MATH-131 (Numerical Methods for Scientists and Engineers) — Worksheet 4

Semester: Spring 2019,   Instructor: Nicholas Knight

Due Feb. 19 at 2359. *Please remember to cite your sources, including collaborators.*

Deliverable: Submit a Live script titled `worksheet4.mlx` via CatCourses (under Assignments). Divide this file into sections, one for each of the following questions, plus an extra (final) section containing all the function definitions. Document each function definition to explain the input and output arguments.

1. Implement fixed-point iteration. Your signature should be

   ```
   function p = fp(g, p0, maxits)
   ```

   where `g` is a function handle and `p0`, `maxits`, and `p` are numbers. The return value `p` is the result of `maxits` applications of `g`, i.e.,

   $$\mathtt{p} = \mathtt{g}(\mathtt{g}(\cdots \mathtt{g}(\mathtt{p0})\cdots)).$$

   Test your code on the following four functions.

   (a) $g\colon x \mapsto (x + 2/x)/2$.
   (This is the 'Babylonian method' for computing $\sqrt{2} = 1.414\cdots$.)

      i. Confirm visually that the fixed-point problem "Solve $g(x) = x$" is equivalent to the root-finding problem "Solve $f(x) = 0$' with $f\colon x \mapsto x^2 - 2$. That is, make a plot showing that $y = f(x)$ intersects $y = 0$ at the same x-values which $y = g(x)$ intersects $y = x$.

      ii. Experiment with the parameters `p0` and `maxits`. How can you pick `p0` to control which of the two fixed points ($x = \pm\sqrt{2}$) the method converges to?

      iii. Try `p0` $= 1.5$ and compare with the bisection method (Worksheet 3) with `a` $= 1$, `b` $= 2$, and `tol` $=$ `eps`. Plot the error (`abs(sqrt(2) - p)`) of the two methods as a function of iteration number. (You will have to modify your `fp` and `bisection` codes to return all of the iterates, not just the last.) Use a `semilogy` plot.

   (b) $g\colon x \mapsto \cos(x)$.

      i. Plot $y = g(x)$ and $y = x$ to visually confirm that g has a unique fixed point.

      ii. Plot the error as a function of iteration number, using the converged value as the exact solution.

      iii. (*Challenge; ungraded*) Estimate the rate at which the error (as a function of iteration number n) converges (to zero) when `p0` $= 1$. Do this by fitting a curve of the form $n \mapsto Cq^n$ to the error.

   (c) $g\colon x \mapsto 2x$.
   This function has a unique fixed point at $x = 0$. For what starting value(s) `p0` does fixed point iteration converge to it? What happens otherwise?

   (d) $g\colon x \mapsto \begin{cases} x/2 & x \neq 0 \\ 1 & x = 0 \end{cases}.$
   This function has no fixed points. What does your `fp` implementation do? If you performed fixed-point iteration in exact (infinite precision) arithmetic, what would happen instead?