Moving a Sofa Around a Corner

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1 Introduction

What is the maximum area of a sofa S that can move around an L-shaped hallway H with unit width? More formally, let this hallway be defined as the subset of the plane $H = \{(x, y) \mid (x \leq 1 \wedge y \leq 1) \wedge (x \geq 0 \vee y \geq 0)\}$ shown in Figure 1. S must be a closed subset of H_L, where $H_L = \{(x, y) \in H \mid x < 0\}$. We want to apply a continuous rigid motion to S to obtain S', with S' being a closed subset of $H_R = \{(x, y) \in H \mid y < 0\}$. Additionally, at any point during this continuous motion, the sofa must remain inside of the hallway. If a sofa satisfies these constraints, we will call it a valid sofa.

2 Warm-up

Let's first examine some simple solutions to this problem. The simplest valid sofa is the unit square, which is clearly able to move through the hallway using only translations. Going one step up from that, we arrive at the semicircle with unit radius, which can clear the right-angle turn by rotations. So, the sequence of transformations that the semicircle undergoes is first a sequence of translations, then rotations, and then translations once again.

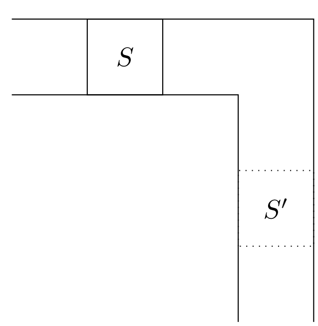

Figure 1: L-shaped hallway H with a trivial sofa S. There exists a sequence of rigid transformations to S that results in S'

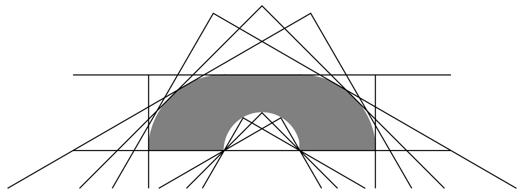

Figure 2: A Hammersley sofa. This is an element of a family of sofas that are constructed with two quarter circles, conjoined by a rectangle with a semicircle cut out of the bottom. The largest Hammersley sofa has an area of $\frac{\pi}{2} + \frac{2}{\pi}$ (pictured).

If we want to obtain a nontrivial solution, we will need to perform these rotations and translations simultaneously. One approach to thinking about this problem is by looking at rigid transformations of the hallway itself, and looking at the joint intersection of all of these transformed hallways. Intuitively, if over the course of these transformations the hallway rotates by a right angle, then the region formed by this intersection is a valid sofa, since the relative motion of the sofa relative to the hallway is continuous and rigid.

One such construction of a valid sofa is done by translating the the inner corner of the hallway on a circular path, and rotating that hallway at a constant rate as these translations are being performed. This results in a shape that consists of two quarter circles, conjoined by a rectangle with a semicircle cut out of the bottom. It turns out that if we adjust the radius of the circle along which we translate the hallway, we can obtain many different sofas that are identical except for the width of the rectangle and semicircle in the middle. In the trivial case, if a radius 0 is used, we get the semicircle.

The largest sofa in this family is known as the Hammersley sofa[3], which has an area of $\frac{\pi}{2} + \frac{2}{\pi}$. Hammersley conjectured that this was an optimal area, in my opinion likely because the construction and result feel natural.

3 Gerver's Sofa

To this day, there is still a large gap between the best known lower and upper bounds. The best known upper bound is 2.37 [4], while the best known lower bound is has an area of around 2.2195, given by Gerver [1]. This solution is reasonably conjectured to indeed be the optimal solution, and I will present a high-level overview of his main theorem and construction.

3.1 A proof sketch

Suppose we had a collection of all closed regions that could move through the hallway in question with continuous, rigid transformations. Call this set $\mathcal{S}$. Gerver argued that this set is compact, and since it is, it must have an element with maximal area. Call the maximal element S^*. Then, there exists a sequence

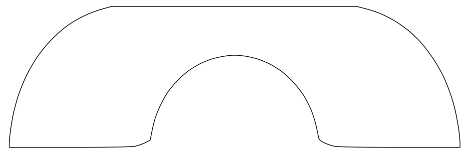

Figure 3: Gerver's Sofa, consisting of 18 straight or curved sections each described by an analytic expression. Many of these cannot be explicitly defined, and they are expressed in terms of constants that also can only be implicitly defined by a system of equations.

of polygons (P_n) such that for all n, P_n has an area greater than or equal to that of S^*. Lastly, we construct an infinite sequence (S_n) with each $S_n \in \mathcal{S}$ such that as n approaches infinity, S_n approximates P_n. Therefore, since $\mathcal{S}$ is compact, it must be that (S_n) has a convergent subsequence, and the element to which it converges must have an area equal to that of S^*.

3.2 Compactness

To prove that $\mathcal{S}$ is compact, let us first define it. Let $p : [0, \frac{\pi}{2}] \to \mathbb{R}^2$ be defined as $p(\theta) = (x(\theta), y(\theta))$. Then, for any given θ we will define $H_{p,\theta}$ to be H rotated about the origin by θ and translated so $(0,0)$ goes to $p(\theta)$. We can then define a sofa $S_p = \bigcap_{\theta \in [0, \frac{\pi}{2}]} H_{p,\theta}$, and $\mathcal{S}$ is the collection of all such S_p.

First, we will prove that $\mathcal{S}$ is bounded. This can be done simply by looking at the intersection between $H_{p,0}$ and $H_{p,\frac{\pi}{4}}$. This intersection alone is trivially bounded by a rectangle of height 1 and length 4, independent of choice of p. The intersection of all $H_{p,\theta}$ must be contained inside of this intersection, so for any p, S_p is bounded. Therefore, all sofas in $\mathcal{S}$ can be bounded by some ball around the origin, which we will say has radius C.

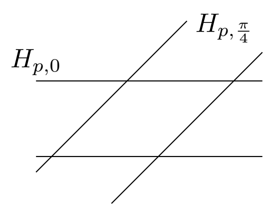

Figure 4: The intersection point between $H_{p,0}$ and $H_{p,\frac{\pi}{4}}$ has edges that intersect at an angle of $\frac{\pi}{4}$.

Next, we can see that $\mathcal{S}$ is compact under the topology induced by what appears to be $p(\theta)$ (details omitted because I do not quite understand this topic). This essentially follows from the fact that if we are given some sequence of sofas $S_1, S_2, \ldots$, with S_i defined by p_i, then p_i is bounded by C. Additionally, because $\mathcal{S}$ is bounded, we can show that each p_i is Lipschitz continuous with a

Lipschitz constant of C.

For any given θ, there is a subsequence $S_1, S_2, \ldots$ such that $p_i(\theta)$ converges, since all p_i is bounded by C. This is true for all $\theta \in \mathbb{Q} \cap \left[0, \frac{\pi}{2}\right]$, since we can continuously take infinite subsequences that converge at each of the countably infinite values of θ. Since the rationals are dense on this interval, and each p_i is Lipschitz continuous, we get that there exists some subsequence of $S_1, S_2, \ldots$ whose path functions converge. Call this point of convergence p_∞, we just need to show that S_∞ defined by p_∞ belongs to $\mathcal{S}$.

This is done quite quickly by noticing that S_∞ is formed by the intersection of closed sets, so it itself is a closed set. If we suppose for contradiction that S_∞ is not in $\mathcal{S}$, then it must be true that for some θ, S_∞ does not lie inside of $H_{p_\infty,\theta}$. Since these sets are both closed, then there must be some $\varepsilon > 0$ such that H contains points ε distance outside of $H_{p_\infty,\theta}$. However, since S_∞ can be approximated arbitrarily closely by S_i in the sequence $S_1, S_2, \ldots,$ this is a contradiction, because it implies there is some S' in $\mathcal{S}$ that lies a nonzero distance outside of $H_{p',\theta}$. Therefore, $\mathcal{S}$ is compact.

3.3 Polygonal Approximations

Suppose S^* exists. Then, we can construct a sequence of polygonal approximations $P_1, \ldots,$ such that as n goes to infinity, P_n approaches S^*. Additionally, for all P_n, the area of P_n is greater than S^*.

This is done quite simply. Suppose that S^* is defined by the path function $p^*(\theta)$. Then, we know that $S^* = \bigcap_{\theta \in \left[0, \frac{\pi}{2}\right]} H_{p^*,\theta}$. Let P_n be defined as the intersection of n of these $H_{p^*,\theta}$, where the angle is evenly distributed on the interval from 0 to $\frac{\pi}{2}$. Formally, $P_n = \bigcap_{k \in [n]} H_{p^*, \frac{k\pi}{2n}}$. It's easy to see that for all n, $S^* \subseteq P_n$, so the area of P_n is greater than that of S^*. Additionally, the polygon P_n can be approximated arbitrarily close by S_n as n goes to infinity (proof omitted), and P_n always has an area greater than or equal to that of S^*, so it must be that there is a sequence of elements of $\mathcal{S}$ that converges at S^*.

3.4 A Numerical Approach

Despite the fact that Gerver proved the existence of an optimal sofa, he never found and proved a sofa to be optimal. In his paper, he presents a sofa that satisfies the optimality condition he discussed, which is necessary but not sufficient to be the optimal valid sofa. Further numerical approaches[2, 5] provide some convincing evidence that Gerver's solution is optimal.

4 Modeling and Cutting

To build the objects, I used Inkscape to create .svg files that contained all of the sofas (the unit square and unit semicircle, Hammersley sofa, Gerver's sofa, Romik's ambidextrous sofa, and some numerical polygonal approximations of the optimal sofa). The scale I used for the model was 2 inches for every 1 unit.

The hallway was roughly 4 units long in both directions, which provably can hold all sofas in $\mathcal{S}$.

Unfortunately, I had trouble printing out Hammersley's sofa, due to issues with the file as it transferred to the laser cutting machine. Modeling Gerver's sofa exactly would have been extremely difficult because it is defined implicitly, based on numerical constants that are also defined implicitly. As a result, I settled for an approximate model, and I traced a to-scale image of Gerver's sofa on Inkscape. This didn't cause any issues with the final product, which almost perfectly moved around the corner as intended.

References

[1] J. L. Gerver. On moving a sofa around a corner. *Geometriae Dedicata*, 42(3):267–283, 1992.

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