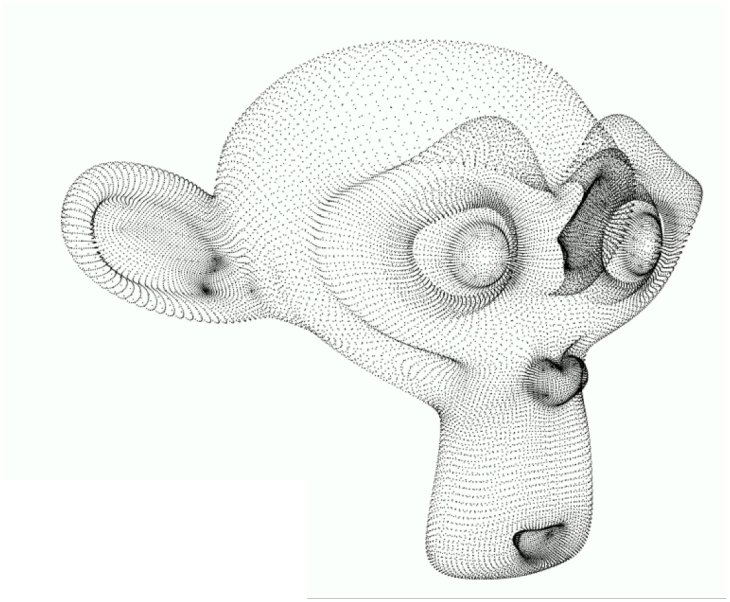
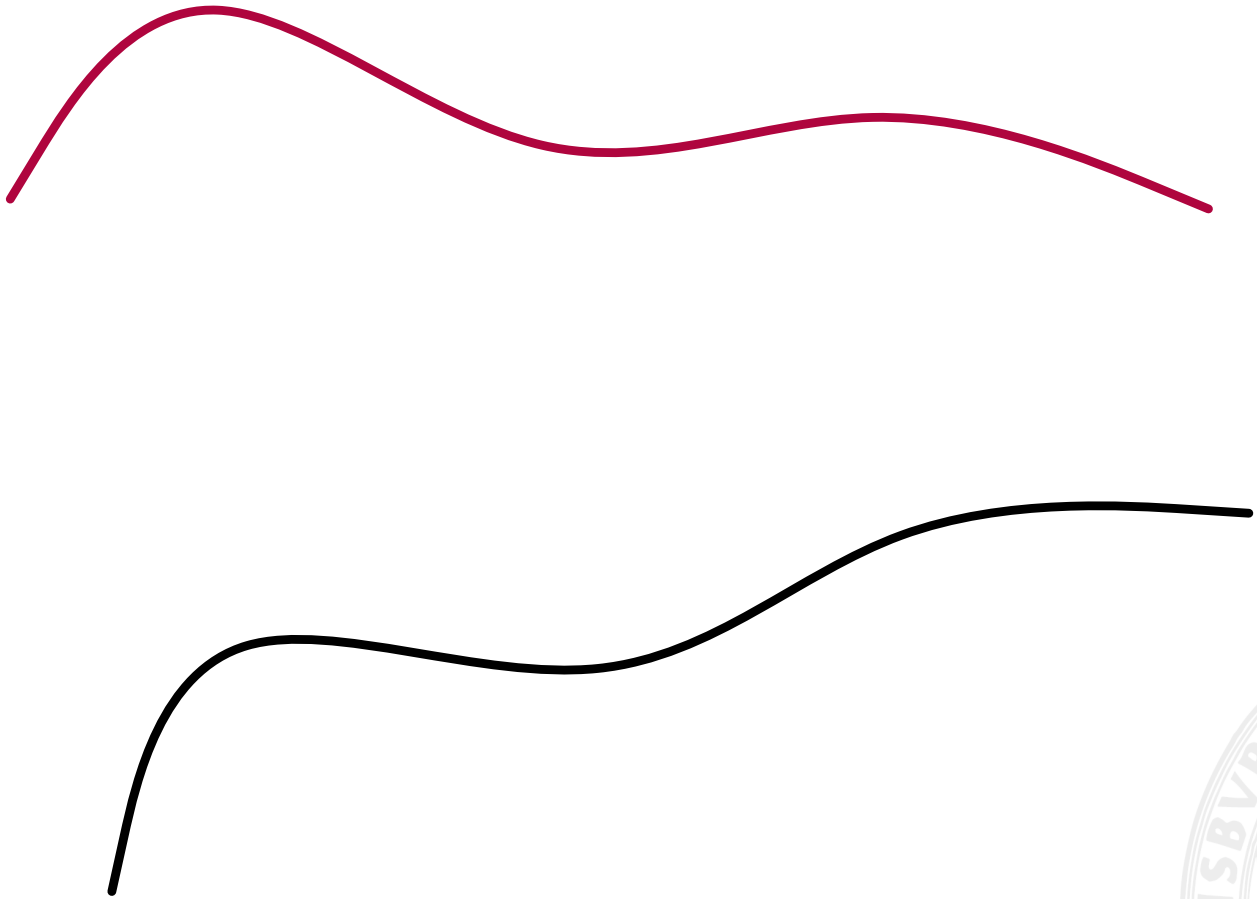
Point Cloud Registration



Least-Squares Fitting & Iterative Closest Point & *k*-D Trees

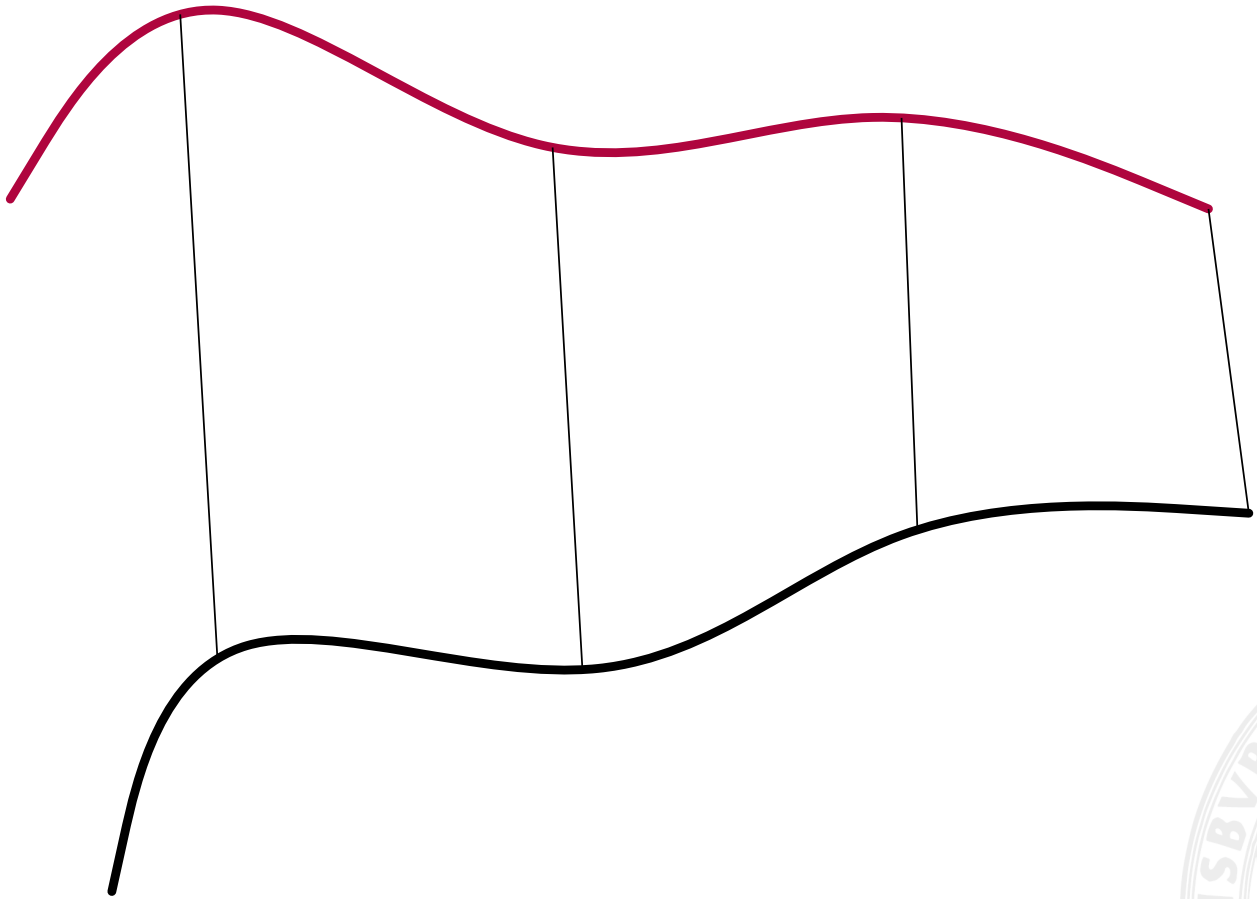
Motivation

Say that two curves are given ...



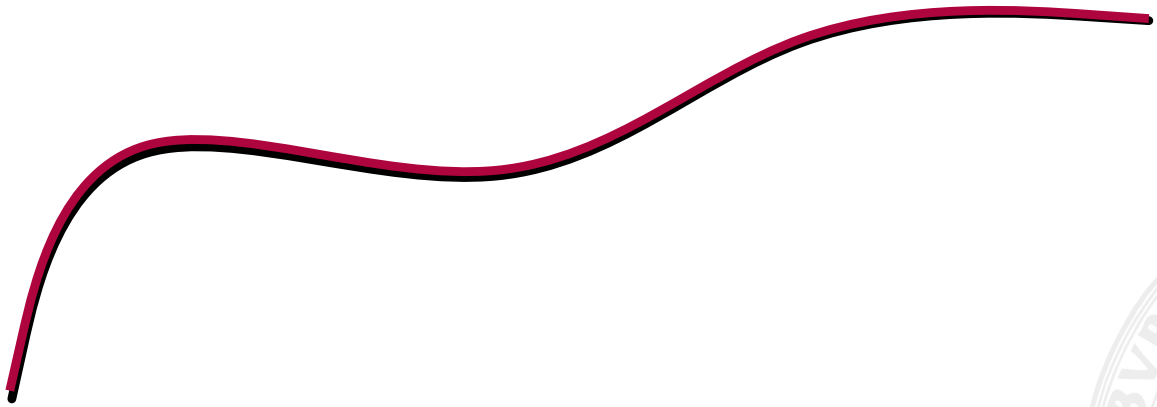
Motivation

... and the (exact) point correspondences are known



Motivation

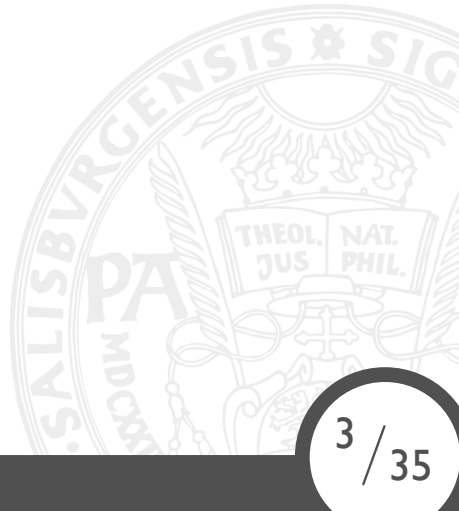
We can find the correct rotation+translation+scaling to align them!



Motivation

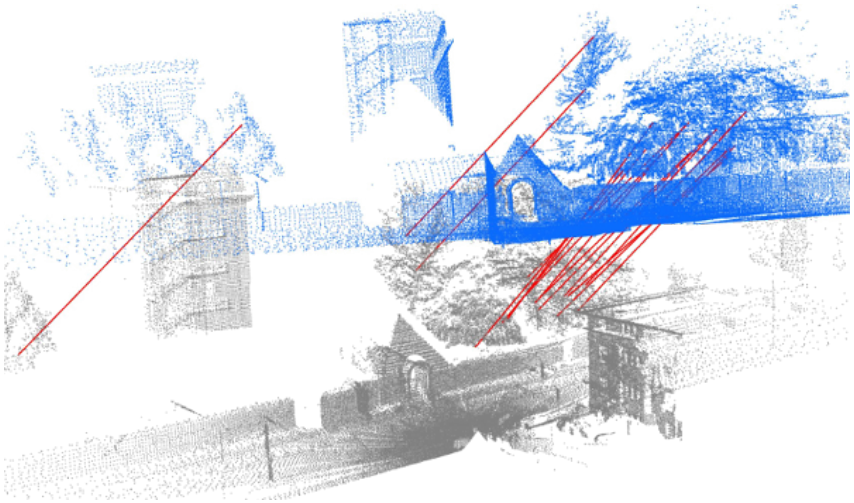
Iterative Closest Point (ICP) Algorithm

If the correspondences are not given, assume that the *closest points* correspond $\rightarrow$ *Iterative closest point (ICP)* algorithm.



Motivation

A more practical example (3D point cloud of the environment):



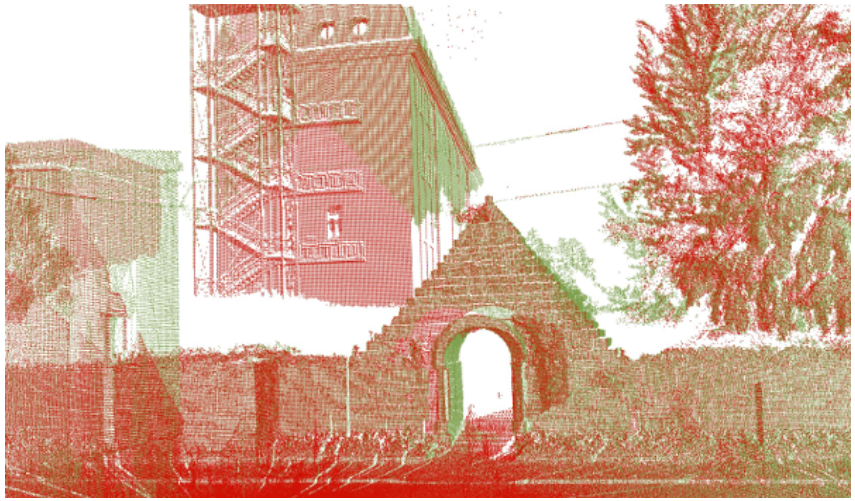


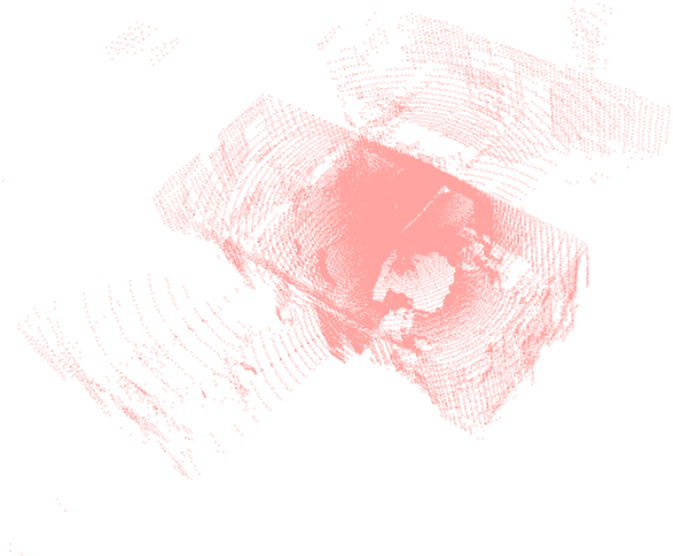
Image source (Point cloud library):

pcl

http://docs.pointclouds.org

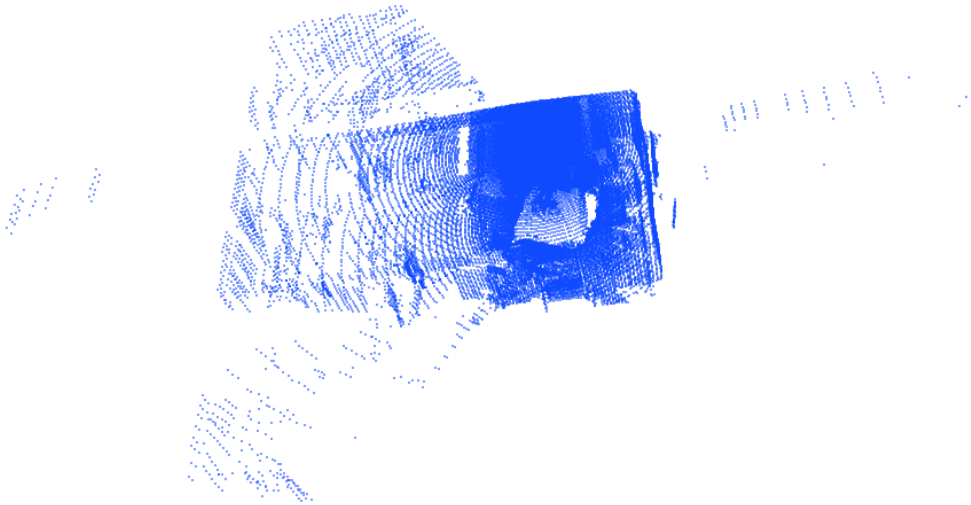
Motivation

Source: Cloud A



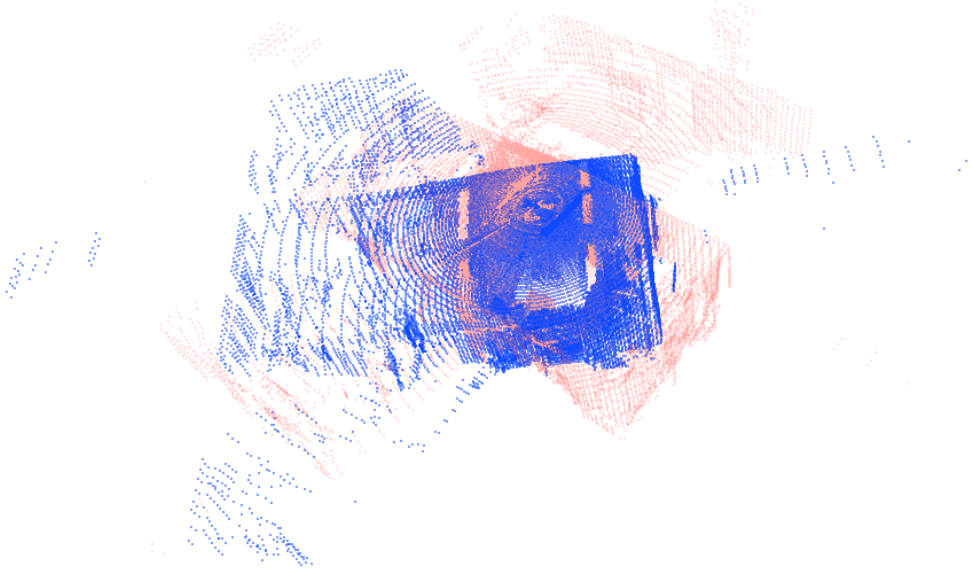
Motivation

Target: Cloud B



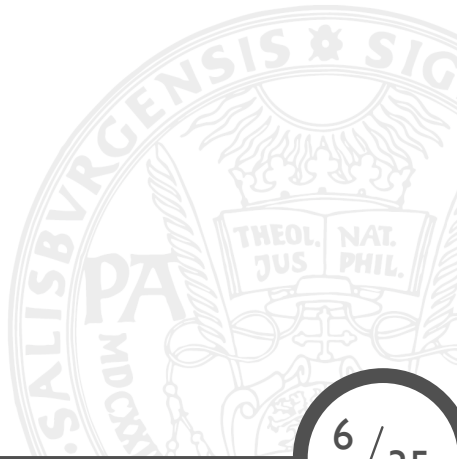
Motivation

How can we “align” B with A?



Least-Squares (LS) Fitting of 3-D Point Sets

Assume both point clouds have an equal number of points.

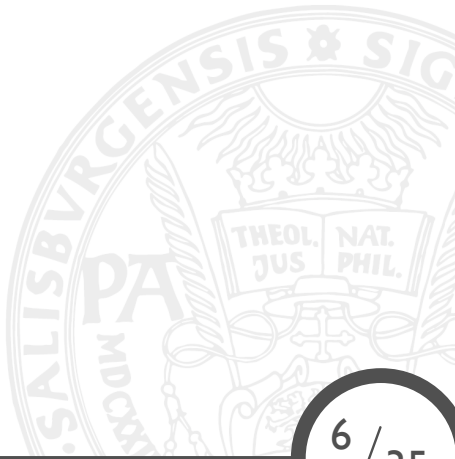


Least-Squares (LS) Fitting of 3-D Point Sets

Assume both point clouds have an equal number of points.

We have ...

- point sets $\{\mathbf{x}_i\}_{i=1}^N$ and $\{\mathbf{y}_i\}_{i=1}^N$ with $\mathbf{x}_i, \mathbf{y}_i \in \mathbb{R}^3$ (same size)
- define: $\mathbf{x}'_i = \mathbf{x}_i - \overline{\mathbf{x}}$ and $\mathbf{y}'_i = \mathbf{y}_i - \overline{\mathbf{y}}$



Least-Squares (LS) Fitting of 3-D Point Sets

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- define: $\mathbf{x}'_i = \mathbf{x}_i - \overline{\mathbf{x}}$ and $\mathbf{y}'_i = \mathbf{y}_i - \overline{\mathbf{y}}$

Optimization problem:

$$\min_{\mathbf{R}\in SO(3), s\in\mathbb{R}, \mathbf{t}\in\mathbb{R}^3} E(\mathbf{R}, \mathbf{t}, s) = \sum_{i=1}^{N} ||\mathbf{y}_i - s\mathbf{R}\mathbf{x}_i - \mathbf{t}||^2$$

where $SO(3) = \{\mathbf{R} \in \mathbb{R}^{3\times 3} : \mathbf{R}^\top\mathbf{R} = \mathbf{I}_3, \det(\mathbf{R}) = +1\}$, s is scaling and $\mathbf{t}$ translation.

Least-Squares (LS) Fitting of 3-D Point Sets

Part I: Let's expand the error term, i.e.,

$$||\mathbf{y}_i - s\mathbf{R}\mathbf{x}_i - \mathbf{t}||^2 = (\mathbf{y}_i - s\mathbf{R}\mathbf{x}_i - \mathbf{t})^\top (\mathbf{y}_i - s\mathbf{R}\mathbf{x}_i - \mathbf{t})$$

Remember, $(a+b+c)^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2$.

Least-Squares (LS) Fitting of 3-D Point Sets

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Remember, $(a+b+c)^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2$.

Consequently, the energy E can be written as

$$\sum_i ||\mathbf{y}_i - s\mathbf{R}\mathbf{x}_i - \mathbf{t}||^2 = \sum_i \mathbf{y}_i^\top \mathbf{y}_i - 2s \sum_i \mathbf{x}_i^\top \mathbf{R}\mathbf{y}_i + s^2 \sum_i \underbrace{\mathbf{x}_i^\top \mathbf{R}^\top \mathbf{R}\mathbf{x}_i}_{\mathbf{x}_i^\top \mathbf{x}_i} -$$

$$2\left(\sum_i \mathbf{y}_i^\top\right)\mathbf{t} + 2s\left(\sum_i \mathbf{x}_i^\top\right)\mathbf{R}^\top\mathbf{t} + N\mathbf{t}^\top\mathbf{t}$$

($\mathbf{R}$ is symmetric, hence $\mathbf{R}^\top = \mathbf{R}$)

Least-Squares (LS) Fitting of 3-D Point Sets

Taking $dE/d\mathbf{t}$ and setting the result to 0 allows us to solve for the optimal translation $\mathbf{t}^*$, i.e.,

Least-Squares (LS) Fitting of 3-D Point Sets

Taking $dE/d\mathbf{t}$ and setting the result to 0 allows us to solve for the optimal translation $\mathbf{t}^*$, i.e.,

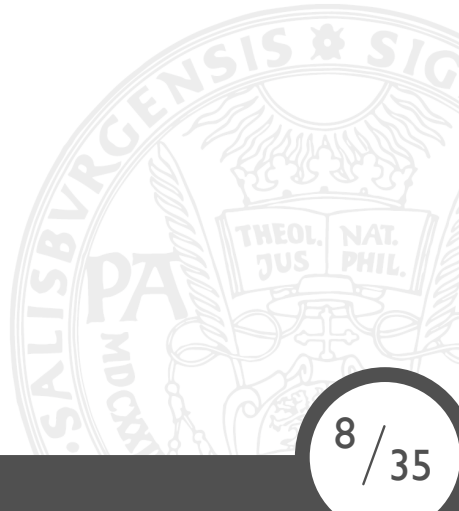
$$2s\mathbf{R}\sum_i \mathbf{x}_i - 2\left(\sum_i \mathbf{y}_i\right) + 2N\mathbf{t} = 0$$

$$\Rightarrow \frac{1}{N}\sum_i \mathbf{y}_i - s\mathbf{R}\frac{1}{N}\sum_i \mathbf{x}_i = \mathbf{t}^*$$

or, equivalently,

$$\mathbf{t}^* = \bar{\mathbf{y}} - s\mathbf{R}\bar{\mathbf{x}}$$

(see definitions from before)



Least-Squares (LS) Fitting of 3-D Point Sets

Let's update the energy functional with this optimal choice $\mathbf{t}^*$, i.e.,

$$
\begin{aligned}
E(\mathbf{R}, \mathbf{t}, s) &= \sum_{i=1}^{N} ||\mathbf{y}_i - \bar{\mathbf{y}} - s\mathbf{R}\mathbf{x}_i + s\mathbf{R}\bar{\mathbf{x}}||^2 \\
&= \sum_{i=1}^{N} ||(\underbrace{\mathbf{y}_i - \bar{\mathbf{y}}}_{\mathbf{y}'_i}) - s\mathbf{R}(\underbrace{\mathbf{x}_i - \mathbf{x}}_{\mathbf{x}'_i})||^2 \\
&= \sum_{i=1}^{N} ||\mathbf{y}'_i - s\mathbf{R}\mathbf{x}'_i||^2
\end{aligned}
$$

Upon expansion, we get

Least-Squares (LS) Fitting of 3-D Point Sets

Let's update the energy functional with this optimal choice $\mathbf{t}^*$, i.e.,

$$\begin{aligned} E(\mathbf{R},\mathbf{t},s) &= \sum_{i=1}^{N} ||\mathbf{y}_i - \bar{\mathbf{y}} - s\mathbf{R}\mathbf{x}_i + s\mathbf{R}\bar{\mathbf{x}}||^2 \\ &= \sum_{i=1}^{N} ||(\underbrace{\mathbf{y}_i - \bar{\mathbf{y}}}_{\mathbf{y}'_i}) - s\mathbf{R}(\underbrace{\mathbf{x}_i - \mathbf{x}}_{\mathbf{x}'_i})||^2 \\ &= \sum_{i=1}^{N} ||\mathbf{y}'_i - s\mathbf{R}\mathbf{x}'_i||^2 \end{aligned}$$

Upon expansion, we get

$$E(\mathbf{R},\mathbf{t},s) = s^2 \sum_i \mathbf{x}_i'^{\top}\mathbf{x}'_i - 2s \sum_i \mathbf{x}_i'^{\top}\mathbf{R}^{\top}\mathbf{y}'_i + \sum_i \mathbf{y}_i'^{\top}\mathbf{y}'_i$$

Least-Squares (LS) Fitting of 3-D Point Sets

What about the scale? Taking E/ds and setting the result to 0 gives

$$s^* = \frac{\sum_i (\mathbf{R}\mathbf{x}'_i)^\top \mathbf{y}'^\top_i}{\sum_i \mathbf{x}'^\top_i \mathbf{x}'_i}$$

Horn[1] symmetrizes the error term w.r.t. scale as

$$||(1/\sqrt{s})\mathbf{y}'_i - \sqrt{s}\mathbf{R}\mathbf{x}'_i||^2$$

which has the "nice" effect that we obtain

$$s^* = \frac{\sum_i \mathbf{y}'^\top_i \mathbf{y}'_i}{\sum_i \mathbf{x}'^\top_i \mathbf{x}'_i}$$

[1] B. Horn. "Closed-form solution of absolute orientation using unit quaternions". In: *Journal of the Optical Society of America A* 4 (1987), pp. 629–642.

Least-Squares (LS) Fitting of 3-D Point Sets

From our error expansion (under the optimal choices for $\mathbf{t}$, s), the minimization problem can now be reduced to

$$\arg \max_{\mathbf{R} \in SO(3)} E(\mathbf{R}, \mathbf{t}^*, s^*) = \arg \max_{\mathbf{R} \in SO(3)} \sum_i \mathbf{x}_i'^\top \mathbf{R}^\top \mathbf{y}_i'$$

with (due to the cyclic property of the trace)

$$\sum_i \mathbf{x}_i'^\top \mathbf{R}^\top \mathbf{y}_i' = \sum_i \mathbf{y}_i'^\top \mathbf{R} \mathbf{x}_i' = \operatorname{tr}\left(R \underbrace{\sum_i \mathbf{x}_i' \mathbf{y}_i'^\top}_{\mathbf{H}} \right) = \operatorname{tr}(\mathbf{RH})$$

Arun et al.[2] show that the optimal $\mathbf{R}$ is

$$\mathbf{R}^* = \mathbf{V}\mathbf{U}^\top \quad \text{with} \quad \mathbf{H} := \sum_i \mathbf{x}_i' \mathbf{y}_i'^\top = \underbrace{\mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top}_{\text{i.e., SVD of } \mathbf{H}}$$

[2] K.S. Arun, T.S. Huang, and S.D. Blostein. "Least-Squares Fitting of Two 3-D Point Sets". In: *Pattern Analysis and Machine Intelligence* 9.5 (1986), pp. 698–700.

Iterative Closest Point (ICP)

Registration WITHOUT point correspondences

Setup: We have point clouds X and Y

- $X = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$
- $Y = \{\mathbf{y}_1, \dots, \mathbf{x}_m\}$

Remark

X and Y might have a different number of points!

Iterative Closest Point (ICP)

Registration without point correspondences[3]

1. Initialize $(\mathbf{R}^*, \mathbf{t}^*, s^*) = (\mathbf{R}_0, \mathbf{t}_0, s_0)$
2. (Select S random points, say $S = 1000$)
3. Use $(\mathbf{R}^*, \mathbf{t}^*, s^*)$ to find matches $\mathbf{y}_k$ for all $n = 1, \ldots, S$

$$k = \arg\min_m ||\mathbf{y}_m - s^*\mathbf{R}^*\mathbf{x}_n - \mathbf{t}^*||^2$$

   and collect all corresponding tuple $\{(\mathbf{x}_i, \mathbf{y}_i)\}_{i=1}^S$
4. Minimize error function (using our least-squares solution)

$$(\mathbf{R}^*, \mathbf{t}^*, s^*) = \arg\min_{\mathbf{R},s,\mathbf{t}} \sum_{k=1}^{S} ||\mathbf{y}_k - s\mathbf{R}\mathbf{x}_k - \mathbf{t}||^2$$

5. Apply transform and repeat steps (2) – (3) until convergence

[3] P.J. Besl and N.D. McKay. "A Method for Registration of 3-D Shapes". In: *Pattern Analysis and Machine Intelligence* 14.2 (1992), pp. 239–256.

Iterative Closest Point (ICP)

Registration without point correspondences

Besl & McKay and Zhang[4] basically introduce the same ICP algorithm, only that

- Besl & McKay use all point correspondences
- Zhang only use correspondences below a distance threshold (e.g., using prior knowledge about the data)

[4] Z. Zhang. "Iterative Point Matching for Registration of Free-Form Curves and Surfaces". In: *IJCV* 13.2 (1994), pp. 119–148.

Iterative Closest Point (ICP)

Tools of the trade ...

To start working with point clouds, I would recommend ...

- Visualization Toolkit (VTK); C++/Python

available at: `http://www.vtk.org/`

- ParaView (based on VTK) for point cloud visualization (C++/Python)

available at: `http://www.paraview.org`

- Point Cloud Library (PCL) for algorithms (C++)

available at: `http://pointclouds.org`

Iterative Closest Point (ICP)

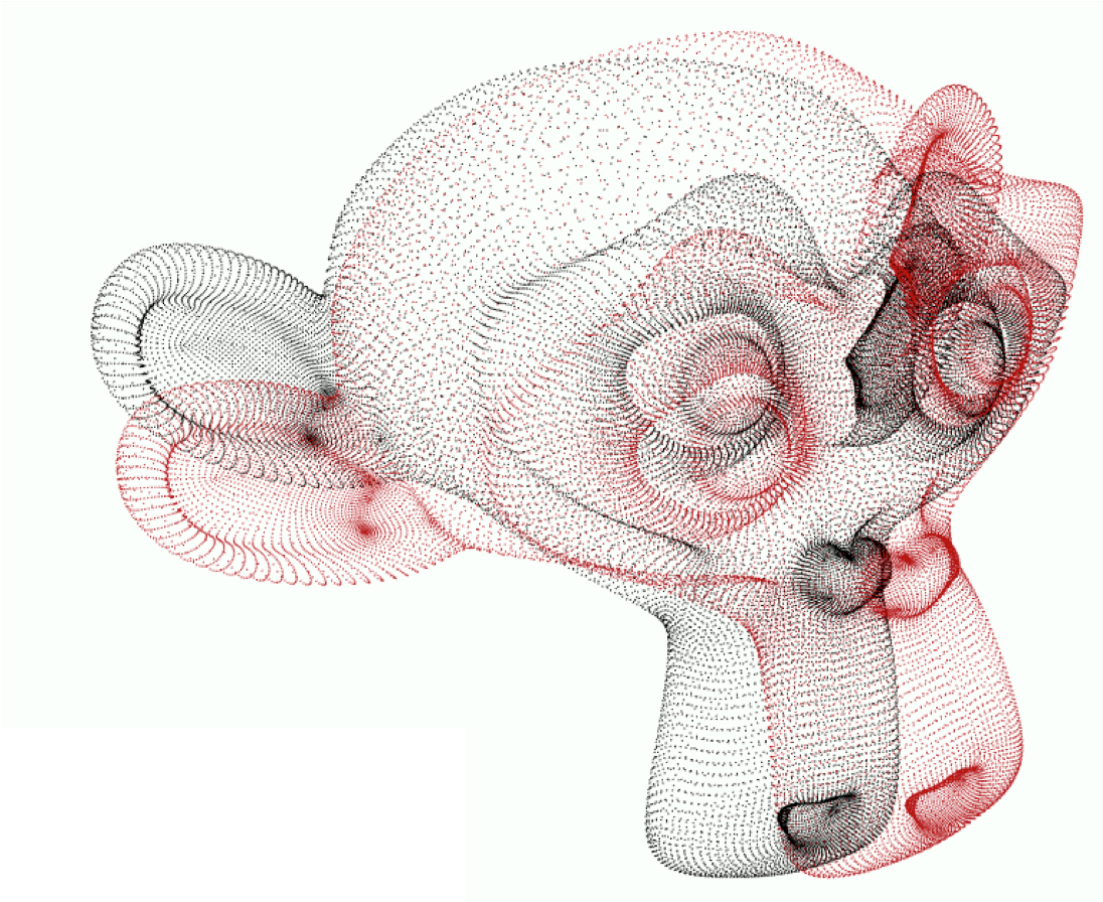
Tools of the trade ...

- Open3D; Python

available at: `http://www.open3d.org`

Iterative Closest Point (ICP)

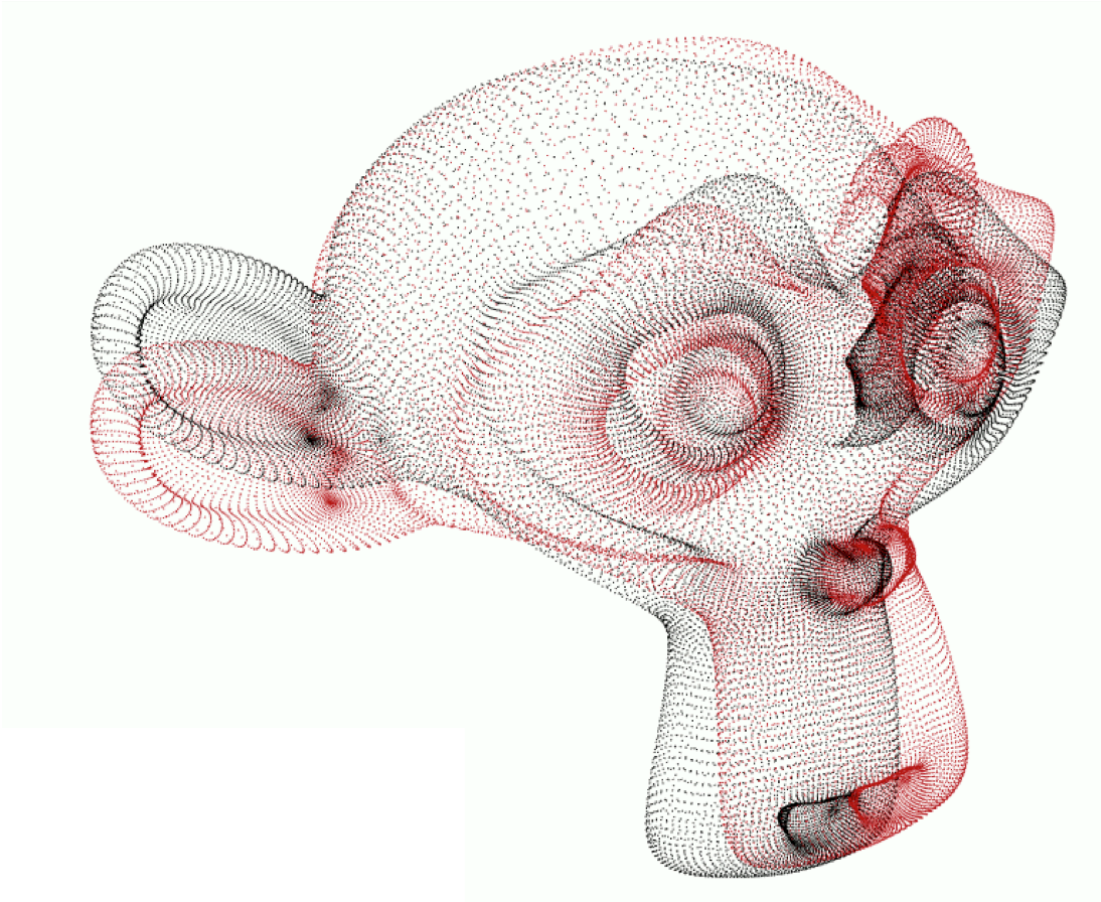
Basic ICP – Example with "Monkey" (using PCL)



Iteration: 0

Iterative Closest Point (ICP)

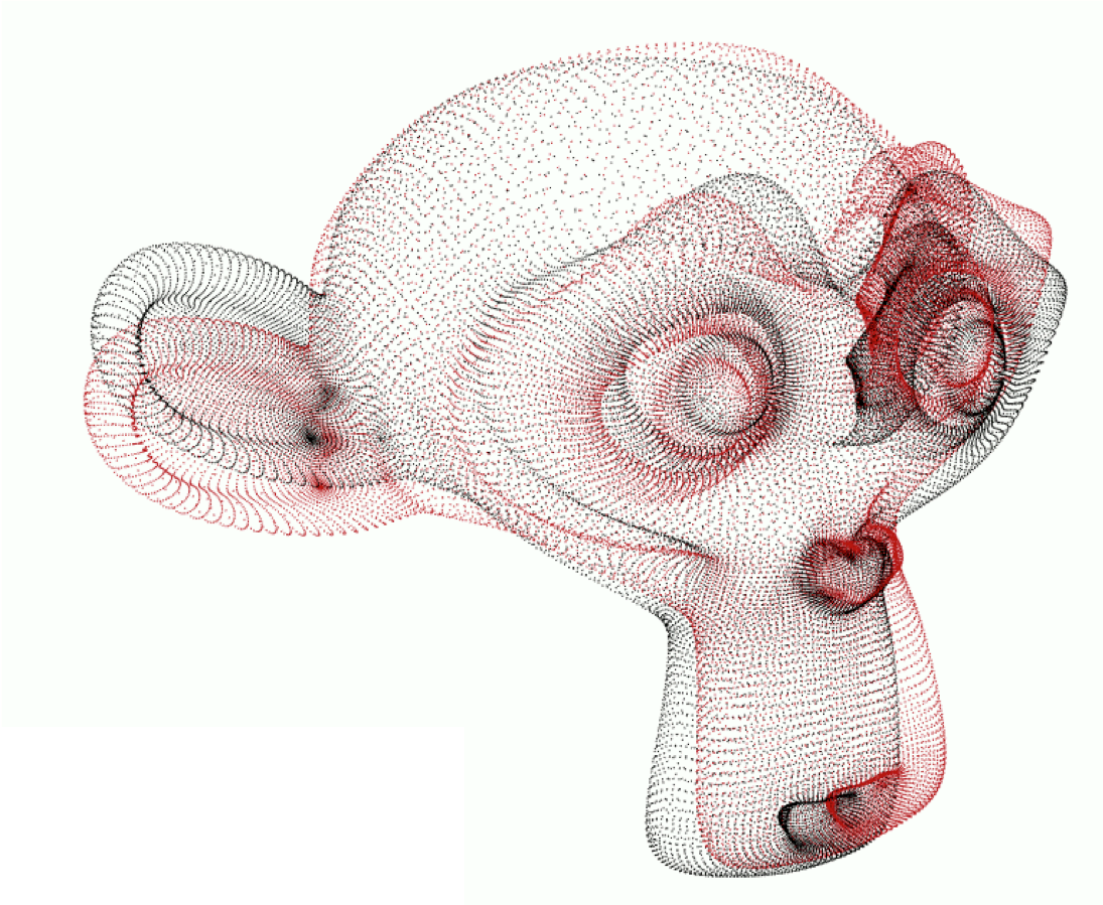
Basic ICP – Example with "Monkey" (using PCL)



Iteration: 3

Iterative Closest Point (ICP)

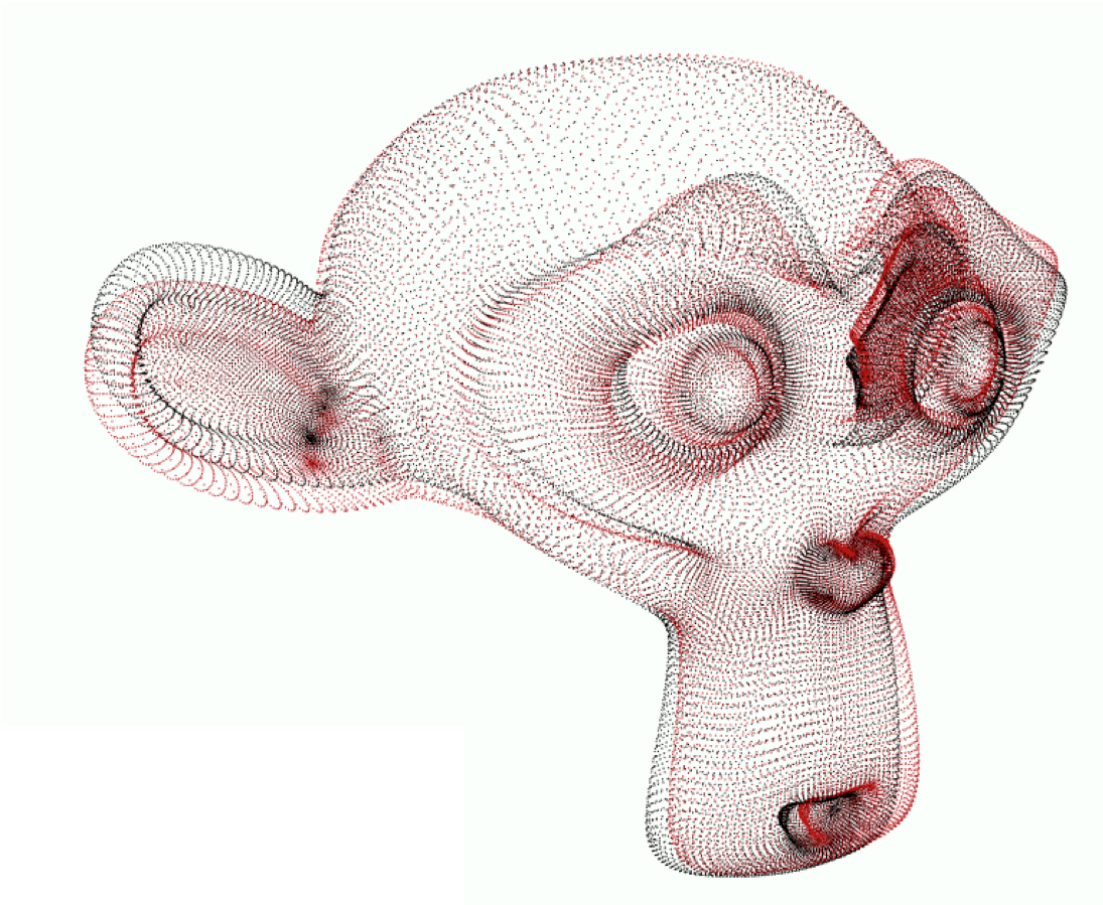
Basic ICP – Example with "Monkey" (using PCL)



Iteration: 5

Iterative Closest Point (ICP)

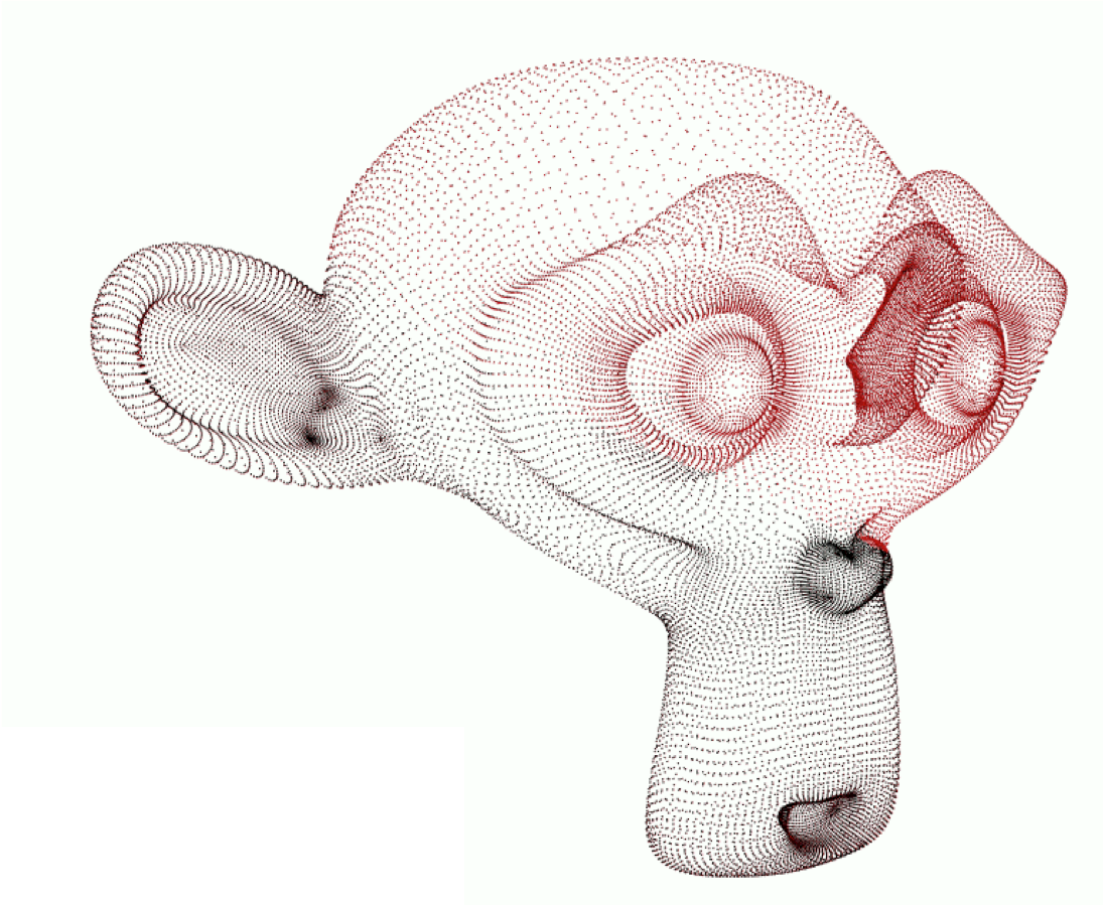
Basic ICP – Example with "Monkey" (using PCL)



Iteration: 9

Iterative Closest Point (ICP)

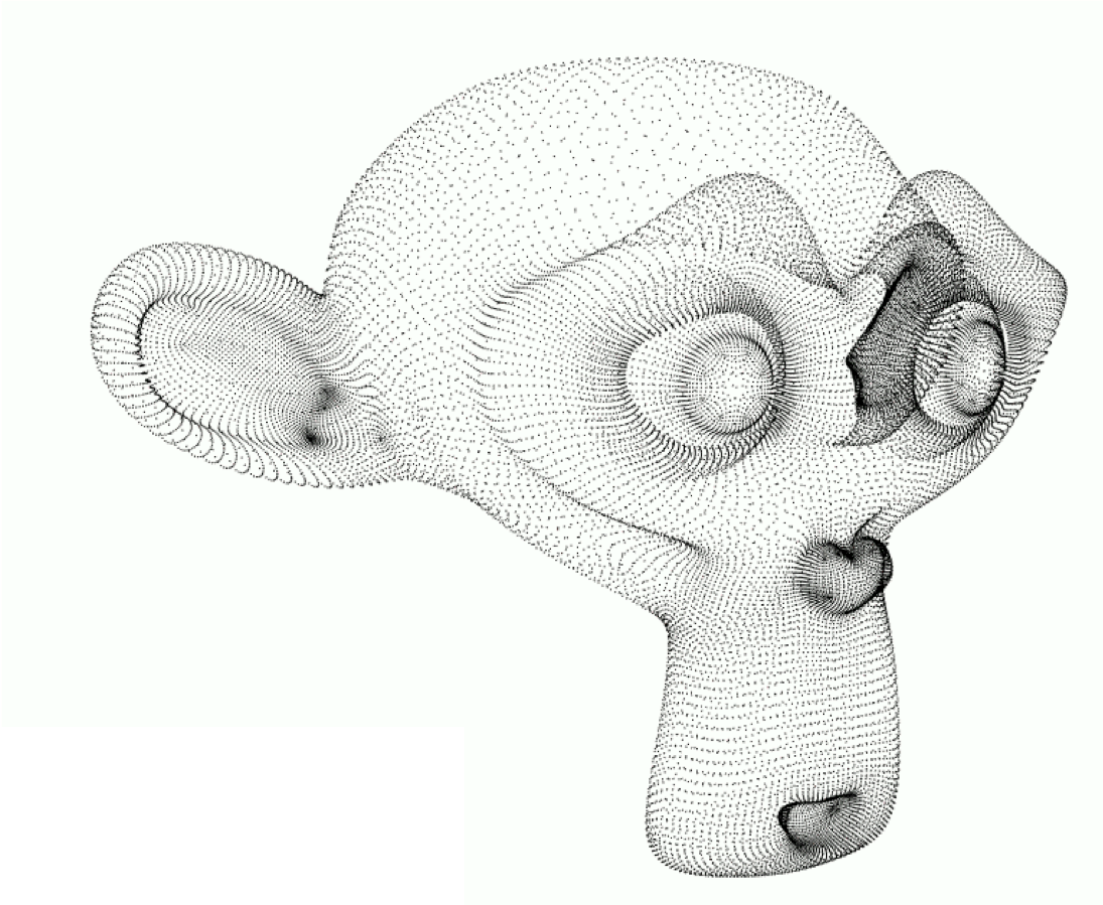
Basic ICP – Example with "Monkey" (using PCL)



Iteration: 25

Iterative Closest Point (ICP)

Basic ICP – Example with "Monkey" (using PCL)

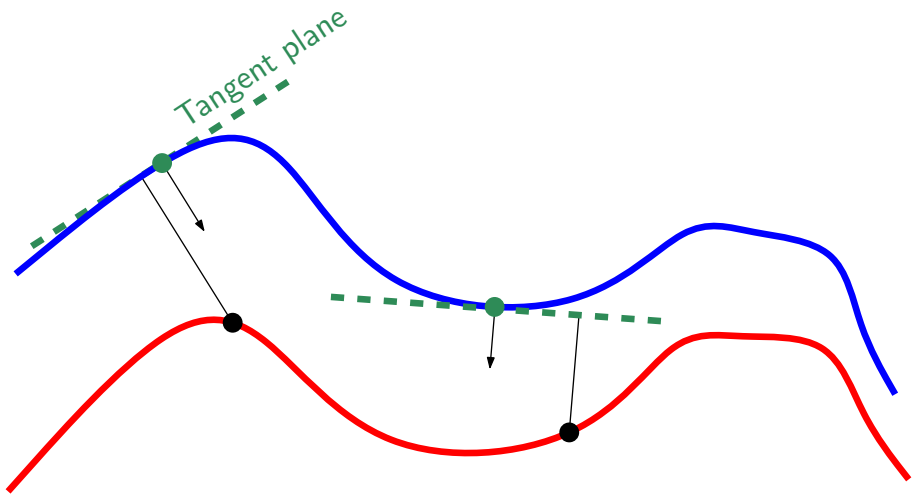


Iteration: Final (Done!)

Iterative Closest Point (ICP)

Variants of ICP with respect to the error metric

One possibility is to replace the point-to-point error metric (as we used before) by a point-to-plane error metric[5]



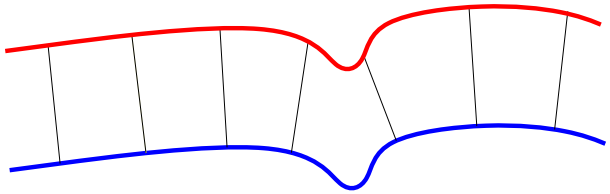
$\Rightarrow$ allows flat regions to "slide" with respect to each other!

[5] Y. Chen and G. Medoni. "Object Modeling By Registration of Multiple Range Images". In: *ICRA*. 1991.

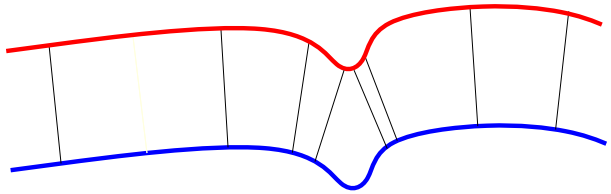
Iterative Closest Point (ICP)

Variants of ICP with respect to the selection of source points

- Use all source points
- Uniform sampling
- Stable sampling[6]
  - Choosing too many points in "featureless regions" is bad
  - Sampling strategy to constrain potentially unstable transformation



Uniform sampling



Stable sampling

We can achieve similar effects as with selection by weighting (however, pre-processing vs. runtime tradeoff)

[6] N. Gelfand et al. "Geometrically Stable Sampling for the ICP Algorithm". In: *3DIM*. 2003.

Iterative Closest Point (ICP)

Variants of ICP with respect to the matching of source points

What about speed? *Finding the closest point* (in a cloud with N points) is the most expensive stage in the ICP algorithm.

- Brute-force $O(N)$
- Better to use spatial data structures (e.g., k-D tree)

In general, closest points can be suboptimal as corresponding points! Alternatives are ...

- matching of colors, e.g., Godin et al.[7]
- matching of normals, e.g., Pulli[8]
- and many others (e.g., consider curvature, local features, etc.)

[7] G. Godin, M. Rioux, and R. Baribeau. "Three-dimensional registration using range and intensity information". In: *SPIE Videometric III*. 1994.

[8] K. Pulli. "Multiview registration for large data sets". In: *3DIM*. 1999.

k-D Trees

Principle

k-D trees are

- spatial data structures introduced by Bentley[9]
- impose a spatial decomposition of *k*-dimensional points that
  - allows efficient search on orthogonal range queries
  - allows efficient nearest neighbor searches

Remark

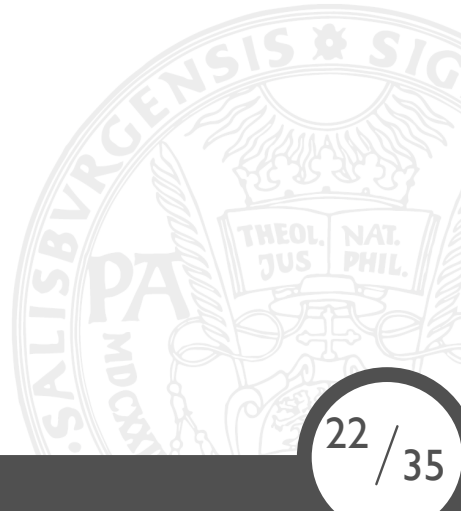
both properties are achieved by "pruning" the search space!

[9] J. Bentley. "Multidimensional binary search trees used for associative searching". In: *Commun. ACM* 18 (1975), 509?517.

k-D Trees

Building a (naive) *k*-D tree

Data is inserted as in the case of a regular binary tree, only that

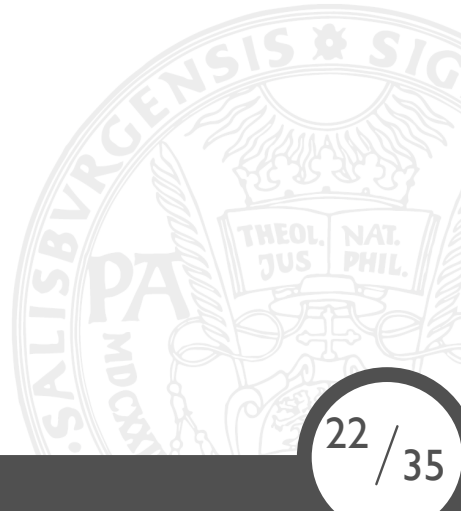


k-D Trees

Building a (naive) *k*-D tree

Data is inserted as in the case of a regular binary tree, only that

- the key that we use changes at each level



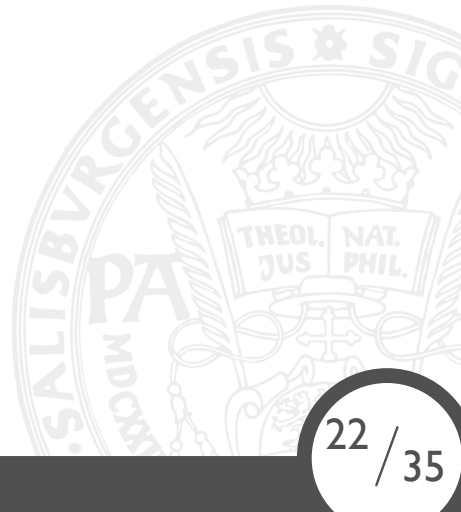
k-D Trees

Building a (naive) k-D tree

Data is inserted as in the case of a regular binary tree, only that

- the key that we use changes at each level
- at each level i, the key is $(i \bmod k)$ where k denotes dimensionality

Example: $a = (1, 2), b = (3, 4), c = (5, 6), d = (7, 8)$



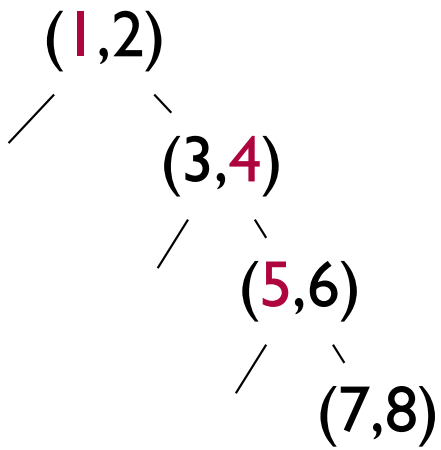
k-D Trees

Building a (naive) k-D tree

Data is inserted as in the case of a regular binary tree, only that

- the key that we use changes at each level
- at each level i, the key is $(i \bmod k)$ where k denotes dimensionality

Example: $a = (1, 2), b = (3, 4), c = (5, 6), d = (7, 8)$

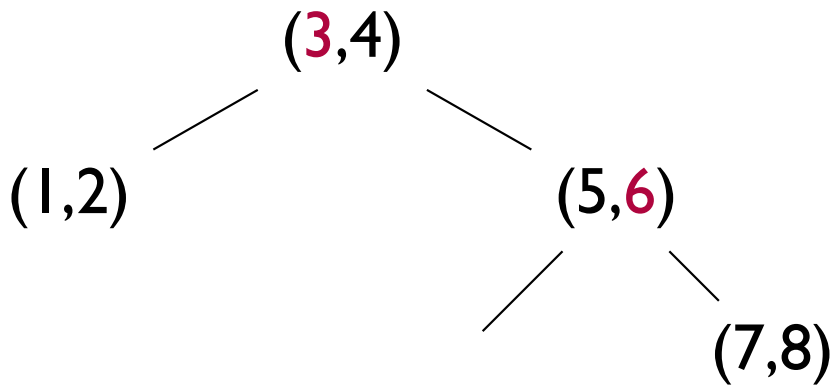


Insertion order: “abcd”

k-D Trees

Building a (naive) k-D tree

Same example: $a = (1, 2), b = (3, 4), c = (5, 6), d = (7, 8)$



Insertion order: “bcad”

k-D Trees

Building a (median) k-D tree (see Friedman et al.[10])

We are given points $\{\mathbf{p}_i\}_{i=1}^{N}$, $\mathbf{p}_i \in \mathbb{R}^k$.

Algorithm (Building a median k-D tree)

1. Select split (i.e., "cutting dimension") coordinate either by
   - 1.1 at level i, the key is $(i \bmod k)$ – cycle through coordinate axes
   - 1.2 find axis with maximal spread and choose this as key at level i
2. Split data according to the median m_c at the key coordinate c:
   - left subtree: $\{\mathbf{p}_i : p_{ic} \leq m_c\}$
   - right subtree: $\{\mathbf{p}_i : p_{ic} > m_c\}$

[10] J. Friedman, J. Bentley, and R. Finkel. "An algorithm for finding best matches in logarithmic expected time". In: *ACM Transactions on Mathematical Software* 3 (1977), pp. 209–226.

k-D Trees

Median k-D tree – Properties

Properties:

- Size: $O(N)$
- Depth: $O(\log N)$
- Construction time: $O(N \log N)$

  (Note: the median can be found in $O(N)$[11])

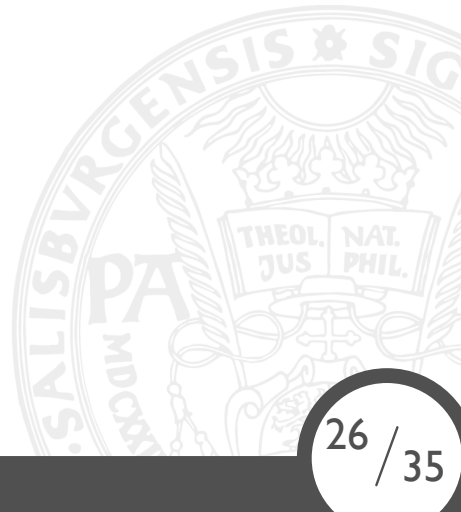
[11] M. Blum et al. "Time bounds for selection". In: *Journal of Computer and System Sciences* 7.4 (1973), pp. 448–461.

k-D Trees

Some remarks ...

Remarks on the choosing "cutting dimension"

- cycle through (using level mod dimension)
  - might produce skinny (elongated) cells
    $\rightarrow$ bad impact on query times



k-D Trees

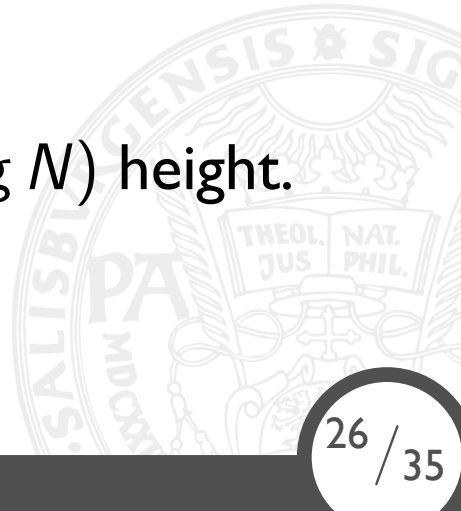
Some remarks ...

Remarks on the choosing "cutting dimension"

- cycle through (using level mod dimension)
  - might produce skinny (elongated) cells
    $\rightarrow$ bad impact on query times
- split along dimension of greatest "spread"
  - e.g., use difference between min. and max. value as statistic
  - Bentley refers to such a tree as an *optimized k-D tree*

Remarks on the choosing "cutting value"

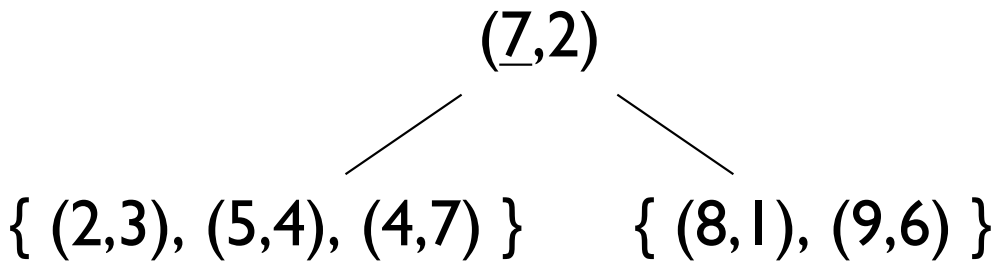
- Choosing the median allows us to guarantee $O(\log N)$ height.



Example from:

http://en.wikipedia.org/wiki/K-d_tree

Data: $(7,2), (5,4), (2,3), (4,7), (9,6), (8,1)$
(our strategy: cycle dimension & split by the *upper* median $x_{\lceil \frac{n+1}{2} \rceil}$)



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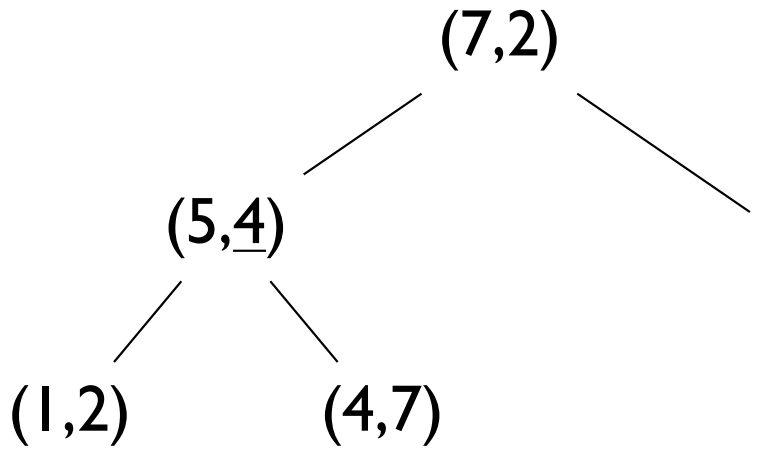

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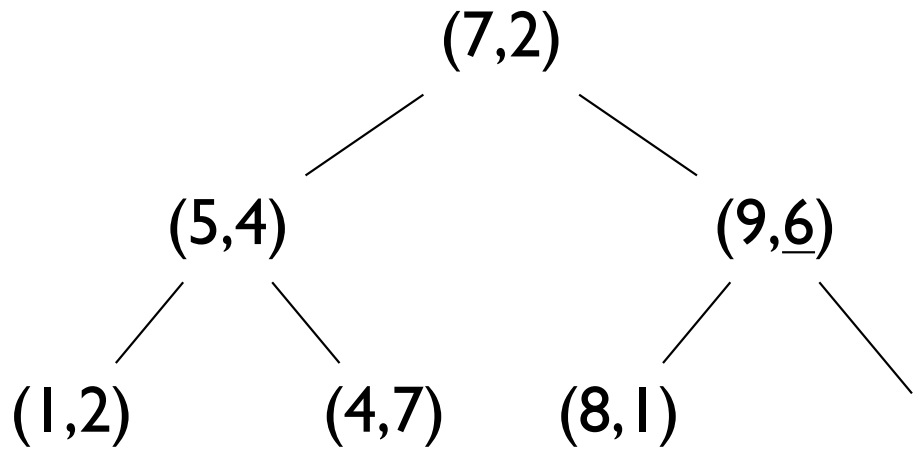
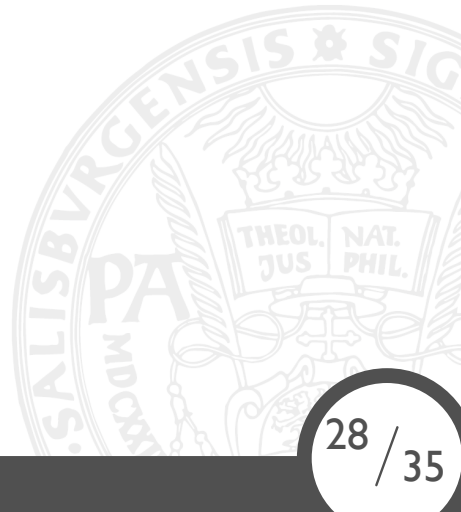


k-D Trees

Nearest neighbor search with a k-D tree

Problem: Given a point $\mathbf{q}$, find the point $\mathbf{p}_i$ that is closest to $\mathbf{q}$



k-D Trees

Nearest neighbor search with a *k*-D tree

Problem: Given a point $\mathbf{q}$, find the point $\mathbf{p}_i$ that is closest to $\mathbf{q}$

First strategy: Find cell that would contain $\mathbf{q}$ and return point in it.

k-D Trees

Nearest neighbor search with a *k*-D tree

Problem: Given a point **q**, find the point $\mathbf{p}_i$ that is closest to **q**

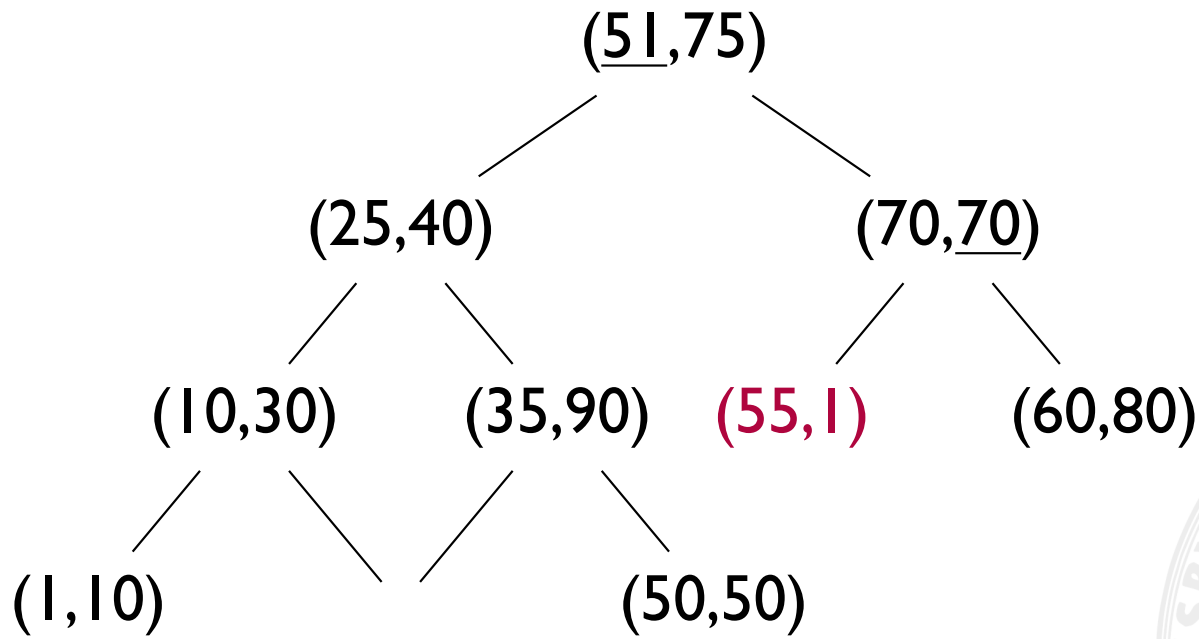
First strategy: Find cell that would contain **q** and return point in it.
DOES NOT WORK!

k-D Trees

Nearest neighbor search with a *k*-D tree

Problem: Given a point $\mathbf{q}$, find the point $\mathbf{p}_i$ that is closest to $\mathbf{q}$

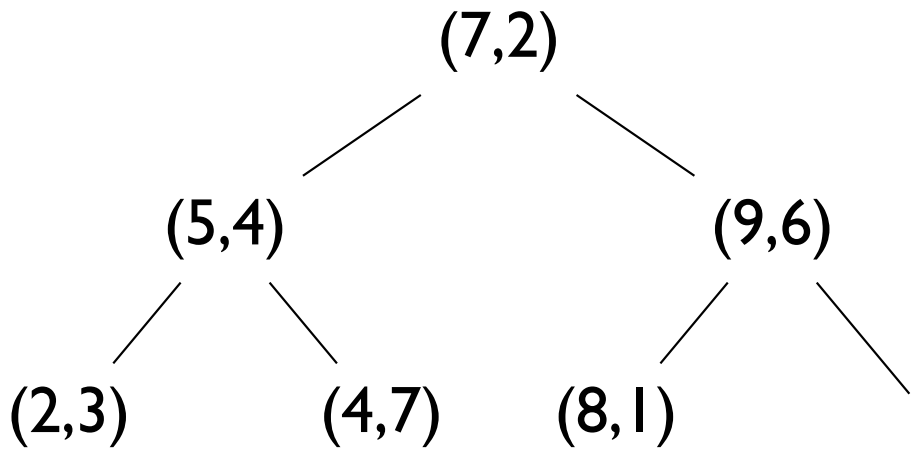
First strategy: Find cell that would contain $\mathbf{q}$ and return point in it.
DOES NOT WORK! In the following example, set $\mathbf{q} = (52, 52)$



k-D Trees

Nearest neighbor search with a k-D tree

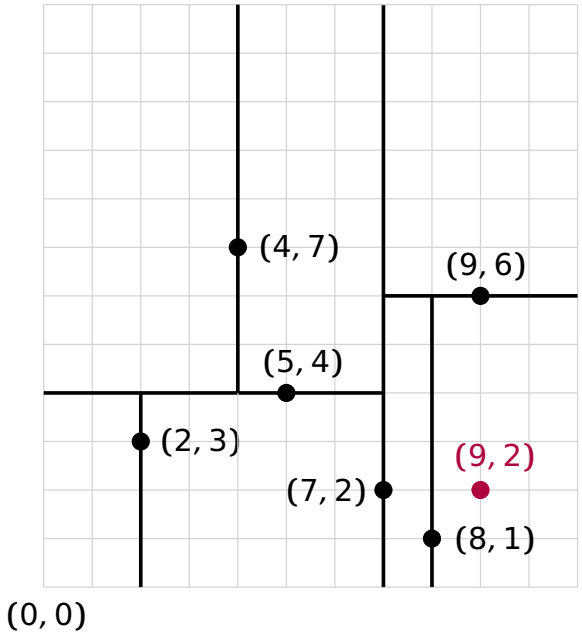
Let's look at our Wikipedia example (median k-D tree):



Task: Search for the nearest-neighbor to $\mathbf{q} = (9, 2)$

k-D Trees

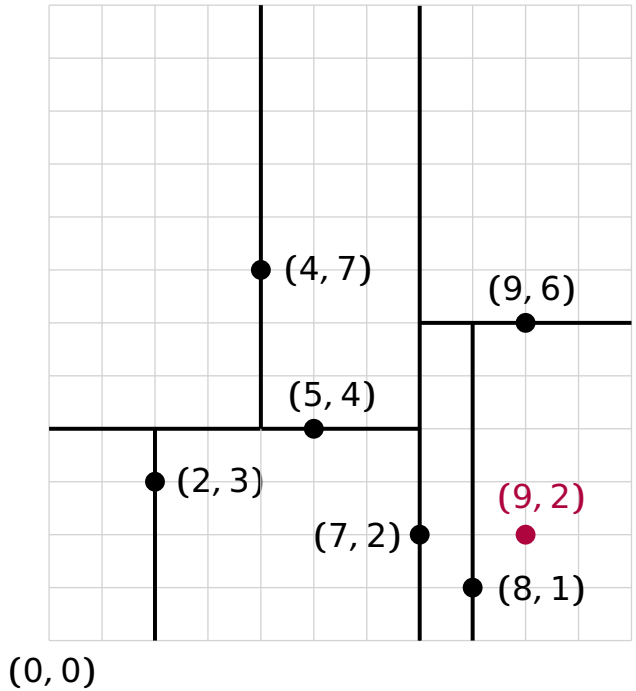
Nearest neighbor search with a *k*-D tree



What's the basic strategy?

- Store the closest point found so far
- Prune subtrees once their bounding boxes cannot contain **q**
- Search subtrees in most promising order

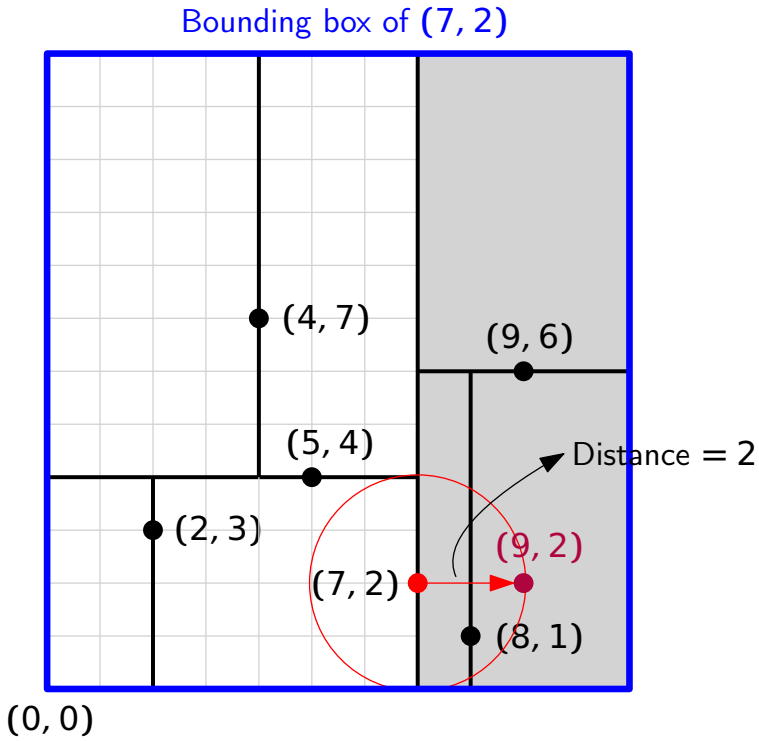
Example



(Symbol legend: → BB dist., → point-to-point dist,, ● current point)

Example

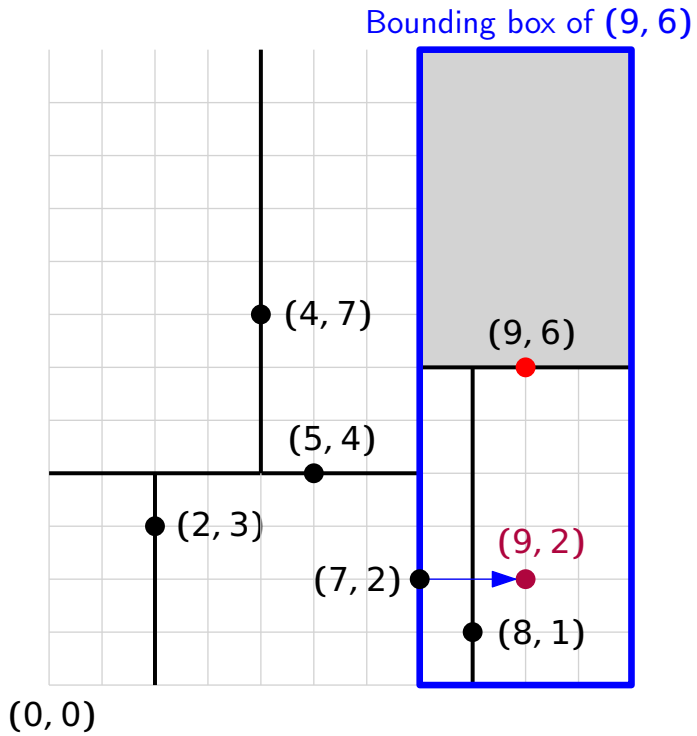
Init. distance at ∞. Distance of $\mathbf{q}$ to BB of $(7, 2)$ is $< \infty$. Store $(7, 2)$ as best point so far. Next, since $9 > 7 \rightarrow$ goto right subtree



(Symbol legend: $\rightarrow$ BB dist., $\rightarrow$ point-to-point dist,, ● current point)

Example

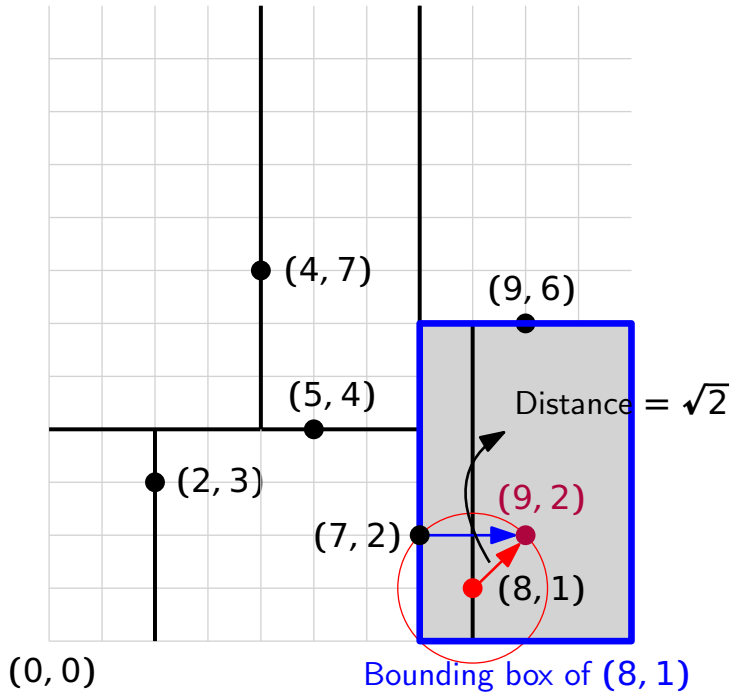
At $(9, 6)$, distance of $\mathbf{q}$ to BB of $(9, 6)$ not > 2, hence, continue;
Distance of $(9, 6)$ to $\mathbf{q}$ not best. Next, $2 < 6 \rightarrow$ goto left subtree



(Symbol legend: → BB dist., → point-to-point dist,, ● current point)

Example

We are now at the left subtree of $(9, 6)$; Distance of **q** to BB of $(8, 1)$ not > 2, hence continue; Distance to $(8, 1)$ is now best!

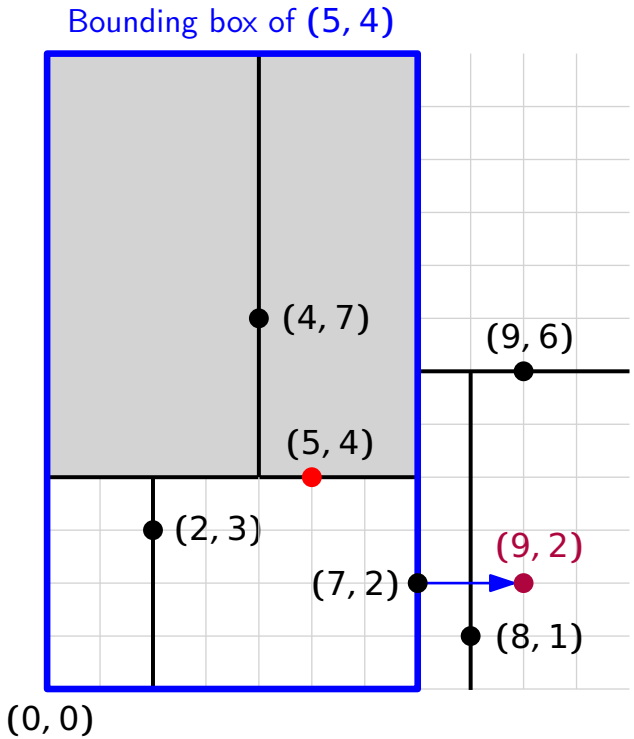


(Also, no need to traverse further since (8,1) is leaf)

(Symbol legend: → BB dist., → point-to-point dist,, ● current point)

Example

We are now in the left subtree of $(7, 2)$ at $(5, 4)$; Now, distance of **q** to BB of $(5, 4) > \sqrt{2}$; hence, return; DONE!



(Symbol legend: → BB dist., → point-to-point dist,, ● current point)

More on k-D trees

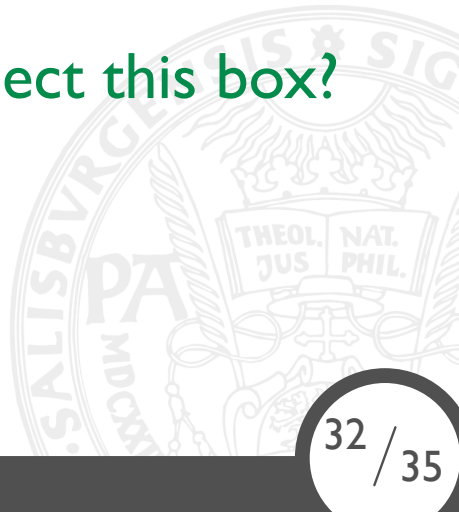
Curse of dimensionality

Consider n points uniformly in $[0,1]^d$ and a space partitioning that splits each box in the middle (round robin on axes).

Assume each box is split t times (tree depth t) $\rightarrow$ each axis is split t/d times (say, t/d is integer) with box dimensions

$$[2^{-t/d}, \ldots, 2^{-t/d}]$$

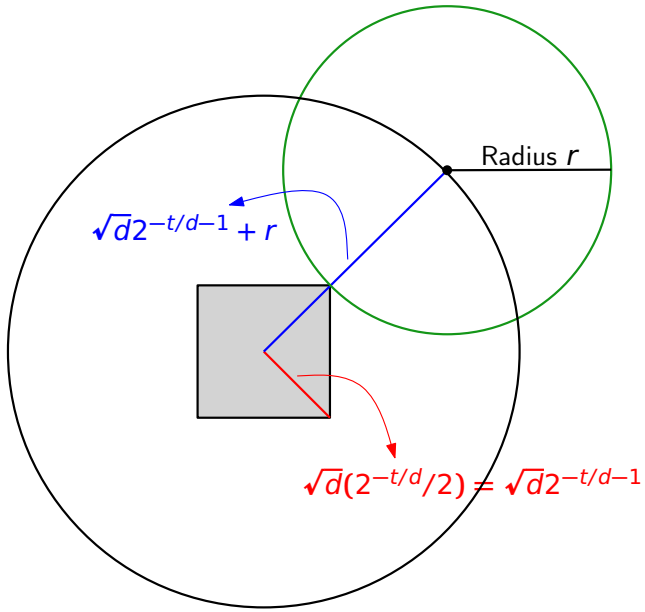
What is the probability that ball of radius r will intersect this box?



More on k-D trees

Curse of dimensionality

The length of an r-agonal of an hypercube is $a\sqrt{d}$ (where a denotes the side-length). Hence, we have

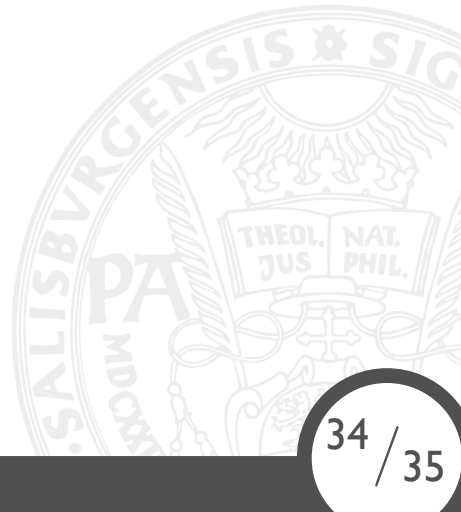


The probability that the box intersects a random query sphere is at most the the volume of a sphere of radius $\sqrt{d}2^{-t/d-1} + r$.

More on k-D trees

Curse of dimensionality

Assume that $\sqrt{d}2^{-t/d-1} \leq r$.



More on k-D trees

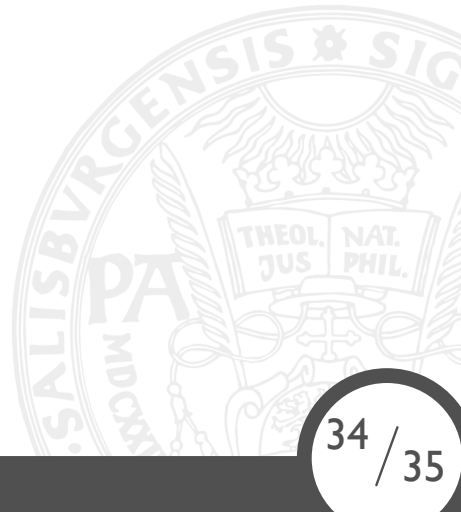
Curse of dimensionality

Assume that $\sqrt{d}2^{-t/d-1} \leq r$.

Hence, the probability is bounded by the volume of a "bounding ball" of radius $2r$. The volume of an $\mathbb{R}^d$-ball of this radius is

$$V_d(r) = \frac{\pi^{d/2}(2r)^d}{(d/2)!} \text{ (if } d \text{ even)}$$

Hence, the dependency of the volume is in $O(r^d)$.



More on k-D trees

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Hence, the dependency of the volume is in $O(r^d)$.

Consequence

As a result, we expect to inspect only a $O(r^d)$ fraction of boxes and, since $r \leq 1$, this leads to significant speedups!

More on k-D trees

Curse of dimensionality

But, we assumed that $\sqrt{d}2^{-t/d-1} \leq r$. The question is, "how does this behave when d is large"?

- Let's consider $t \approx \log(n)$ (which we have in a balanced tree), which gives boxes of volume $\approx 1/n$[12]

[12] since $(2^{-\log(n)/d})^d$ (i.e., box vol.) reduces to $2^{-\log(n)}$ and by letting log be approximated by $\log_2$, we get $\approx 1/n$

More on k-D trees

Curse of dimensionality

But, we assumed that $\sqrt{d}2^{-t/d-1} \leq r$. The question is, "how does this behave when d is large"?

- Let's consider $t \approx \log(n)$ (which we have in a balanced tree), which gives boxes of volume $\approx 1/n$[12]
- We get $\sqrt{d}2^{-\log(n)/d-1} \leq r$; requiring that $r \leq 1$, leads to

$$\begin{aligned} \sqrt{d}2^{-\log(n)/d}2^{-1} &\leq 1 \\ 2^{-\log(n)/d} &\leq 2/\sqrt{d} \\ 2^{\log(n)/d} &\geq \sqrt{d}/2 \\ \Rightarrow n \geq (\sqrt{d}/2)^d & \end{aligned}$$

or, in other words, n needs to be exponential in d!

[12] since $(2^{-\log(n)/d})^d$ (i.e., box vol.) reduces to $2^{-\log(n)}$ and by letting log be approximated by $\log_2$, we get $\approx 1/n$

More on *k*-D trees

Curse of dimensionality

Summary

We only gain if the number of points n is exponential in the number of dimensions d.

What can we do about that problem?

- Locality-Sensitive-Hashing (LSH)[13]
- Randomized *k*-D trees[14]
- Hierarchical *k*-means trees (in FLANN)[15]

[13] A. Adoni and P. Indyk. "Near-Optimal Hashing Algorithms for Near Neighbor Problem in High Dimensions". In: *FOCS*. 2006.

[14] C. Silpa-Anan and R. Hartley. "Optimised KD-trees for fast image descriptor matching". In: *CVPR*. 2008.

[15] M. Muja and D.G. Lowe. "Fast Approximate Nearest Neighbors with Automatic Algorithm Configuration". In: *VISAPP*. 2009.