ON THE IMAGE OF A NONCOMMUTATIVE POLYNOMIAL

Let n be a (fixed) integer ≥ 2, and let F be a field. We will be concerned with the following problem:

Which subsets of $M_n(F)$ are images of noncommutative polynomials?

This question was raised by I. Kaplansky. By the image of a (noncommutative) polynomial $f = f(x_1, \ldots, x_d)$ we mean the set $\mathrm{im}(f) = \{f(a_1, \ldots, a_d) \,|\, a_1, \ldots, a_d \in M_n(F)\}$. A necessary condition for a subset S of $M_n(F)$ to be equal to $\mathrm{im}(f)$ for some f is that S is closed under conjugation by invertible matrices, i.e., $tSt^{-1} \subseteq S$ for every invertible $t \in M_n(F)$. Chuang proved that if F is a finite field, $0 \in S$, and if we consider only polynomials with zero constant term, then this condition is also sufficient. This is not true for infinite fields. Say, the set of all square zero matrices cannot be the image of a polynomial. We will give some examples of sets that can appear as images of noncommutative polynomials. We will also present Lvov's conjecture for multilinear Lie polynomials.