Voltage Response and Impedance of RC, RL and LC circuits

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Abstract

The purpose of the first part of this experiment was to analyze the behaviour of Inductors and Capacitors in AC circuits. RC, RL and LC circuits were used to find the time constants associated with each circuit.In section 1, Using a DC switch circuit the the time constant for the RC circuit was calculated to be $0.33 \pm (0.5 * 10^{-5})$ seconds.
In section 2, Using a square wave generator the time constant for the RC circuit was $(0.7 \pm 0.3) * 10^-5$ seconds, time constant for the RL circuit was calculated to be 0.08 ± 0.002 milli-Seconds and the time constant for the LC circuit was 25 ± 0.0037 micro-second .
Section 3 of the experiment deals with the impedance of RC and LC circuit. The impedance of both circuits was plotted as a function of frequency.

Introduction

This experiment seeks to find the transient behavior of an LCR (DC) circuit as well as to find the impedance of an AC circuit. The transient behavior of an LCR circuit can be derived from Kirchhoff's second law, which states that the sum of all voltages in a closed circuit is zero. Z_{Device} indicates either a capacitor or inductor.

$$V_{Input} - V_{Resistor} - V_{Z_{Device}} = 0 \tag{1}$$

The models equations for the Voltages across capacitor, inductor and resistor in RC, RL, and LC circuits can be derived from the previous ODE and equations((14),(15),(16)) which can be found in section 3 of the appendix. V_0 is the input voltage, t is time, and ϕ is the phase constant. In an RC circuit, when the capacitor is charging the voltage across the capacitor and resistor is given by:

$$V_{capacitor_{RC}} = V_0 e^{-t/\tau} \tag{2}$$

$$V_{resistor_{RC}} = V_0(1 - e^{-t/\tau}) \tag{3}$$

and when the capacitor is discharging:

$$V_{capacitor_{RC}} = V_0(1 - e^{-t/\tau}) \tag{4}$$

$$V_{resistor_{RC}} = V_0(e^{-t/\tau}) \tag{5}$$

The time constant of an RC circuit is given by $\tau = RC$.
In an RL circuit:

$$V_{Resistor_{RL}} = RV_0(1 - e^{-t/\tau_{RL}}) \tag{6}$$

is the voltage of the resistor when the inductor is driving voltage(V_L is decreasing), and

$$V_{Resistor_{RL}} = RV_0(e^{-t/\tau_{RL}}) \tag{7}$$

when the inductor is resisting the voltage (V_L is increasing). The time constant of the RL circuit is $\tau_{RL} = L/R$
In an LC circuit:

$$V_{Capacitor_{LC}} = V_0(1 - sin(t/\tau_C + \phi)) \tag{8}$$

The time constant for LC circuit is $\tau_{LC} = \sqrt{LC}$
This experiment will also analyse and find the impedance of an RC and RL circuit (AC current). The impedance of an LCR circuit is given by equation (9), so the impedance of RC and RL circuit can be found by removing the component that is not present.

$$Z_{LCR} = R + j\left(\omega L - \frac{1}{\omega C}\right) \tag{9}$$

$$Z_{RC} = R + j\left(-\frac{1}{\omega C}\right) \tag{10}$$

$$Z_{RL} = R + j\left(\omega L\right) \tag{11}$$

The phase between the Voltage and current is given by the following equation.

$$\phi(\omega) = \arctan \frac{\omega L - \frac{1}{\omega C}}{R} \tag{12}$$

${\chi_{reduced}}^2$ values will be calculated to analyse the goodness of fit for every useful data and their curve-fit.

$${\chi_r}^2 = \frac{1}{N-m} \sum_i^N \left(\frac{y_i - y(x_i)}{u(y_i)} \right)^2 \tag{13}$$

N is number of observations and m is the number of parameters. A perfect fit is indicated by χ_R^2 value of 1, an over-fit model or lack of data is indicated by χ_R^2 value <1, and a poor model is indicated by χ_R^2 value > 1

Methods and Procedures

All three sections of the experiment were performed using a Keysight DSOX1202G Oscilloscope connected to RC, RL and LC circuits. The resistor, capacitor and inductor values were measured using a Meterman 37xr multimeter. For the first section, a circuit as shown in Figure 9 was constructed using 1.5 V DC battery, a 449 $\pm$ 3 $k\Omega$ resistor and a 1.0 $\pm$ 0.04 μF capacitor. The experiment was performed by pressing and releasing the switch at regular interval. The oscilloscope was connected across the capacitor and the voltage data as a function of time was recorded for the capacitor. The data for the capacitor was curve fitted using Equation 2 and Equation 4 and the time constant (RC) was obtained.

The second section of experiment contain three circuits connected to a square wave generator. The first circuit is an RC circuit as shown in Figure 10 with resistor value of 503 $\pm$ 4 $k\Omega$ and the capacitor value of 22.5 $\pm$ 0.8 nF and the same equations as section 1 were used to obtain the time constant.
The RL circuit with a resistor of value 503 $\pm$ 4 Ω and the inductor of value 44.3 $\pm$ 5 mH was constructed using the circuit diagram in Figure 11. The voltage across the resistor was calculated as function of time and fitted to Equation 6 and Equation 7 equation and the time constant (L/R) for the circuit was obtained.
The LC circuit as shown in Figure 12 was constructed with an inductor of value 44.3 $\pm$ 5 mH and a capacitor of value 22.5 $\pm$ 0.8 nF.
In the third section of the experiment an RC and RL circuit was connected to a sine wave generator and the phase difference between the current and the voltage and current as a function of frequency was obtained using the oscilloscope. The Impedance as a function of frequency was plotted for both circuits.
The time constants were obtained using python's curve fit function and the plots were created using the Matplotlib library. The uncertainties were calculated using the Multi-meter specifications [1] and Equation 8.

Results and Analysis

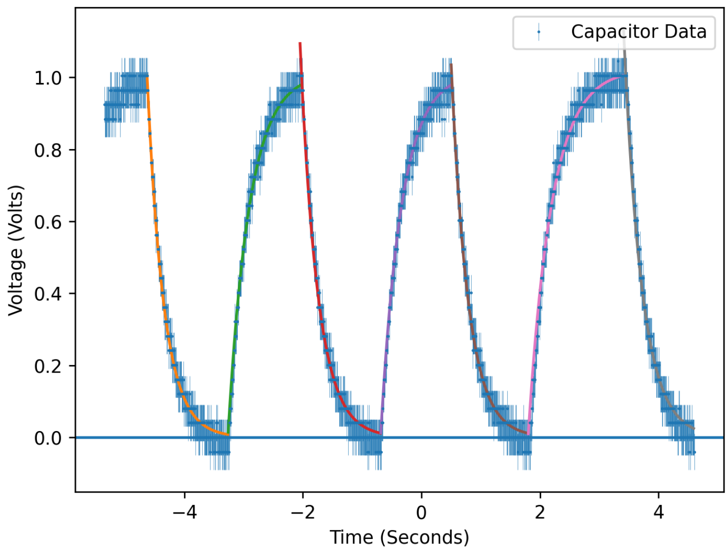
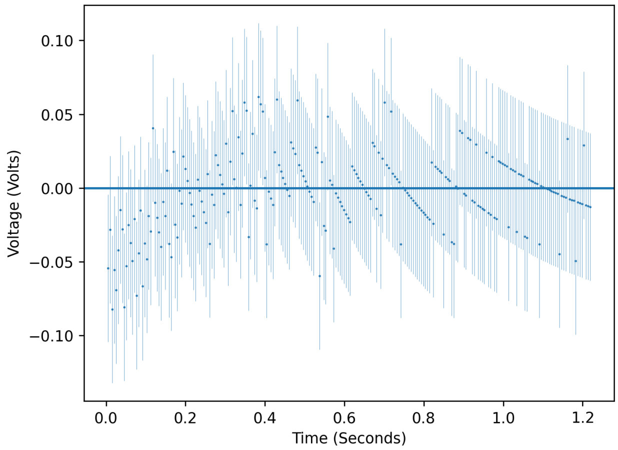
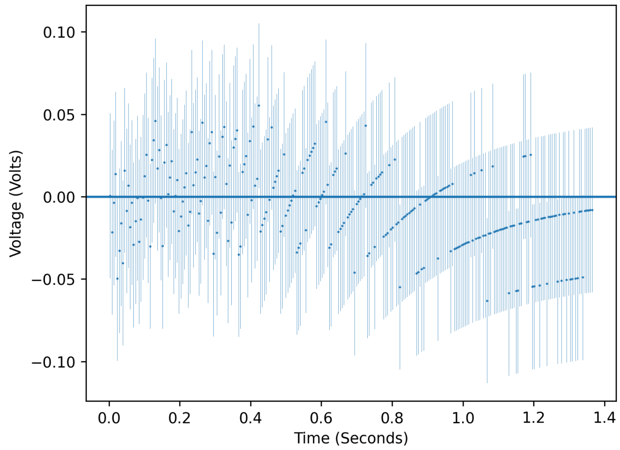


Figure 1: Capacitor Voltage with device error of the RC circuit plotted as a function of time. The Charging curves are fitted using Equation 2 equation and the Discharging curves are fitted using Equation 4. y = 0 references the 0 Volts line for better clarity.

For the section 1 of the experiment consisting of the RC circuit, The above graph gives seven values of the time constant (RC) from the seven different curves. These values were averaged to obtain the time constant value of $0.33 \pm (0.5 * 10^{-5})$ seconds. The value obtained from this curve fit is smaller than the theoretical RC value of the circuit of 0.45 ± 0.05 seconds.



(a) Residual Plot for the first Discharging Curve



(b) Residual plot for the first Charging curve

Figure 2: The above graphs show residual plots for the first charging and first discharging curve of the capacitor in the RC circuit. The curve are shown in Figure 1

The above graph (a) shows the residual plot for the first discharge curve. The ${\chi_r}^2$ value associated with this plot is calculated to be 0.24. This value is very small because the device uncertainty of the oscilloscope is relatively large. This value also supports the fact that the error bars for almost all the data-points overlap the curve. For the same reason the ${\chi_r}^2$ value for the Charging curve residual (b) is 0.28.

For the second section of the experiment, In part 1, the source of voltage was a square wave of frequency 1kHz and Amplitude of 4 V. The voltage across the resistor in the RC circuit as a function of time is given by Figure 3

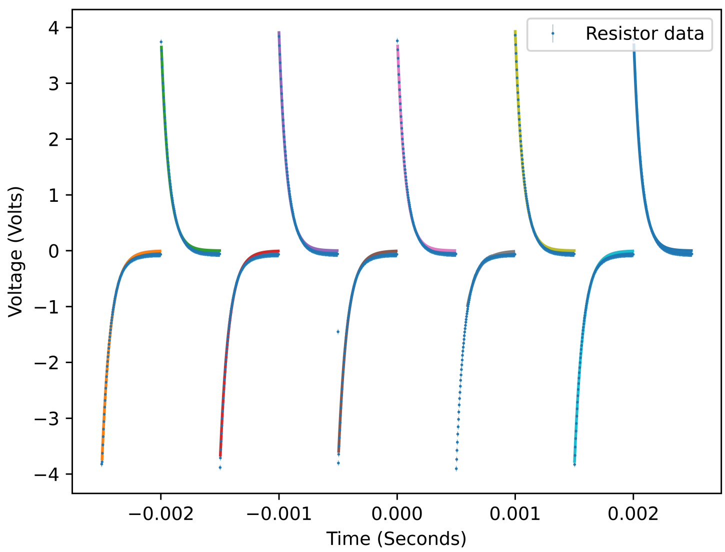
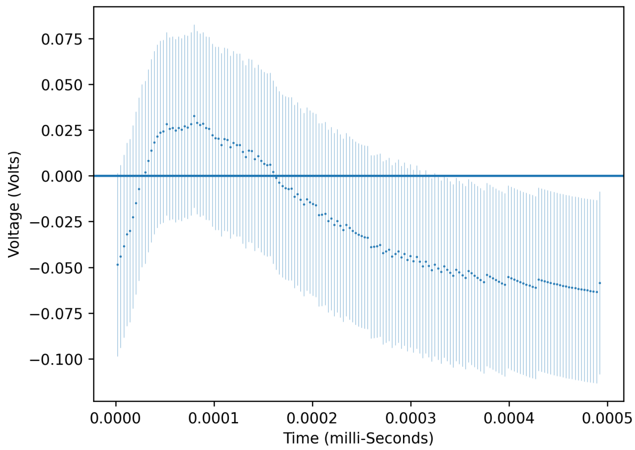
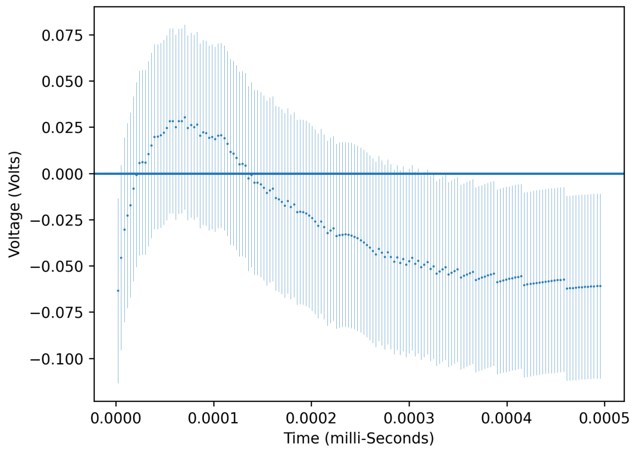


Figure 3: Resistor Voltage with device error plotted as a function of time in an RC circuit driven by a square wave generator. The Charging curves are fitted using Equation 3 equation and the Discharging curves are fitted using Equation 3.

The theoretical time constant for the circuit is $(1.1 \pm 0.04) * 10^{-5}$ seconds. The value of time constant obtained from the curves is obtained to be $(0.7 \pm 0.3) * 10^-5$ seconds. The value is smaller than the theoretical value but falls within the error range. The residual plots for first rise and first fall curves is displayed in Figure 4.



(a) Residual Plot for the first Charging Curve

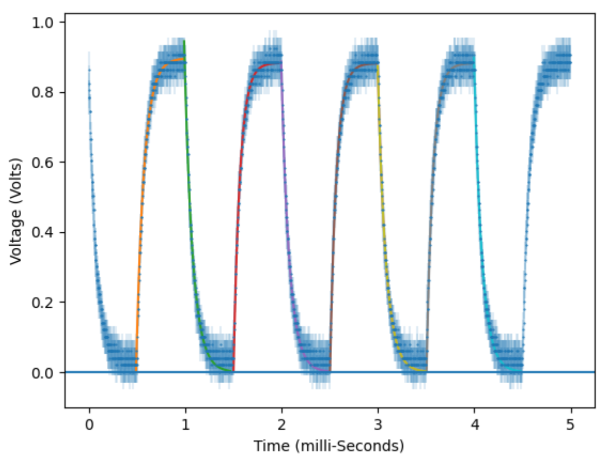


(b) Residual plot for the first Discharging curve

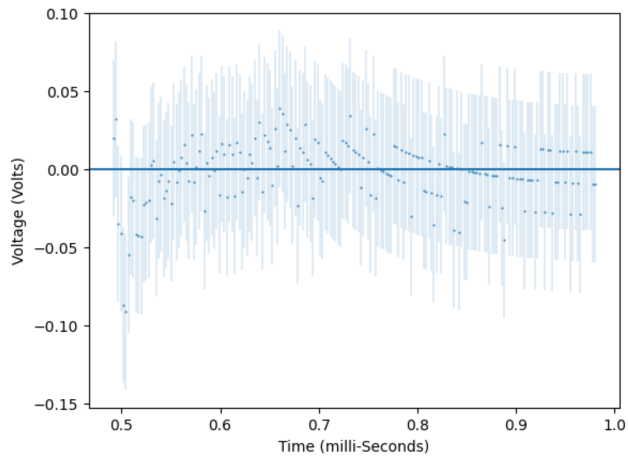
Figure 4: The above graphs show residual plots for the first charging and first discharging curve for a RC circuit driven by a square wave generator

The residual graph for when the circuit is charging (4(a)) has related ${\chi_r}^2$ value of 0.66 and ${\chi_r}^2$ value for when the circuit is discharging (4(b)) is 0.67. The small ${\chi_r}^2$ values are due to high relative uncertainty in the data and the fact that error bars of most of the points on the graph overlap the zero line.

The second part of section 2 deals with the 11 RL circuit. The Voltage across the resistor is plotted as a function of time in Figure 5(a) using two functions for voltage increase (Equation 7) and decrease (Equation 6). The value of the time constant obtained from the graph is 0.08 ± 0.002 milli-Seconds. This value is almost exactly the same as the theoretical time constant of 0.08 ± 0.005 milli-Seconds for the RL circuit. The ${\chi_r}^2$ value for the residual plot in Figure 5(b) was calculated to be 0.25. Again this value was very small because of the fact that the curve fits the data really well with all error bars overlapping the curve.



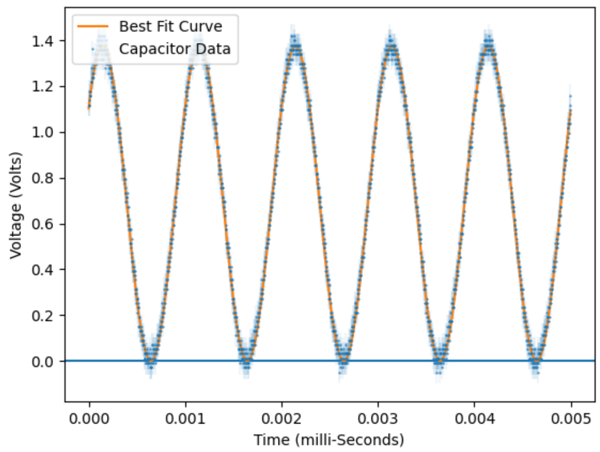
(a) Voltage across the Resistor in the RL circuit



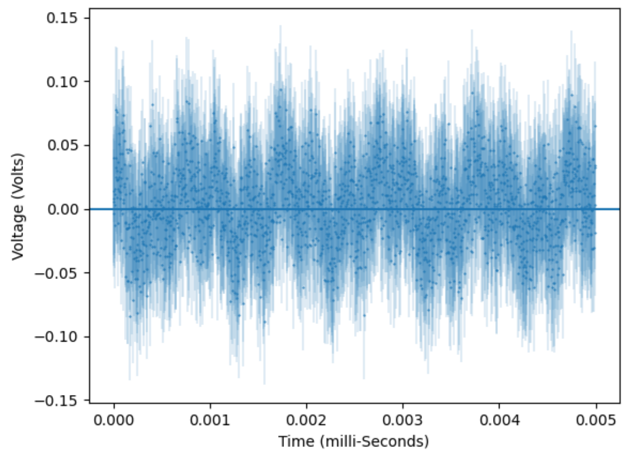
(b) Residual plot for the first rise curve

Figure 5: The first graph shows the voltage across the resistor in the RL circuit driven by the square wave generator. The second graph is the residual plot for the first rise curve fitted to Equation 7 model.

For the LC circuit, the voltage across the capacitor as a function of time is shown in Figure 6(a). The data was fitted to the Equation 8 model to obtain the time constant ($\sqrt{LC}$) for the circuit. The value obtained from the curve fit for the time constant was 25 ± 0.0037 micro-seconds. This value was smaller than the expected theoretical value of 32 ± 0.02 micro-seconds. The residual plot for this curve is shown in Figure 6(b). Since the uncertainty in the data was very small the ${\chi_r}^2$ value of 0.37 obtained for this residual plot was also very small. This value is backed by the fact that the data points were very close to the curve.



(a) Voltage across the Capacitor in the LC circuit with the zero line to signifying the zero voltage



(b) Residual plot for LC circuit curve

Figure 6: The first graph shows the voltage across the capacitor in the LC circuit driven by the square wave generator. The second graph is the residual plot for the data fitted to Equation 8 model.

The biggest source of uncertainty in our data was the device uncertainty from the oscilloscope or the multimeter. Sample uncertainty calculations are shown in Appendix - Section 3. The uncertainties in the data were relatively high for these experiments giving small ${\chi_r}^2$ values across the board. Thought these values were supported by the residual plots as the data overlapped the curve. Section 3 of this experiment deals with the phase voltage and impedance of RC and RL circuits as a function of frequency. The phase angle vs frequency plot for both circuits is shown in Figure 8.

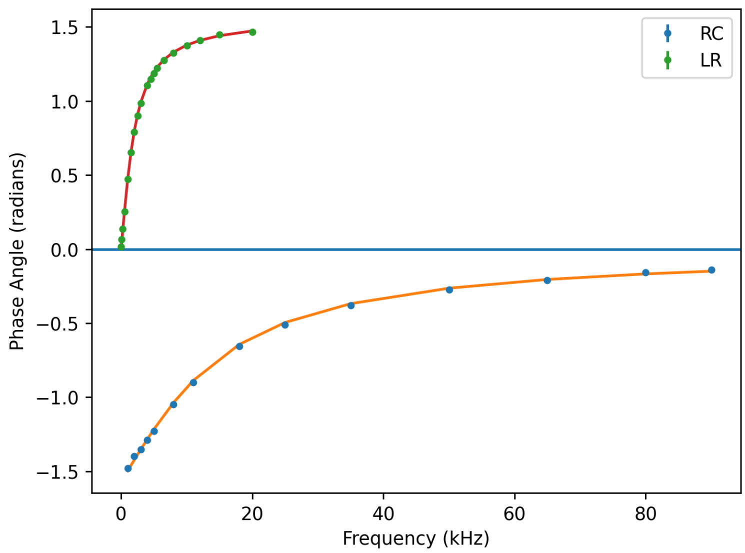
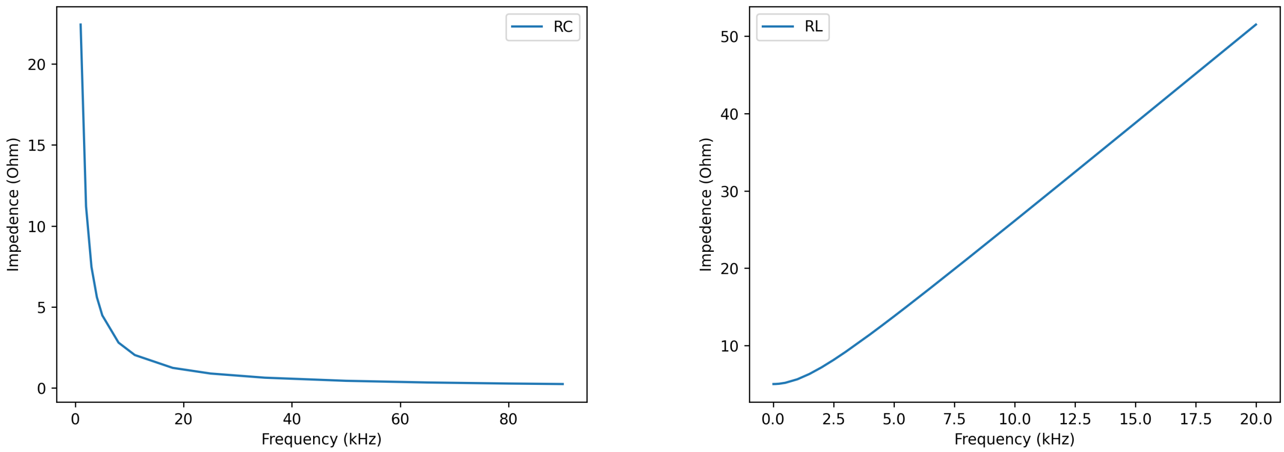


Figure 7: Phase Angle as a function of frequency plotted for RC and LR circuits using Equation 12.

Using the frequency values and the associated R, C and L values, calculated Impedance as a function of frequency for both the RC and RL circuit is shown in Figure 8.



(a) Impedance of the RC circuit plotted as a function of frequency

(b) Impedance of the RL circuit plotted as a function of frequency

Figure 8: The first graph shows the impedance of the RC circuit calculated using Equation 10. The second graph shows the Impedance of the RL circuit calculated using Equation 11.

The impedance if the RL circuit rises quickly with frequency and follows an almost straight line. The Impedance of the RC circuit decreases with frequency and reaches a saddle point at around 10 kHz and then decreases very slowly. Both the curves loosly follow the theoretical pattern expected from the circuits. The RL circuit is ideally expected to be a straight line. The fit for the RL data is close but not fully straight especially at smaller frequency. The RC circuit Impedance response is as expected with a sudden drop and then the Impedance reaches a saturation point as frequency increases.

Conclusion

In section one of the experiment, the time-constant of an RC circuit was solved for with a 1.5 V DC battery. The analysis consisted of calculating the time-constants and their uncertainties by curve-fitting the datas and graphs, and finding residual plots and χ^2_R values to analyse the quality of fit. The analysis showed that the time-constant from the section one data ($0.33 \pm (0.5 \times 10^{-5})$ seconds) was slightly less than the theoretical value (0.45 ± 0.05 seconds) from measuring the resistance and capacitance, and the χ^2_R value for the data and Curve is small at 0.28, which indicates that more data could be used to improve the quality of fit. In conclusion, the calculated values were close to the theoretical values but noticeably different due to internal resistance of the circuit and oscilloscope (which affects oscilloscope reading).

Section two of the experiment consisted of calculating the time-constant with voltage supplied from an oscilloscope in three sections: part one concerns with finding the time-constant of an RC circuit, part two with RL circuit, and part three is solving for the time-constant of an LC circuit. The calculated and theoretical value of the time-constant in part one (RC circuit) are $(1.1 \pm 0.01) \times 10^{-5}$ and $(0.7 \pm 0.3) \times 10^{-5}$, in part two (RL circuit) they are 0.08 ± 0.002 milli-Seconds and 0.08 ± 0.005 milli-Seconds respectively, and 25 ± 0.0037 micro-Seconds and 32 $\pm$ 0.02 micro-Seconds in part three.

The calculated fit parameters of the three sections were close to but often smaller than the theoretical value. Besides part two having a good χ^2_R value of 0.68, the other two sections had smaller χ^2_R values (part 1: 0.24, part 3: 0.37, around half the value of part 2) due to the occasional high uncertainties and a close fit, which is supported by their respective residual plots. Most of the residuals and their uncertainty cross the zero line, meaning most of the data and the uncertainty pass through the curve-fits.

In section 3, the curve-fit of the phase angle - frequency plot overlapped closely. The impedance of the RC and RL circuit was shown to follow the theoretical graph of the circuits, though not perfectly.

Some shortcomings that have been demonstrated in the experiment are that the calculated parameters often had a smaller value than the theoretical value. The quality of the device also played a part, since certain sections of the recorded data was much noisier than the other sections.

References

[1] https://www.transcat.com/media/pdf/37xr$_{m}eterman.pdf$
Data sheet for the Meterman 37XR multimeter

Appendix

Section 1: Circuit Diagrams

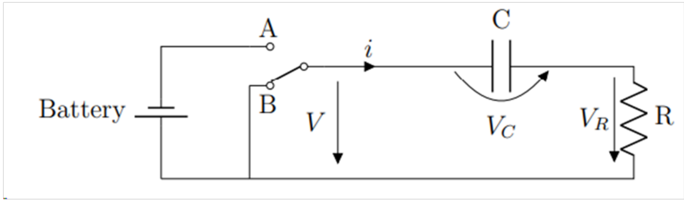


Figure 9: Experiment 1 - RC circuit

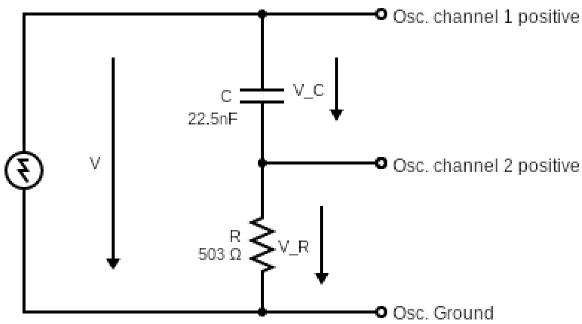


Figure 10: Experiment 2 - RC circuit

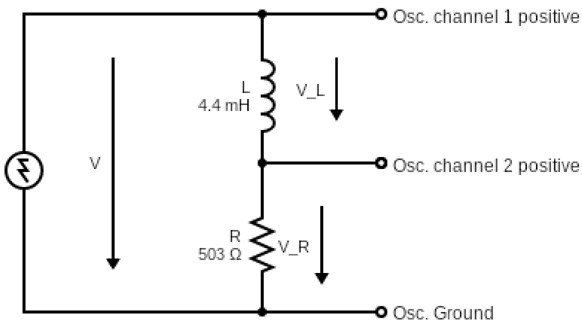


Figure 11: Experiment 2 - RL circuit

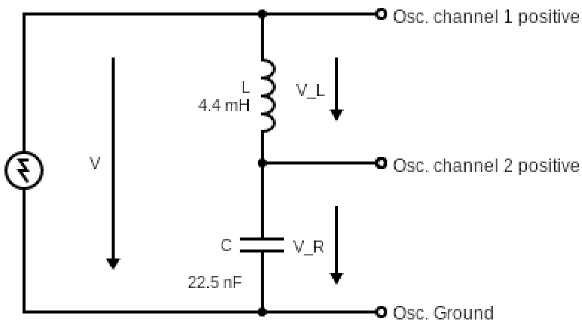


Figure 12: Experiment 2 - LC circuit

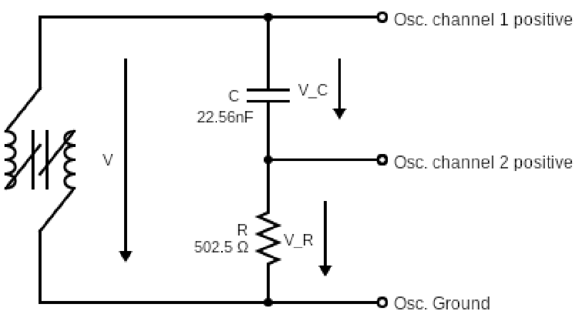


Figure 13: Experiment 3 - RC circuit

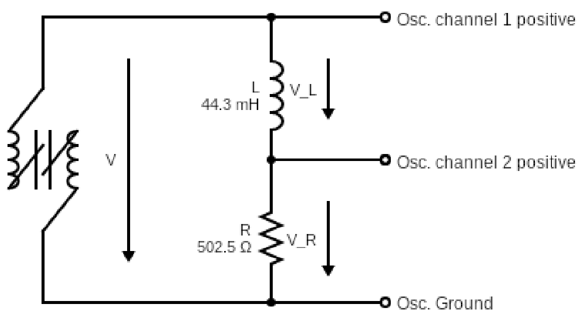


Figure 14: Experiment 3 - RL circuit

Section 2- Equation Derivations: Three equations to know for deriving the voltage expressions:

$$V_R(t) = RI(t) \tag{14}$$

$$V_L(t) = L\frac{dI(t)}{dt} \tag{15}$$

$$V_C(t) = \frac{Q(t)}{C} \tag{16}$$

The three circuit equations used to set up an ODE, to solve for the voltage across the electrical components:

RC circuit transient behavior:

$$V_{Input} - V_{Resistor} - V_{Capacitor} = 0 \tag{17}$$

$$RC\frac{dQ(t)}{dt} + Q(t) = V_{Input}C \tag{18}$$

RL circuit transient behavior:

$$V_{Input} - V_{Resistor} - V_{Inductor} = 0 \tag{19}$$

$$\frac{L}{R}\frac{dI(t)}{dt} + I(t) = \frac{V_{Input}}{R} \tag{20}$$

LC circuit transient behavior:

$$V_{Input} - V_{Inductor} - V_{Capacitor} = 0 \tag{21}$$

$$LC\frac{d^2Q}{dt^2} + Q(t) = V_{Input}C \tag{22}$$

Section 3: Sample Uncertainty Calculations

(a) For the multimeter:

For a resistor value of 503 ohms the error is calculated using the formula specified in the data-sheet for the multimeter [1].

$$\begin{aligned}
x &= 503.3\Omega \\
\Delta x &= x * 0.5\% + (0.1 * 8) \\
\Delta x &= 503.3 * 0.5\% + (0.1 * 10) \\
\Delta x &= 3.5 \\
\Delta x &= 4\Omega
\end{aligned}$$

(b) For the oscilloscope:

For a resolution of 0.5 V the uncertainty in voltage given by the resolution divided by the number of divisions.

$$\begin{aligned}
\Delta V &= resolution/divisions \\
\Delta V &= 0.5V/10 \\
\Delta V &= 0.05V
\end{aligned}$$

Section 4: Raw Data

Please follow the link to access the raw data for this experiment.

$https://q.utoronto.ca/groups/538315/files/folder/LCR\%20DATA$