Y_{t-1} - \phi_2 Y_{t-2} + \eta_t$

-> projection with the ARMA(1,1) model:

$Y_t = \gamma_1 Y_{t-1} + \varepsilon_t + \gamma_2 \varepsilon_{t-1}$

$Y_{t+1} = \gamma_1 Y_t + \underbrace{\varepsilon_{t+1}}_{\text{future shock}} + \gamma_2 \varepsilon_t^{\in I_t}$ ← only depends on information available at time t!

$\hat{E}(Y_{t+1} | I_t) = \hat{Y}_{t+1|t} = \gamma_1 Y_t + \varepsilon_t$

$Y_{t+2} = \gamma_1 Y_{t+1} + \underbrace{\varepsilon_{t+2} + \gamma_2 \varepsilon_{t+1}}_{\text{future shocks}} = \gamma_1 (\gamma_1 Y_t + \varepsilon_{t+1} + \gamma_2 \varepsilon_t) + \varepsilon_{t+2}$

$+ \gamma_2 \varepsilon_{t+1} = \underbrace{\gamma_1^2 Y_t}_{\in I_t} + \gamma_1 \varepsilon_{t+1} + \underbrace{\gamma_1 \gamma_2 \varepsilon_t}_{\in I_t} + \varepsilon_{t+2} + \gamma_2 \varepsilon_{t+1}$

$= \boxed{\underbrace{\gamma_1^2 Y_t + \gamma_1 \gamma_2 \varepsilon_t}_{\in I_t} + \underbrace{(\gamma_1 + \gamma_2) \varepsilon_{t+1} + \varepsilon_{t+2}}_{\text{future shocks}}}$

$\hat{Y}_{t+2|t} = \gamma_1^2 Y_t + \gamma_1 \gamma_2 \varepsilon_t$

$Y_{t+3} = \gamma_1 Y_{t+2} + \varepsilon_{t+3} + \gamma_2 \varepsilon_{t+2} = \gamma_1 (\gamma_1^2 Y_t + \gamma_1 \gamma_2 \varepsilon_t + (\gamma_1 + \gamma_2) \varepsilon_{t+1} + \varepsilon_{t+2})$

$+ \varepsilon_{t+3} + \gamma_2 \varepsilon_{t+2} = \gamma_1^3 Y_t + \gamma_1^2 \gamma_2 \varepsilon_t + (\gamma_1^2 + \gamma_2 \gamma_1) \varepsilon_{t+1} + \gamma_1 \varepsilon_{t+2}$

$+ \varepsilon_{t+3} + \gamma_2 \varepsilon_{t+2} = \underbrace{\gamma_1^3 Y_t + \gamma_1^2 \gamma_2 \varepsilon_t}_{\in I_t} + \underbrace{(\gamma_1^2 + \gamma_1 \gamma_2) \varepsilon_{t+1} + (\gamma_1 + \gamma_2) \varepsilon_{t+2}}_{\text{future shocks}}$

$+ \underbrace{\varepsilon_{t+3}}_{\text{future shocks}}$

$\hat{Y}_{t+3|t} = \gamma_1^3 Y_t + \gamma_1^2 \gamma_2 \varepsilon_t$

...

$\hat{Y}_{t+3|t} = \varphi_1^3 Y_t + \varphi_1^2 \varphi_2 \mathcal{E}_t$

...

$\lim_{h \to \infty} \hat{Y}_{t+h|t} \longrightarrow 0.$ If the forecast horizon 'approximates infinity', than the prediction will converge to zero.

~~It's hard to give a~~

where $\hat{Y}_{t+h|t} = \varphi_1^h \cdot Y_t + \varphi_1^{h-1} \cdot \varphi_2 \mathcal{E}_t$

if $|\varphi_1| < 1$ than limit will go to zero.

and $\varphi_1 = 0.2$ $|0.2| < 1$ ✓

$|\varphi_2| = |0.1| < 1$ ✓

AR(2) model → prediction / forecast:

$Y_t = \phi_1 Y_{t-1} - \phi_2 Y_{t-2} + \eta_t$

$Y_{t+1} = \boxed{\phi_1 Y_t - \phi_2 Y_{t-1} + \eta_{t+1}}$ → $\hat{Y}_{t+1|t} = \phi_1 Y_t - \phi_2 Y_{t-1}$

future error

$Y_{t+2} = \phi_1 Y_{t+1} - \phi_2 Y_t + \eta_{t+2} = \phi_1(\phi_1 Y_t - \phi_2 Y_{t-1} + \eta_{t+1}) - \phi_2 Y_t + \eta_{t+2}$

$= \phi_1^2 Y_t - \phi_1\phi_2 Y_{t-1} + \phi_1 \eta_{t+1} - \phi_2 Y_t + \eta_{t+2} =$

$\boxed{(\phi_1^2 - \phi_2) Y_t - \phi_1\phi_2 Y_{t-1} + \phi_1 \eta_{t+1} + \eta_{t+2}}$

$\hat{Y}_{t+2|t} = (\phi_1^2 - \phi_2) Y_t - \phi_1\phi_2 Y_{t-1}$

$Y_{t+3} = \phi_1 Y_{t+2} - \phi_2 Y_{t+1} + \eta_{t+3} = \phi_1 \left[(\phi_1^2 - \phi_2) Y_t - \phi_1\phi_2 Y_{t-1} + \phi_1 \eta_{t+1} \right.$

$\left. + \eta_{t+2} \right] - \phi_2 (\phi_1 Y_t - \phi_2 Y_{t-1} + \eta_{t+1}) + \eta_{t+3} =$

$= \underbrace{(\phi_1^3 - \phi_1\phi_2) Y_t} \underline{- \phi_1^2 \phi_2 Y_{t-1}} + \phi_1^2 \eta_{t+1} + \phi_1 \eta_{t+2} -$

$- \phi_1\phi_2 Y_t \underline{+ \phi_2^2 Y_{t-1}} - \phi_2 \eta_{t+1} + \eta_{t+3} =$

$(\phi_1^3 - 2\phi_1\phi_2) Y_t + (\phi_2^2 - \phi_1^2\phi_2) Y_{t-1} + (\phi_1^2 - \phi_2) \eta_{t+1} + \phi_1 \eta_{t+2} + \eta_{t+3}$

$$y_{t+3} = \boxed{\underbrace{(\phi_1^3 - 2\phi_1\phi_2) y_t + (\phi_2^2 - \phi_1^2\phi_2) y_{t-1}}_{\in I_t} + (\phi_1^2 - \phi_2)\eta_{t+1} + \phi_1 \eta_{t+2} + \eta_{t+3}}$$

$$\hat{y}_{t+3|t} = (\phi_1^3 - 2\phi_1\phi_2) y_t + (\phi_2^2 - \phi_1^2\phi_2) y_{t-1}$$

$$y_{t+4} = \phi_1 y_{t+3} - \phi_2 y_{t+2} + \eta_{t+4} = \phi_1 \big((\phi_1^3 - 2\phi_1\phi_2) y_t + (\phi_2^2 - \phi_1^2\phi_2) \ldots$$

$$+ (\phi_1^2 - \phi_2)\eta_{t+1} + \phi_1\eta_{t+2} + \eta_{t+3}\big) - \phi_2\big((\phi_1^2 - \phi_2) y_t -$$

$$- \phi_1\phi_2 y_{t-1} + \phi_1\eta_{t+1} + \eta_{t+2}\big) + \eta_{t+4} =$$

$$(\phi_1^4 - 2\phi_1^2\phi_2) y_t + (\phi_1\phi_2^2 - \phi_1^3\phi_2) y_{t-1} + (\phi_1^3 - \phi_1\phi_2)\eta_{t+1} + \phi_1^2\eta_{t+2}$$

$$+ \phi_1\eta_{t+3} - (\phi_1^2\phi_2 - \phi_2^2) y_t + \phi_1\phi_2^2 y_{t-1} - \phi_1\phi_2\eta_{t+1} - \phi_2\eta_{t+2}$$

$$+ \eta_{t+4} =$$

$$\boxed{\underbrace{(\phi_1^4 - 3\phi_1^2\phi_2 + \phi_2^2) y_t + (2\phi_1\phi_2^2 - \phi_1^3\phi_2) y_{t-1}}_{\in I_t} + \underbrace{(\phi_1^3 - 2\phi_1\phi_2)\eta_{t+1} + (\phi_1^2 - \phi_2)\eta_{t+2} + \phi_1\eta_{t+3} + \eta_{t+4}}_{\text{future shocks}}}$$

$$\hat{y}_{t+4|t} = (\phi_1^4 - 3\phi_1^2\phi_2 + \phi_2^2) y_t + (2\phi_1\phi_2^2 - \phi_1^3\phi_2) y_{t-1}$$

$$y_{t+5} = \phi_1 y_{t+4} - \phi_2 y_{t+3} + \eta_{t+5} = \phi_1\big((\phi_1^4 - 3\phi_1^2\phi_2 + \phi_2^2) y_t + (2\phi_1\phi_2^2 - \phi_1^3\phi_2) y_{t-1}$$

$$+ (\phi_1^3 - 2\phi_1\phi_2)\eta_{t+1} + (\phi_1^2 - \phi_2)\eta_{t+2} + \phi_1\eta_{t+3} + \eta_{t+4}\big) -$$

$$- \phi_2\big((\phi_1^3 - 2\phi_1\phi_2) y_t + (\phi_2^2 - \phi_1^2\phi_2) y_{t-1} + (\phi_1^2 - \phi_2)\eta_{t+1} + \phi_1\eta_{t+2} + \eta_{t+3}\big) + \eta_{t+5}$$

$$= (\phi_1^5 - 3\phi_1^3\phi_2 + \phi_1\phi_2^2) y_t + (2\phi_1^2\phi_2^2 - \phi_1^4\phi_2) y_{t-1} + (\phi_1^4 - 2\phi_1^2\phi_2)\eta_{t+1}$$

$$+ (\phi_1^3 - \phi_1\phi_2)\eta_{t+2} + \phi_1^2\eta_{t+3} + \phi_1\eta_{t+4} - (\phi_1^3\phi_2 - 2\phi_1^2\phi_2) y_t -$$

$$- (\phi_2^3 - \phi_1^2\phi_2^2) y_{t-1} - (\phi_1^2\phi_2 - \phi_2^2)\eta_{t+1} - \phi_1\phi_2\eta_{t+2} - \phi_2\eta_{t+3}$$

$$+ \eta_{t+5}$$

$$y_{t+5} = (\phi_1^5 - 4\phi_1^3\phi_2 + \cancel{3}\, 2\phi_1^2\phi_2 + \phi_1\phi_2^2)\, y_t + (3\phi_1^2\phi_2^2 - \phi_1^4\phi_2 - \phi_2^3)\, y_{t-1}$$
$$+ \text{ future shocks}$$

$$\hat{y}_{t+5|t} = (\phi_1^5 - 4\phi_1^3\phi_2 + 2\phi_1^2\phi_2 + \phi_1\phi_2^2)\, y_t + (3\phi_1^2\phi_2 - \phi_1^4\phi_2 - \phi_2^3)\, y_{t-1}$$

All in all, it is hard to find a formula for describing the exact $\hat{y}_{t+h|t}$.

But $\hat{y}_{t+h|t}$ depends on y_t and y_{t-1} ($y_t, y_{t-1} \in I_t$).

However, if h (forecasting period is large enough, then

$$\lim_{h \to \infty} \hat{y}_{t+h|t} \longrightarrow 0.$$

Also, it is not surprising that the point estimates converge to 0. (y_t and y_{t-1} depends on the power of ϕ_1 and ϕ_2 which goes to 0

$|\phi_1| < 1$ and $|\phi_2| < 1$

$|0.32| < 1$ $\quad |-0.03| < 1$

Otherwise $\phi_1^5 - 4\phi_1^3\phi_2 + 2\phi_1^2\phi_2^2 + \phi_1\phi_2^2 \le \underbrace{\phi_1^5 + 0 + 2\phi_1^4 + \phi_1^3}$

($\underset{0.32}{\phi_1}, \underset{0.03}{\phi_2} > 0$; $|\phi_1| > \phi_2$)

magnitude of the coefficient is proportional with ϕ_1^h if you want to forecast $\hat{y}_{t+h|t}$

$$\phi_1^h \xrightarrow[h \to \infty]{} 0$$

Similarly,

$3\phi_1^2\phi_2 - \phi_1^4\phi_2 - \phi_2^3 \le 3\phi_1^3$

$\sim (h-2)\phi^{h-2}$

proportional with

0

if $h \to \infty$