24.118: Paradox and Infinity

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May 10, 2014

1. (a) Yes. There are no non-empty subsets of an empty set, so it is trivially well ordered.

   (b) No. $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$
   There is no least element between $\emptyset$ and $\{\{\emptyset\}\}$ because neither is a subset of the other.

   (c) No. $\mathcal{P}(\mathcal{P}(\emptyset)) - \{\emptyset\} = \{\emptyset, \{\emptyset\}\} - \{\emptyset\} = \{\{\emptyset\}\}$
   Set containing one element is trivially well ordered as there is only one subset and thereby the one element is the least element.

   (d) No. $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) - \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} - \{\{\emptyset\}\} = \{\emptyset, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$
   There still is no ordering between $\emptyset$ and $\{\{\emptyset\}\}$ as described in `(b)`.

   (e) Yes. $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) - \{\{\{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} - \{\{\{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$
   Every subset comparison has a least element.

2. (a) Proof by contradiction, assume that there is some α that is a smaller ordinal than $\emptyset$. Then by our definition α must be an element of $\emptyset$. But $\emptyset$ has no elements. Thus we arrive at a contradiction and can conclude that there is no smaller ordinal than $\emptyset$.

   (b) Proof by contradiction, assume that there is some α containing infinitely many members that is a smaller ordinal than ω. Then ω must contain some element N that is not within α such that α could be represented as $1_0, 2_0, 3_0, ...$ and that ω can be represented as $1_0, 2_0, 3_0, ..., N$. By definition, however, α is infinite so any N chosen will be a member of that ordinal. Thus α and ω contain the same elements, so we can conclude that no infinite sequence can be smaller than ω.

3. (a) Map element 0_0 from ordinal ω to the element ω from the ordinal ω'. Then map each element n_0 from ordinal ω to the element $(n-1)_0$ from the ordinal ω'. Pairing elements in this manner would give the following 1-to-1 correspondence between ω and ω'

$$\begin{array}{ccccc} 0_0 & 1_0 & 2_0 & 3_0 & ... \\ | & | & | & | & \\ \omega & 0_0 & 1_0 & 1_0 & ... \end{array}$$

   (b) The shape of ω is $|||...$ while the shape of ω' is $|||...|$ so they are not isomorphic. As was shown with the 1-to-1 correspondence above, the positions had to be rearranged to get from one ordering to the other (specifically element ω from ω' was brought from the last position to the first position).

4. (i) True
   (ii) False
   (iii) True
   (iv) True
   (v) True
   (vi) False
   (vii) True
   (viii) False
   (ix) False
   (x) True

5. (i) |||...|||...|||...
   (ii) |||...|||...|||...
   (iii) |||...|||...|||...|||... ...
   (iv) |||...|||...|||...|||... ... |||...|||...|||...|||... ... |||...|||...|||...|||... ... .
   (v) Take Ω to be |||...|||...|||...|||... ... |||...|||...|||...|||... ... |||...|||...|||...|||... ... . from above. Then we have $\Omega\Omega\Omega$ ...