Cartan Connection for Schrödinger equation. The nature of vacuum.

@QM foundations & nature of time seminar
arXiv:2004.04622 [math-ph]

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Outline

Part I - Heuristics

Hydrodynamics analogy

Figure: Couder et al. jumping droplets [1]. More - last time Jarek's talk. Fig. from: https://www.spektrum.de

Hydrodynamics analogy

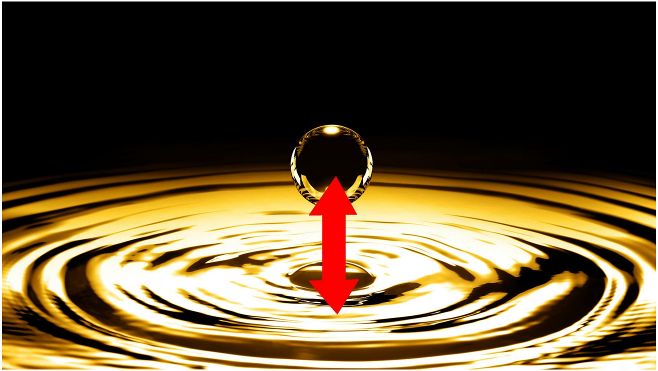

Figure: Can we find such two interacting 'quantities' in the Schrödinger equation?

What we need? What we wish for?

- We need two 'parts':
  - 'small' - particle/wave that **evolves**;
  - 'big' - background/'vacuum' that is 'potential' and reacts **instantaneously** for boundary conditions changes;
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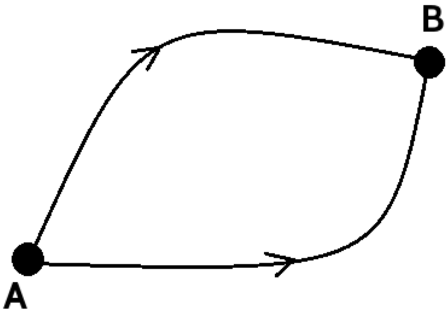
Idea

Weyl's orignal idea - the length of object change when it moves through space-time.



Obviously this is not true for length and gravity, however, for phase the idea gave rise to gauge filed theories.
Check what happens if we scale wave function instead of length.

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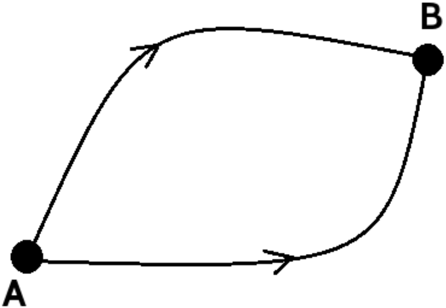

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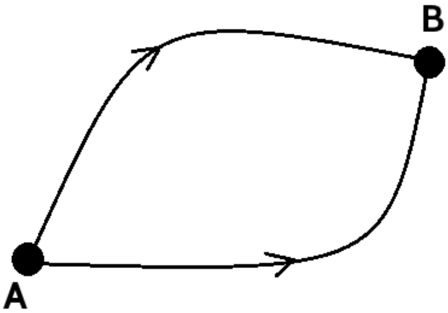

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Substitute $u(x,t) \to n(x)u(x,t)$ in 1D Schrödinger equation

$$i\partial_t u + \partial_x^2 u = Vu, \tag{1}$$

which gives

$$n(i\partial_t u + \partial_x^2 u) = -2\frac{dn}{dx}\partial_x u + u\left(Vn - \frac{d^2n}{dx^2}\right). \tag{2}$$

We can split it in many ways. One choice is:

$$\begin{cases} i\partial_t u + \partial_x^2 u + \frac{2}{n}\frac{dn}{dx}\partial_x u = 0 \\ \frac{1}{n}\frac{d^2n(x)}{dx^2} = V(x). \end{cases} \tag{3}$$

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- The first equation - evolutionary equation interacts with scale by gradient term.

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Idea

There is also an interesting coincidence. Start with

$$\frac{1}{n}\frac{d^2n(x)}{dx^2} = V(x). \tag{5}$$

Substitute $\omega = \frac{d}{dx}\ln(n)$ (the derivative of phase of n), then we get

$$\frac{d}{dx}\omega + \omega\omega = V. \tag{6}$$

This reminds the definition of curvature equation for connection ω:

$$d\omega + \omega \wedge \omega = \Omega, \tag{7}$$

however now multiplication is antisymmetric $\wedge$, and ω has values in some Lie algebra of (Lie) gauge group. Intuition suggests the right direction, but the details are more involved.

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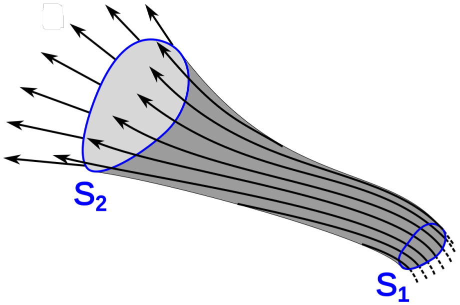
What does the Schrödinger equation describe?
The continuity equation:

$$\nabla_a T^{ab} = 0, \quad b = 1 \dots n. \tag{8}$$



What does the Schrödinger equation describe?
For a vector: $u^a := T^{ab}$ and defining (n-1)-form:

$$\lambda = u_b \star dx^b, \tag{9}$$

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$$d^\nabla \lambda = 0. \tag{10}$$

For the Schrödinger equation, the form is

$$\lambda = in(x)u(x,t)dx - \left(n(x)\partial_x u(x,t) - \frac{dn(x)}{dx}\right)dt, \tag{11}$$

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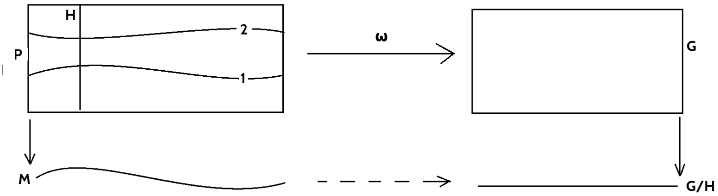

- M - manifold on which we are 'living';
- P - principal bundle = manifold M and all possible reference frames (element of gauge group H); Change of reference frame results in vertical move along fibre H, e.g., from the section 1 to the section 2.
- G - full group that corresponds to P (sometimes only $\mathfrak{g}$ is provided);
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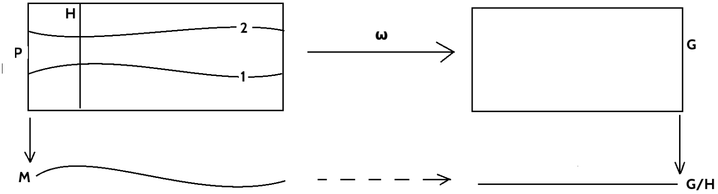

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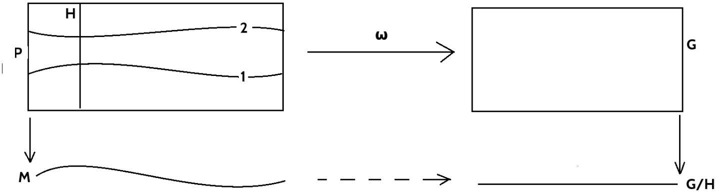

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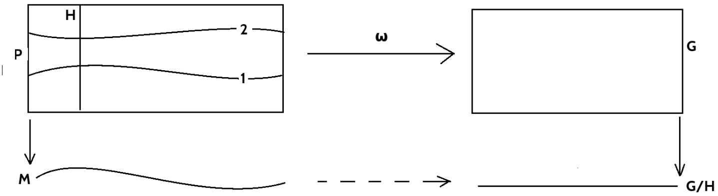

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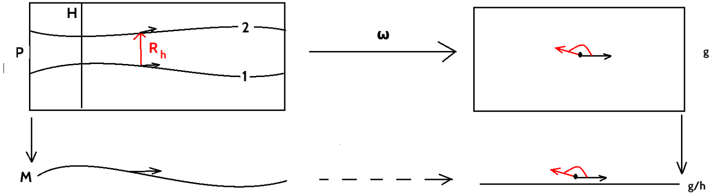
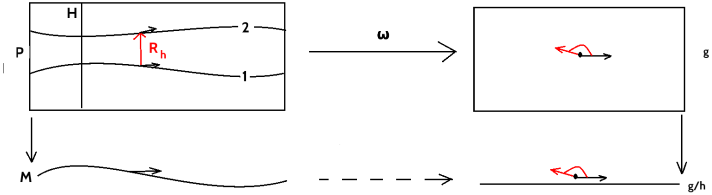


- $\omega_p : T_pP \xrightarrow{\cong} \mathfrak{g}$ - Cartan connection;
  - $R_h^*\omega = Ad(h^{-1})\omega$, $h \in H$ - if you goes vertically ($R_hp = hp$) in P, then you change the frame of reference (adjoint action).
  - $\omega(R_{h*}X) = X$ for $X \in \mathfrak{h}$ - shift vertical vector to the origin.



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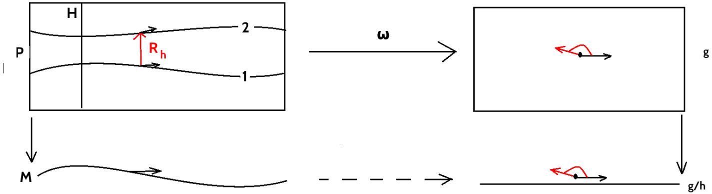

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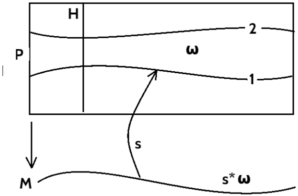

- $\theta := s^*\omega$ is a (local) Cartan connection (Cartan gauge) on M;
- The transformation law:

$$\theta' = Ad(h^{-1})\theta + h^*\omega_{MC}, \tag{13}$$

  where the Maurer-Cartan form is $h^*\omega_{MC} = h^{-1}dh$ for a matrix group.
- Physicists usually call connection its pullback along any section s (choice of reference frame).
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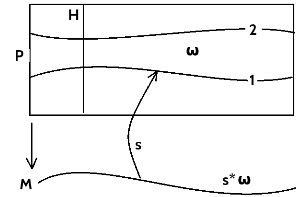

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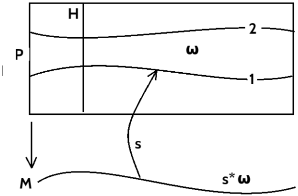

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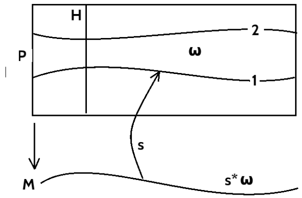

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- We want to rewrite the Schrödinger equation as a continuity equation:
$$\nabla_i u^i = \partial_{x^i} u^i + \omega^i_j(\partial_{x^i}) u^j = 0, \tag{14}$$
where $[\omega^i_j]$ is connection with values of some Lie algebra.
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The plan

- Step 1. We must 'flatten' Schrödinger equation to the 1st order equation. Introduce the jet space.
- Step 2. Select a group G.
- Step 3. Calculate all possible coefficients of connection.
- Step 4. Determination of the gauge group H - subgroup of G.
- Step 5. Torsion-free condition.

We present it for dim$= 1$, and for general case see [2].

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Step 1. Introduce the jet space.

Introduce first jet space, where v_1 is a new variable, that projects to the condition $\partial_x u = v_1$. However in jest space v_1 variable is not necessary associated with derivative of $u = v$.

The equation flattens:

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Lift scaling $u \to gu$ to the jest space:

$$g = \begin{bmatrix} a & 0 \\ b & a \end{bmatrix}. \tag{16}$$

The transformation is

$$\begin{bmatrix} a & 0 \\ b & a \end{bmatrix} \begin{bmatrix} v \\ v_1 \end{bmatrix} = \begin{bmatrix} av \\ bv + av_1 \end{bmatrix}, \tag{17}$$

which for $b = \partial_x a$ is a standard transformation of derivative. But we keep it general - so called Lie-Bäcklund group.

Under this element the equation transforms to

$$i\partial_t v + \partial_x v + v\frac{1}{a}(i\partial_t a + \partial_x b - Va) + v_1\frac{1}{a}(b + \partial_x a) = 0. \qquad (18)$$

Step 3. Calculate all possible coefficients of connection.

Compare:

$$i\partial_t v + \partial_x v + v\frac{1}{a}(i\partial_t a + \partial_x b - Va) + v_1\frac{1}{a}(b + \partial_x a) = 0, \quad (19)$$

with

$$\nabla_i u^i = \partial_{x^i} u^i + \omega^i_j(\partial_{x^i}) u^j = 0. \quad (20)$$

Since connection form has values in scaling Lie algebra

$$\omega = \begin{bmatrix} \alpha & 0 \\ \beta & \alpha \end{bmatrix}, \quad (21)$$

where $\alpha = \alpha_t dt + \alpha_x dx$ and $\beta = \beta_t dt + \beta_x dx$, we have

$$i\partial_t v + \partial_x v + v(\alpha_t + \beta_x) + v_1 \alpha_x = 0. \quad (22)$$

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vs

$$i\partial_t v + \partial_x v + v(\alpha_t + \beta_x) + v_1\alpha_x = 0. \qquad (24)$$

We have the constraints on connection coefficients

$$\begin{cases} \alpha_t + \beta_x = \frac{1}{a}(i\partial_t a + \partial_x b - Va) \\ \alpha_x = \frac{1}{a}(b + \partial_x a) \end{cases} \qquad (25)$$

Step 3. Calculate all possible coefficients of connection.

Besides, we embed the connection in the Weyl structure (additional scaling of the whole equation).
The full Cartan connection matrix is:

$$\omega_W = \begin{bmatrix} \epsilon & 0 & 0 \\ dv & \frac{1}{a}(b+\partial_x a)dx + \frac{1}{a} i\partial_t a dt & 0 \\ dv_1 & \frac{1}{a}(\partial_x b - Va)dx & \frac{1}{a}(b+\partial_x a)dx + \frac{1}{a} i\partial_t a dt \end{bmatrix}, \tag{26}$$

for some 1-form ϵ to be determined later.

Step 4. Determination of the gauge group H.

For $h \in H$ we have

$$h = \begin{bmatrix} e & 0 \\ f & e \end{bmatrix}. \tag{27}$$

The Maurer-Cartan form for this element is

$$h^* \omega_{MC} = h^{-1} dh = \begin{bmatrix} d\ln(e) & 0 \\ d\frac{f}{e} & d\ln(e) \end{bmatrix} \tag{28}$$

Transformation of equation under h must give Maurer-Cartan form, since we want to restore connection transformation

$$\omega' = \underbrace{Ad(h^{-1})\omega}_{\text{linear}} + \underbrace{h^* \omega_{MC}}_{\text{differential}}. \tag{29}$$

Linear part transforms correctly. 'Differential' part results from derivatives in equations and must coincide with the Maurer-Cartan form.

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Therefore we compare:

$$i\partial_t(hv)+\partial_x(hv_1) = i\partial_t v+\partial_x v_1+v\frac{1}{e}(i\partial_t e + \partial_x f) + v_1\frac{1}{e}(f + \partial_x e) = 0, \tag{30}$$

with

$$i\partial_t v + \partial_x v_1 + v(\frac{1}{e}\partial_t e + \partial_x(f/e)) + v_1\frac{1}{e}\partial_x e = 0. \tag{31}$$

This gives constraints for H - gauge group manifold:

$$\begin{cases} f = (\lambda - 1)\partial_x e \\ (i-\lambda)\partial_t \ln(e) - (\lambda - 1)(\partial_x \ln(e))^2 = \partial_x^2 \ln(e) \end{cases} \tag{32}$$

for $\lambda \in \mathbb{R}$.

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Step 5. Torsion-free condition.

Curvature is defined as

$$\Omega = d\omega_W + \omega_W \wedge \omega_W = \begin{bmatrix} d\epsilon + \epsilon \wedge \epsilon & 0 \\ d\theta + \theta \wedge \epsilon + \omega \wedge \theta & d\omega + \omega \wedge \omega \end{bmatrix}. \tag{33}$$

Torsion usually vanishes - it is sensible assumption for physics (and Euclidean space).

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that is:

$$0 = \frac{1}{a}(\partial_x b - Va). \tag{35}$$

For $b = \partial_x a$ we get original condition

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Vanishing torsion also determines the Weyl structure:

$$\epsilon = -\alpha = -(\partial_x \ln(a)dx + \frac{1}{a} i\partial_0 a dt). \tag{37}$$

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Summary

- Properly factorized, the Schrödinger equation has geometric meaning as a continuity equation.
- The scaling/background has geometric meaning as a Cartan connection for scaling group.
- Knowing (non-unique) Cartan connection we can do differential geometry in jet spaces.
- In jet space, gradient coordinates are not connected with derivatives of wave functions, so the probability fluxes are not coupled with wave function. Virtual particles flows?
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Questions, comments, discussion.

Thank You for Your Attention