IEOR4375 Final Project

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1 Introduction

In this study, the main object is to price the following $T-$claim:

$$\boldsymbol{\chi}(N, k, k', \Delta) = N \cdot \max\left[0, \left(k - \frac{S(T)}{S(0)}\right) \cdot \left(\frac{L(T,T,T+\Delta)}{L(0,T,T+\Delta)} - k'\right)\right] \tag{1}$$

where $S(t)$ is the EUROSTOXX50 spot price quantoed from EUR into USD and $L(t,T,T+\Delta)$ is the 3 month SOFR forward rate. The choices of models for these two main parts of the claim affect the pricing result the most. Thus the assumptions and methodologies for modeling these two parts will be discussed separately in detail in the following sections. Throughout this write-up, we will refer to them as *stock price* (foreign and quantoed) and *(forward) term rate ratio.*

2 Assumptions

2.1 General

Assumption 1 (Filtration)**.** We will assume the quote time, t, satisfies $t \in [0,T]$. That is, when pricing said $T-$claim at t, we make all estimations and calibrations with filtration $\{\mathcal{F}_t\}$.

2.2 Stock Price

Assumption 2 (Quantoed Stock)**.** For stock price, we retain the assumptions put forth in [Bjork] chapter 17, namely Proposition 17.4 and 17.5, which assumes GBM form for stock and exchange rate (and thus their quotient) dynamics under $\mathbb{P}$ and $\mathbb{Q}$ measure:

$$\begin{aligned} d\tilde{S}_f &\overset{\mathbb{P}}{=} \tilde{S}_f(\alpha_f + \alpha_X + \sigma_f\sigma_X)dt + \tilde{S}_f(\sigma_f + \sigma_X)d\bar{W} \\ d\tilde{S}_f &\overset{\mathbb{Q}}{=} \tilde{S}_f r_d dt + \tilde{S}_f(\sigma_f + \sigma_X)dW \end{aligned}$$

where

- S_f denotes foreign stock, and $\tilde{S}_f$ denotes quantoed foreign stock;
- α_f, α_X denotes real-world drift of foreign stock and exchange rate (USD/EUR) respectively;
- σ_f, σ_X denotes volatility of foreign stock and exchange rate respectively;
- r_d denotes domestic short rate;
- $d\bar{W}$ and dW are BMs associated with $\mathbb{P}$ and $\mathbb{Q}$ measures respectively.

Additionally, as suggested by [Bjork] Remark 17.2.4, we treat the invariant diffusion term as

$$\sigma_f + \sigma_X = \sqrt{\delta_f^2 + \delta_X^2 + 2\delta_f \delta_X \rho_{fX}} \tag{2}$$

for ease of calibration. δ_f, δ_X, ρ_{fX} are idiosyncratic diffusion coefficients for foreign stock and exchange rate (domestic/foreign), and their correlation coefficient, which can be estimated via simple regressions.

Under the Assumption 2 stated above specifying GBM form, we can explicitly write out the integral form

$$\frac{S(T)}{S(0)} = \exp\left[\int_0^T r_d(s)ds - \frac{1}{2}\int_0^T \sigma_S^2(s)ds + \int_0^T \sigma_S(s)dW_s\right] \tag{3}$$

$$\text{where} \quad \sigma_S(t,S) = \sigma_f(t,S) + \sigma_X(t,S)$$

which approximates its discretized form in actual implementation:

$$S(T) = S(0)\exp\left[\sum_{k=0}^{T/\Delta}(r_d(k\Delta)\cdot\Delta) - \frac{1}{2}\left(\sum_{k=0}^{T/\Delta}\sigma_S^2(k\Delta)\cdot\Delta\right) + \sum_{k=0}^{T/\Delta}(\sigma_S(k\Delta)\cdot\Delta)\right] \tag{4}$$

2.3 Forward Term Rate

Due to a lack to access to commercially published SOFR term rates (*e.g.* CME SOFR 3M Term), the author of this study chose to use Monte Carlo simulations to approximate the true market level of the forward term rate according to its definition based on the absence of arbitrage:

$$L(t,T,T+\Delta) = \frac{p(t,T) - p(t,T+\Delta)}{\Delta p(t,T+\Delta)} \qquad \text{([Bjork] \textbf{Definition 22.2})}$$

where all bond price terms, $p(t,T)$ and $p(t,T+\Delta)$, are determined by the spot short rate. Under this assumption, we in fact do not observe $L(0,T,T+\Delta)$ at $t=0$. Instead, we use the filtration $\{\mathcal{F}_0\}$, which contains information before $t=0$, to calibrate the short rate model of choice, and use the estimated parameters to obtain an estimation for $L(0,T,T+\Delta)$, which henceforth stays constant throughout the pricing routine. On the other hand, $L(T,T,T+\Delta)$ ought to be simulated using parameters calibrated on $\{\mathcal{F}_t\}$ where t is the quotation time of the $T-$claim. Under the assumption that $t=0$ as in this study, one must form an expectation of the parameter for $L(T,T,T+\Delta)$.

3 Data

Data in the project includes input data and real market data. Input data are: T maturity in years assumed to be 3, $\triangle$ is a period of 3-month and T + $\triangle$ is the settlement date. k and k' are the strike prices, which are tested using multiple values for contract pricing. The market data used include:

- Interest rate: Given the SOFR term rate is based on expectations in the future and not obtained readily from another source. We calculate the forward term rate and the term rate ratio $\frac{L(T,T,T+\triangle)}{L(0,T,T+\triangle)}$ in terms of bond price and reduced to an expression with SOFR spot rate. SOFR spot rate is downloaded from Federal Reserve Bank of New York[1]. To process this data, we use a moving average approach to replace the original data to avoid 0 daily changes. This is due to numerical instability induced by the lack of variance between day-to-day SOFR spot rates. The moving average will add daily variance to the rate for future data simulations. A fixed-length rolling window is set to 90 days as default but will be later tested using different values. To improve the accuracy or follow an easier approach, one might consider using SOFR term forward rate from data sources including CME and Bloomberg.

- Federal Fund rate: To simulate SOFR spot rate, we will also utilize FFR, the federal funds rate, since the SOFR spot rate does not show a mean-reverting nature and will result in the negative parameters if the models including CIR and Vaicek applied directly. We obtain FFR from St. Louis FED [2]. With the data, we calculate the difference between SOFR and FFR, which is then processed with the moving average approach described above. We then use the historical difference between these two rates to calibrate the parameter set for our simulation model. Once the path of the difference is simulated, we transfer back the result by adding back the federal fund rate. Since FFR is unknown for any t > 0, we assume the rate to be a constant with a default value equal to the FFR at time 0, which is the current rate.

- EUROSTOXX50: To obtain the S(t) quantoed from EUR into USD, we use the stock index of Eurozone stocks through Yahoo finance library with ticker "ˆSTOXX50E" . This will obtain S(t) $\forall t \leq 0$. $\forall t > 0$, dynamic of S(t) and $\frac{S(T)}{S(0)}$ will be simulated using GBM and exchanged rate [3].

- Exchange rate: Similarly, the exchange rate is obtained by the yahoo finance library with ticker "EURUSD=X". Note that this exchange rate is in terms of EUR/USD, while our derivations assumed a exchange rate of USD/EUR. Because of this discrepancy, we took the reciprocal first of the raw downloaded data before fitting.

Then with the exchange rate and $S(t)$, we can obtain the corresponding idiosyncratic diffusion coefficient σ_f, σ_X and ρ_{fX}, which are used to calculate σ_s detailed in section 2.2

[1] https://www.newyorkfed.org/markets/reference-rates/sofr-averages-and-index
[2] https://fred.stlouisfed.org/series/FEDFUNDS
[3] See formula (3)

4 Methodology

4.1 Calibration

4.1.1 Quanto Stock Price

Note that equation (4) shows that the simulation scheme of the quantoed stock requires two main components: short rate r_d sample paths, and calibrate quanto volatility σ_S. We will defer the details on short rate calibration to a later section and focus on the calibration of σ_S now.

Equation (2), the proxy of [Bjork] Remark 17.2.4, shows that the computation for σ_S is equivalent to the estimation of three parameters: δ_f, δ_X, and ρ_{fX}, the idiosyncratic diffusion coefficients and correlation coefficient of foreign stock and exchange rate. Since GBM dynamics are assumed for both foreign stock and exchange rate, the estimation of these three parameters is nothing more than the estimation of the standard deviations and correlation coefficients of log returns. Due to the triviality of these computations, we decided to not provide a verbose block of pseudocode for quanto calibration.

4.1.2 Forward Term Rate

Recall that [Bjork] **Definition 22.2** points out that forward term rate $L(t, T, T + \Delta)$ is a function of bond prices $p(t, T)$ and $p(t, T + \Delta)$, which are functions of short rate $r(t)$. This implies that the simulation of $L(t, T, T + \Delta)$ requires the following components: r_t, $r(s; \mathbf{\Theta}_s)$ for $s \in [t, T + \Delta]$. For SOFR rates, however, the rate level is heavily affected by the interest rate target set by the Federal Reserve and demonstrates abrupt jumps in the time series. This severely affects the calibration fitting of most short-rate models that assume the mean-reversion property. Therefore, we proceed with our calibration routine by fitting our models on the difference between effective FFR and the SOFR rate, as in most cases, one obtains knowledge of the current FFR before the calibration processes. Consequently, to simulate $\dfrac{L(T, T, T + \Delta)}{L(0, T, T + \Delta)}$, we need to calibrate two sets of parameters,

$$\mathbf{\Theta}^0 \triangleq \{\mathbf{\Theta}_s \mid s \in [0, T]\}$$
$$\mathbf{\Theta}^T \triangleq \{\mathbf{\Theta}_s \mid s \in [T, T + \Delta]\}$$

to simulate $r(t) - r_f(t)$ needed for evaluating the denominator and the nominator respectively. This study focuses on two mean-reverting constant parameter models, Vasicek and CIR and their calibration schemes are discussed separately as follows:

- **Vasicek**. Recall that Vasicek model is defined by three parameters a, b, σ, specified as follows:

$$dr = (b - ar)dt + \sigma dW \qquad \text{([Bjork]24.4)}$$

which can be rewritten as

$$dr = bdt - ardt + \sigma\sqrt{dt}z$$
$$\text{where} \quad z \sim \mathcal{N}(0, 1)$$

This implies that parameters can be estimated via OLS regression

$$dr = \beta_0 + \beta_1 r + \epsilon$$

and the best parameter estimates are given by

$$\begin{aligned}
\hat{b} &= \hat{\beta}_0/dt \\
\hat{a} &= -\hat{\beta}_1/dt \\
\hat{\sigma} &= \sqrt{\mathrm{Var}(\epsilon)/dt}
\end{aligned}$$

- **CIR**. Recall that CIR model is defined by three parameters a, b, σ, specified as follows:

$$dr = a(b - r)dt + \sigma\sqrt{r}dW \qquad \text{([Bjork]24.5)}$$

which can be rewritten as

$$\frac{dr}{\sqrt{r}} = \frac{abdt}{\sqrt{r}} - a\sqrt{r}dt + \sigma\sqrt{dt}z$$
$$\text{where} \quad z \sim \mathcal{N}(0, 1)$$

Which implies that parameters can be estimated via OLS regression (no intercept)

$$\frac{dr}{\sqrt{r}} = \beta_1 \cdot \frac{1}{\sqrt{r}} + \beta_2\sqrt{r} + \epsilon$$

and the best parameter estimates are given by

$$\begin{aligned}
\hat{a} &= -\hat{\beta}_2/dt \\
\hat{b} &= \hat{\beta}_1/(\hat{a}dt) \\
\hat{\sigma} &= \sqrt{\mathrm{Var}(\epsilon)/dt}
\end{aligned}$$

As these calibration schemes have been discussed, we address here another point of interest: although we require short rate simulations for $s \in [T, T + \Delta]$, we only have information up to $t = 0$. This means that we need to form some expectation of the parameter set $\mathbf{\Theta}^T$. In the process of producing this study, several attempts had been made regarding forming $\mathrm{E}(\mathbf{\Theta}^T)$, but most approaches fail due to he sheer length of the forecast horizon (*e.g.* ARIMA(p, d, q) models converging to simple mean). As a result, the authors make the assumption that the parameter set $\mathbf{\Theta}^T$ converges to the long-run trend reflected in all data available. Hence, the simple psedocode for calibrating both parameter sets goes as follows:

Algorithm 1 `calibrate_short_rate`: calibrate short rate parameter set $\mathbf{\Theta}$

.

Set: model, horizon, dt, $\mathbf{X} = [(\mathbf{r} - \mathbf{r}_f), d(\mathbf{r} - \mathbf{r}_f)]$
if horizon == 0 **then**
  $\mathbf{X} \leftarrow \mathbf{X}(-252 :, :)$ ▷ Only one year of data
else if horizon == T **then**
  $\mathbf{X} \leftarrow \mathbf{X}$ ▷ All data
end if
if model == 'vasicek' **then**
  $\beta_0, \beta_1, \boldsymbol{\epsilon} \leftarrow$`regress`$(\mathbf{X}(:, 1), \underline{(\mathbf{X}(:, 0))}$, `intercept` $=$ True$)$
  $b, a, \sigma \leftarrow \beta_0/dt, -\beta_1/dt, \sqrt{\mathrm{Var}(\boldsymbol{\epsilon})/dt}$
else if model == 'cir' **then**
  $\mathbf{X}(:, 2) \leftarrow \sqrt{\mathbf{X}(:, 0)}$
  $\beta_0, \beta_1, \boldsymbol{\epsilon} \leftarrow$`regress`$\left(\frac{\mathbf{X}(:,1)}{\mathbf{X}(:,2)}, (\frac{1}{\mathbf{X}(:,2)}, \mathbf{X}(:, 2))\right.$, `intercept` $=$ False$\left.\right)$
  $a, \sigma \leftarrow -\beta_2/dt, \sqrt{\mathrm{Var}(\boldsymbol{\epsilon})/dt}$
  $b \leftarrow \beta_1/(a \cdot dt)$
end if
return (a, b, σ)

Finally, the transformation from short-rate sample paths and then to forward-term rates is a sequence of simple discrete integral, exponential transformation, and function calls. We abbreviate the pseudocode here, but the reader can refer to the implementation for further details.

4.2 Simulation

With calibration schemes for both quantoed stock and short rate, we can simulate the quanto value and short rate used for the pricing routine.

- The simple pseudocode to simulate short rate goes as follows:

Algorithm 2 `sim_short_rate`: simulate short rate using Gaussian.

Set: model, $\mathbf{r}_f = [r_f(t)]$, a, b, σ, n, T, dt, r_0, $\mathbf{dr} = [\mathbf{0}]_{ncm}$, $\mathbf{r} = [\mathbf{0}]_{n\times(m+1)}$
$m \leftarrow \text{int}(T/dt)$
$\mathbf{r}(:, 0) \leftarrow r_0$
$\mathbf{r}_f \leftarrow [\mathbf{r}_f]_{n\times m}$
$\mathbf{Z} \leftarrow$ `randstdnormal`(n, m)
if model == 'cir' **then**
  for $t \leftarrow 1$ to m **do**
    $increment \leftarrow a \times (b - \mathbf{r}[:, t])$
    $\mathbf{dr}(:, t) \leftarrow dt \cdot increment + \sigma \cdot \mathbf{Z}(:, t) \cdot \sqrt{dt} * \sqrt{\mathbf{r}(:, t)}$
    $\mathbf{r}(:, t+1) \leftarrow \mathbf{r}(:, t) + \mathbf{dr}(:, t)$
  end for
else if model == 'vasicek' **then**
  for $t \leftarrow 1$ to m **do**
    $increment \leftarrow b - a \cdot \mathbf{r}(:, t)$
    $\mathbf{dr}(:, t) \leftarrow dt \cdot increment + \sigma \cdot \mathbf{Z}(:, t) \cdot \sqrt{dt}$
    $\mathbf{r}(:, t+1) \leftarrow \mathbf{r}(:, t) + \mathbf{dr}(:, t)$
  end for
end if
return $\mathbf{r} + \mathbf{r}_f$

4.3 Main Pricing Routine

Finally, we present the final pricing scheme using all aforementioned algorithms, and abbreviated `calibrate_quanto`, and `eval_quanto` that are explained in sections 4.1.1 and 4.2.1:

Algorithm 3 `main`: compute the price of the T-claim $\boldsymbol{\chi}$

Set: model, N, k, k', n, dt, T, Δ

$\mathbf{X}, \mathbf{ffr} \leftarrow$ `read_rate_data()` ▷ $\mathbf{X}$ contains difference between SOFR and FFR

$\mathbf{S} \leftarrow$ `read_quanto_data()`

$\boldsymbol{\Theta}_0 \leftarrow$ `calibrate_short_rate`(model, `horizon` $= 0, dt, \mathbf{X}$) ▷ calibrate $[0, T]$

$\boldsymbol{\Theta}_T \leftarrow$ `calibrate_short_rate`(model, `horizon` $= T, dt, \mathbf{X}$) ▷ calibrate $[T, T+\Delta]$

$r_0 \leftarrow \mathbf{X}(0, -1)$ ▷ Most recent r observation as r_0

$\mathbf{r}^0 \leftarrow$ `sim_short_rate`(model, $a_0, b_0, \sigma_0, n, T+\Delta, dt, r_0$)

$\mathbb{E}(r_T) \leftarrow$ `mean`($\mathbf{r}^0$, `axis` $= 0$)(`int`$(T/dt) - 1$) ▷ use simulated r_T to form expectation

$\mathbf{r}^T \leftarrow$ `sim_short_rate`(model, $a_T, b_T, \sigma_T, n, \Delta, dt, \mathbb{E}(r_T)$)

$\mathbf{L}_0 \leftarrow$ `forward_term_rate`($\mathbf{r}^0 + \mathbf{ffr}(-1), 0, T, \Delta, dt$) ▷ Use newest FFR observation to recover simulated SOFR rates

$\mathbf{L}_T \leftarrow$ `forward_term_rate`($\mathbf{r}^T + \mathbf{ffr}(-1), (T-T), (T-T), \Delta, dt$) ▷ simulate starting at T

$\mathbf{L} \leftarrow \mathbf{L}_T / \mathbf{L}_0$

$\sigma_S \leftarrow$ `calibrate_quanto`($\mathbf{S}$)

$\mathbf{r}_d \leftarrow \mathbf{r}^0(:, :$ `int`$(T/dt) + 1) + \mathbf{ffr}(-1)$ ▷ simulated $\mathbf{r}$ $\forall t \in [0, T]$

$\mathbf{Q} \leftarrow$ `eval_quanto`($\mathbf{r}_d, \sigma_S, dt$) ▷ integration according to equation (4)

discount $\leftarrow$ `bond_price`($\mathbf{r}_d, 0, T, dt$) ▷ get stochastic discount factor

$\boldsymbol{\chi} \leftarrow \text{discount} \cdot N \cdot \max\left((k - \mathbf{Q}) \cdot (\mathbf{L} - k'), 0\right)$

return $\boldsymbol{\chi}$ ▷ $\dim(\boldsymbol{\chi}) = (n, 1)$

The returned result is the claim's value at $t = 0$ of sample size n. To obtain an evaluation, one can compute the mean and standard deviation of the output vector.

5 Result and Conclusion

By comparing the model Vasicek and CIR run on $(k, k') = (0.8, 1.5)$ and assumed federal funds rate of 0.05, the average claim price and the standard deviation are:

	Vasicek	CIR
Claim Price	0.0901523149822069	0.08811083487059998
Claim Std	0.09369932915958011	0.09283253090669259

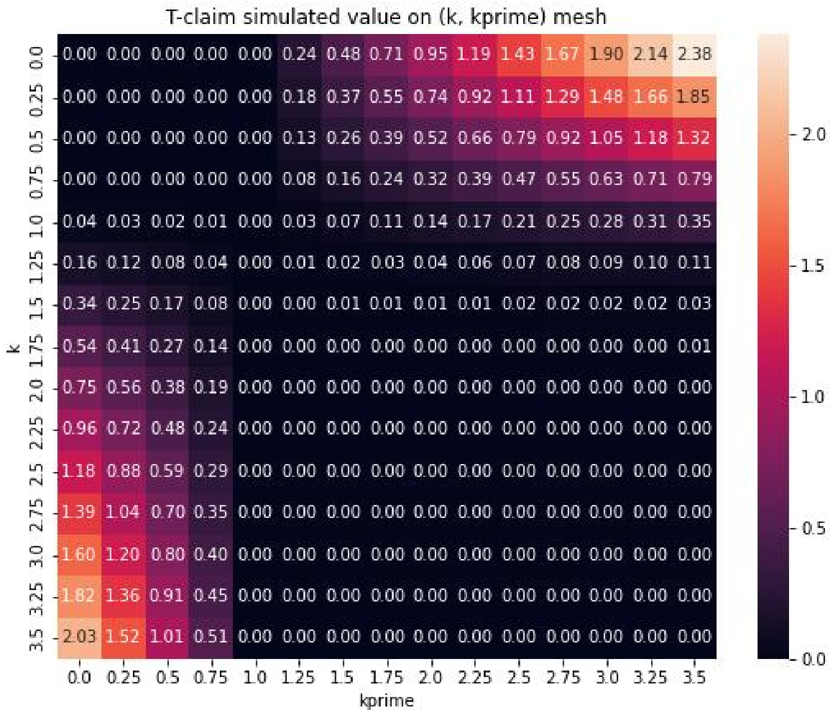
We notice that the variance of the claim price for CIR is smaller than Vasicek.

This is caused by the smaller variance in short rate in CIR. In addition to the mean-reverting nature of CIR, volatility is proportional to the square root of the interest rate, limiting the variance. This prevents negative interest rates, which can occur in the Vasicek model.

Thus we will proceed with CIR in later simulations and comparisons.

With the CIR model, simulations will be run for different combinations of k and k prime and moving average window. To show the change of the price of T-claim with respect to the change in k and k', we run the main pricing routine on a mesh of k and k' values in $(0, 3.5) \times (0, 3.5)$ with an increment of 0.25. The result is shown in the below heatmap generated from the output in Figure 1.

Figure 1: Heatmap: $(k, k') \in (0, 3.5) \times (0, 3.5)$



where we effectively observe that the price of the T-claim attains a higher value when exactly one of k, k' attains a high value and the other attains a low value. This agrees with our intuition regarding the form of the payout: when k is close to zero, the first component in the product within max is very likely to be negative. Hence the payout would attain a higher value should the second component also attain a negative value, that is, when k' gets bigger; on the other hand, when k' is close to 0, the second component in the product within max is very likely to be positive. In this situation, the payout would attain a higher value should the first component attain a positive value, that is, when k is big.

It is also interesting to observe from the heat map that the price of the claim is much more sensitive to the change in k or k' if the other one is very close to 0. This is because, under model assumptions, both the term rate ratio and the quanto ratio are strictly positive, which results in the fact that if k or k' is 0, the difference between the quanto ratio or the term rate ratio (respectively) and their respective strike would have a probability of 1 attaining their respective sign. Given this almost surety, the other strike would have a huge effect on the claim price, should it have the same sign as the other component.

Finally, below is the simulation run on different moving average window lengths on parameters $(k, k') = (2, 0)$, where the smallest variance is attained with a 180 sized window.

	moving average window	claim price	std
0	30	0.904178	0.213444
1	90	0.904504	0.21373
2	180	0.905695	0.212996
3	360	0.903351	0.213763

Figure 2: Comparison: different MA windows

This result also shows that the moving average transformation of the original data set is stable, as the resulting claim price does not change much up to the second moment in terms of its distributional properties.

References

[Bjork] Björk, Tomas. Arbitrage theory in continuous time. Oxford University Press, 2009.