DS 592 Introduction to Sequential Decision Making **Homework #1**
Spring 2025 Due:

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Definitions

i. **Statement:** $\mathbb{P}(\emptyset) = 0$
`https://statproofbook.github.io/P/prob-emp`

ii. **Statement:** $(A \cap B) \cup (A^c \cap B) = B$

Proof. Let A and B be events in $\mathcal{F}$, so

$$\begin{aligned}
&1. \quad (A \cap B) \cup (A^c \cap B) = B \cap (A \cup A^c) && \text{(Distributive Law)}\\
&2. \quad = B \cap \Omega && \text{(Complement Law)}\\
&3. \quad = B && \text{(Identity Law)}
\end{aligned}$$

$$\therefore \quad (A \cap B) \cup (A^c \cap B) = B$$

Therefore, $(A \cap B) \cup (A^c \cap B) = B$ □

iii. **Statement:** $\forall A \in \mathcal{F} \land \mathbb{P}(A) = 0 \implies \forall B \in \mathcal{F}, \mathbb{P}(A \cap B) = 0$
`https://statproofbook.github.io/P/prob-mon.html`

iv. **Statement:** $\forall A, B \in \mathcal{F}, \mathbb{P}(A \cap B) = \mathbb{P}(B) - \mathbb{P}(A^c \cap B)$

v. **Statement:** $\forall A \in F, \mathbb{P}(A) = 1 - \mathbb{P}(A^c)$
`https://statproofbook.github.io/P/prob-comp.html`

Question 1

(a) We need to prove that events $\emptyset, \Omega \in \mathcal{F}$ are independent of any other event.

Proof. Assume that the event $\emptyset \in \mathcal{F}$ is not independent of some event A

$$\begin{aligned}
&1. \quad \exists A \in \mathcal{F}, \mathbb{P}(\emptyset \cap A) \neq \mathbb{P}(\emptyset) \cdot \mathbb{P}(A) \equiv \exists A \in \mathcal{F}, \mathbb{P}(\emptyset) \neq \mathbb{P}(\emptyset) \cdot \mathbb{P}(A) && \text{(Null Law)}\\
&2. \quad \equiv \exists A \in \mathcal{F}, 0 \neq 0 \cdot \mathbb{P}(A) && (\mathbb{P}(\emptyset) = 0)\\
&3. \quad \equiv 0 \neq 0 && (0 \cdot \text{a} = 0)
\end{aligned}$$

$$\therefore \quad \exists A \in \mathcal{F}, \mathbb{P}(\emptyset \cap A) \neq \mathbb{P}(\emptyset) \cdot \mathbb{P}(A) \equiv 0 \neq 0.$$

Since $0 \neq 0$ is a false statement, the initial assumption was wrong.
Therefore, $\emptyset$ is independent of any other event. □

Proof. Assume that the event $\Omega \in \mathcal{F}$ is not independent of some event A.

$$
\begin{array}{llr}
1. & \exists A \in \mathcal{F}, \mathbb{P}(\Omega \cap A) \neq \mathbb{P}(\Omega) \cdot \mathbb{P}(A) \equiv \exists A \in \mathcal{F}, \mathbb{P}(A) \neq \mathbb{P}(\Omega) \cdot \mathbb{P}(A) & \text{(Identity Law)} \\
2. & \equiv \exists A \in \mathcal{F}, \mathbb{P}(A) \neq 1 \cdot \mathbb{P}(A) & \text{(Second Axiom of Probability)} \\
3. & \equiv \exists A \in \mathcal{F}, \mathbb{P}(A) \neq \mathbb{P}(A) & (1 \cdot \mathrm{a} = \mathrm{a})
\end{array}
$$

$$\therefore \exists A \in \mathcal{F}, \mathbb{P}(\Omega \cap A) \neq \mathbb{P}(\Omega) \cdot \mathbb{P}(A) \equiv \exists A \in \mathcal{F}, \mathbb{P}(A) \neq \mathbb{P}(A)$$

Since $\mathbb{P}(A) \neq \mathbb{P}(A)$ is a false statement, the initial assumption was wrong.

Therefore, Ω is independent of any other event. □

(b) We need to prove that any event $A \in \mathcal{F}$ with $\mathbb{P}(A) \in \{0, 1\}$ is independent of any other event.

Proof. Assume that there exists a pair of events $A, B \in \mathcal{F}$ with $\mathbb{P}(A) \in \{0, 1\}$ that are not independent of each other.

Case $\mathbb{P}(A) = 0$**:**

$$
\begin{array}{llr}
1. & \exists A, B \in \mathcal{F}, \mathbb{P}(A \cap B) \neq \mathbb{P}(A) \cdot \mathbb{P}(B) \equiv \exists A, B \in \mathcal{F}, 0 \neq \mathbb{P}(A) \cdot \mathbb{P}(B) & (\text{Statement}_{iii}) \\
2. & \equiv \exists B \in \mathcal{F}, 0 \neq 0 \cdot \mathbb{P}(B) & (\text{Statement}_{i}) \\
3. & \equiv 0 \neq 0 & (0 \cdot \mathrm{a} = 0)
\end{array}
$$

$$\therefore \quad \exists A, B \in \mathcal{F}, \mathbb{P}(A \cap B) \neq \mathbb{P}(A) \cdot \mathbb{P}(B) \equiv 0 \neq 0$$

Case $\mathbb{P}(A) = 1$**:**

$$
\begin{array}{llr}
1. & \exists A, B \in \mathcal{F}, \mathbb{P}(A \cap B) \neq \mathbb{P}(A) \cdot \mathbb{P}(B) \equiv \exists A, B \in \mathcal{F}, \mathbb{P}(B) - \mathbb{P}(A^c \cap B) \neq \mathbb{P}(A) \cdot \mathbb{P}(B) & (\text{Statement}_{iv}) \\
2. & \equiv \exists A, B \in \mathcal{F}, \mathbb{P}(B) - 0 \neq \mathbb{P}(A) \cdot \mathbb{P}(B) & (\text{Statement}_{iii}) \\
3. & \equiv \exists A, B \in \mathcal{F}, \mathbb{P}(B) \neq \mathbb{P}(A) \cdot \mathbb{P}(B) & \text{(Simplification)} \\
4. & \equiv \exists B \in \mathcal{F}, \mathbb{P}(B) \neq 1 \cdot \mathbb{P}(B) & (\mathbb{P}(A) = 1) \\
5. & \equiv \exists B \in \mathcal{F}, \mathbb{P}(B) \neq \mathbb{P}(B) & \text{(Simplification)}
\end{array}
$$

$$\therefore \quad \exists A, B \in \mathcal{F}, \mathbb{P}(A \cap B) \neq \mathbb{P}(A) \cdot \mathbb{P}(B) \equiv \mathbb{P}(B) \neq \mathbb{P}(B)$$

Since when $\mathbb{P}(A) = 0$ the conclusion $0 \neq 0$ is a false statement and when $\mathbb{P}(A) = 1$ the conclusion $\mathbb{P}(B) \neq \mathbb{P}(B)$ is a false statement, the initial assumption was wrong.

Therefore, if $\mathbb{P}(A) \in \{0, 1\}$, the event $A \in \mathcal{F}$ is independent of any other event in the sample space. □

(c) We need to find extra information about an event $A \in \mathcal{F}$ that is independent of its complement.

$$
\begin{array}{lrr}
1. & \mathbb{P}(A \cap A^c) = \mathbb{P}(A) \cdot \mathbb{P}(A^c) & \text{(Given)} \\
2. & \mathbb{P}(\emptyset) = \mathbb{P}(A) \cdot \mathbb{P}(A^c) & \text{(Complement Law)} \\
3. & 0 = \mathbb{P}(A) \cdot \mathbb{P}(A^c) & (\text{Statement}_{i}) \\
4. & 0 = \mathbb{P}(A) \cdot \Big(1 - \mathbb{P}(A)\Big) & (\text{Statement}_{v}) \\
5. & 0 = \mathbb{P}(A) - \mathbb{P}^2(A) & \text{(Distribute product)} \\
6. & 0 = x - x^2 & (x = P(A)) \\
7. & x - x^2 = 0 &
\end{array}
$$

If we solve for x; $x_1 = 0, x_2 = 1$.

If $x \in \{0, 1\}$ and $x = \mathbb{P}(A)$, $\mathbb{P}(A) \in \{0, 1\}$.

Therefore, as proved in the previous question, if $\mathbb{P}(A) \in \{0, 1\}$ event A is independent of any other event, which also applies here.

An event that is independent of its complement must also be independent of any other event.

(d) We need to find extra information about an event $A \in \mathcal{F}$ that is independent of itself.

1.	$\mathbb{P}(A \cap A) = \mathbb{P}(A) \cdot \mathbb{P}(A)$	(Given)
2.	$\mathbb{P}(A) = \mathbb{P}(A) \cdot \mathbb{P}(A)$	(Idempotent Law)
3.	$\mathbb{P}(A) = \mathbb{P}^2(A)$	(Rewriting)
4.	$x = x^2$	($x = \mathbb{P}(A)$)
5.	$x - x^2 = 0$	

If we solve for x; $x_1 = 0, x_2 = 1$

If $x \in \{0, 1\}$ and $x = \mathbb{P}(A)$, $\mathbb{P}(A) \in \{0, 1\}$.

Therefore, as proved in the previous question, if $\mathbb{P}(A) \in \{0, 1\}$ event A is independent of any other event, which also applies here.

An event that is independent of itself must also be independent of any other event.

(e) X_1 - Heads/Tails (H / T) on the first coin flip.

X_2 - Heads/Tails (H / T) on the second coin flip.

$$\Omega = \Big\{HH, HT, TH, TT\Big\}$$

$$\mathcal{F} = \left\{\begin{matrix} \emptyset & \Omega & \{HH\}_A & \{HT\}_B \\ \{TH\}_C & \{TT\}_D & \{HH, HT\}_E & \{HH, TH\}_F \\ \{HH, TT\}_G & \{HT, TH\}_H & \{HT, TT\}_I & \{TH, TT\}_J \\ \{HH, HT, TH\}_K & \{HH, HT, TT\}_L & \{HH, TT, TH\}_M & \{HT, TH, TT\}_N \end{matrix}\right\}$$

The letters in the subscript correspond to the name of that event.

We need to find all pairs that are independent.

First, $\forall A \in \mathcal{F}, A \perp\!\!\!\perp \emptyset$, therefore any pair of events where one of the elements is the $\emptyset$ satisfies the answer.

Analogous, $\forall A \in, A \perp\!\!\!\perp \Omega$, therefore any pair of events where one of the elements is the $\emptyset$ satisfies the answer.

All sets with 1 element have a probability of $\frac{1}{4}$

All sets with 2 elements have a probability of $\frac{1}{2}$

All sets with 3 elements have a probability of $\frac{3}{4}$

Let $Z = \left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right\}$

Now, let $x_1, x_2, x_3 \in Z$.

There is only 1 combination of x_1, x_2, x_3 that makes the statement $x_1 \cdot x_2 = x_3$ true. The values are $x_1 = x_2 = \frac{1}{2}$ and $x_3 = \frac{1}{4}$.

Which also implies that the only way to have a pair of events be independent of each other, is for them to have the above said probabilities, since otherwise it is mathematically impossible.

Therefore, if a pair contains a set of length 1 or 3, that pair doesn't satisfy the answer.

Now we are left with sets of length 2, we also know that the intersection must have a probability of $\frac{1}{4}$ which is unique to sets of length 1, therefore, the pair is suitable only if the intersection of the events results in a set of length 1.

This leaves us with the following pairs:

1) (E , F)
2) (E , G)
3) (E , H)
4) (E , I)
5) (F , G)
6) (F , H)

7) (F , J)
8) (G , I)
9) (G , J)
10) (H , I)
11) (H , J)
12) (I , J)

Therefore, the only pairs that would be independent are the ones mentioned above and all possible pairs of events given one of the element is $\emptyset$ or Ω.

(f) X_1 - outcome of independent roll of an unbiased three-sided die.

X_2 - outcome of independent roll of an unbiased three-sided die.

$$\Omega = \left\{ \begin{array}{ccccc} & 1 & 2 & 3 & X_2 \\ 1 & (1,1) & (1,2) & (1,3) & \\ 2 & (2,1) & (2,2) & (2,3) & \\ 3 & (3,1) & (3,3) & (3,3) & \\ X_1 & & & & \end{array} \right\}$$

We need to show that events $\{X_1 \leq 2\}$ and $\{X_1 = X_2\}$ are independent of each other.

Let $A = \{X_1 \leq 2\}$ and $B = \{X_1 = X_2\}$

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

If we look at Ω we can see the following:

$\mathbb{P}(A) = \frac{2}{3}$

$\mathbb{P}(B) = \frac{1}{3}$

$\mathbb{P}(A \cap B) = \frac{2}{9}$

Therefore,

$$\frac{2}{9} = \frac{2}{3} \cdot \frac{1}{3}$$

$$\frac{2}{9} = \frac{2}{9}$$

Since the equation above is indeed true, then the events must be independent.

Question 2

Given:

Proof. Let $X_0 = 0$ and for $j \geq 0$ let X_{j+1} be chosen uniformly over the real interval $[X_j, 1]$. Then, for $k \geq 0$ Y_k the sequence $Y_k = 2^k \cdot (1 - X_k)$ is a martingale.

$$\forall \{a, b\} \in \mathbb{C}, x \sim Uniform(a, b) \implies \mathbb{E}[x] = \frac{a+b}{2}$$

Therefore,

$$\mathbb{E}[X_{k+1} | \mathcal{F}_k] = \frac{X_k + 1}{2}$$

Adaptedness For a sequence Y_k to be a martingale, the following must be true: A sequence must be $\mathcal{F}_k$ measurable. First, X_k is measurable by $\mathcal{F}_k$, Y_k is also measurable by $\mathcal{F}_k$.

Integrability

For a sequence Y_k to be a martingale, the following must be true: $\mathbb{E}[Y_K] < \infty$

$$\begin{aligned}
&1. && 0 \leq X_k \leq 1 \implies 0 \geq -X_k \geq -1 && \text{(Negate both sides)} \\
&2. && \implies 1 \geq 1 - X_k \geq 0 && \text{(Add 1 to all terms)} \\
&3. && \implies 2^k \geq 2^k \cdot (1 - X_k) \geq 0 && \text{(Multiply by } 2^k\text{)} \\
&4. && \implies 2^k \geq Y_k \geq 0 && \text{(Given)} \\
& \therefore && 0 \leq X_k \leq 1 \implies 2^k \geq Y_k \geq 0 && \text{(Final bound on } Y_k\text{)}
\end{aligned}$$

As we can see, Y_k is bounded by 0 and 2^k, therefore is integrable.

The Martingale Property

For a sequence Y_k to be a martingale, the following must be true: $\mathbb{E}[Y_{k+1}|Y_k] = \mathcal{F}_k$

$$\begin{aligned}
&1. && \mathbb{E}[Y_{k+1}|\mathcal{F}_k] = \mathbb{E}[2^{k+1} \cdot (1 - X_{k+1})|\mathcal{F}_k] && \text{(Given)} \\
&2. && = \mathbb{E}[2^{k+1} - 2^{k+1} \cdot X_{k+1}|\mathcal{F}_k] && \text{(Distribute product)} \\
&3. && = \mathbb{E}[2^{k+1}|\mathcal{F}_k] - \mathbb{E}[2^{k+1} \cdot X_{k+1}|\mathcal{F}_k] && \text{(Linearity of expectation)} \\
&4. && = 2^{k+1} - 2^{k+1} \cdot \mathbb{E}[X_{k+1}|\mathcal{F}_k] && \text{(Constant factor rule)} \\
&5. && = 2^{k+1} - 2^{k+1} \cdot \frac{X_k + 1}{2} && \text{(Given)} \\
&6. && = 2^{k+1} - 2^k \cdot (X_k + 1) && \text{(Simplify fraction)} \\
&7. && = 2^{k+1} - 2^k \cdot X_k - 2^k && \text{(Distribute } 2^k\text{)} \\
&8. && = 2^k \cdot (2 - X_k - 1) && \text{(Factor out } 2^k\text{)} \\
&9. && = 2^k \cdot (1 - X_k) && \text{(Simplify expression)} \\
&10. && = Y_k && (Y_k = 2^k(1 - X_k)) \\
& \therefore && \mathbb{E}[Y_{k+1}|\mathcal{F}_k] = Y_k &&
\end{aligned}$$

Therefore, since all the properties stand true, the sequence Y_k is a martingale. $\square$

Question 3

Given:

X_1 - independent roll of an unbiased six-sided die
X_2 - independent roll of an unbiased six-sided die
$X = X_1 + X_2$

(a) $\mathbb{E}[X|X_1 \text{ is even}]$

$$\mathbb{E}[X|X_1 \text{ is even}]$$

$X = X_1 + X_2$

$$\mathbb{E}[X_1 + X_2|X_1 \text{ is even}]$$

Linearity of expectation.

$$\mathbb{E}[X_1|X_1 \text{ is even}] + \mathbb{E}[X_2|X_1 \text{ is even}]$$

$$\left(\sum_{x=1}^{6} x \cdot \mathbb{P}\Big(X_1 = x|X_1 \text{ is even}\Big)\right) + \left(\sum_{x=1}^{6} x \cdot \mathbb{P}\Big(X_2 = x|X_1 \text{ is even}\Big)\right)$$

$X_1 \perp\!\!\!\perp X_2 \implies \mathbb{P}(X_2|X_1 \text{ is even}) = \mathbb{P}(X_2)$

$$\left(\sum_{x=1}^{6} x \cdot \mathbb{P}\Big(X_1 = x|X_1 \text{ is even}\Big)\right) + \left(\sum_{x=1}^{6} x \cdot \mathbb{P}\Big(X_2 = x\Big)\right)$$

Let A $= X_1$ is even

$$\begin{aligned}
&1. & \mathbb{E}[X|A] &= \mathbb{E}[X_1 + X_2|A] && \text{(Given)}\\
&2. & &= \mathbb{E}[X_1|A] + \mathbb{E}[X_2|A] && \text{(Linearity of Expectation)}\\
&3. & &= \mathbb{E}[X_1|A] + \mathbb{E}[X_2] && (X_2 \perp\!\!\!\perp X_1)\\
&4. & &= \left(\sum_{x=1}^{6} x \cdot \mathbb{P}\Big(X_1 = x|A\Big)\right) && \text{(Definition of Expectation)}\\
& & &+ \left(\sum_{x=1}^{6} x \cdot \mathbb{P}\Big(X_2 = x|A\Big)\right)\\
&5. & &= 1 \cdot \mathbb{P}\Big(X_1 = 1|A\Big) + 2 \cdot \mathbb{P}\Big(X_1 = 2|A\Big)\\
& & &+ 3 \cdot \mathbb{P}\Big(X_1 = 3|A\Big) + 4 \cdot \mathbb{P}\Big(X_1 = 4|A\Big)\\
& & &+ 5 \cdot \mathbb{P}\Big(X_1 = 5|A\Big) + 6 \cdot \mathbb{P}\Big(X_1 = 6|A\Big)\\
& & &+ 1 \cdot \mathbb{P}\Big(X_2 = 1\Big) + 2 \cdot \mathbb{P}\Big(X_2 = 2\Big) + 3 \cdot \mathbb{P}\Big(X_2 = 3\Big)\\
& & &+ 4 \cdot \mathbb{P}\Big(X_2 = 4\Big) + 5 \cdot \mathbb{P}\Big(X_2 = 5\Big) + 6 \cdot \mathbb{P}\Big(X_2 = 6\Big)\\
&6. & &= 1 \cdot 0 + 2 \cdot \frac{1}{3} + 3 \cdot 0 + 4 \cdot \frac{1}{3} + 5 \cdot 0 + 6 \cdot \frac{1}{3} && \text{(Using given probabilities)}\\
& & &+ 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}
\end{aligned}$$

7. $= \frac{4}{6} + \frac{8}{6} + \frac{12}{6} + \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6}$ (Rewriting fractions)

8. $= \frac{4+8+12+1+2+3+4+5+6}{6}$ (Combine numerators)

9. $= \frac{45}{6}$ (Summation)

10. $= \frac{15}{2}$ (Simplify fraction)

11. $= 7.5$ (Final calculation)

$$\therefore \qquad \mathbb{E}[X|X_1 \text{ is even}] = 7.5$$

(b) $\mathbb{E}[X|X_1 = X_2]$

1. $\mathbb{E}[X|X_1 = X_2] = \mathbb{E}[X_1 + X_2|X_1 = X_2]$ (Given)

2. $= \mathbb{E}[X_1|X_1 = X_2] + \mathbb{E}[X_2|X_1 = X_2]$ (Linearity of Expectation)

3. $$= \left(\sum_{x=1}^{6} x \cdot \mathbb{P}\Big(X_1 = x|X_1 = X_2\Big)\right) + \left(\sum_{x=1}^{6} x \cdot \mathbb{P}\Big(X_2 = x|X_1 = X_2\Big)\right)$$ (Definition of Expectation)

4. $$\begin{aligned}
&= 1 \cdot \mathbb{P}\Big(X_1 = 1|X_1 = X_2\Big) + 2 \cdot \mathbb{P}\Big(X_1 = 2|X_1 = X_2\Big) \\
&+ 3 \cdot \mathbb{P}\Big(X_1 = 3|X_1 = X_2\Big) + 4 \cdot \mathbb{P}\Big(X_1 = 4|X_1 = X_2\Big) \\
&+ 5 \cdot \mathbb{P}\Big(X_1 = 5|X_1 = X_2\Big) + 6 \cdot \mathbb{P}\Big(X_1 = 6|X_1 = X_2\Big) \\
&+ 1 \cdot \mathbb{P}\Big(X_2 = 1|X_1 = X_2\Big) + 2 \cdot \mathbb{P}\Big(X_2 = 2|X_1 = X_2\Big) \\
&+ 3 \cdot \mathbb{P}\Big(X_2 = 3|X_1 = X_2\Big) + 4 \cdot \mathbb{P}\Big(X_2 = 4|X_1 = X_2\Big) \\
&+ 5 \cdot \mathbb{P}\Big(X_2 = 5|X_1 = X_2\Big) + 6 \cdot \mathbb{P}\Big(X_2 = 6|X_1 = X_2\Big)
\end{aligned}$$

5. $$\begin{aligned}
&= 1 \cdot \frac{\mathbb{P}\Big(\{X_1 = 1\} \cap \{X_1 = X_2\}\Big)}{\mathbb{P}\Big(X_1 = X_2\Big)} + 2 \cdot \frac{\mathbb{P}\Big(\{X_1 = 2\} \cap \{X_1 = X_2\}\Big)}{\mathbb{P}\Big(X_1 = X_2\Big)} \\
&+ 3 \cdot \frac{\mathbb{P}\Big(\{X_1 = 3\} \cap \{X_1 = X_2\}\Big)}{\mathbb{P}\Big(X_1 = X_2\Big)} + 4 \cdot \frac{\mathbb{P}\Big(\{X_1 = 4\} \cap \{X_1 = X_2\}\Big)}{\mathbb{P}\Big(X_1 = X_2\Big)} \\
&+ 5 \cdot \frac{\mathbb{P}\Big(\{X_1 = 5\} \cap \{X_1 = X_2\}\Big)}{\mathbb{P}\Big(X_1 = X_2\Big)} + 6 \cdot \frac{\mathbb{P}\Big(\{X_1 = 6\} \cap \{X_1 = X_2\}\Big)}{\mathbb{P}\Big(X_1 = X_2\Big)} \\
&+ 1 \cdot \frac{\mathbb{P}\Big(\{X_2 = 1\} \cap \{X_1 = X_2\}\Big)}{\mathbb{P}\Big(X_1 = X_2\Big)} + 2 \cdot \frac{\mathbb{P}\Big(\{X_2 = 2\} \cap \{X_1 = X_2\}\Big)}{\mathbb{P}\Big(X_1 = X_2\Big)} \\
&+ 3 \cdot \frac{\mathbb{P}\Big(\{X_2 = 3\} \cap \{X_1 = X_2\}\Big)}{\mathbb{P}\Big(X_1 = X_2\Big)} + 4 \cdot \frac{\mathbb{P}\Big(\{X_2 = 4\} \cap \{X_1 = X_2\}\Big)}{\mathbb{P}\Big(X_1 = X_2\Big)} \\
&+ 5 \cdot \frac{\mathbb{P}\Big(\{X_2 = 5\} \cap \{X_1 = X_2\}\Big)}{\mathbb{P}\Big(X_1 = X_2\Big)} + 6 \cdot \frac{\mathbb{P}\Big(\{X_2 = 6\} \cap \{X_1 = X_2\}\Big)}{\mathbb{P}\Big(X_1 = X_2\Big)}
\end{aligned}$$

$$
\begin{aligned}
6.\quad & = 1 \cdot \frac{\mathbb{P}\Big(\{X_1 = 1\} \cap \{X_1 = X_2\}\Big)}{\frac{1}{6}} + 2 \cdot \frac{\mathbb{P}\Big(\{X_1 = 2\} \cap \{X_1 = X_2\}\Big)}{\frac{1}{6}} \\
& + 3 \cdot \frac{\mathbb{P}\Big(\{X_1 = 3\} \cap \{X_1 = X_2\}\Big)}{\frac{1}{6}} + 4 \cdot \frac{\mathbb{P}\Big(\{X_1 = 4\} \cap \{X_1 = X_2\}\Big)}{\frac{1}{6}} \\
& + 5 \cdot \frac{\mathbb{P}\Big(\{X_1 = 5\} \cap \{X_1 = X_2\}\Big)}{\frac{1}{6}} + 6 \cdot \frac{\mathbb{P}\Big(\{X_1 = 6\} \cap \{X_1 = X_2\}\Big)}{\frac{1}{6}} \\
& + 1 \cdot \frac{\mathbb{P}\Big(\{X_2 = 1\} \cap \{X_1 = X_2\}\Big)}{\frac{1}{6}} + 2 \cdot \frac{\mathbb{P}\Big(\{X_2 = 2\} \cap \{X_1 = X_2\}\Big)}{\frac{1}{6}} \\
& + 3 \cdot \frac{\mathbb{P}\Big(\{X_2 = 3\} \cap \{X_1 = X_2\}\Big)}{\frac{1}{6}} + 4 \cdot \frac{\mathbb{P}\Big(\{X_2 = 4\} \cap \{X_1 = X_2\}\Big)}{\frac{1}{6}} \\
& + 5 \cdot \frac{\mathbb{P}\Big(\{X_2 = 5\} \cap \{X_1 = X_2\}\Big)}{\frac{1}{6}} + 6 \cdot \frac{\mathbb{P}\Big(\{X_2 = 6\} \cap \{X_1 = X_2\}\Big)}{\frac{1}{6}} \\
7.\quad & = \frac{\frac{1}{36}}{\frac{1}{6}} \cdot (1+2+3+4+5+6+1+2+3+4+5+6) && \text{(Substituting probabilities)} \\
8.\quad & = \frac{1}{6} \cdot 2 \cdot (1+2+3+4+5+6) && \text{(Simplifying fractions)} \\
9.\quad & = \frac{1}{6} \cdot 42 && \text{(Summation)} \\
10.\quad & = 7 && \text{(Final calculation)}
\end{aligned}
$$

$$\therefore \quad \mathbb{E}[X|X_1 = X_2] = 7$$

(c) $\mathbb{E}[X_1|X = 9]$

$$
\begin{aligned}
1.\quad \mathbb{E}[X_1|X = 9] & = \sum_{x=1}^{6} x \cdot \mathbb{P}\Big(X_1 = x|X = 9\Big) && \text{(Given)} \\
2.\quad & = 1 \cdot \mathbb{P}\Big(X_1 = 1|X = 9\Big) + 2 \cdot \mathbb{P}\Big(X_1 = 2|X = 9\Big) \\
& + 3 \cdot \mathbb{P}\Big(X_1 = 3|X = 9\Big) + 4 \cdot \mathbb{P}\Big(X_1 = 4|X = 9\Big) \\
& + 5 \cdot \mathbb{P}\Big(X_1 = 5|X = 9\Big) + 6 \cdot \mathbb{P}\Big(X_1 = 6|X = 9\Big) \\
3.\quad & = 1 \cdot \frac{\mathbb{P}\Big(\{X_1 = 1\} \cap \{X = 9\}\Big)}{\mathbb{P}\Big(X = 9\Big)} + 2 \cdot \frac{\mathbb{P}\Big(\{X_1 = 2\} \cap \{X = 9\}\Big)}{\mathbb{P}\Big(X = 9\Big)} && \text{(Bayes' Rule)} \\
& + 3 \cdot \frac{\mathbb{P}\Big(\{X_1 = 3\} \cap \{X = 9\}\Big)}{\mathbb{P}\Big(X = 9\Big)} + 4 \cdot \frac{\mathbb{P}\Big(\{X_1 = 4\} \cap \{X = 9\}\Big)}{\mathbb{P}\Big(X = 9\Big)} \\
& + 5 \cdot \frac{\mathbb{P}\Big(\{X_1 = 5\} \cap \{X = 9\}\Big)}{\mathbb{P}\Big(X = 9\Big)} + 6 \cdot \frac{\mathbb{P}\Big(\{X_1 = 6\} \cap \{X = 9\}\Big)}{\mathbb{P}\Big(X = 9\Big)} \\
4.\quad & = 1 \cdot \frac{0}{\mathbb{P}\Big(X = 9\Big)} + 2 \cdot \frac{0}{\mathbb{P}\Big(X = 9\Big)} + 3 \cdot \frac{\frac{1}{6} \cdot \frac{1}{6}}{\mathbb{P}\Big(X = 9\Big)} \\
& + 4 \cdot \frac{\frac{1}{6} \cdot \frac{1}{6}}{\mathbb{P}\Big(X = 9\Big)} + 5 \cdot \frac{\frac{1}{6} \cdot \frac{1}{6}}{\mathbb{P}\Big(X = 9\Big)} + 6 \cdot \frac{\frac{1}{6} \cdot \frac{1}{6}}{\mathbb{P}\Big(X = 9\Big)} && \text{(Substituting probabilities)}
\end{aligned}
$$

5. $$= \frac{1}{\mathbb{P}\Big(X=9\Big)} \cdot \left(1 \cdot 0 + 2 \cdot 0 + 3 \cdot \left(\frac{1}{6}\right)^2 + 4 \cdot \left(\frac{1}{6}\right)^2 + 5 \cdot \left(\frac{1}{6}\right)^2 + 6 \cdot \left(\frac{1}{6}\right)^2\right)$$

6. $$= \frac{1}{\mathbb{P}\Big(X=9\Big)} \cdot \left(0 + 0 + \frac{3}{36} + \frac{4}{36} + \frac{5}{36} + \frac{6}{36}\right) \qquad \text{(Simplifying fractions)}$$

7. $$= \frac{1}{\mathbb{P}\Big(X=9\Big)} \cdot \frac{3+4+5+6}{36}$$

8. $$= \frac{1}{\mathbb{P}\Big(X=9\Big)} \cdot \frac{18}{36}$$

9. $$= \frac{18}{36 \cdot \mathbb{P}\Big(X=9\Big)}$$

10. $$= \frac{18}{36 \cdot \frac{1}{9}} \qquad \text{(Substituting } \mathbb{P}(X=9) = \frac{1}{9}\text{)}$$

11. $$= \frac{18 \times 9}{36} \qquad \text{(Multiplication)}$$

12. $$= \frac{9}{2} \qquad \text{(Simplification)}$$

13. $$= 4.5 \qquad \text{(Final computation)}$$

$$\therefore \qquad \mathbb{E}[X_1|X=9] = 4.5$$

(d) $\mathbb{E}[X_1 - X_2|X=k]$ for $k \in \{1, ..., 12\}$

1. $$\mathbb{E}[X_1 - X_2|X=k] = \mathbb{E}[X_1 + X_2 - 2 \cdot X_2|X=k] \qquad \text{(Rewriting } X_1 - X_2\text{)}$$
2. $$= \mathbb{E}[X|X=k] - \mathbb{E}[2 \cdot X_2|X=k] \qquad \text{(Linearity of Expectation)}$$
3. $$= \mathbb{E}[X|X=k] - 2 \cdot \mathbb{E}[X_2|X=k] \qquad \text{(Factor out constant)}$$
4. $$= k - 2 \cdot \mathbb{E}[X_2|X=k] \qquad (\mathbb{E}[X|X=k] = k)$$
5. $$= k - 2 \cdot \frac{k}{2} \qquad (\mathbb{E}[X_2|X=k] = \frac{k}{2})$$
6. $$= k - k \qquad \text{(Simplification)}$$
7. $$= 0$$

$$\therefore \qquad \mathbb{E}[X_1 - X_2|X=k] = 0.$$

Programming Question

Link to script: `https://github.com/rlaziz/CDSDS592/blob/main/HW/HW1/programming-assignment.py`