Destroying randomness and genericity using symmetric differences

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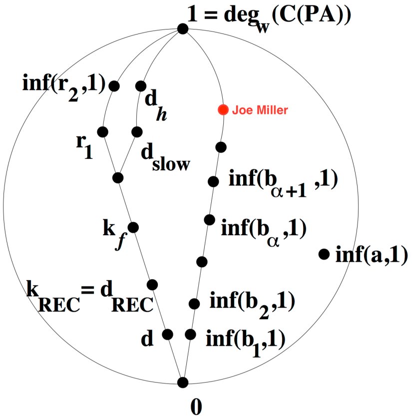
A degree with a name

Joint work with



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Theorem. (Nies) *A set A is K-trivial if and only if A is low for Martin-Löf randomness.*

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Uniformly?

Can the use of A be slow growing, for example, the identity?

Stabiliser

Definition. Let $\mathcal{C} \subseteq 2^\omega$. Then the *stabiliser* of $\mathcal{C}$ is the set

$$\{A \in 2^\omega \mid \forall X \in \mathcal{C}(A \triangle X \in \mathcal{C})\}.$$

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Name comes from group theory: $\triangle$ is an action of $(2^\omega, \triangle)$ on 2^ω.

Stabilisers in randomness

What do stabilisers look like for notions of randomness? Note that

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Stabilisers have also been studied by Kihara and Miyabe, for their connections to null-additivity.

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Even the answer to the following questions is unclear:

- Are the Martin-Löf-stabilisers degree-invariant?
- What is the cardinality of the Martin-Löf-stabiliser?

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Kihara and Miyabe also have partial results that strongly suggest that a set A is Schnorr-stabilising if and only if A is uniformly low for Schnorr randomness.

Back to Martin-Löf randomness

Theorem. *A set A is Martin-Löf-stabilising if and only if A is K-trivial.*

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$$\exists^\infty n(K(A \upharpoonright n) \leq K(n) + c).$$

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Proof. Let Y be a noncomputable c.e. set which is low for K (Muchnik). Let $X \leq_T Y$ be 1-generic. Because $A \triangle X$ is 1-generic, it is infinitely often K-trivial by a result of Barmpalias and Vlek. Now

$$\begin{aligned} K(A \upharpoonright n) \leq K^X(A \upharpoonright n) + O(1) &\leq K^X((X \triangle A) \upharpoonright n) + O(1) \\ &\leq K((X \triangle A) \upharpoonright n) + O(1). \end{aligned}$$

□

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$$n \notin A \wedge \exists \tau \succeq \sigma_s 0^n 1(\{s\}^\tau(s)\downarrow)$$

or

$$n \in A \wedge \forall \tau \succeq \sigma_s 0^n 1(\{s\}^\tau(s)\uparrow)$$

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- Code A into $X \triangle A$, which we can do efficiently infinitely often.
- If some $n \in \omega$ is unsuitable to code whether we forced the jump or not, then $A \upharpoonright (n+1)$ *also* has a short description!

Open questions

- What about other randomness notions, such as computable randomness? Schnorr randomness?
- Is there any randomness notion for which stabilising and uniformly low do not coincide?
- Are there any randomness notions for which stabilising is not degree-invariant?
- (With Jason Rute) What is special about symmetric difference? Can we replace it by addition? Other group operations?

More details

For more details, see:

R. Kuyper and J. S. Miller, Nullifying randomness and genericity using symmetric difference, submitted.

Weak 2-randomness

By essentially the same proof, we get:

Theorem. *For any set A, the following are equivalent:*

- *A is weakly-2-random-stabilising,*
- *A is (weakly-2-random,Martin-Löf)-stabilising,*
- *A is K-trivial.*

Some results of Kihara and Miyabe (2)

Theorem. *A set A is uniformly low for Schnorr randomness if and only if for every Schnorr null set N, the set*

$$\{X \triangle A \mid X \in N\}$$

is Schnorr null.

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$$B \triangle P \subseteq Q_m,$$

where Q_m is the complement of the mth component U_m of the universal Martin-Löf test.

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Proof. Assume not. Let P_0 be an arbitrary nonempty Π^0_1-class only containing Martin-Löf random sets. By finite extension, build $X \in P_0$ such that $B \triangle X \in \bigcap_{m \in \omega} U_m$. That is, at step $s+1$, apply the hypothesis to $P = P_0 \cap [\![\sigma_s]\!]$ to find $\sigma_{s+1} \succeq \sigma_s$ such that $B \triangle \sigma_{s+1} \in U_{s+1}$. $\square$

The proof for Martin-Löf randomness

Let P and Q_m be as in the previous lemma. We will be building a KC set L and a Π^0_1-class R. By the recursion theorem, we know an index e for R. We will guarantee that $\mu(R) \geq 1 - 2^{-e-m-2}$, which ensures that $Q_m \subseteq R$. It now suffices to show that A is K-trivial for all A such that $A \triangle P \subseteq R$, which is what we will do.

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The idea is that, whenever we see that our enemy ensures that $\sigma \triangle P \subseteq R$, we use the measure it provides to enumerate a request for $(K(|\sigma|), \sigma)$. Of course, we need to make sure that the measure provided for different such pairs is independent. We ensure this by building R in a highly independent way.

Construction of R

For every pair of a string $\sigma \in 2^{<\omega}$ and every $k \in \omega$, reserve an exclusive interval $f(\sigma, k)$ of ω of size $(k + e + m + 2)$. Whenever $K_{s+1}(n) < K_s(n)$, for every string $\sigma \in 2^n$, remove from R all paths $X \succeq \sigma$ such that X is constantly 0 on the block $f(\sigma, k)$. Call this removed set $V_{k,n}$.

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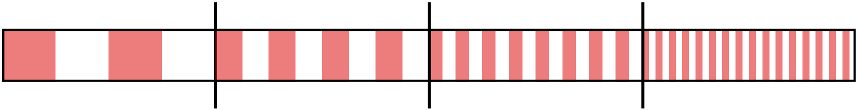

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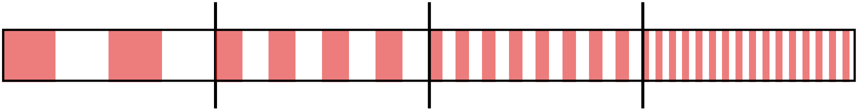

$$\lambda(\overline{R}) \leq \sum_{k:\mathcal{U}(k)\downarrow} 2^{-k-e-m-2} = 2^{-e-m-2}\Omega \leq 2^{-e-m-2}$$

Construction of L

For all σ, if $k = K_s(|\sigma|)$, and we see a string $\tau \succeq \sigma$ such that $\tau \triangle V_{n,k} \subseteq \overline{P}_s$, enumerate a request $(K_s(|\sigma|), \sigma)$.

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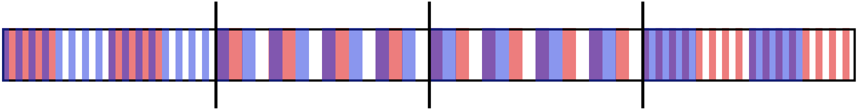
Why is this a KC set? Whenever $(k, \sigma) \neq (k', \sigma')$, the complements of the sets $\sigma \triangle V_{k,|\sigma|}$ and $\sigma' \triangle V_{k',|\sigma'|}$ are independent.



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Thus,

$$\prod_{(k,\sigma) \in L} (1 - \mu(\sigma \triangle V_{k,|\sigma|})) \geq \mu(P) > 0,$$

and hence

$$\sum_{(k,\sigma) \in L} 2^{-k} = c \sum_{(k,\sigma) \in L} \mu(V_{k,|\sigma|}) < \infty.$$