1.

Basis)

Let $S = \phi$. Then $2^S = \{\phi\}$.

$|S| = 0$, $|2^S| = 1$

$2^{|S|} = 2^0 = 1 = |2^S|$

$\therefore$ Basis holds.

Inductive Assumption)

Let $|S| = n$, $n \geq 0$ be a finite integer.

Assume $|2^S| = \sum_{k=0}^{n} C(n,k) = 2^n = 2^{|S|}$.

Inductive Step)

When $|S|$ is $n+1$ which is a finite integer,

$$|2^S| = \sum_{k=0}^{n+1} C(n+1,k) = C(n+1,0) + C(n,0) + 2\sum_{k=1}^{n-1} C(n,k) + C(n,n) + C(n+1,n+1)$$

$$= 2C(n,0) + 2\sum_{k=1}^{n-1} C(n,k) + 2C(n,n)$$

$$= 2\sum_{k=0}^{n} C(n,k) = 2 \cdot 2^n = 2^{n+1} \text{ by inductive assumption.}$$

$(\because C(n+1,0) = C(n,0) = C(n,n) = C(n+1,n+1) = 1)$

We get $|2^S| = 2^{|S|} = 2^{n+1}$ when $|S| = n+1$.

$\therefore$ If our claim is true for n, it must be also true for $n+1$. Since n can be any finite number, the statement must be true for any finite set S. ■

2.

$G = \{\{S\}, \{a\}, S, P\}$, $P: S \to aSa \mid aa$

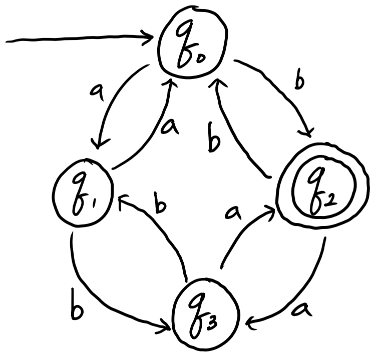
$S \Rightarrow aa$, $S \Rightarrow aSa \Rightarrow aaaa$, $S \Rightarrow aSa \Rightarrow aaSaa \Rightarrow aaaaaa$.

So, $S \Rightarrow aSa \Rightarrow aaSaa \Rightarrow aaaSaaa \overset{*}{\Rightarrow} a^nSa^n \Rightarrow a^{2n+2}$

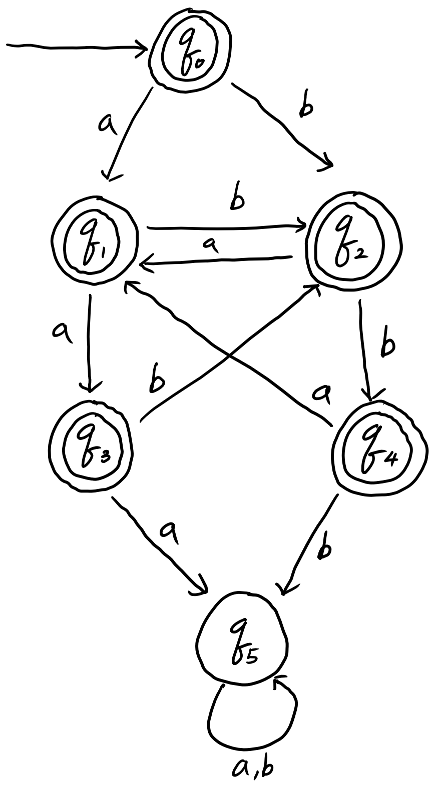
$\therefore S \overset{*}{\Rightarrow} a^nSa^n \Rightarrow a^{2n+2}$. $(a^naaa^n = a^{2n+2})$

Thus, $L(G) = \{a^{2n+2} : n \geq 0\}$

3.



4.

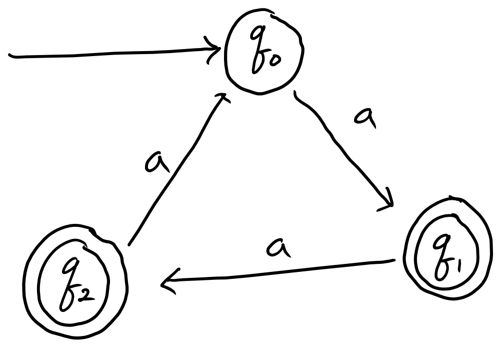


5.

$L = \{a^n : n \text{ is not a multiple of } 3\}$

L is regular if and only if there is a dfa M such that $L = L(M)$.

So, the dfa for $\Sigma = \{a\}$ that accepts the of strings which length is not a multiple of 3 will make the language L regular.



Since we have constructed a dfa for the language L, the languae L is regular.