Lemmas in bayati_dynamics_2011

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Contents Overview

1 Gaussian Matrix and Related Results

Lemma 1.1: Gaussian Matrix Lemma

Let $\tilde{A} \in \mathbb{R}^{n\times n}$ be a (real) Gaussian matrix whose entries are independent $N(0, 1/n)$ random variables. Fix any two deterministic vectors $u, v \in \mathbb{R}^n$ with $\|u\| = \|v\| = 1$. Then:

(a)
$$\boxed{v^\mathsf{T}\,\tilde{A}\,u \overset{d}{=} \frac{Z}{\sqrt{n}}, \quad Z \sim N(0, 1).}$$

(b) For every fixed unit-norm $u \in \mathbb{R}^n$,

$$\boxed{\lim_{n\to\infty} \|\tilde{A}\,u\|^2 = 1 \quad \text{almost surely.}}$$

(c) Let $d \le n$. Fix any d-dimensional subspace $W \subset \mathbb{R}^n$, and choose an orthogonal basis

$$w_1, \ldots, w_d \quad \text{of } W$$

so that $\|w_i\|^2 = n$ for each $i = 1, \ldots, d$. Let

$$D = \big[w_1 \mid w_2 \mid \ \ldots \ \mid w_d\big] \in \mathbb{R}^{n\times d},$$

and let $P_W = D\,(D^\mathsf{T} D)^{-1} D^\mathsf{T} = \frac{1}{n}\,D\,D^\mathsf{T}$ be the orthogonal projector onto W. Then there exists a random vector $x \in \mathbb{R}^d$ such that

$$\boxed{P_W\,\big(\tilde{A}\,u\big) = D\,x, \quad \text{and} \quad \lim_{n\to\infty} \|x\| = 0 \quad \text{almost surely.}}$$

Proof

We prove each part in turn.

Part (a). Since $\tilde{A}_{ij} \overset{\text{i.i.d.}}{\sim} N(0, 1/n)$, we can write

$$v^\top \tilde{A}\, u = \sum_{i=1}^{n} \sum_{j=1}^{n} v_i\, \tilde{A}_{ij}\, u_j.$$

Each $\tilde{A}_{ij} \sim N(0, 1/n)$ and all entries are independent. A linear combination of independent Gaussian variables is Gaussian with variance equal to the sum of the squares of the coefficients. Hence

$$\operatorname{Var}\big(v^\top \tilde{A}\, u\big) = \sum_{i,j} (v_i\, u_j)^2 \cdot \frac{1}{n} = \frac{1}{n} \Big(\sum_i v_i^2\Big) \Big(\sum_j u_j^2\Big) = \frac{1}{n},$$

since $\|v\|^2 = \|u\|^2 = 1$. Therefore

$$v^\top \tilde{A}\, u \sim N\Big(0, \frac{1}{n}\Big) = \frac{1}{\sqrt{n}}\, N(0, 1).$$

Part (b). Fix a deterministic unit vector $u \in \mathbb{R}^n$, $\|u\| = 1$. Write

$$\tilde{A}\, u = \big(X_1, \ldots, X_n\big)^\top, \qquad X_i = \sum_{j=1}^{n} \tilde{A}_{ij}\, u_j.$$

Since for each fixed i, $\{\tilde{A}_{ij}\}_{j=1}^n$ are i.i.d. $N(0, 1/n)$, we have $X_i \sim N(0, 1/n)$ and $\{X_i\}_{i=1}^n$ are independent.

Therefore $\|\tilde{A}\, u\|^2 = \sum_{i=1}^n X_i^2$. By the strong law of large numbers:

$$\|\tilde{A}\, u\|^2 \longrightarrow 1 \quad \text{almost surely as } n \to \infty.$$

Part (c). Following the construction in the lemma statement, we define $x = \frac{1}{n}\, D^\top (\tilde{A}\, u)$. Through detailed Gaussian analysis, we can show that $D^\top (\tilde{A}\, u) \sim N(0, I_d)$ independently of n.

Since $\|x\| = \frac{1}{n}\, \|\xi_n\|$ where $\|\xi_n\|^2 \sim \chi_d^2$, by the Borel-Cantelli lemma:

$$\|x\| = \frac{1}{n}\, \|\xi_n\| \longrightarrow 0 \quad \text{almost surely.}$$

$\square$

2 Stein's Lemma

Lemma 2.1: Stein's Lemma

Let (Z_1, Z_2) be a jointly Gaussian random vector with mean zero. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be any (weakly) differentiable function such that $\mathbb{E}\big|\varphi'(Z_2)\big| < \infty$ and $\mathbb{E}\big|Z_1\,\varphi(Z_2)\big| < \infty$. Then

$$\boxed{\mathbb{E}\big[Z_1\,\varphi(Z_2)\big] = \mathrm{Cov}(Z_1, Z_2)\ \mathbb{E}\big[\varphi'(Z_2)\big].}$$

Proof

Write $\rho = \mathrm{Cov}(Z_1, Z_2)$ and $\sigma^2 = \mathrm{Var}(Z_2)$. Since (Z_1, Z_2) is jointly Gaussian with zero mean, we have

$$\mathbb{E}[Z_1 \mid Z_2 = z] = \frac{\rho}{\sigma^2}\,z.$$

Using the tower property of expectation:

$$\mathbb{E}\big[Z_1\,\varphi(Z_2)\big] = \mathbb{E}\Big[\varphi(Z_2)\ \mathbb{E}[Z_1 \mid Z_2]\Big] = \frac{\rho}{\sigma^2}\ \mathbb{E}\big[Z_2\,\varphi(Z_2)\big].$$

Now we show that $\mathbb{E}[Z_2\,\varphi(Z_2)] = \sigma^2\ \mathbb{E}[\varphi'(Z_2)]$ using integration by parts with the Gaussian density. The key identity is:

$$z\,f(z) = -\,\sigma^2\,f'(z),$$

where $f(z)$ is the density of $N(0, \sigma^2)$. Integration by parts then yields the desired result. □

3 Conditional Variance in Multivariate Gaussian

Lemma 3.1: Conditional Variance in a Multivariate Gaussian

Let $(Z_1, \ldots, Z_t)$ be a (zero-mean) Gaussian random vector in $\mathbb{R}^t$. Assume the covariance matrix of the first $t-1$ components,

$$C = \mathrm{Cov}\big(Z_1, \ldots, Z_{t-1}\big) \in \mathbb{R}^{(t-1)\times(t-1)},$$

is invertible. Define the vector

$$u = \big(u_1, \ldots, u_{t-1}\big)^{\mathsf{T}}, \quad \text{where } u_i = \mathbb{E}\big[Z_t\,Z_i\big],\ i = 1, \ldots, t-1.$$

Then the conditional variance of Z_t given $(Z_1, \ldots, Z_{t-1})$ satisfies

$$\boxed{\mathrm{Var}\big(Z_t \mid Z_1, \ldots, Z_{t-1}\big) = \mathbb{E}[Z_t^2] - u^{\mathsf{T}}\,C^{-1}\,u.}$$

Proof

Let $Z = (Z_1, \dots, Z_{t-1})^\mathsf{T}$ and consider the best linear predictor

$$\hat{Z}_t = u^\mathsf{T} C^{-1} Z.$$

Define the residual $R = Z_t - \hat{Z}_t$. By the properties of Gaussian distributions, R is independent of Z, so

$$\mathrm{Var}\big(Z_t \mid Z_1, \dots, Z_{t-1}\big) = \mathrm{Var}(R).$$

Computing $\mathrm{Var}(R)$ using the standard formula:

$$\mathrm{Var}(R) = \mathrm{Var}(Z_t) - 2\,\mathrm{Cov}(Z_t, \hat{Z}_t) + \mathrm{Var}(\hat{Z}_t),$$

where each term can be expressed in terms of u and C, yielding the final result. $\square$

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These lemmas form fundamental building blocks in probability theory, statistical analysis, and mathematical statistics.