Numerical analysis and simulation of a shallow water system on variable topography

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Abstract

Shallow water waves with variable bottom topography have many applications, and can be mathematically approached in many ways. The topographies that occur in nature can be very complicated in structure, but exploring a domain with a complicated topographical structure is difficult, so only two simple mathematical functions are investigated. The shallow water equations are used for this problem since they can handle variable bottom topography, and are fairly easy to work with. There are a few different waves that occur in shallow water systems with variable bottom topography, and as the parameters are adjusted the system produces different waves, which are bifurcations. In this thesis the domain of three bifurcation parameters is investigated. The parameters are initial height, initial velocity, and the slope or step size of the topography. Bifurcations caused by changing the initial conditions or topography can be identified by tracking entropy, and a resulting bifurcation diagram was constructed.

Contents

1 Introduction

The field of fluid dynamics is an incredibly deep and useful to study due to the vast amount of applications. The study of free surface flows is an important piece in the world of fluid dynamics, and there is an immense amount of knowledge about free surface flows on flat bottom topography. However, free surface flows on a non-flat surface are less studied (see e.g. [2, 3, 4, 5, 6]). Many applications can be approached when a non-flat topography is considered. One example that cannot be approached with flat topography is the shoaling problem, which is the problem where waves propagate from deep to shallow water.

A few notable researchers that have analytically and numerically investigated shallow water systems with topography have done so with some different models and approaches. The Su-Gardner equations, the variable Korteweg-deVries equations, the Boussinesq equations, the shallow water equations, and linear wave approximations have all been used to model shallow water systems with variable topography [1]. Many of the models used consider dispersion, and for my problem I an ignoring dispersion since I am only investigating the first wave in my system. The main parameters of undular bores generated in transcritical shallow water flow over localized topography has been successfully solved analytically using the Su-Gardner equations [2]. Solitons have been created by propagating surface water waves with a current over an uneven bottom [3]. The inverse problem, which is to determine the topography based on the surface profile, have also been studied in [4] by introducing a new variable into the shallow water equations and reconstructing the new partial differential equation (PDE). Once the PDE has been reconstructed the system can then be numerically solved to determine the topography. Invariant imbedding methods have been shown to be an accurate way of solving the influence of bottom topography on the propagation of linear shallow water waves [5]. The variable Korteweg-deVries equation has been derived and shown to be accurate in modeling slowly-varying periodic waves over variable topography and slowly varying solitary waves over topography [6].

Many other studies on fluid dynamics have shown that the addition of variable topography has a very significant effect on the dynamics and surface profile of the system. Thin film flows is one such field that has had research done on the effect of topography. Free surface defects on thin film flows has been shown to be completely controllable by changing the substrate topography [7]. The inverse problem of reconstructing the substrate topography from the free surface has also been shown to be possible [7]. In [7] they also identified many applications that can benefit from their research; some examples include the micromanufacturing of patterned functional layers, and surface defect location and characterization.

2 Background

A model will need to be constructed in order to simulate a shallow water system, and there are many different ways shallow water systems can be modeled. Many of the equations that model shallow water flow are derived from conservation laws, and then modified to fit the exact problem. In this thesis I decided to use the shallow water equations due to their simplicity and accuracy (see Model section for more info). The shallow water equations with variable topography have been numerically solved for many different problems [8]. One notable problem that is similar to the setup being considered in this thesis is the shoaling problem. The shoaling problem consists of a wave going from deep water to shallow water, which can be as simple as a linear slope or a much more complicated profile. One example where shoaling is applicable is with tsunamis. A tsunami that is not near to the shore is a very fast, small amplitude, and long wavelength wave, which means the shallow water equations are applicable here. As the tsunami approaches the shore the system gains a forcing parameter due to the bathymetry, or topography, of the ocean applying a force on the wave, which then changes the wave into the more well known and destructive form of a tsunami. Having solutions to both the near shore and non-near shore problem can help predict what the tsunami will do, which can in turn help people minimize the damage caused by those destructive waves and, both of those problems can be modeled by the shallow water equations. Wave shoaling has been solved analytically and numerically with the shallow water equations [9].

Understanding what types of waves that my numerical experiments create is necessary for quantifying when and where the waves happen. The numerical experiments that I ran create shock waves and rarefaction waves. One of the waves collapses on itself, and my model will capture this as a nearly vertical jump in height since my model will not allow for two fluid heights at one point. The collapsing wave comes in two different forms in nature, and both of them have a distinct numerical form. One collapsing wave has the recirculating piece above the current or above the surface, and the other is in the current or below the surface. Numerically, the above surface shock has the jump occur before the end of the changing piece of topography, $|\frac{d}{dx}B(x)|(x = \text{shock}) > 0$, and the below surface shock occurs at the very end of the changing piece of the topography, $|\frac{d}{dx}B(x)|(x = \text{shock}) = 0$ (see figures 2 and 9 in the results section for examples). The second case will be a wave that increases in height over a distance instead of nearly instantaneously, and will be numerically represented as a strictly increasing function, in height, between two points, and constant elsewhere. The third will be a rarefaction wave, and this can be seen numerically by surface profile decreasing in proportion to the slope of the topography. These waves are of particular interest because if the initial fluid height or the topography is changed enough the resulting system will produce a different wave, which is a bifurcation. Understanding when the system bifurcates will give insight into how the topography effects the surface profile.

The model I am using requires multiple inputs in order to be solvable. Two of the necessary pieces are the initial conditions and the topographies. Both can be taken from real world systems with the use of a mathematical approximation. The types of topographies that are interesting to this problem are those that show up in nature and in human-made structures. Many of the topographies that show up in nature are incredibly complicated and do not have a nice mathematical model and thus using these topographies would prove to be very difficult. So, I eliminated any complicated structure from consideration for modeling, and doing so reduces the problem down to localized configurations involving a linear change, and a step change in the bottom profile. A combination of step and linear profiles can, in principle, be used to approximate any topography; however, since the system is nonlinear, the resulting surface profile cannot be obtained by linear superposition. Linear and step pieces do represent a few natural and human-made surfaces and constructing methods for analyzing them can help for future projects on more complicated topographies.

Since all shallow water systems contain water they have some amount of energy within them, which is known as entropy. The amount of entropy within a system can change over time from heat dissipation, water leaving or entering at the boundaries, and through shocks doing work. Entropy can be written as potential energy plus kinetic energy where kinetic energy is commonly written as $\frac{1}{2}mv^2$ where m is mass and v is the velocity, and potential energy is commonly formulated as mgh where g is the gravitational constant, m is mass, and h is height. In [10] the authors investigate the entropy evolution in a one dimensional periodic layered medium, and are able to accurately predict if the system formed a shock by looking at the entropy dissipation of the system. In [10] the authors are also able to identify that the entropy dissipation is a function of shock speed. They also explore the theory numerically by seeing how much the entropy dissipation changes due to grid refinement, and find that entropy dissipation can be significantly reduced with a finer grid. For my thesis

I am using the same ideas as presented in [10] to detect if a shock formed and the size of the shock.

3 Model system

Solving a shallow water system with variable bottom topography can be done with either a linear approach or a nonlinear approach. Both methods involve an approximation in the form of a model. For linear methods the model is usually easy to implement and compute; however, linear methods are also less accurate for certain parameter ranges in nonlinear systems. For shallow water waves the linear method requires the Froude number to be less than one [2], where the Froude number is a dimensionless number that is a ratio of inertial and gravitational forces, $\frac{|u|}{\sqrt{gh}}$ [11]. Nonlinear approaches use a more complicated model, and can be very difficult to work with both analytically and numerically. Nonlinear models tend to be less limiting for fluid dynamics problems, and can describe some phenomena much more accurately [2]. The parameters in this thesis are widely varied and go outside of the range of Froude numbers for where a linear approach is accurate, and since nonlinear approaches do not restrict the Froude number I used a nonlinear model.

Within the set of nonlinear models there are still many choices; however, there is one model that stands out for the type of problem that I studied, which is the shallow water equations. These equations can be derived from one of the most general models for fluids, which is the Navier-Stokes equations. The Navier-Stokes equations are constructed from assuming conservation of mass, momentum, and energy. However, these equations are difficult to use and since the system that is being investigated in this thesis does not need to be very general I can make some assumptions to simplify the problem. By depth integrating the Navier-Stokes equations I arrive at the shallow water equations. The shallow water equations in one dimension will be used to study this problem, and in dimensionless form they are,

$$\begin{bmatrix} h \\ uh \end{bmatrix}_t + \begin{bmatrix} uh \\ hu^2 + h^2 \end{bmatrix}_x = \begin{bmatrix} 0 \\ hB_x \end{bmatrix} \tag{1}$$

with the following dimensionless conditions,

$$\frac{u_0 t_0}{x_0} = 1 \quad \frac{gh_0 t_0^2}{2x_0^2} = 1 \quad \frac{-gt_0^2}{x_0} = 1 \tag{2}$$

where $h = h(x,t)$ is the height of a water column above $z = 0$, $u = u(x,t)$ is the fluid velocity, B is the bathymetry or topography, and g is the acceleration due to gravity. A visual representation of a shallow water system with the shallow water equations parameters, wavelength λ, and amplitude a can be seen below.

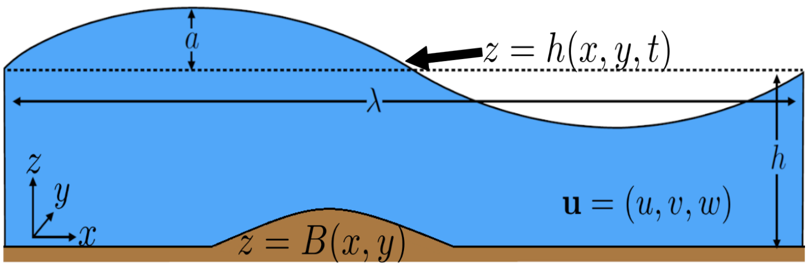


Figure 1: Model picture of a shallow water system

A derivation for the shallow water equations can be seen in [11]. Since the shallow water equations are derived from the Navier-Stokes equations, they form a set of conservation laws in the appropriate variables but, the variables do not yield a conservative form due to the $-ghB_x$ term. The system (1) is nonlinear and hyperbolic, which means (1) satisfies the requirement of not restricting the Froude number. The conservation laws, or assumptions, made to construct these equations, not including the topography term, are: conservation of mass, conservation of momentum, and that the depth is small relative to the wavelength. These assumptions mean mass will neither be created nor destroyed within the system, there are no external forces acting on the system and, the depth being small relative to wavelength means $h/\lambda \approx 0$ where h is the depth and λ is the wavelength. All of these assumptions will hold in the systems being considered in this thesis because all of the mass within the system stays within the system, no external forces are being applied to change the momentum, and all initial conditions considered will be consistent with small depth relative to the wavelength. However, breaking waves are not perfectly represented with my model since shocks have zero wavelength and the shallow water equations assume long waves. So, the shallow water equations cannot model every piece of what happens with a breaking wave, and instead represent a breaking wave as a step. To include all of the other details about a breaking wave, like the mixing of air and water, one would have to use a much more complicated model, and that is beyond the scope of this thesis. Dispersive phenomena also occur in the general system I am modeling; however, including those phenomena would increase the complexity of the problem, and I am just looking at breaking and standing waves, which are not dispersive phenomena. With all of that considered the shallow water equations are a decent model to use for this thesis, and the equations are reasonable to numerically solve, as discussed in the next section.

4 Research question

The purpose of this thesis is to add additional understanding of how variable bottom topography can be controlled to produce specific types of waves, and to apply a method of mathematically identifying bifurcations. For each shallow water system the exact structure varies; however, there are a few rudimentary parameters that are present in all shallow water systems, and they are inflow, outflow and bathymetry. The inflow and outflow consists of boundary conditions for fluid velocity and fluid height. By controlling these parameters, I investigated the parameter ranges for where each type of wave occurs, and I identified when the system bifurcates by tracking entropy.

5 Approach and methodology

A lot of the research done on the shallow water equations with variable topography problem was done on the shoaling problem. However, this problem only considers a single wave interacting with the batymetry, which does not apply to a problem that has a constant inflow of water, and the initial conditions do not necessarily contain any waves. So, the results from research done on the shoaling problem does not directly apply to my problem because there is an inflow, and the initial conditions do not produce waves without variable topography or some other forcing function. To create inflow boundary conditions I used methods described in [11]. The shallow water equations do not place any restrictions on the types of initial conditions that are used in this thesis, but the range of initial conditions need to be restricted in order to make the problem manageable. To prevent the numerical solver from having to run for long periods of time, which would give more time for numerical error to grow, I specified fluid height and velocity initial conditions that are somewhat close to what the system is going to evolve into. The surface profile generally evolves into a flat section followed by a change due to the topography followed by another flat section. There are a few different things that can be caused by the bathymetry, and two of those things are opposites of each other, and thus I chose the most simple and easily iteratable initial condition for height, a flat surface. For the velocity initial condition similar things happen because the only thing in the system that will cause a change in velocity is the topography, which is a result of boundary conditions not causing any waves and the initial height also having no waves. And thus, a zero-slope line for initial velocity is ideal.

There are many ways to numerically solve the shallow water equations, or partial differential equations in general. The more common solvers include finite differences, finite elements, pseudospectral methods, and Riemann solvers. Finite difference methods are numerical methods where the differential equation is approximated with difference equations. For partial differential equations, these methods require setting up a spatial grid, $x_0, x_2, ..., x_j$ and a temporal grid, $t_0, t_1, ..., t_k$ then using either a forward difference or a backward difference to create a recurrence equation, which is then solved to approximate the next step [12]. Since finite differences require $j * k$ points and only taking one step for each iteration makes a computationally intense method, and thus is not ideal for this thesis. Another possibility is the class of pseudospectral methods. Fourier pseudospectral is one of the most common implementations, this method makes use of the Fast Fourier Transform to transform the partial differential equation into ordinary differential equations (ODEs) which can then be solved through any of the usual ODE solving methods [13]. The pseudospectral method could be used for this thesis; however, it does not allow for discontinuities, like shocks. Riemann solvers allow for discontinuities unlike pseudospectral methods. Riemann solvers solve the Riemann problem which consists of an initial value problem where there are two constant states separated by a jump discontinuity. An initial value problem is an ordinary differential equation with a point defined in the domain, and a jump discontinuity occurs when the limit of the function is not equal on both sides of the point. There are many different types of Riemann solvers. A few of the more notable ones include Godunov's exact solver, Roe solvers, HLL solvers, and f-wave solvers [14]. All of these make use of the semi-analytic solution around a point. The point, i, is found by using the averaged value of semi-analytic solution at points $i - \frac{1}{2}$ and $i + \frac{1}{2}$ [14]. Since the Riemann solver allows for discontinuous states I am able to implement discontinuous topography and have discontinuous solutions, and thus I used a Riemann solver for this thesis.

CLAWPACK, a package originally developed by Randall J. LeVeque in 1994, is a partial differential equation solving software package that implements four different Riemann solvers for the shallow water equations. The f-wave solver with bathymetry is the solver that I used because it allows for variable bathymetry unlike the other three shallow water Riemann solvers in CLAWPACK. The f-wave method is described in [15] as a high-resolution wave-propagation algorithm in which waves are based directly on a decomposition of flux differences $f_i(Q_i) - f_{i-1}(Q_{i-1})$ into eigenvectors of an approximate Jacobian matrix, where $f(q)$ is a spatially discretized flux function. Using a Riemann solver like the f-wave method is particularly advantageous because I will need to implement step functions along with step initial conditions, which are not continuous, and Riemann solvers can handle discontinuous states and topography. Also, shock waves by definition are discontinuous features so, by using a Riemann solver I was able to create solutions that are accurate, and close to what happens in nature. CLAWPACK also allows for custom boundary conditions, which I will need to use to create inflow conditions.

As discussed earlier, the topographies that are of interest need to be simplified in order to have an easily understood mathematical function approximate the topography. The mathematical function that I decided on has an obvious way to make a slight change to the topography to find bifurcations, and that slight change is very easy to iterate. The two functions that reasonably approximate most topographies are steps and linear slopes. Many objects have a structure that, in the right reference frame, is nearly linear. The step function represents any sort of abrupt drop in the topography, whether that is a completely vertical drop or an extremely steep slope function. The step function can be applied to weirs, ledges in a river, the end of a spillway, Levee walls, etc. Waterfalls and waterfall-like objects may have a step topography, but they would require two free surfaces instead of one, which cannot be done with my model. The linear slope function approximates any sort of drop over space. Linear slopes can be applied to the shoaling problem, a gradient in a river, any sort of rock or object that has a slanted but relatively unchanging surface, spillway ramps that connect to the bottom of the river, etc. The way those two topographies can be varied is also quite simple. For the linear function the slope is the parameter that can be varied, and for the step function the step size is varied. Slowly changing the slope or step size of the topography will cause a change in the flow velocity after the topography due to the system gaining more or less kinetic energy. When the change is large enough a different wave will form, and this is a bifurcation.

The boundary conditions for my system are constructed in a way that produced an accurate representation of the desired system. For my problem I am assuming the inflow is constant in time since in the and the outflow lets waves pass without reflection. Since the topogrpahy I am using is localized and I am not placing it near the boundaries the resulting shockwaves are not touching the boundaries, and thus the outflow condition is just zero-order extrapolation. Zero order extrapolation is a method that mirrors the system then takes the value from the nearest point and uses that value as the value on the boundary. Ghost cells are values that are added on to the edge of the domain, used for computation, and reset at each iteration to create the effect of what a boundary condition would be doing. The inflow condition is created by just setting the ghost cells on the inflow side to be a fixed value. The inflow and outflow conditions are constructed as described in LeVeque's Finite Volume Methods book [11].

The initial conditions also had a restriction placed upon them. The initial velocity condition cannot have velocities that cause the system to flow towards the inflow boundary condition. The initial height cannot be at or below the topography because that either produces a trivial solution or a solution that arises from impossible to create conditions. There were also some numerical issues that occurred when the initial velocity was 2.5 units or more above the highest point in the topography. If the initial height was also around 2.5 units or more above the higher point in the topography then making a small iteration made no change in the resulting system, and thus making it pointless to test any more values in that range. All of those restrictions on the initial height placed the parameter range in $[0.1, 3.0]$ units above the topography, and the initial velocity is restricted between $[0.1, 2.5]$ as well. Iterating these initial conditions is fairly straightforward as well since the only parameter that can be changed in the magnitude. So, I picked a set of values in each parameter ranges that closely explored the ranges that cause a larger change, and more loosely explored the ranges with larger values since those caused less change.

I also needed to implement a way of detecting which type of wave occurred and when the system bifurcated. The method I decided on using is an entropy based method. Entropy based methods are a robust way of detecting if shocks occur, and this is done by tracking the entropy of a system as explained in [10]. How entropy is represented in my system is described in the background section of this thesis. In [10] they are using a different set of equations and as such I cannot directly use their result relating entropy dissipation to shock speed; however, their results show that by just looking at the entropy dissipation shocks can be detected. By directly calculating entropy and subtracting out the change in entropy coming in at the boundaries of the system, due to water flowing out, one can identify if a shock occurred and the size or speed of a shock. Putting the 3 necessary pieces together for calculating the entropy loss can be seen in equation 3 where D is the domain and ∂D is the boundary.

$$\int_D \text{entropy} - \int_{\partial \mathrm{D}} \text{entropy flux} - \int_D \text{initial conditions} \tag{3}$$

Since entropy is the summation of kinetic and potential we have entropy $= \frac{1}{2}mu^2 + mgh$. The entropy equation cannot be directly computed since we do not know what m is; however, the mass of water in one dimension is proportional to the height $m \propto ch$ where c is a proportionality constant. Plugging in the value for m gives, entropy $= c(\frac{1}{2}hu^2 + gh^2)$ and entropy flux $= \frac{\partial}{\partial t}c(\frac{1}{2}hu^2 + gh^2) = c(\frac{1}{2}\frac{dh}{dt}u^2 + hu\frac{du}{dt} + 2gh\frac{dh}{dt})$. Now that we have all the necessary info equation 3 can be updated.

$$\int_D c(\frac{1}{2}hu^2 + gh^2) - \int_{\partial D} c(\frac{1}{2}\frac{dh}{dt}u^2 + hu\frac{du}{dt} + 2gh\frac{dh}{dt}) - \int_D \text{initial conditions} \quad (4)$$

All of equation 4 can directly be computed from data except for the derivatives in the entropy flux piece, which will be computed from a simple differencing scheme. These equations can then be used to plot an entropy dissipation curve with a given set of parameters for initial height, initial velocity, and slope. The critical points on an entropy plot indicate a change in the system, which also indicates bifurcations. The critical points points of an entropy dissipation curve for a given set of parameters for initial height, initial velocity, and slope will indicate bifurcations.

6 Results and discussion

The method of using entropy to identify when shocks occur and the size of the shock was successful since the entropy loss plots helped identify bifurcations. By varying the topography slope in the range of $[\frac{1}{16}, 6]$, $h_0 \in [0.65, 3]$, $u_0 \in [0.25, 3]$, and topography step $\in [0.1, 1.5]$, I was able to identify where bifurcations occurred and the ranges for which each phenomenon would occur within the ranges I tested. There were 5 different outcomes that occurred in the system. The first outcome was where the system has relatively high energy, and for linear topography this is when the initial height and velocity is $h_0 \in (2, 3)$ and $u_o \in (1.75, 2.5)$ respectively, and for step topography $h_0 \in (1.0, 3.0)$ and $u_0 \in (.75, 2.5)$. The surface profile that is created in this case mirrors the non-constant piece of the topography; this is what I have been referring to as a standing wave. Where mirroring is referring to a function of the form, $f(x) = -f(x) + c$. The second phenomenon occurred in linear topography and step topography systems, and is a shock wave that occurred some distance after the start of the changing piece of the topography, and this wave is the breaking wave. Third, a shock occurred that had a vertical piece, a sloped piece, was steady, and was between the start and end of the sloped piece of topography (see figure 8 or 9 for an example), and did not occur in step topography systems. The fourth phenomenon was a rarefaction wave, and moved outside of the observed domain relatively quickly (approx between 0.1 and 10 time units). Rarefaction waves in these systems are the only non-steady wave because the shock piece is moving in the positive $x-$direction. The last phenomenon is the system becoming numerically unstable due to initial velocity being relatively high, which is approximately $u_o = 3.0$ or larger, or a dry state occurring.

Plots of the simulations I ran have a few distinct pieces. The first is the orange curve which represents velocity, and a positive velocity indicates flow going to the right or positive the x direction. The second is the blue curve which represents the fluid height above $z = 0$. The third piece is the topography, which is represented by the green curve. And the last piece is the title which lists the initial conditions to the system. The shocks can be identified by looking for a significant change in the blue curve, and this usually appears slightly after the point $x = 0$ due to this being the point when the bottom topography starts changing. In figure 2, the shock occurs at the point $x = 0.5$ as indicated by the vertical piece on the fluid height curve.

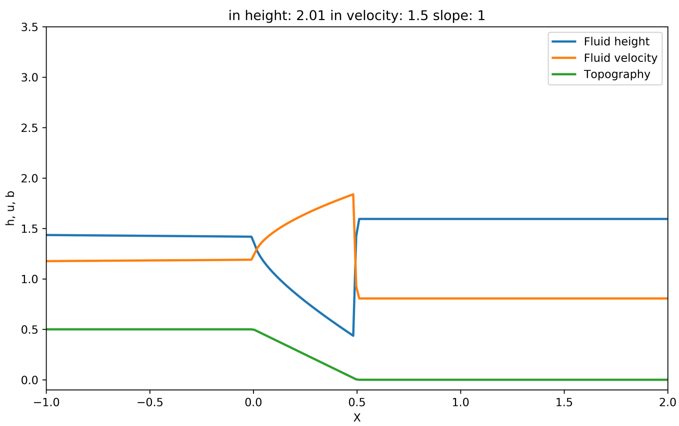


Figure 2: Clean shock example

All phenomena except for rarefaction waves are steady, and can be seen clearly on a contour plot. A system is steady on a contour plot if the function does not change in x as the time axis, t, increases. After around $t = 70$ on figure 3 the solution becomes steady. On

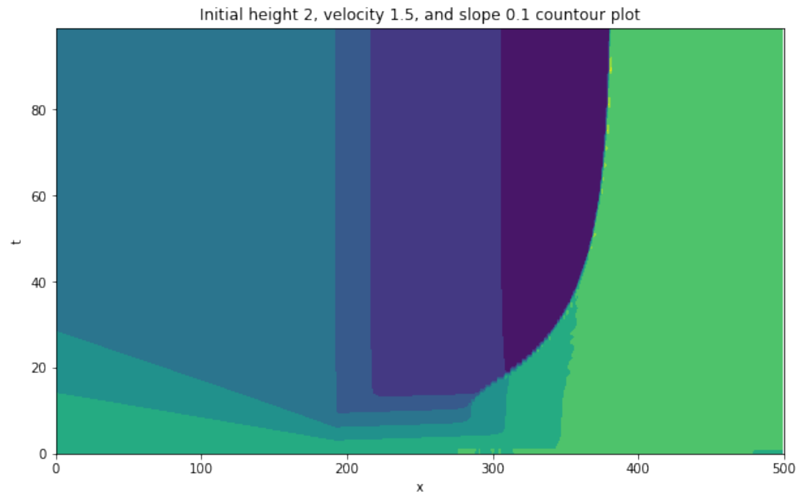


Figure 3: Contour plot of the simulation from figure 2

Validating the inflow condition can be done by testing to see if there is no inflow when the initial height starts out constant, and validating the outflow can be done by just checking to see if waves pass without reflecting. For shallow water systems a cosine wave initial condition for height, $h_0 = cos(x)$ when $x = [\frac{-pi}{2}, \frac{pi}{2}]$, with strictly positive initial velocity, $u_0 > 0$ will produce a wave that propagates in the positive $x-$direction until it reaches the boundaries. Figure 4 shows a wave that propagates until it reaches the boundary, and then leaves the system, as expected. Figure 5 is a simulation where the initial conditions, and topography are constant, and produces a constant plot for all time, as expected.

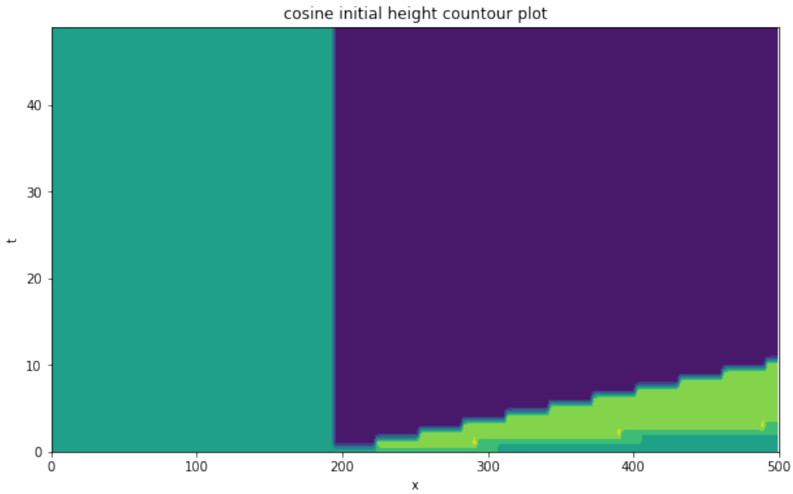


Figure 4: Contour plot of a cosine initial height

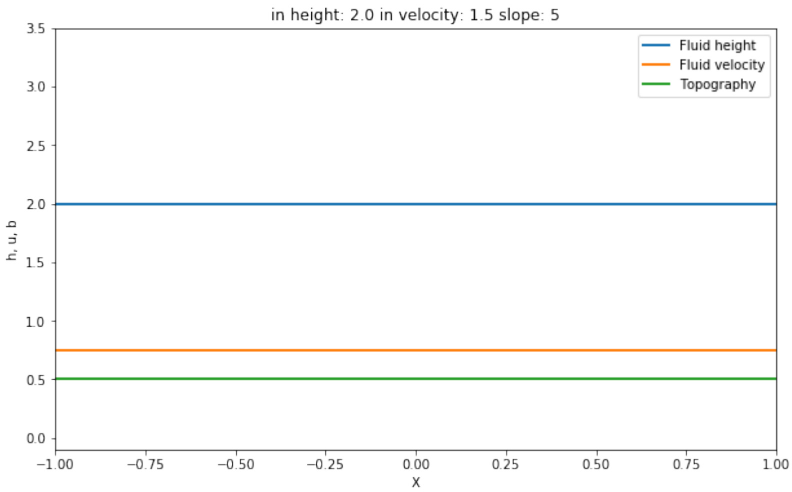


Figure 5: Constant topography and initial condition simulation

If one of the parameters (height, velocity, slope, topography) changes enough we expect a bifurcation to occur, and we can also expect the entropy loss to be different. By using an entropy vs one of the parameters plot one could identify if and when this bifurcation occurs. To identify a bifurcation on an entropy vs slope plot one needs to look for critical points because these points indicate a change in entropy loss, which would mean a different phenomenon could be occurring. However, critical points do not guarantee a bifurcation due to potential numerical error. With initial height 2.3 and initial velocity 1.5 we can see the entropy plot (figure 6) is indicating upwards of four bifurcations.

On figure 6 there are a few points that could indicate bifurcations. The y-axis on the figure is the relative entropy loss, and this means at the value $y = -1$ the largest amount of entropy was lost. Anything above the $y = -1$ point had an entropy loss of the y-value relative to the maximum. Since the structure of figure 6 is fairly jagged there are quite a few possible points that could be bifurcations. Although, since a few of the critical points have another critical point very close by, and might only be a critical point due to numerical error, the list can be reduced. There are most likely only a few points that will produce a bifurcations, and it is likely that there is just one critical point in the range slope $= [0.5, 0.75]$, and slope $= [1, 1.5]$. Looking at the solution between those points we can see some different behaviour. Figure 7 shows a significant shock with a small sloped piece. Figure 8 shows a shock and a strange spike before the linear piece, and figure 9 shows a much smaller shock with a much longer sloped piece.

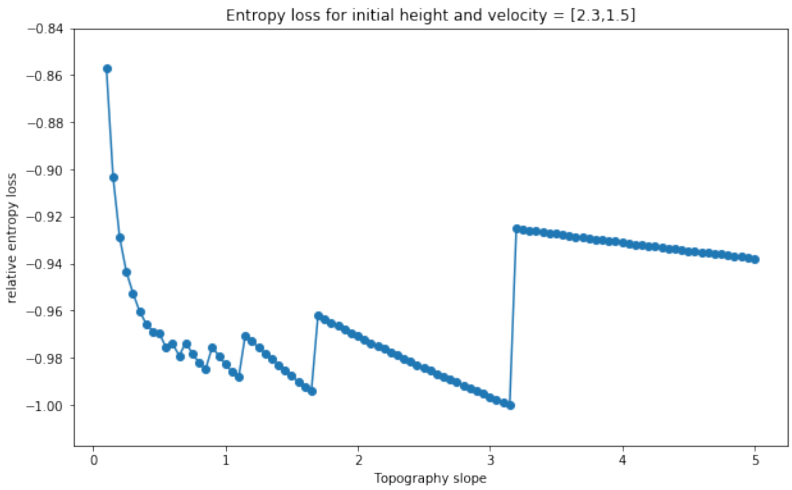


Figure 6: Entropy plot that shows a few bifurcations

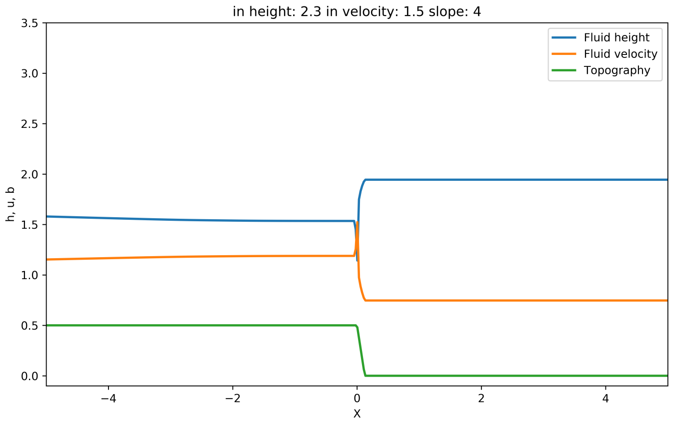


Figure 7: Shock with a small linear piece

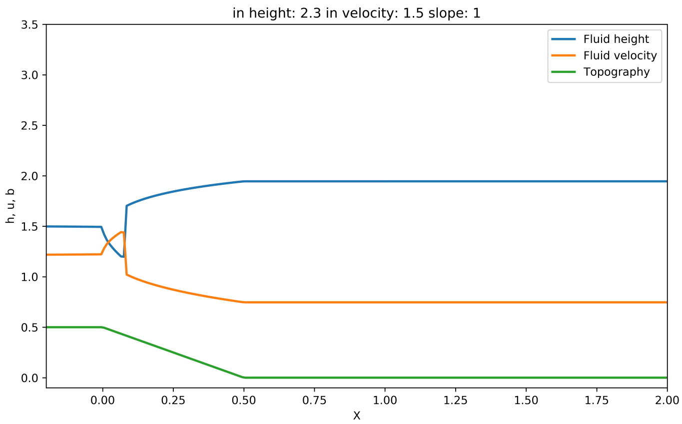


Figure 8: Above surface shock

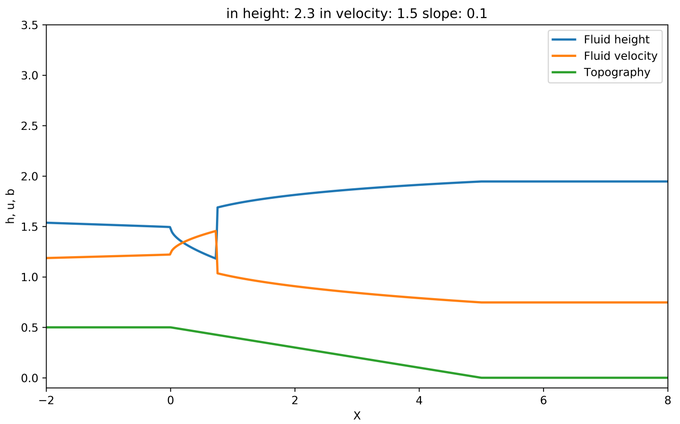
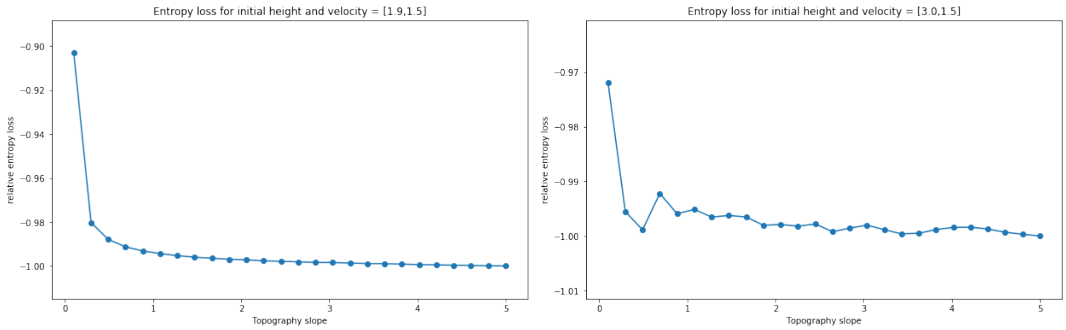


Figure 9: Clean above surface shock

We can also expect to see bifurcations occurring when we change initial height or initial velocity, and these bifurcations can be identified by looking for when the structure of the plot changes. For example, a smooth exponential-like curve to a jagged curve. Changing the initial height and keeping initial velocity and slope the same also produces bifurcations, which can be seen by the radically different structures on figure 10 compared to figure 6.



(a) Decreased height entropy plot (b) Increased height entropy plot

Figure 10: Increased and decreased initial height entropy plots

Looking at figures 11, 12, and 13 we can see three distinct structures which means there are bifurcation points between each of the initial heights.

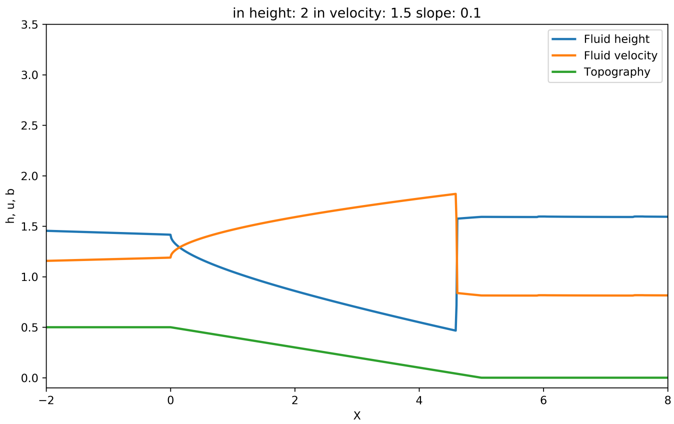


Figure 11: Clean shock wave

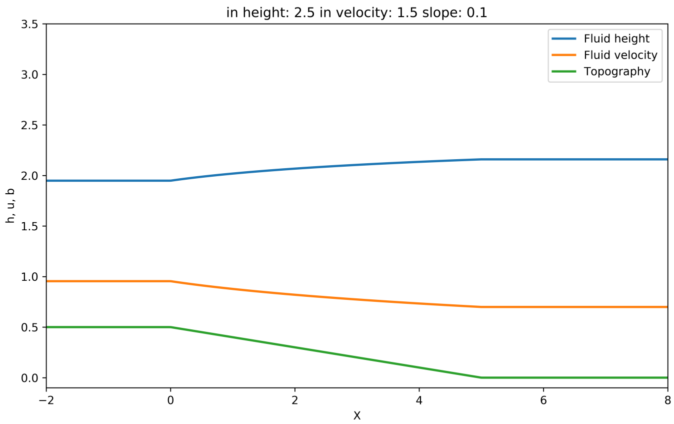


Figure 12: Standing wave

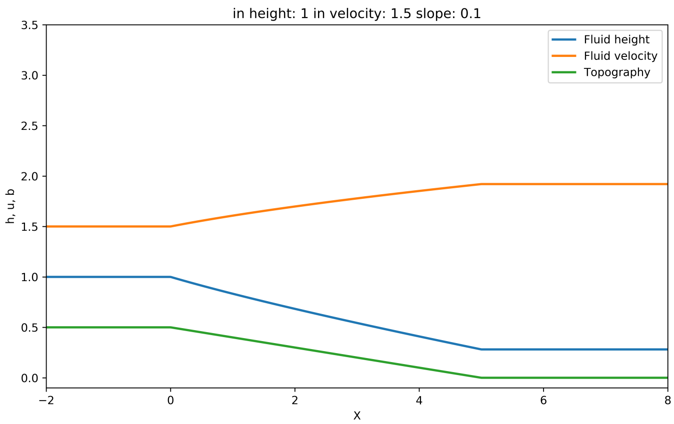


Figure 13: Rarefaction wave

In order to see the full structure of when bifurcations occur we need to look at a bifurcation diagram with the three bifurcation parameters but, the domain for that would be three-dimensional. We need to look at either a projection or just fix a value. Looking at a fixed value is a lot easier to understand since many projections make the axis very complicated. Fixing the initial velocity parameter makes the most sense because velocity tends to work similarly on the system as initial height, and a slope vs initial height bifurcation diagram can show all of the possible desired states. Figure 14 and 16 were created by using the entropy loss plots to give an approximation for when the system bifurcated, then manually checking around a bifurcation to figure out what phenomenon occurred (identifying which phenomenon occurred is explained in the background section), and to more precisely identify the parameter values that cause the system to bifurcate. The dots on figures 14 and 16 are the points that I manually tested.

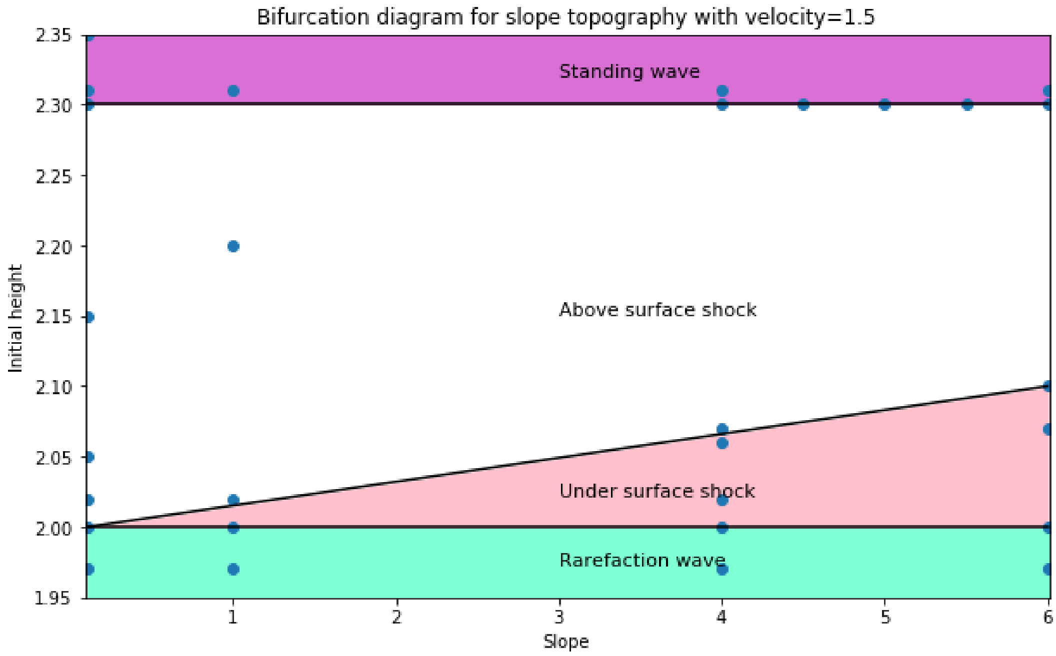


Figure 14: Bifurcation plot with initial velocity fixed at 1.5

Figure 14 shows that there are some very clear regions where specific phenomenon occur, and thus specific parameter ranges for each type of phenomenon can be written as some basic equations. Bifurcation diagrams can be made for other velocity values, and those diagrams have a the same ordering for where the regions are placed.

Looking at the entropy loss values for the domain in figure 14 roughly shows how the entropy plots can be used as an indicator for bifurcations. In figure 15 the white lines are taken from the regions in figure 14. The regions for rarefaction waves, standing waves, and shock waves are very distinct; although, the differences in entropy loss between the two types of shock waves are not noticeable for this set of parameters.

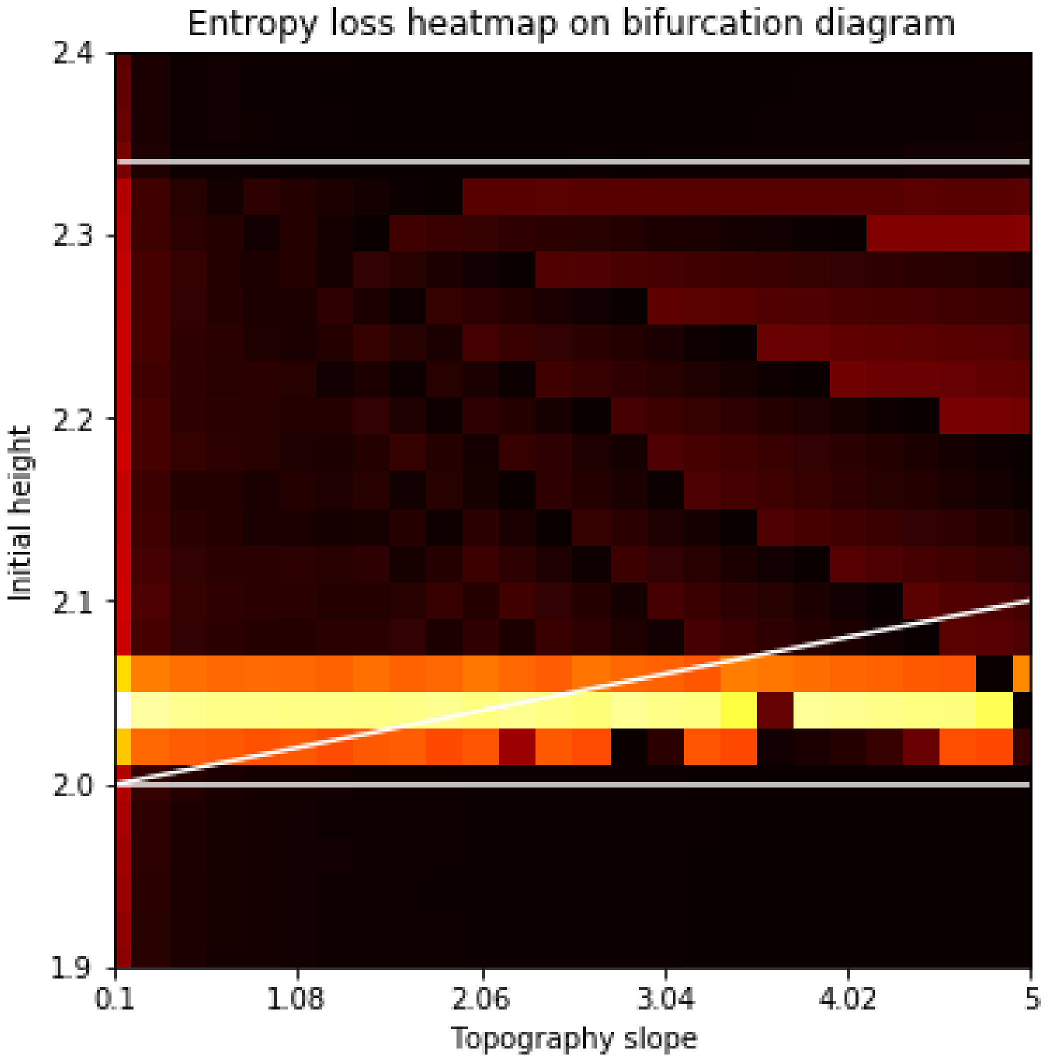


Figure 15: Entropy loss heatmap for figure 14

When using a step topography instead of a sloped topography the phenomena were reduced to rarefaction waves, shocks, and standing waves. Overall, the step topography had similar results to the slope topography since the rarefaction waves occurred when the system had smaller initial heights, standing waves occurred when the system had higher initial heights, and shocks were in-between. Figure 16 shows the bifurcation diagram for step topography with the initial velocity fixed at 1.0.

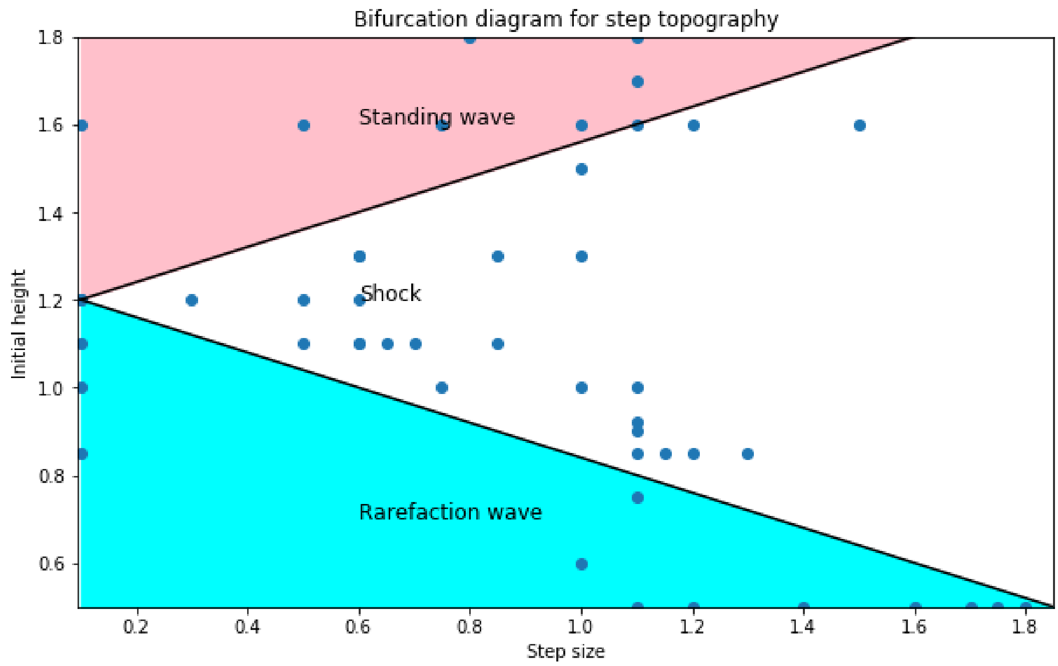


Figure 16: Bifurcation plot with initial velocity fixed at 1.0

7 Conclusions

The knowledge gained from free surface flows has been studied and used in many different applications, and each system has its own set of complexities involved. The research done on shallow water systems with non-flat topography exists but, was lacking a close analysis on bifurcations caused by changing the topography or initial conditions. With a proper model and parameters the system can be explored and analyzed to gain insights on the parameter ranges that create each type of phenomenon.

While my model is not perfect, and there are many types of topographies that I did not do analysis on I was able to approach some of the rudimentary system that can then be built upon in the future. My approach was able to build an understanding of what types of surface profile can occur, show how entropy can be used as an indicator of bifurcations, and create a multi-parameter bifurcation plot showing where each type of phenomenon occurred.

7.1 Future work

My implementation of tracking entropy has some bug that is causing the entropy loss to be much larger then it should be. The issues with entropy did not prevent results but, it made the data less accurate especially for step topography, and for this reason there is no entropy loss heatmap for the step topography bifurcation diagram. Obtaining experimental data for this problem would help build another layer to the analysis, and can be done without constructing new experiments since there are some very controlled systems out there like the Boise Whitewater Park. Lastly, modeling more complicated topography will increase the range of applicability of this topic.

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