1. Let X and Y be two decision problems. Suppose we know that X reduces to Y in polynomial time. State whether the following statements are true or false and give a brief explanation.

a. If Y is NP-complete then so is X.

False. We cannot certain that X is NP-complete. X can be in P, or NP.

b. If X is NP-complete then so is Y.

False. If X is NP-complete, then Y can be NP- complete or NP-Hard.

c. If Y is NP-complete and X is in NP then X is NP-complete.

False. X can be NP or NP-complete.

d. If X is NP-complete and Y is in NP then Y is NP-complete.

True. For instance, if 3-SAT is reduced to Y and Y is in NP, then Y is NP-complete.

e. If X is in P, then Y is in P.

False. We cannot certain Y because Y is difficult than X so that Y can be P or NP.

f. If Y is in P, then X is in P.

True. X is not difficult than Y so that X is in P.

g. X and Y can't both be in NP

False. X and Y can both be in NP. For instance, 3-SAT can be reduced to 4-SAT, and both are in NP.

2. Two well-known NP-complete problems are 3-SAT and TSP, the Traveling Salesman Problem. The 2-SAT problem is a SAT variant in which each clause contains at most two literals. 2-SAT is known to have a polynomial-time algorithm. State whether the following statements are true or false and give a brief explanation.

a. 3-SAT $\leq_p$ TSP.

True. We already know that 3-SAT and TSP are NP-complete so that 3-SAT can be reduced to TSP.

b. If $P \neq NP$, then 3-SAT $\leq_p$ 2-SAT.

False. 3-SAT $\leq_p$ 2-SAT means NP can be reduced to P so that P = NP. This is contradiction of the statement.

c. If TSP $\leq_p$ 2-SAT, then P = NP.

True. 3-SAT $\leq_p$ 2-SAT means NP can be reduced to P so that P = NP.

3. A Hamiltonian path in a graph is a simple path that visits every vertex exactly once. Show that HAM-PATH = { (G, u, v): there is a Hamiltonian path from u to v in G } is NP-complete. You may use the fact that HAM-CYCLE is NP-complete.

1) Prove Hamiltonian path is in NP.

We can verify the certificate of Hamiltonian path in polynomial time. By using adjacency matrix, we can verify whether the certificate is Hamiltonian path or not in $O(n^2)$. (n is number of vertices).

Since the certificate of Hamiltonian path can be verified in polynomial time, Hamiltonian path is in NP.

2) Prove that Hamiltonian cycle(NP-complete) can be reduced to Hamiltonian Path.

We will transform Hamiltonian cycle into Hamiltonian path by adding two new vertices.

First, we will choose any two adjacent vertices of a graph which is Hamiltonian cycle, and connect each new vertex to each chosen vertex. This takes polynomial time by using adjacency matrix ($O(n^2)$, n is number of vertices of the graph).

Then, newly added vertices will be starting point and end point of the path. Since, in Hamiltonian cycle, every vertex can be visited only once, there is Hamiltonian path between the starting point and the end point so that there is Hamiltonian path from the starting point and the end point. Moreover, if we do the same process to a graph which is not Hamiltonian cycle, there is no Hamiltonian path in the graph which is result of transformation. Thus, we can conclude that if there is a Hamiltonian cycle in the original graph, then there is a Hamiltonian path in the new graph.

If there is Hamiltonian path in the graph which is the result of the transformation, then there exists Hamiltonian cycle between the starting point and the end point. Thus, we can conclude that if there is Hamiltonian path in the new graph, then there is a Hamiltonian cycle in the original graph.

(Note: The original graph is Hamiltonian cycle.)

Since Hamiltonian path is in NP and Hamiltonian cycle is reduced to Hamiltonian path, Hamiltonian path is NP-complete.

4. K-COLOR. Given a graph G = (V,E), a k-coloring is a function c: V -> {1, 2, … , k} such that c(u) ≠ c(v) for every edge (u,v) ∈ E. In other words, the number 1, 2, .., k represents the k colors and adjacent vertices must have different colors. The decision problem K-COLOR asks if a graph can be colored with at most K colors. The 3-COLOR decision problem is NP-complete by using a reduction from SAT. Use the fact that 3- COLOR is NP-complete to prove that 4-COLOR is NP-complete.

1) Prove 4-COLOR is in NP.

We can verify the certificate of 4-COLOR by searching the graph and checking the adjacent vertices having different colors and using 4 colors. This process takes $O(n^2)$ by using adjacency matrix.

Since the certificate of 4-COLOR can be verified in polynomial time, 4-COLOR is in NP.

(Note: n is number of vertices of the graph.)

2) Show that 3-COLOR can be reduced to 4-COLOR.

After proving 4-COLOR is in NP, we have to show that 3-COLOR can be reduced to 4-COLOR in order to prove 4-COLOR is in NP-complete.

Let there is a graph which vertices are colored in three colors.

We will denote a graph which can be colored in 3 colors as G, and a graph which can be colored in 4 colors as G'.

We will transform G into G' by connecting one new vertex to all vertices of G. This transformation takes polynomial time ($O(n)$, n is number of vertices).

If G can be colored in 3 colors, then G' can be colored in 4 colors. The reason is that if the vertex which is adjacent to other vertices is colored in a unique color, and the other vertices is always colored in one of the other 3 colors.

If G' can be colored in 4 colors, then G can be colored in 3 colors. The reason is that the newly added vertex is colored in a unique color, and the rest of the vertices are colored in one of the other 3 colors. So, G can be colored in 3 colors.

Since 4-COLOR problem is in NP and 3-COLOR is reducible to 4-COLOR, 4-COLOR is NP-complete.