Lorenz System

M537 Final Project - Group D

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Abstract

This report applies concepts from Math 537 to explore nonlinear dynamical systems and bifurcation phenomena through computational and analytical methods. The project will focus on investigating the Lorenz system, a three-dimensional (3-D) system of ordinary differential equations, originally derived by Edward Lorenz in 1963 to model atmospheric convection (2), which exhibits chaotic behavior and sensitivity dependent on initial conditions. Through stability analysis of fixed points and bifurcation theory, we aim to characterize the system's dynamical behavior across varying parameters. This includes exploring saddle-node, transcritical, pitchfork, and Hopf bifurcations as presented in Strogatz's *Nonlinear Dynamics and Chaos* (Chapter 8, Sections 8.1-8.3). Furthermore, we employ **XPP-AUTO** software to explore bifurcations, phase portraits, and nullclines, providing both computational and visual insights into the system's dynamics. Numerical simulations reveal the system's transition to chaos and the emergence of the characteristic Lorenz attractor. Following analytical and computational exploration, we discuss applications of the Lorenz system in weather prediction, fluid dynamics, and broader implications for understanding deterministic chaos in physical systems.

Introduction

The Lorenz system, introduced by meteorologist Edward Lorenz in 1963, is one of the most known examples of deterministic chaos in nonlinear dynamics. Originally derived for the purpose of modeling atmospheric convection in Rayleigh-Bénard flow (2), the system consists of three ordinary differential equations, coupled to describe the evolution of convective motion in a heated fluid dynamics. Despite the visual simplicity of the equations, the Lorenz system holds rich dynamics, including a sensitive dependence on initial conditions, which holds great significance in the exploration of chaotic behavior.

The motivation stems from the attempt to understand how small shifts or perturbations of atmospheric conditions can lead to dramatically different weather outcomes. Mathematically, the system provides a framework for studying how deterministic equations can produce unpredictable long-term behavior.

In this report, we investigate the Lorenz system's dynamics across varying parameter values with an emphasis on bifurcation phenomena. We can examine the stability of fixed points and identify the bifurcations that dictate transitions between qualitatively different behaviors. More specifically, we study Hopf bifurcations, which mark the transition from steady convection to oscillatory behavior and saddle-node bifurcations that create or destroy equilibrium states. Through phase plane analysis, nullcline geometry, and bifurcation diagrams generated using *XPP-AUTO*, we can trace the system's evolution and characterize the emergence of the iconic Lorenz attractor.

Model Formulation

The Lorenz system is a nonlinear system of three coupled, first-order ordinary differential equations originally derived by Edward Lorenz in 1963 as a simplified model for atmospheric convection. The equations are:

$$\begin{aligned}
\dot{x} &= \sigma(y - x), \\
\dot{y} &= rx - y - xz, \\
\dot{z} &= xy - bz,
\end{aligned}$$

where the dot notation $(\dot{x}, \dot{y}, \dot{z})$ denotes time derivatives.

(a) **Variables:**

- $x(t)$: intensity of convective motion (horizontal fluid velocity),
- $y(t)$: horizontal temperature variation,
- $z(t)$: vertical temperature variation.

(b) **Parameters:**

- $\sigma > 0$: Prandtl number (ratio of momentum diffusivity to thermal diffusivity),
- $r > 0$: Rayleigh number (proportional to temperature difference across the fluid layer),
- $b > 0$: geometric factor related to the physical dimensions of the system.

(c) **Control Parameter:**

- The primary control parameter in the Lorenz system is **r**, which governs the emergence of instability and bifurcation behavior.
- Varying **r** changes the qualitative dynamics of the system, including transitions from fixed points to oscillatory and chaotic behavior.

Default Parameter Values:
The classical Lorenz parameter values are:

$$\sigma = 10, \qquad b = \frac{8}{3}, \qquad r = 28$$

These were chosen by Lorenz to reflect the behavior of fluids such as air under heating.

Analysis

Consider the Lorenz system:

$$\begin{aligned}\dot{x} &= \sigma(y-x),\\ \dot{y} &= rx - y - xz,\\ \dot{z} &= xy - bz\end{aligned}$$

(a) **Equilibria**

$$\begin{aligned}\text{To find equilibrium points, we set:}\quad & \dot{x}=\dot{y}=\dot{z}=0\\ \text{From }\ \dot{x}=0:\quad & \sigma(y-x)=0 \quad\to\quad \underline{y=x}\\ \text{From }\ \dot{y}=0\ \text{ with }\ y=x:\quad & rx-x-xz=0 \quad\to\quad \underline{x(r-1-z)=0}\end{aligned}$$

(i) <u>If $x=0$, then from $\dot{x}=0$, we have:</u> $\quad y=x \;\to\; y=0$

From $\dot{z}=0$, we have:

$$\dot{z}=xy-bz \implies xy-bz=0 \implies 0-bz=0 \implies z=0$$

So one equilibrium point is:

$$\boxed{E_0=(0,0,0)}$$

(ii) <u>If $x\neq 0$, then from $\dot{y}=0$ and $y=x$, we have:</u> $\quad x(r-1-z)=0 \;\to\; z=r-1$

Now substituting into $\dot{z}=0$ using $y=x$ and $z=r-1$:

$$\dot{z}=xy-bz \implies xy-bz=0 \implies x^2=b(r-1) \implies x=\pm\sqrt{b(r-1)}$$

Since $y=x$, the two nontrivial equilibria are:

$$E_1=\Big(\sqrt{b(r-1)},\ \sqrt{b(r-1)},\ r-1\Big) \quad\text{and}\quad E_2=\Big(-\sqrt{b(r-1)},\ -\sqrt{b(r-1)},\ r-1\Big)$$

These equilibria exist only when r $>$ 1.

(b) **Jacobian Matrix**

We compute the Jacobian matrix $J(x,y,z)$ by taking the partial derivatives of the vector field:

$$J(x,y,z)=\frac{\partial(\dot{x},\dot{y},\dot{z})}{\partial(x,y,z)}=\begin{bmatrix}\frac{\partial\dot{x}}{\partial x} & \frac{\partial\dot{x}}{\partial y} & \frac{\partial\dot{x}}{\partial z}\\ \frac{\partial\dot{y}}{\partial x} & \frac{\partial\dot{y}}{\partial y} & \frac{\partial\dot{y}}{\partial z}\\ \frac{\partial\dot{z}}{\partial x} & \frac{\partial\dot{z}}{\partial y} & \frac{\partial\dot{z}}{\partial z}\end{bmatrix} \implies J(x,y,z)=\begin{bmatrix}-\sigma & \sigma & 0\\ r-z & -1 & -x\\ y & x & -b\end{bmatrix}$$

(c) **At Equilibrium** $E_0=(0,0,0)$

$$J(E_0)=\begin{bmatrix}-\sigma & \sigma & 0\\ r & -1 & 0\\ 0 & 0 & -b\end{bmatrix}$$

To find eigenvalues, compute the characteristic polynomial:

$$\det(J-\lambda I)=\det\begin{pmatrix}-\sigma-\lambda & \sigma & 0\\ r & -1-\lambda & 0\\ 0 & 0 & -b-\lambda\end{pmatrix}$$

$$\begin{aligned}\det(J-\lambda I) &= (-\sigma-\lambda)\Big((-1-\lambda)(-b-\lambda)-0\Big)-\sigma\Big((r)(-b-\lambda)-0\Big)+0\cancel{\Big((r)(0)-(-1-\lambda)(0)\Big)}\\ &= (-\sigma-\lambda)(-1-\lambda)(-b-\lambda)-\sigma(r)(-b-\lambda)\\ &= (-b-\lambda)\Big((-\sigma-\lambda)(-1-\lambda)-\sigma(r)\Big)\\ &= (-b-\lambda)\Big(\lambda^2+(\sigma+1)\lambda+\sigma-\sigma r\Big)\\ \det(J-\lambda I) &= (-b-\lambda)\Big(\lambda^2+(\sigma+1)\lambda+\sigma(1-r)\Big)=0\\ &\implies \boxed{(\lambda+b)\Big(\lambda^2+(\sigma+1)\lambda+\sigma(1-r)\Big)=0}\end{aligned}$$

Thus, the characteristic polynomial is: $\quad (\lambda + b)\left(\lambda^2 + (\sigma+1)\lambda + \sigma(1-r)\right) = 0$

This yields two equations: $\quad \underline{\lambda + b = 0 \quad \text{or} \quad \lambda^2 + (\sigma+1)\lambda + \sigma(1-r) = 0}$

Solving for roots by isolating and using the quadratic formula, we can find three eigenvalues. That is, the eigenvalues of the Jacobian at the origin are:

$$\boxed{\lambda_1 = -b, \quad \lambda_{2,3} = \frac{-(\sigma+1) \pm \sqrt{(\sigma+1)^2 - 4\sigma(1-r)}}{2}}$$

Qualitative Phase Space Analysis

(a) **The sign of the discriminant** $\left[(\sigma+\mathbf{1})^{\mathbf{2}} - \mathbf{4}\sigma(\mathbf{1}-\mathbf{r})\right]$ **determines whether the roots are:**

- (i) Real and distinct (if discriminant > 0),
- (ii) Real and repeated (if discriminant $= 0$),
- (iii) Complex conjugates (if discriminant < 0).

(b) **Stability of** E_0

- (i) If $r < 1$: all eigenvalues have negative real parts $\implies$ E_0 is a **stable node**.
- (ii) If $r = 1$: one eigenvalue is zero $\implies$ **pitchfork bifurcation** occurs.
- (iii) If $r > 1$: one eigenvalue becomes positive $\implies$ E_0 becomes a **saddle** with an unstable direction.

(c) **Stability of** E_1 **and** E_2

At E_1 and E_2, the Jacobian becomes more complicated. Its characteristic polynomial is a cubic whose roots depend on the parameters σ, r, and b.

For the classical Lorenz parameters:

$$\sigma = 10, \quad b = \frac{8}{3}$$

A **Hopf bifurcation** occurs when:

$$r_{\text{Hopf}} \geq \frac{\sigma(\sigma+b+3)}{\sigma-b-1} \approx \underline{24.74}$$

- (i) If $1 < r < r_{\text{Hopf}}$, then E_1 and E_2 are **stable spirals**.
- (ii) If $r = r_{\text{Hopf}}$, A **Hopf bifurcation** occurs, in which the nonzero equilibria lose stability, and a periodic orbit emerges. As r increases further, global bifurcations lead to the formation of the **chaotic attractor**.
- (iii) If $r > r_{\text{Hopf}}$, the equilibria become **unstable**.

Bifurcation Analysis

As stated before, the equilibrium points are:

$$\begin{aligned}
E_0 &= (0,0,0), && \text{for } x = 0 \\
E_1 &= \left(\sqrt{b(r-1)},\ \sqrt{b(r-1)},\ r-1\right), && \text{for } x \neq 0 \\
E_2 &= \left(-\sqrt{b(r-1)},\ -\sqrt{b(r-1)},\ r-1\right), && \text{for } x \neq 0
\end{aligned}$$

The Hopf bifurcation only occurs when $r_H \geq \frac{\omega+b+3}{\omega=b-1} \approx 24.74$

- (i) $1 < r < r_H :\ E_{1,2}$ are spiral stable
- (ii) $r = r_H$: A **Hopf bifurcation** occurs, nonzero equilibria lose stability, and a periodic orbit emerges. As r increases, global bifurcations form the **chaotic attractor**.

(iii) $r > r_H$: equilibria become unstable

From this, you can analyze that the Hopf bifurcation will be supercritical. Supercritical is when a parameter passes through a critical point, transitioning from stable to unstable. We have shown this above. In the Lorenz model, $r = r_H$ is unstable, and the equilibria lose stability without a cycle replacing it. This leads to chaos theory.

As r increases, the nonzero equilibria $E_{\pm}$ gradually weaken and lose their stability. At $r_H \approx 24.74$, a critical point, a Hopf bifurcation occurs, making the system unstable. Complex eigenvalue pairs cross the imaginary axis. An unstable periodic orbit forms at this point. The trajectories beyond the bifurcation move unpredictably and become chaotic. This is the chaotic dynamics of the Lorenz system.

Simulation Results

We can analyze the vector field structure using nullclines and 2D plots of the phase space using XPP-Auto. A **nullcline** is a surface (or curve) where a particular time derivative vanishes.

For example:

- $\dot{x} = 0 \to y = x$
- $\dot{y} = 0 \to z = r - 1 - \frac{y}{x}$
- $\dot{z} = 0 \to xy = bz$

We analyze these intersections to identify equilibrium points and assess local vector field behavior.

(i) For $r = 27, \ \sigma = 10, \ b = 8/3$

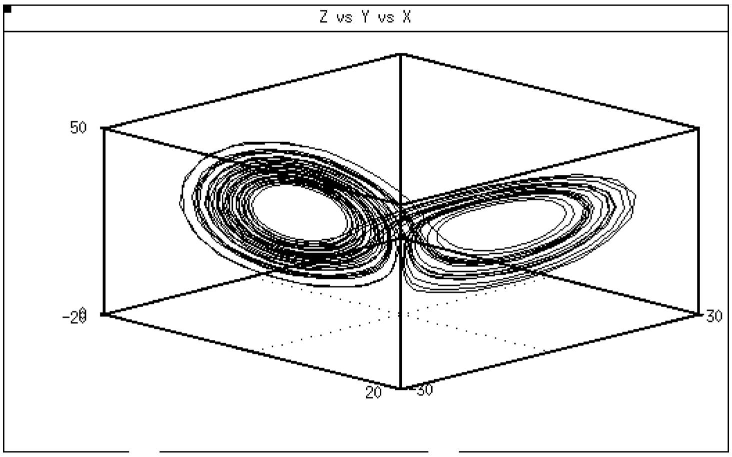

Figure 1: 3D Phase Portrait.

The figure above illustrates a 3-dimensional phase portrait of the Lorenz system. The trajectory spirals around two lobes, often called the Lorenz attractor's 'wings.' These two lobes are centered on unstable equilibria that help structure the flow. Due to tiny differences in initial conditions, the trajectories diverge exponentially, so the orbit never repeats itself, yet remains confined to a butterfly shape. This plot shows how the variables x, y, and z evolve and settle into this complex attractor, rather than approaching a steady point or periodic orbit over time.

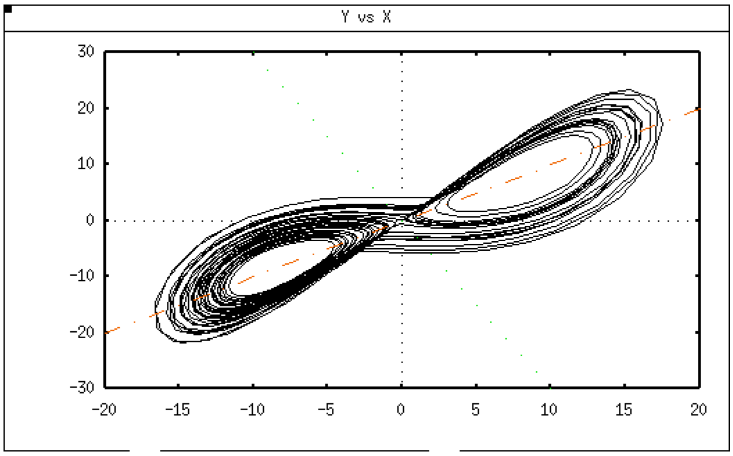

Figure 2: 2D Phase Portrait with Nullcline.

This next figure above is a 2-dimensional phase portrait of the Lorenz system on the x-y plane. Projecting the 3-dimensional phase portrait onto only two variables still displays the butterfly-shaped structure made of two swirling lobes. Each lobe corresponds to one of the system's unstable equilibrium points, and the trajectory loops around one lobe before unpredictably switching to the other lobe. The dense overlapping curves further demonstrate that the orbit never repeats itself, but the trajectories are still confined to a characteristic butterfly-like shape. The orange dashed line is a nullcline that represents the unstable eigenvector direction near the equilibrium, which illustrates how trajectories are pushed away and stretched before swirling around the lobes.

(ii) For $r = \{0,\ 0.5,\ 1\},\ \sigma = 10,\ b = 8/3$

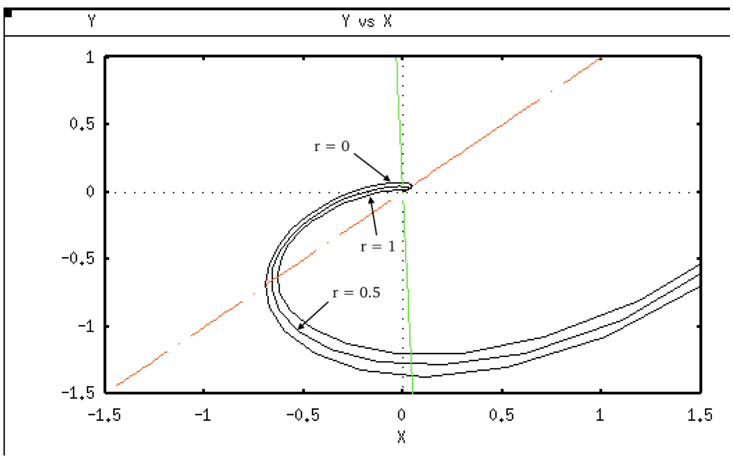

Figure 3: Zoomed-in plot at the origin of $r = 0,\ 0.5,\ 1$.

This plot above is a simulation of the phase lines for the Lorenz system with r-values of 0, 0.5, and 1. This plot is zoomed in on the origin to illustrate the tiny perturbations at the very beginning of every phase. Further emphasizing the nature of this system and how small changes in the initial conditions can result in drastically different phase behaviors as $t \to \infty$.

(iii) For $r = 24.74,\ \sigma = 10,\ b = 8/3$

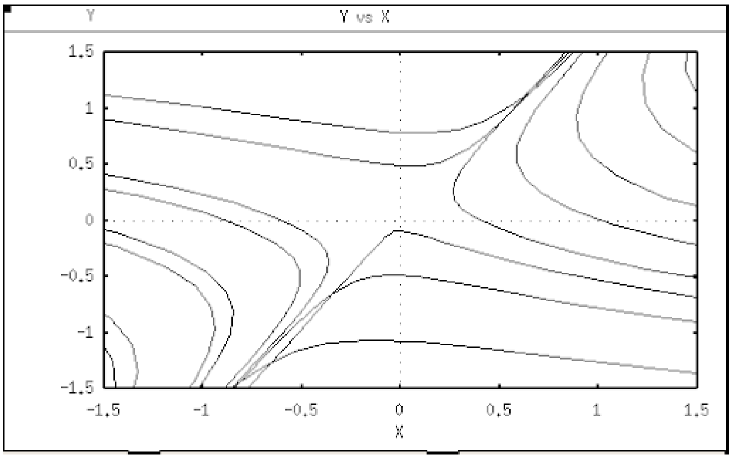

Figure 4: Zoomed-in plot at the origin of $r = 24.74$.

This figure shows a two-dimensional projection of the Lorenz system onto the $x - y$ plane, with contour lines illustrating the structure of the vector field near the origin. The contours give a sense of how trajectories are guided through this region of phase space. Close to the origin, the curves change direction and exhibit a mild rotational pattern, suggesting a shift in the local behavior of the system. This is consistent with what is expected near a Hopf bifurcation, where the nature of the equilibrium changes and oscillatory behavior begins to appear. Farther from the center, the contours become smoother and more evenly spaced, indicating a more regular flow. Overall, the plot offers a localized view of how the system's dynamics evolve in this parameter setting.

Discussion

The Lorenz System was originally developed by Edward Lorenz to create a mathematical model on atmospheric convection. (1)(Lorenz, 1963) This model uses the system to describe the intensity of the convection $\dot{x}$ and the temperature variation in both the vertical $\dot{z}$ and horizontal $\dot{y}$ axis. The parameters that this model used were σ which represented the Prandtl number, r which represented the Rayleigh number, and β is related to the physical properties of the region we are considering. (2)(Sparrow, 1982) The Prandtl number is a scalar property that is the ratio of the momentum diffusivity to the thermal diffusivity.(3)(Coulson, Richardson, 1999) In simpler terms this is a representation of how momentum and heat diffuse through a fluid. The Rayleigh number represented by r is another scalar property that is used quite often in convection problems. This number tells whether or not buoyancy driven natural convection plays an important role in heat transfer. (4)(Zeneli. etc, 2021) The last parameter used in this β describes the geometry of the region. This is used only in the $\dot{z}$ equation, as this details the vertical temperature distortion. (1)(Lorenz, 1963) Furthermore, all of the parameters were all taken to be positive. This is not the only application of the Lorenz system, this system has been used in different fields apart from meteorology. The next application uses the Lorenz system in electrical circuits.

First discussed in a paper from Pecora and Carroll in 1990, the idea of a chaotic system having a self-synchronization property. This paper demonstrated that two chaotic systems can converge to identical trajectories and become in sync with each other.(6)(Pecora & Carroll, 1990) In 1993 Cuomo and Oppenheim sought to apply what Pecora and Carroll proposed by implementing their concepts into an electronic circuit. Looking at the Lorenz system, they found that $\dot{y}$ was decomposable into two stable subsystems, a drive system and a stable response system. These two systems can synchronize when coupled with a common drive signal.(5)(Cuomo & Oppenheim, 1993)

$$\dot{u} = \sigma(v - u), \tag{1}$$
$$\dot{v} = ru - v - 20uw, \tag{2}$$
$$\dot{w} = 5uv - bw. \tag{3}$$

Directly implementing the original Lorenz system would lead to some issues since equations give a very wide and dynamic range of values that exceed power supply limits. To mediate this problem Cuomo and Oppenheim performed a transformation of variables. They defined $u = \frac{x}{10}$, $v = \frac{y}{10}$, and $w = \frac{z}{20}$. (5)(Cuomo & Oppenheim, 1993) This transformed system is referred to as the transmitter and was implemented into an analog circuit.

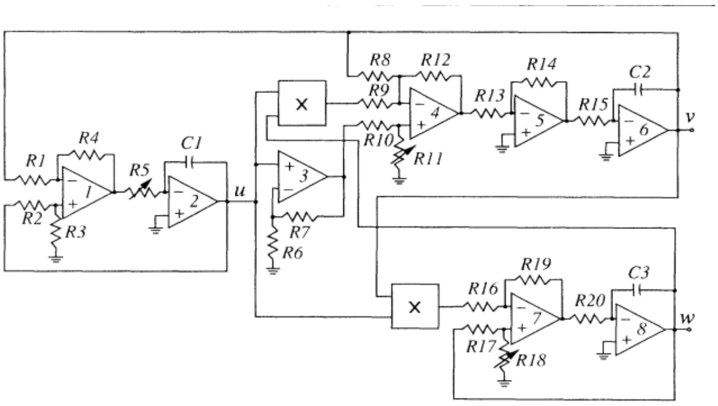

Figure 5: Lorenz-based chaotic circuit (5)(Cuomo & Oppenheim, 1993)

As seen in the diagram above this is an exact implementation of the equation. One thing to note is that the coefficients σ, ρ, and β can be varied by adjusting the resistors, R_5, $R_1$1, and $R_1$8 respectively. Now this is not the only system, there still needs to be a response system which will synchronize with the chaotic signals from the transmitter.

$$\dot{u}_r = \sigma(v_r - u_r), \tag{4}$$

$$\dot{v}_r = ru - v_r - 20uw_r, \tag{5}$$

$$\dot{w}_r = 5uv_r - bw_r. \tag{6}$$

The equations are almost identical to each other, with slight variations in the variables used. Let us use the u,v, and w variables from the transmitter equations as a vector d and the variables from the receiver v_r, u_r, and w_r as a vector r. We can define the dynamical errors e as $d - r$. This error dynamics show that the synchronization of the Lorenz system is a result of a stable error dynamics between the transmitter and the receiver. (5)(Cuomo & Oppenheim, 1993) A potential use for this technique is in signal masking and recovery. A masking signal is added to the transmitter to the information signal and the masking signal can be removed at the receiver. The idea would be to remake the masking signal at the receiver and subtract it from the received signal to recover the information.

Conclusion

In this project, we analyzed the Lorenz system, three nonlinear ODEs derived from Rayleigh–Bénard convection, to understand how simple deterministic equations can produce such complex behavior. Through equilibrium analysis, we identified the fixed points E_0 and $E\pm$ and determined how their stability changes with the Rayleigh parameter r. By examining bifurcations, particularly the saddle-node structure of the nonzero equilibria and the supercritical Hopf bifurcation near $r \approx 24.74$, we observed how the system transitions from stable steady states to oscillatory motion and ultimately to chaos. Computational tools such as phase portraits, nullcline geometry, and bifurcation diagrams generated in *XPP-AUTO* allowed us to visualize these transitions and capture the formation of the butterfly–shaped Lorenz attractor, highlighting the system's sensitive dependence on initial conditions. Together, these results illustrate the depth of nonlinear dynamical systems and the mechanisms through which chaos emerges. Possible extensions of this work include studying embedded periodic orbits, comparing the Lorenz system to other chaotic models, investigating parameter sensitivity in greater detail, or exploring applications such as chaotic synchronization and secure communication.

References

1. Lorenz, E. N. (1963). Deterministic Nonperiodic Flow. Journal of the Atmospheric Sciences, 20(2), 130–141. https://doi.org/10.1175/1520-0469(1963)020%3C0130:dnf%3E2.0.co;2

2. Sparrow, C. (1982). Introduction and Simple Properties. Applied Mathematical Sciences, 1–12. https://doi.org/10.1007/978-1-4612-5767-7_1

3. Coulson, J. M., & Richardson, J. F. (1999). Coulson & Richardson's chemical engineering. Volume 1 : fluid flow, heat transfer and mass transfer. Butterworth-Heinemann.

4. Zeneli, M., Nikolopoulos, A., Karellas, S., & Nikolopoulos, N. (2021). Numerical methods for solid-liquid phase-change problems. Ultra-High Temperature Thermal Energy Storage, Transfer and Conversion, 165–199. https://doi.org/10.1016/b978-0-12-819955-8.00007-7

5. Cuomo, K. M., & Oppenheim, A. V. (1993). Circuit implementation of synchronized chaos with applications to communications. Physical Review Letters, 71(1), 65–68.

   https://acoustique.ec-lyon.fr/chaos/CuomoOppenheim93.pdf

6. Pecora, L. M., & Carroll, T. L. (1990). Synchronization in chaotic systems. Physical Review Letters, 64(8), 821–824. https://doi.org/10.1103/physrevlett.64.821

7. Bard, E. — xpp: A Differential Equations Solver and Graph Plotter. https://sites.pitt.edu/~phase/bard/bardware/xpp/xpp.html