RESEARCH STATEMENT

RUTGER KUYPER

1. Introduction

My current research is on three different but related topics in mathematical logic: two of them are in computability theory, namely the Medvedev and Muchnik lattices and algorithmic randomness; the third topic is probability logic and has connections to the various subfields of logic.

The *Medvedev* and *Muchnik lattices* are two algebraic structures which naturally arise from concepts studied in classical computability theory. They were introduced by Medvedev [20] in 1955 and Muchnik [21] in 1963 after an informal idea of Kolmogorov [11] in 1932, in an attempt to characterise Brouwer's *intuitionistic logic* using computational concepts. Informally, Brouwer's intuitionistic logic is classical logic as we know it, except that proofs by contradiction are not allowed. Several attempts have been made to connect this constructive logic with computation from a classical viewpoint, with the Kolmogorov–Medvedev–Muchnik approach being one of them, and two other approaches being the realisability approach initiated by Kleene [10] and continued by various others (see van Oosten [22] for an overview), and the Curry–Howard correspondence used in computer science being another one. Skvortsova, Sorbi and Terwijn have closely studied the connections between the Medvedev and Muchnik lattices and intuitionistic logic as well as the structural properties of these lattices, and I am continuing this line of research [12, 14, 16]. Let me emphasise that our study is from a classical viewpoint. Two recent surveys on the subject are Sorbi [25] and Hinman [6].

In *algorithmic randomness*, we study notions of effective randomness. That is, we study what it means for an infinite binary sequence to be random. I am especially (but certainly not exclusively) interested in the interplay of randomness with other parts of (effective) mathematics, for example the connections between effective randomness and the closely related notion of effective genericity on one hand, and effective analysis on the other hand. I have studied this in Kuyper and Terwijn [18]. Another example of this is my work with Hirschfeldt, at the University of Chicago, and with Jockusch and Schupp, at the University of Illinois at Urbana-Champaign, on the interplay of algorithmic randomness and the recent computability-theoretic notion of *coarse reducibility* [7]. Coarse reducibility is one of the notions recently introduced by Jockusch and Schupp to study the concept of being 'almost computable', based on the notion of generic-case computability from complexity theory introduced by Kapovich, Myasnikov, Schupp and Shpilrain [8]. An introduction to the field of algorithmic randomness can be found in Downey and Hirschfeldt [4].

In *probability logic*, one combines logic with concepts from probability theory. There are many different approaches to this (for example, Steinhorn [28] and Keisler [9]). The specific logic I am studying is called *ε-logic*, introduced by Terwijn [29]. The main motivation is that we want this logic to be *learnable* in a sense closely related to Valiant's celebrated notion of *pac-learning* [31]: given some unknown structure and some oracle that allows us to randomly sample elements from the structure and that tells us all the basic properties of our sample, we want to decide if a given (first-order) expression holds in the structure or not after taking only a finite sample. In classical first-order logic we cannot do this, because to decide if a universal statement holds, we cannot get around checking all elements instead of only finitely many. In ε-logic, this problem is solved by interpreting universal quantifiers not as 'for all' but as 'with probability at least $1 - \varepsilon$', while keeping the definition of the existential quantifier as it is classically. Terwijn [29] has shown that this logic is indeed learnable. I am investigating both the model-theoretic properties of this logic (Kuyper and Terwijn [17]) as well as the computational properties of this logic (Kuyper [13, 15]).

2. The Medvedev and Muchnik lattices

I am fascinated by the connections the Medvedev and Muchnik lattices make between intuitionistic logic on one hand and computability on the other hand, while using algebraic logic as an intermediary. I study these structures to obtain a better understanding of the connection between the notions of reasoning and computing.

As mentioned above, Medvedev and Muchnik based their structures on an informal idea of Kolmogorov. Kolmogorov's idea was that intuitionistic logic works very much like a *calculus of problems*. Here the definition of a 'problem' is left informal, but should be seen as the kind of problem a mathematician would study. As a simple example given by Kolmogorov, a problem could be to, given $a, b, c \in \mathbb{Q}$, find a solution of $ax^2 + bx + c = 0$. Kolmogorov notes that there are natural operations on problems: for example, there is an 'and' operation, written as $\wedge$, where to solve the problem $\mathcal{A} \wedge \mathcal{B}$ means to solve both $\mathcal{A}$ and $\mathcal{B}$, and there is an implication operation $\rightarrow$, where to solve $\mathcal{A} \rightarrow \mathcal{B}$ means to give a procedure to convert a solution of $\mathcal{A}$ into a solution of $\mathcal{B}$.

Kolmogorov's idea was formalised by Medvedev as follows: problems are subsets $\mathcal{A}$ of $\mathbb{N}^{\mathbb{N}}$ (where $\mathbb{N}^{\mathbb{N}}$ denotes the set of functions from the natural numbers to itself); a solution of $\mathcal{A}$ is then any function $f \in \mathcal{A}$. He also gives a pre-ordering $\mathcal{A} \leq_{\mathcal{M}} \mathcal{B}$ on $\mathcal{P}(\mathbb{N}^{\mathbb{N}})$, which should be interpreted as '$\mathcal{A}$ is solvable from $\mathcal{B}$': we say that $\mathcal{A} \leq_{\mathcal{M}} \mathcal{B}$ or $\mathcal{A}$ *Medvedev reduces to* $\mathcal{B}$ if there is a single Turing functional Γ such that $\mathcal{A} \subseteq \Gamma(\mathcal{B})$, which roughly means that there is a single effective computational procedure which converts every element of $\mathcal{B}$ into an element of $\mathcal{A}$. If we now say that $\mathcal{A}$ and $\mathcal{B}$ are *Medvedev-equivalent* if both $\mathcal{A} \leq_{\mathcal{M}} \mathcal{B}$ and $\mathcal{B} \leq_{\mathcal{M}} \mathcal{A}$, then we let the *Medvedev lattice* $\mathcal{M}$ be the set of equivalence classes of $\mathcal{P}(\mathbb{N}^{\mathbb{N}})$ under Medvedev equivalence, and we call the elements of $\mathcal{M}$ the *Medvedev degrees*. It turns out that this structure is a lattice, i.e. any two elements $\mathcal{A}, \mathcal{B}$ have a least upper bound $\mathcal{A} \oplus \mathcal{B}$ and a greatest lower bound $\mathcal{A} \otimes \mathcal{B}$. In fact, the Medvedev lattice even turns out to be a Brouwer algebra, i.e. there is also an implication operator $\mathcal{A} \rightarrow \mathcal{B}$ satisfying

$$\mathcal{A} \oplus \mathcal{C} \geq_{\mathcal{M}} \mathcal{B} \Leftrightarrow \mathcal{C} \geq_{\mathcal{M}} \mathcal{A} \rightarrow \mathcal{B}.$$

Brouwer algebras, introduced by McKinsey and Tarski [19], have natural associated propositional theories between intuitionistic propositional logic IPC and classical propositional logic CL. Therefore, there is also a natural way to assign such a theory $\mathrm{Th}(\mathcal{M})$ to the Medvedev lattice $\mathcal{M}$. Given Kolmogorov's motivation, one would hope that this theory is exactly (the theory of) IPC, which would then give us a way to assign computational semantics to IPC. Unfortunately, this turns out not to be the case: Medvedev already noticed that $\mathrm{Th}(\mathcal{M})$ contains the weak law of the excluded middle $\neg p \vee \neg\neg p$, which is not a part of IPC. Sorbi [24] later showed that in fact $\mathrm{Th}(\mathcal{M}) = \mathrm{IPC} + \neg p \vee \neg\neg p$.

However, the story does not end there. For any Brouwer algebra $\mathscr{B}$, if $x \in \mathscr{B}$, one can consider the quotient lattice $\mathscr{B}/x$ of $\mathscr{B}$ by the principal filter $\{y \in \mathscr{B} \mid y \geq x\}$, which turns out to again be a Brouwer algebra. The following remarkable result is one of the main motivations of my research.

Theorem 2.1. (Skvortsova [23]) *There is an* $\mathcal{A} \in \mathcal{M}$ *such that* $\mathrm{Th}(\mathcal{M}/\mathcal{A}) = \mathrm{IPC}$.

However, the main ideological objection to this result is the collection $\mathcal{A}$ which she constructed, because it is constructed in an ad hoc manner and has no natural interpretation, be it computationally motivated or arising from some other natural concept studied in mathematics.

Let us briefly turn to the Muchnik lattice $\mathcal{M}_w$. The definition of $\mathcal{M}_w$ is very much like that of $\mathcal{M}$, except that we now say that $\mathcal{A} \leq_w \mathcal{B}$ if for every $f \in \mathcal{B}$ there exists a Turing functional Γ such that $\Gamma(f) \in \mathcal{A}$. Thus, the Muchnik lattice is a non-uniform version of the Medvedev lattice. Everything that has been said above about $\mathcal{M}$ also holds for $\mathcal{M}_w$: it is a Brouwer algebra, its theory is $\mathrm{IPC} + \neg p \vee \neg\neg p$, and there is a factor of it which captures exactly IPC (Sorbi and Terwijn [27]), albeit also unnatural.

Terwijn [30] asked if there are any natural factors of the Medvedev or Muchnik lattice which yield IPC as their theory. It turns out that for the Muchnik lattice, there are several natural classes which capture IPC and which can be defined in terms of familiar notions from computability theory.

Theorem 2.2. (Kuyper [14]) *Let* $\mathcal{A}$ *be the class of low functions or the class of* Δ^0_2 *functions of 1-generic or computable Turing degree, and let* $\overline{\mathcal{A}} = \mathbb{N}^{\mathbb{N}} \setminus \mathcal{A}$. *Then*

$$\mathrm{Th}(\mathcal{M}/\overline{\mathcal{A}}) = \mathrm{IPC}.$$

Assuming the continuum hypothesis, the same holds for $\mathcal{A}$ *the class of hyperimmune-free functions or* $\mathcal{A}$ *the class of computably traceable functions.*

Question 2.3. Is the continuum hypothesis required in the previous statement?

Even for the Medvedev lattice, Skvortsova's result can be polished. Unfortunately for the Medvedev lattice the result is clearly less satisfying than for the Muchnik lattice, so this is a matter subject to ongoing research.

Theorem 2.4. (Kuyper [16]) *Let A be a computably independent set (for example, let A be 1-random or 1-generic). Let $A^{[i]} = \{n \mid \langle i,n\rangle \in A\}$ where $\langle i,n\rangle$ denotes some fixed computable pairing function. Let $\mathcal{A} = \left\{i^\frown f \mid f \geq_T A^{[i]}\right\}$. Then*

$$\mathrm{Th}(\mathcal{M}/\mathcal{A}) = \mathrm{IPC}.$$

Another important question is exactly which logics between IPC and CL can be obtained from a factor of the Medvedev lattice. Sorbi and Terwijn [26] have shown that infinitely many of such logics can be obtained, while not all can. The following theorem answered one of the open questions from their paper, and shows that a large class of logics cannot be obtained.

Theorem 2.5. (Kuyper [16]) *For every non-trivial $\mathcal{A}$,*

$$\mathrm{Th}(\mathcal{M}/\mathcal{A}) \subseteq \mathrm{IPC} + \neg p \vee \neg\neg p.$$

Problem 2.6. Give a complete classification of the intermediate logics which can be obtained from a factor of $\mathcal{M}$.

Finally, I am currently studying extensions of the Medvedev lattice to first-order logic [12]. This uses the concept of a *first-order hyperdoctrine* from categorical logic. Again, we get a structure which does not capture intuitionistic first-order logic IQC but a proper extension of it, and there is again a natural notion of factor. As in the propositional case, we can ask if we can get every first-order intuitionistic theory from a factor of this Medvedev hyperdoctrine. This turns out to be false.

Theorem 2.7. (Kuyper [12]) *There is no factor of the Medvedev hyperdoctrine which interprets Heyting arithmetic, i.e. the intuitionistic analogue of Peano arithmetic.*

Problem 2.8. Characterise which first-order intuitionistic theories can be obtained from a factor of the Medvedev hyperdoctrine ([12] contains some partial results).

Aside from the questions mentioned in this section, I also want to investigate the connections between the Medvedev and Muchnik lattices and the field of algorithmic randomness discussed in the next section by studying how the classes of various randomness notions interact with each other as elements of the Medvedev and Muchnik lattices.

3. Algorithmic randomness

I am interested in the connections between algorithmic randomness on one hand, and various other parts of (effective) mathematics on the other hand. I am currently looking at this from two perspectives.

First, I am looking at connections between algorithmic randomness and computable analysis. In algorithmic randomness, several different notions of randomness are studied, where a real $x \in [0,1]$ is said to be random if it is not in any 'effective' set of measure 0. There are various ways to define 'effective', which yield different notions of randomness, of which Martin-Löf randomness is the most well-studied. Recently, several researchers have studied connections between notions of randomness and the differentiability of functions. For example, Brattka, Miller and Nies [2], building on work by Demuth [3], have shown that a real number $x \in [0,1]$ is Martin-Löf random if and only if every computable function of bounded variation is differentiable at x.

Instead of taking a measure-theoretic approach at defining randomness, one can also take a Baire category-based approach and say that x is random if it is not in any 'effective' meagre sets. In computability theory, such reals are usually called *generic*, with 1-genericity roughly being the equivalent of Martin-Löf randomness (which is also called 1-randomness). Given the Erdös-Sierpinski duality (which roughly states that in the non-effective case meagre and measure 0 satisfy very similar properties), one would expect genericity and randomness to have similar properties. In the effective settings this turns out to hold often, but not always. It turns out that there is also a connection between differentiable functions and 1-genericity.

Theorem 3.1. (Kuyper and Terwijn [18]) *Let $x \in [0,1]$. Then x is 1-generic if and only if for every differentiable, computable function $f : [0,1] \to \mathbb{R}$ we have that f' is continuous at x.*

A related notion is the weaker notion of weak 1-genericity.

Problem 3.2. Find a similar characterisation for weak 1-genericity using notions from computable analysis.

I have also recently studied *coarse reducibility*, and its connections with algorithmic randomness. We say that $D \in 2^{\mathbb{N}}$ is a *coarse representation* of $X \in 2^{\mathbb{N}}$ if the density ρ of the symmetric difference $X \triangle D$ is 0, i.e. if

$$\lim_{n\to\infty} \frac{|(X \triangle D) \upharpoonright n|}{n} = 0.$$

We say that X is *coarsely reducible* to Y, or $X \leq_c Y$, if, using the notation from the previous section, the set of coarse representations of X Medvedev-reduces to the set of coarse representations of Y. In Hirschfeldt, Jockusch, Kuyper and Schupp [7] it is shown that there is a natural embedding E of the Turing degrees into the degree structure induced by the pre-order $\leq_c$, where the basic idea is to code a set in a highly redundant way. Therefore one would expect this embedding not to be surjective, which turns out to be true in a strong way. Given $X \in 2^{\mathbb{N}}$, let $X^{\mathfrak{c}}$ be the set of those $A \in 2^{\mathbb{N}}$ which are (non-uniformly) computable from every coarse representation of X. The following facts hold about $X^{\mathfrak{c}}$, where K-triviality is a well-studied notion in algorithmic randomness.

Theorem 3.3. (Hirschfeldt, Jockusch, Kuyper and Schupp [7]) *Let $X \in 2^{\mathbb{N}}$ be 1-random. Then every element of $X^{\mathfrak{c}}$ is K-trivial. If X is even weak 2-random, then $X^{\mathfrak{c}}$ only contains computable elements. However, if X is Δ^0_2 and 1-random, then $X^{\mathfrak{c}}$ contains a non-computable element. Finally, there is a K-trivial A which is not in $X^{\mathfrak{c}}$ for any 1-random.*

As mentioned in the previous section, one of the future directions of my research is to get a better understanding of how randomness behaves within the Medvedev and Muchnik lattices. Furthermore, I want to continue and expand the research outlined above.

4. ε-logic

In computational learning theory, Valiant [31] introduced the celebrated pac-learning model. As mentioned in the introduction, trying to apply this concept to first-order logic is troublesome, because taking just a finite sample from a first-order model is not enough to decide the truth of a statement. It is therefore interesting to study in which ways we can obtain a logic which is learnable. In [32] Valiant introduced a learnable logic, but his logic has several shortcomings: for example, Valiant only deals with finite models, and the syntax is different from regular first-order logic. Terwijn [29] introduced ε-logic, a learnable logic in the general syntax of first-order logic. Informally, this logic is obtained by changing the reading of 'for all' into 'for many'.

An *ε-model* consists of a pair $(\mathfrak{M}, \mathcal{D})$, where $\mathfrak{M}$ is a first-order model (i.e. $\mathfrak{M}$ consists of a set called the universe, together with a collection of functions and relations on the universe) and $\mathcal{D}$ is a probability measure on the universe of $\mathfrak{M}$ satisfying some additional measurability conditions. Now, what it means for a first-order statement to *ε-hold* in $(\mathfrak{M}, \mathcal{D})$ is defined inductively. For the quantifiers we say that $(\mathfrak{M}, \mathcal{D}) \models_\varepsilon \forall x \varphi(x)$ if and only if the set $\{a \in \mathfrak{M} \mid (\mathfrak{M}, \mathcal{D}) \models_\varepsilon \varphi(a)\}$ has $\mathcal{D}$-measure at least $1 - \varepsilon$, but on the other hand we still say that $(\mathfrak{M}, \mathcal{D}) \models_\varepsilon \exists x \varphi(x)$ if and only if there exists an $a \in \mathfrak{M}$ such that $(\mathfrak{M}, \mathcal{D}) \models_\varepsilon \varphi(a)$. The other connectives behave syntactically the same as in classical logic. In particular, $(\mathfrak{M}, \mathcal{D}) \models_\varepsilon \neg\forall x \varphi(x)$ holds if and only if $(\mathfrak{M}, \mathcal{D}) \models_\varepsilon \exists x \neg\varphi(x)$, and similarly with $\forall$ and $\exists$ interchanged.

The variable ε should be seen as an error parameter: when we look at the universal quantifier, we allow at most ε-measure many counterexamples. Note that the case $\varepsilon = 1$ is trivial; when not mentioned explicitly we consider $\varepsilon \in [0, 1) \cap \mathbb{Q}$. Also note that ε-logic is an example of a *paraconsistent* logic, i.e. a statement φ and its negation $\neg\varphi$ can hold at the same time. Indeed, the negation of a formula $\forall x \varphi(x)$ is $\exists x \neg\varphi(x)$, and it can certainly be the case that $\varphi(x)$ holds for almost all x but not for all x.

As mentioned above, Terwijn [29] has shown that this logic is indeed learnable, in some technical sense which is closely related to Valiant's pac-model. Thus, this logic accomplishes our original goal. I am studying various properties of this logic. First, I have looked at the model theory for this logic, which turns out to be very different from classical model theory. As an example, let us consider the downwards Löwenheim-Skolem theorem. In classical logic, this says that any theory which is consistent (i.e. has some model) also has a countable model. This turns out to be false for ε-logic.

Theorem 4.1. (Kuyper and Terwijn [17]) *Let $\varepsilon \in [0, 1) \cap \mathbb{Q}$. There is an ε-consistent formula φ which has no countable ε-model.*

However, a very closely related version does hold: for every ε-model there is always an equivalent model of cardinality exactly 2^ω, i.e. the cardinality of the real numbers. In fact, we can even say something about the measure.

Theorem 4.2. (Kuyper and Terwijn [17]) *Let $\varepsilon \in [0,1) \cap \mathbb{Q}$ and let Γ be an ε-consistent theory. There is an ε-model on $[0,1]$ plus atoms with the Lebesgue measure which also satisfies Γ.*

Question 4.3. Let λ_ε be the Löwenheim number of ε-logic, i.e. the least cardinal κ such that every ε-consistent sentence has a model of cardinality at most κ. What is λ_ε? It is known that $\aleph_1 \leq \lambda_\varepsilon \leq 2_0^{\aleph}$. Furthermore, $\lambda_\varepsilon = 2^{\aleph_0} > \aleph_1$ and $\lambda_\varepsilon = \aleph_1$ are both consistent with ZFC [17].

Another surprising result is that compactness fails, unless $\varepsilon = 0$.

Theorem 4.4. (Kuyper and Terwijn [17]) *Let $\varepsilon \in (0,1) \cap \mathbb{Q}$. Then ε-logic is not compact, i.e. there is an infinite set of sentences Γ such that every finite subset is ε-satisfiable but Γ is not.*

Theorem 4.5. (Kuyper [15]) *0-logic is compact.*

Aside from the model theory, I also study the computational complexity of ε-logic [13, 15]. The known results are summarised in Table 1.

	$\varepsilon \in (0,1) \cap \mathbb{Q}$	$\varepsilon = 0$
ε-satisfiability	Σ^1_1-complete [15]	decidable [15]
ε-validity	Π^1_1-hard [13]	Σ^0_1-complete [15]

Table 1. Complexity of validity and satisfiability in ε-logic.

Question 4.6. What is the precise complexity of ε-validity for $\varepsilon \in (0,1) \cap \mathbb{Q}$?

Recently several other people have looked at ε-logic, e.g. Yang [33], or related probability logics, e.g. Goldbring and Towsner [5]. Aside from the interesting questions about ε-logic that are still open, more research into how these different probability logics relate to each other also looks fruitful.

References

1. J. Barwise and S. Feferman (eds.), *Model-theoretic logics*, Perspectives in Mathematical Logic, vol. 8, Springer-Verlag, 1985.
2. R. Brattka, J. S. Miller, and A. Nies, *Randomness and differentiability*, to appear in Transactions of the American Mathematical Society.
3. O. Demuth, *The differentiability of constructive functions of weakly bounded variation on pseudo numbers*, Commentationes Mathematicae Universitatis Carolinae **16** (1975), no. 3, 583–599, In Russian.
4. R. G. Downey and D. R. Hirschfeldt, *Algorithmic randomness and complexity*, Springer, 2010.
5. I. Goldbring and H. Towsner, *An approximate logic for measures*, Israel Journal of Mathematics **199** (2014), no. 2, 867–913.
6. P. G. Hinman, *A survey of Mučnik and Medvedev degrees*, The Bulletin of Symbolic Logic **18** (2012), no. 2, 161–229.
7. D. R. Hirschfeldt, C. G. Jockusch, Jr., R. Kuyper, and P. E. Schupp, *Coarse reducibility and algorithmic randomness*, in preparation.
8. I. Kapovich, A. Myasnikov, P. Schupp, and V. Shpilrain, *Generic-case complexity, decision problems in group theory and random walks*, Journal of Algebra **264** (2003), 665–694.
9. H. J. Keisler, *Probability quantifiers*, in Barwise and Feferman [1].
10. S. C. Kleene, *On the interpretation of intuitionistic number theory*, The Journal of Symbolic Logic **10** (1945), no. 4, 109–124.
11. A. Kolmogorov, *Zur Deutung der intuitionistischen Logik*, Mathematische Zeitschrift **35** (1932), no. 1, 58–65.
12. R. Kuyper, *First-order logic in the Medvedev lattice*, submitted.
13. ———, *Computational hardness of validity in probability logic*, Proceedings of Logical Foundations of Computer Science (S. Artemov and A. Nerode, eds.), Lecture Notes in Computer Science, vol. 7734, Springer, 2013, pp. 252–265.

14. ———, *Natural factors of the Muchnik lattice capturing IPC*, Annals of Pure and Applied Logic **164** (2013), no. 10, 1025–1036.
15. ———, *Computational aspects of satisfiability in probability logic*, Mathematical Logic Quarterly **60** (2014), no. 6, 444–470.
16. ———, *Natural factors of the Medvedev lattice capturing IPC*, Archive for Mathematical Logic **53** (2014), no. 7, 865–879.
17. R. Kuyper and S. A. Terwijn, *Model theory of measure spaces and probability logic*, The Review of Symbolic Logic **6** (2013), no. 3, 367–393.
18. ———, *Effective genericity and differentiability*, Journal of Logic & Analysis **6** (2014), no. 4, 1–14.
19. J. C. C. McKinsey and A. Tarski, *On closed elements in closure algebras*, The Annals of Mathematics Second Series **47** (1946), no. 1, 122–162.
20. Yu. T. Medvedev, *Degrees of difficulty of the mass problems*, Doklady Akademii Nauk SSSR, (NS) **104** (1955), no. 4, 501–504.
21. A. A. Muchnik, *On strong and weak reducibilities of algorithmic problems*, Sibirskii Matematicheskii Zhurnal **4** (1963), 1328–1341.
22. J. van Oosten, *Realizability: an introduction to its categorical side*, Studies in Logic and the Foundations of Mathematics, vol. 152, Elsevier, 2008.
23. E. Z. Skvortsova, *A faithful interpretation of the intuitionistic propositional calculus by means of an initial segment of the Medvedev lattice*, Sibirskii Matematicheskii Zhurnal **29** (1988), no. 1, 171–178.
24. A. Sorbi, *Embedding Brouwer algebras in the Medvedev lattice*, Note Dame Journal of Formal Logic **32** (1991), no. 2, 266–275.
25. ———, *The Medvedev lattice of degrees of difficulty*, Computability, Enumerability, Unsolvability: Directions in Recursion Theory (S. B. Cooper, T. A. Slaman, and S. S. Wainer, eds.), London Mathematical Society Lecture Notes, vol. 224, Cambridge University Press, 1996, pp. 289–312.
26. A. Sorbi and S. A. Terwijn, *Intermediate logics and factors of the medvedev lattice*, Annals of Pure and Applied Logic **155** (2008), 69–85.
27. ———, *Intuitionistic logic and Muchnik degrees*, Algebra Universalis **67** (2012), 175–188.
28. C. I. Steinhorn, *Borel structures and measure and category logics*, in Barwise and Feferman [1].
29. S. A. Terwijn, *Probabilistic logic and induction*, Journal of Logic and Computation **15** (2005), no. 4, 507–515.
30. ———, *The Medvedev lattice of computably closed sets*, Archive for Mathematical Logic **45** (2006), no. 2, 179–190.
31. L. G. Valiant, *A theory of the learnable*, Communications of the ACM **27** (1984), no. 11, 1134–1142.
32. ———, *Robust logics*, Artificial Intelligence **117** (2000), 231–253.
33. G. Yang, *Computabilities of validity and satisfiability in probability logics over finite and countable models*, arXiv:1410.3059.

Radboud University Nijmegen, Department of Mathematics, P.O. Box 9010, 6500 GL Nijmegen, the Netherlands.
E-mail address: `mail@rutgerkuyper.com`