Project 2 gradient computation

Date

1 Markowitz Gradient

1.1 Element-wise approach

The objective function is given by

$$\text{minimize} \;\; \left(-\sum_{j=1}^{n} \bar{r}_j x_j \right) \;+\; \theta \left(\frac{1}{T} \sum_{t=1}^{T} \left[\sum_{j=1}^{n} (r_{j,t} - \bar{r}_j) x_j \right]^{\pi} \right)^{1/\pi}$$

Differentiate the objective function with respect to x_j yields

$$\frac{\partial \text{obj}}{\partial x_j} = -\bar{r}_j + \frac{\theta}{\pi} \left(\frac{1}{T} \sum_{t=1}^{T} \left[\sum_{j=1}^{n} (r_{j,t} - \bar{r}_j) x_j \right]^{\pi} \right)^{-(1-1/\pi)} \left(\frac{\pi}{T} \sum_{t=1}^{T} \left[\sum_{j=1}^{n} (r_{j,t} - \bar{r}_j) x_j \right]^{(\pi-1)} (r_{j,t} - \bar{r}_j) \right)$$

1.2 Matrix calculus approach

Or, alternatively, following matrix notations, the objective function can be rewritten as

$$\text{minimize} \;\; -\bar{\mathbf{r}}'\mathbf{x} + \frac{\theta}{T^{1/\pi}} ||(\mathbf{r} - \bar{\mathbf{r}})'\mathbf{x}||_{\pi}$$

where $|| \cdot ||^{\pi}$ is the L^{π} norm. Here we must investigate how to take the derivative of a L^p norm with respect to the function argument. According to Wikipedia - Norm,

$$\frac{\partial ||\mathbf{x}||_p}{\partial \mathbf{x}} = \frac{\mathbf{x} \circ |\mathbf{x}|^{p-2}}{||\mathbf{x}||_p^{p-1}}$$

where $\circ$ is element-wise matrix multiplication, and $|\mathbf{x}| = (|x_1|, \ldots, |x_n|)$ is the element-wise absolute value. In other words,

$$\begin{aligned} \mathbf{x} \circ |\mathbf{x}|^{p-2} &= [x_1, \ldots, x_n] \circ [|x_1|^{p-2}, \ldots, |x_n|^{p-2}] \\ &= [(x_1 \cdot |x_1|^{p-2}), \ldots, (x_n \cdot |x_n|^{p-2})] \end{aligned}$$

Therefore, if we were to take derivative of the objective function with respect to $\mathbf{x}$ can be obtained via chain rule:

$$\frac{\partial \text{obj}}{\partial \mathbf{x}} = -\bar{\mathbf{r}} + \frac{\theta}{T^{1/\pi}} \cdot \frac{[(\mathbf{r} - \bar{\mathbf{r}})'\mathbf{x}] \circ |(\mathbf{r} - \bar{\mathbf{r}})'\mathbf{x}|^{\pi-2}}{||(\mathbf{r} - \bar{\mathbf{r}})'\mathbf{x}||_{\pi}^{\pi-1}} \cdot (\mathbf{r} - \bar{\mathbf{r}})$$