hw2_zhengyi

Notations defined by prof

n assets, first T days of the data; weights x_i: **no constraints**

on each day, a position is taken at the open and closed at noon

$p^0_{j,t}$: price of asset j on day t at the open;

$p^1_{j,t}$: price of asset j on day t at noon.

$r_{j,t} = \frac{p^1_{j,t} - p^0_{j,t}}{p^0_{j,t}}$ = return earned by asset j on day t;

$\bar{r_j} = \frac{1}{T}\sum_{i=1}^{T} r_{j,t}$ = average return earned by asset j over T days.

Variables we used for coding

$$x = \begin{bmatrix} x_1 \\ ... \\ x_j \\ ... \\ x_n \end{bmatrix} \in R^n$$

$$ret = \begin{bmatrix} r_{1,1} & ... & r_{1,t} & ... & r_{1,T} \\ & & ... & & \\ r_{j,1} & ... & r_{j,t} & ... & r_{j,T} \\ & & ... & & \\ r_{n,1} & ... & r_{n,t} & ... & r_{n,T} \end{bmatrix} \in R^{n \times T}$$

$$ret\_bar = \begin{bmatrix} \bar{r_1} \\ ... \\ \bar{r_j} \\ ... \\ \bar{r_n} \end{bmatrix} \in R^n$$

$$excess\_ret = \begin{bmatrix} r_{1,1} - \bar{r_1} & ... & r_{1,t} - \bar{r_1} & ... & r_{1,T} - \bar{r_1} \\ & & ... & & \\ r_{j,1} - \bar{r_j} & ... & r_{j,t} - \bar{r_j} & ... & r_{j,T} - \bar{r_j} \\ & & ... & & \\ r_{n,1} - \bar{r_n} & ... & r_{n,t} - \bar{r_n} & ... & r_{n,T} - \bar{r_n} \end{bmatrix} \in R^{n \times T}$$

$$daily\_excess\_ret = excess\_ret^T x =$$

$$\begin{bmatrix} r_{1,1} - \bar{r_1} & ... & r_{j,1} - \bar{r_j} & ... & r_{n,1} - \bar{r_n} \\ & & ... & & \\ r_{1,t} - \bar{r_1} & ... & r_{j,t} - \bar{r_j} & ... & r_{n,t} - \bar{r_n} \\ & & ... & & \\ r_{1,T} - \bar{r_1} & ... & r_{j,T} - \bar{r_j} & ... & r_{n,T} - \bar{r_n} \end{bmatrix}_{T \times n} \begin{bmatrix} x_1 \\ ... \\ x_j \\ ... \\ x_n \end{bmatrix}_{n \times 1} = \begin{bmatrix} \sum_{j=1}^{n}(r_{j,1} - \bar{r_1})x_j \\ ... \\ \sum_{j=1}^{n}(r_{j,t} - \bar{r_t})x_j \\ ... \\ \sum_{j=1}^{n}(r_{j,T} - \bar{r_T})x_j \end{bmatrix} \in R^T$$

Problem formulation

portfolio optimization problem formulation:

$$\min_{(x_1,\ldots,x_n)} \left(-\sum_{j=1}^{n} \bar{r}_j x_j\right) \; + \; \theta \left(\frac{1}{T}\sum_{t=1}^{T}\left(\sum_{j=1}^{n}(r_{j,t}-\bar{r}_j)x_j\right)^{\pi}\right)^{1/\pi}$$

parameters: $\theta \geq 0$ risk aversion parameter; $\pi > 0$

Function

$f : R^n \to R$

$$f(x_1, ..., x_n) = \left(-\sum_{j=1}^{n} \bar{r}_j x_j\right) \; + \; \theta \left(\frac{1}{T}\sum_{t=1}^{T}\left(\sum_{j=1}^{n}(r_{j,t}-\bar{r}_j)x_j\right)^{\pi}\right)^{1/\pi}$$

$$f = ret^T x + \theta\left(\frac{1}{T}\sum_{t=1}^{T} daily\_excess\_ret^{\pi}\right)^{\frac{1}{\pi}}$$

Gradient

$\nabla f : R^n \to R^n$

the j-th gradient: $\frac{\partial f}{\partial x_j} = -\bar{r_j} +$

$$\frac{\theta}{\pi}\left(\frac{1}{T}\sum_{t=1}^{T}\left(\sum_{j=1}^{n}(r_{j,t}-\bar{r_j})x_j\right)^{\pi}\right)^{\frac{1-\pi}{\pi}}\left(\frac{1}{T}\sum_{t=1}^{T}\pi(r_{j,t}-\bar{r_j})\left(\sum_{j=1}^{n}(r_{j,t}-\bar{r_j})x_j\right)^{\pi-1}\right)$$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \ldots \\ \frac{\partial f}{\partial x_j} \\ \ldots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} =$$

$$\begin{bmatrix} -\bar{r_1} + \frac{\theta}{\pi}\left(\frac{1}{T}\sum_{t=1}^{T}\left(\sum_{j=1}^{n}(r_{j,t}-\bar{r_j})x_j\right)^{\pi}\right)^{\frac{1-\pi}{\pi}}\left(\frac{1}{T}\sum_{t=1}^{T}\pi(r_{1,t}-\bar{r_1})\left(\sum_{j=1}^{n}(r_{j,t}-\bar{r_j})x_j\right)^{\pi-1}\right) \\ \ldots \\ -\bar{r_j} + \frac{\theta}{\pi}\left(\frac{1}{T}\sum_{t=1}^{T}\left(\sum_{j=1}^{n}(r_{j,t}-\bar{r_j})x_j\right)^{\pi}\right)^{\frac{1-\pi}{\pi}}\left(\frac{1}{T}\sum_{t=1}^{T}\pi(r_{j,t}-\bar{r_j})\left(\sum_{j=1}^{n}(r_{j,t}-\bar{r_j})x_j\right)^{\pi-1}\right) \\ \ldots \\ -\bar{r_n} + \frac{\theta}{\pi}\left(\frac{1}{T}\sum_{t=1}^{T}\left(\sum_{j=1}^{n}(r_{j,t}-\bar{r_j})x_j\right)^{\pi}\right)^{\frac{1-\pi}{\pi}}\left(\frac{1}{T}\sum_{t=1}^{T}\pi(r_{n,t}-\bar{r_n})\left(\sum_{j=1}^{n}(r_{j,t}-\bar{r_j})x_j\right)^{\pi-1}\right) \end{bmatrix}$$

$$= \begin{bmatrix} -\bar{r_1} \\ \dots \\ -\bar{r_j} \\ \dots \\ -\bar{r_n} \end{bmatrix}_{n\times 1} + \frac{\theta}{\pi}\left(\frac{1}{T}\sum_{t=1}^{T}\left(\sum_{j=1}^{n}(r_{j,t}-\bar{r_j})x_j\right)^{\pi}\right)^{\frac{1-\pi}{\pi}}\frac{\pi}{T}$$

$$\begin{bmatrix} r_{1,1}-\bar{r_1} & \dots & r_{1,t}-\bar{r_1} & \dots & r_{1,T}-\bar{r_1} \\ & & \dots & & \\ r_{j,1}-\bar{r_j} & \dots & r_{j,t}-\bar{r_j} & \dots & r_{j,T}-\bar{r_j} \\ & & \dots & & \\ r_{n,1}-\bar{r_n} & \dots & r_{n,t}-\bar{r_n} & \dots & r_{n,T}-\bar{r_n} \end{bmatrix}_{n\times T} \begin{bmatrix} \left(\sum_{j=1}^{n}(r_{j,1}-\bar{r_1})x_j\right)^{\pi-1} \\ \dots \\ \left(\sum_{j=1}^{n}(r_{j,t}-\bar{r_t})x_j\right)^{\pi-1} \\ \dots \\ \left(\sum_{j=1}^{n}(r_{j,T}-\bar{r_T})x_j\right)^{\pi-1} \end{bmatrix}_{T\times 1}$$

$$= -ret\_bar + \frac{\theta}{T}\left(\frac{1}{T}\sum_{t=1}^{T}(daily\_excess\_ret)^{\pi}\right)^{\frac{1-\pi}{\pi}} excess\_ret \cdot (daily\_excess\_ret)^{\pi-1} \in R^n$$

Code

```
def eval_func(theta, pi, ret, x):
    # ret: (n,T)
    # x: (n,)
    ret_bar = np.mean(ret, axis=1) # ret_bar: (n,)
    excess_ret = ret-np.repeat(ret_bar[:,None], T, axis=1) # excess_ret: (n,T)
    daily_excess_ret = excess_ret.T@x # daily_excess_ret: (T,)
    f1 = -ret_bar.T@x # f1: (n,)
    f2 = theta*np.mean(daily_excess_ret**pi)**(1/pi) # f2: (n,)
    return f1+f2

def eval_grad(theta, pi, ret, x):
    # ret: (n,T)
    # x: (n,)
    ret_bar = np.mean(ret, axis=1) # ret_bar: (n,)
    excess_ret = ret-np.repeat(ret_bar[:,None], T, axis=1) # excess_ret: (n,T)
    daily_excess_ret = excess_ret.T@x # daily_excess_ret: (T,)

    g1 = -ret_bar # g1: (n,)
    c = (theta/T)*np.mean(daily_excess_ret**pi)**(1/pi) # a constant multiplier
    g2 = excess_ret@(daily_excess_ret**(pi-1)) # g2: (n,)
    return g1+c*g2
```