Homework 3 < Part 1 >

Problem 1

$- \ \epsilon_i = f(x) - h_i(x) \quad \Rightarrow \quad E[\epsilon_i(x)^2] = E[(f(x) - h_i(x))^2]$

$$E_{avg} = \frac{1}{M}\sum_{i=1}^{M} E(\epsilon_i(x)^2)$$

$$h_{agg}(x) = \frac{1}{M}\sum_{i=1}^{M} h_i(x)$$

$$E_{agg}(x) = E\left[\left\{\frac{1}{M}\sum_{i=1}^{M} h_i(x) - f(x)\right\}^2\right] = E\left[\left\{\frac{1}{M}\sum_{i=1}^{M} \epsilon_i(x)\right\}^2\right]$$

$E(\epsilon_i(x)) = 0$ for all i

$E(\epsilon_i(x)\epsilon_j(x)) = 0$ for all $i \neq j$

$$E_{agg} = \frac{1}{M} E_{avg}$$

$$\cancel{= \frac{1}{M^2}\sum_{i=1}^{M} E[\epsilon_i(x)^2] = \frac{1}{M^2}\sum_{i=1}^{M} E[f^2(x) - 2f(x)h_i(x) + h_i^2(x)]}$$

$$E_{agg} = E\left[\left\{\frac{1}{M}\sum_{i=1}^{M} \epsilon_i\right\}^2\right]$$

$$= \frac{1}{M^2}[E(\epsilon_1^2) + E(\epsilon_2^2) + \cdots + E(\epsilon_M^2)]$$ due to $E[\epsilon_i(x)\epsilon_j(x)] = 0$

$$= \frac{1}{M^2}[\ E[\Sigma \epsilon_i^2]\ ]$$

$$= \frac{1}{M}\left[\frac{1}{M}E[\Sigma \epsilon_i^2]\right] = \frac{1}{M}E_{avg}$$

So $E_{agg} = \frac{1}{M}E_{avg}$

ex)

$E[(\epsilon_1 + \epsilon_2)^2]$

$= E[\epsilon_1^2 + 2\epsilon_1\epsilon_2 + \epsilon_2^2]$

$= E[\epsilon_1^2] + E[\epsilon_2^2] + 2\underbrace{\cancel{E[\epsilon_1\epsilon_2]}}_{0}$

$E[(\epsilon_1 + \epsilon_2 + \epsilon_3)^2]$

$= E[\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \cancel{2\epsilon_1\epsilon_2}^{\to 0} + \cancel{2\epsilon_2\epsilon_3}^{\to 0} + \cancel{2\epsilon_3\epsilon_1}^{\to 0}]$

$= E[\epsilon_1^2] + E[\epsilon_2^2] + E[\epsilon_3^2]$

Problem 2

$$E_{avg} = \frac{1}{M}\sum_{i=1}^{M} E(\epsilon_i(x)^2)$$

$$E_{agg}(x) = E\left[\left\{\frac{1}{M}\sum_{i=1}^{M} h_i(x) - f(x)\right\}^2\right] = \underline{E\left[\left\{\frac{1}{M}\sum_{i=1}^{M}\epsilon_i(x)\right\}^2\right]}$$

$$E_{avg} = \frac{1}{M}\sum_{i=1}^{M} E\{\epsilon_i(x)^2\}$$

$$f\left(\sum_{i=1}^{M}\lambda_i(x_i)\right) \leq \sum_{i=1}^{M}\lambda_i f(x_i)$$

$$E\left[\left(\frac{1}{M}\sum_{i=1}^{M}\epsilon_i(x)\right)^2\right] = \left(\frac{1}{M}\right)^2 * E\left[\left(\sum_{i=1}^{M}\epsilon_i(x)\right)^2\right]$$

$$\frac{1}{M}E\left(\frac{1}{M}(\epsilon_1(x)+\epsilon_2(x)+\cdots+\epsilon_M(x))^2\right)$$

$2ab \leq a^2+b^2$

$$\leq \frac{1}{M}E(\epsilon_1^2(x)+\epsilon_2^2(x)+\cdots+\epsilon_M^2(x))$$

$(a+b)^2 \leq a^2+b^2$

$a^2+b^2+2ab \leq (a^2+b^2)2$

$$\leq \frac{1}{M}E\left(\sum_{i=1}^{M}\epsilon_i^2(x)\right)$$

So $E_{agg} \leq E_{avg}$

Problem 3

$$D_1 = \frac{1}{N}$$

$$D_2 = \frac{1}{N} * \frac{1}{Z_1} * e^{-a_1 h_1(i) y(i)}$$

$$\vdots$$

$$D_T = \frac{1}{N \prod_{i=1} \frac{1}{Z_i}} * \exp\left(-\sum_{i=1}^{T-1} a_i h_i(i) y(i)\right)$$

$$D_{T+1} = \frac{1}{N \prod_{i=1}^{T} \frac{1}{Z_i}} * \exp\left(-y(i) \sum_{i=1}^{T} a_i h_i(i)\right) \Rightarrow \frac{D_t(i)}{Z_t} * e^{-a_t h_t(i) y(i)}$$

$$H(x) = \operatorname{sign}\left(\sum_{t=1}^{T} a_t h_t(x)\right)$$

if $h_t(x_i) = y_i \rightarrow +1$

$h_t(x_i) \neq y_i \Rightarrow -1$

$$\epsilon_t = \sum_{h_t(x_i) \neq y_i} D_t(i)$$

total h_t error $= \frac{1}{2} - \gamma_t$

$$\text{error} = \frac{1}{N} \sum_{i} \begin{cases} 1 & \text{if } y_i \neq H(x_i) \\ 0 & \text{else} \end{cases}$$

$$= \frac{1}{N} \sum_i \begin{cases} 1 & \text{if } y_i f(x_i) \leq 0 \\ 0 & \text{else} \end{cases} \leq \underbrace{\frac{1}{N} \sum_i \exp(-y_i f(x_i))}$$

$$= \sum_i D_{T+1}(i) \prod_t Z_t$$

$$= \prod_t Z_t$$

$$Z_t = \sum_i D_t(i) \times \begin{cases} e^{-\alpha_t} & \text{if } h_t(x_i) = y_i \\ e^{a_t} & \text{if } h_t(x_i) \neq y_i \end{cases}$$

$$= \sum_{i, h_t(x_i) = y_i} D_t(i) * e^{-a_t} + \sum_{i, h_t(x_i) \neq y_i} D_t(i) * e^{a_t}$$

$$= e^{-a_t} * \sum_{i, h_t(x_i) = y_i} D_t(i) + e^{a_t} * \sum_{i, h_t(x_i) \neq y_i} D_t(i)$$

$$= e^{-a_t} * (1 - \epsilon_t) + \epsilon_t * e^{a_t}$$

set $\alpha_t = \frac{1}{2} \ln\left(\frac{1 - \epsilon_t}{\epsilon_t}\right)$

$$= \cancel{} \quad \cancel{}$$

$$= 2\sqrt{\epsilon_t (1 - \epsilon_t)} \leftarrow \epsilon_t = \frac{1}{2} - \gamma_t$$

$$Z_t = 2\sqrt{\left(\frac{1}{2} - \gamma_t\right)\left(\frac{1}{2} + \gamma_t\right)}$$

$$= \frac{\cancel{2}\sqrt{1 - 4\gamma_t^2}}{\cancel{2}} = \sqrt{1 - 4\gamma_t^2}$$

$$Z_t \leq \sqrt{e^{-4\gamma_t^2}} = e^{-2\gamma_t^2}$$

$$\text{Error} \leq \prod_t Z_t$$

$$\text{Error} \leq \prod_t e^{-2\gamma_t^2}$$

$$\text{Error} \leq e^{-2 \sum_{i=1}^{T} \gamma_t^2}.$$