DS 592: Introduction to Sequential Decision Making

Lecturer: Aldo Pacchiano | Homework 1
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Problem 1

The point of this exercise is to understand independence more deeply. Solve the following problems:

(a) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Show that $\emptyset$ and Ω (which are events) are independent of any other event. What is the intuitive meaning of this?

(b) Continuing the previous part, show that any event $A \in \mathcal{F}$ with $\mathbb{P}(A) \in \{0, 1\}$ is independent of any other event.

(c) What can we conclude about an event $A \in \mathcal{F}$ that is independent of its complement, $A^c = \Omega \setminus A$? Does your conclusion make intuitive sense?

(d) What can we conclude about an event $A \in \mathcal{F}$ that is independent of itself? Does your conclusion make intuitive sense?

(e) Consider the probability space generated by two independent flips of unbiased coins with the smallest possible σ-algebra. Enumerate all pairs of events A, B such that A and B are independent of each other.

(f) Consider the probability space generated by the independent rolls of two unbiased three-sided dice. Call the possible outcomes of the individual dice rolls 1, 2, and 3. Let X_i be the random variable that corresponds to the outcome of the ith dice roll ($i \in \{1, 2\}$). Show that the events $\{X_1 \leq 2\}$ and $\{X_1 = X_2\}$ are independent of each other.

Problem 2

Let $X_0 = 0$ and for $j \geq 0$ let X_{j+1} be chosen uniformly over the real interval $[X_j, 1]$. Show that, for $k \geq 0$, the sequence

$$Y_k = 2^k(1 - X_k)$$

is a martingale.

Problem 3

Suppose that we independently roll two standard six-sided dice. Let X_1 be the number that shows on the first die, X_2 the number on the second die, and X the sum of the numbers on the two dice.

(a) What is $\mathbb{E}[X \mid X_1 \text{ is even}]$?

(b) What is $\mathbb{E}[X \mid X_1 = X_2]$?

(c) What is $\mathbb{E}[X_1 \mid X = 9]$?

(d) What is $\mathbb{E}[X_1 - X_2 \mid X = k]$ for k in the range $[2, 12]$?

Programming assignment

Feel free to use any programming language of your choice for this exercise.

Consider the following 1 small, fully connected neural network. We initiate a random walk within this network, beginning at either node 11 or node 12. At each node, there is an equal probability of proceeding to any of its adjacent nodes. It is important to note that the graph is directed. The task is to ascertain the probability of terminating the random walk at each of the final nodes 41, 42, and 43 for various distributions of initial starting points. Denote X_0 as the starting point of the random walk and $\pi_0 = [\mathbb{P}(X_0 = \{11\}), \mathbb{P}(X_0 = \{12\})]$ and X_4 as the endpoints, termed $\pi_4 = [\mathbb{P}(X_4 = \{41\}), \mathbb{P}(X_4 = \{42\}), \mathbb{P}(X_4 = \{43\})]$. You are required to calculate π_4 for the following π_0:

1. $\pi_0 = [1/2, 1/2]$.
2. $\pi_0 = [0.4, 0.6]$.
3. $\pi_0 = [0.1, 0.9]$.

Bonus

Validate your findings by employing Monte Carlo simulations. In particular, execute a high volume of random walks to confirm that the empirical distribution $\hat{\pi}_4$ aligns closely with the distribution derived in the initial query. You may choose to present either the computed $\hat{\pi}_4$ or graph the change in $\|\pi_4 - \hat{\pi}_4\|_2$ as the quantity of samples varies.

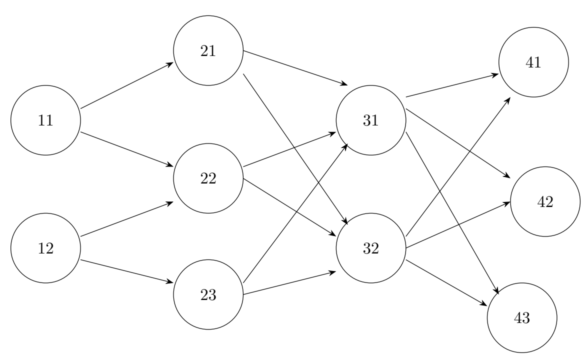

Figure 1: