STAT 135
8. The likelihood ratio test

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Likelihood ratio

Likelihood ratio

Assume that we have some IID data $X_1, \ldots, X_n$ from a distribution with density function $f_\theta(x)$. The likelihood function for $X_1, \ldots, X_n$ is $f_\theta(x_1, \ldots, x_n)$.

We want to to test:

$$H_0 : \theta = \theta_0$$

Against the ***simple*** alternative hypothesis:

$$H_1 : \theta = \theta_1$$

The **likelihood ratio (LR)** is the ratio of the likelihoods under each hypothesis:

Likelihood ratio = $\Lambda = \dfrac{lik(\theta_0)}{lik(\theta_1)} = \dfrac{f_{\theta_0}(x_1, \ldots, x_n)}{f_{\theta_1}(x_1, \ldots, x_n)}$ $\begin{matrix} \longleftarrow \text{Likelihood under } H_0 \\ \longleftarrow \text{Likelihood under } H_1 \end{matrix}$

The LR is an intuitive measure of how plausible H_0 is vs H_1

A smaller LR would imply that… H_1 is more likely than H_0

Likelihood ratio coin flip example

You think that your friend's coin is a trick coin. Suppose that you toss a coin 5 times to see test this theory.

We want to to test: $H_0 : p = 0.5$ against $H_1 : p = 0.7$

Suppose that your five tosses resulted in: HHHTH

The likelihood under H_0 is $P(HHHTH \mid p = 0.5) = 0.5^5 = 0.03125$

The likelihood under H_1 is $P(HHHTH \mid p = 0.7) = 0.7^4 \times 0.3 = 0.07203$

The likelihood ratio is $\dfrac{P(HHHTH \mid H_0)}{P(HHHTH \mid H_1)} = \dfrac{0.03125}{0.07203} = 0.434$

i.e., the coin being fair is less than half as likely as the coin being a trick coin, given the data

Likelihood ratio example

Assume that we have some IID data $X_1, \ldots, X_n$ from a $N(\mu, \sigma^2)$ distribution. We want to test

$$H_0 : \mu = \mu_0 \qquad H_1 : \mu = \mu_1$$

$$f_\mu(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \qquad lik(\mu) = f_\mu(x_1, \ldots, x_n) = (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2}$$

The likelihood ratio is

$$\Lambda = \frac{lik(\mu_0)}{lik(\mu_1)} = \frac{(2\pi\sigma^2)^{-n/2}\exp\{-\frac{1}{2\pi\sigma^2}\sum_i (x_i - \mu_0)^2\}}{(2\pi\sigma^2)^{-n/2}\exp\{-\frac{1}{2\pi\sigma^2}\sum_i (x_i - \mu_1)^2\}}$$

$$= \exp\left\{-\frac{1}{2\sigma^2}\left(\sum_i (x_i - \mu_0)^2 - \sum_i (x_i - \mu_1)^2\right)\right\} = \exp\left\{\frac{n(\mu_0 - \mu_1)}{\sigma^2}\left(\bar{x} - \frac{\mu_0 + \mu_1}{2}\right)\right\}$$

Step-by-step of simplification

The likelihood ratio is

$$\Lambda = \frac{lik(\mu_0)}{lik(\mu_1)} = \frac{(2\pi\sigma^2)^{-n/2}\exp\{-\frac{1}{2\pi\sigma^2}\sum_i (x_i-\mu_0)^2\}}{(2\pi\sigma^2)^{-n/2}\exp\{-\frac{1}{2\pi\sigma^2}\sum_i (x_i-\mu_1)^2\}}$$

$$= \exp\left\{-\frac{1}{2\sigma^2}\left(\sum_i (x_i-\mu_0)^2 - \sum_i (x_i-\mu_1)^2\right)\right\}$$

$$= \exp\left\{-\frac{1}{2\sigma^2}\left(\cancel{\sum_i x_i^2} - 2n\bar{x}\mu_0 + n\mu_0^2 - \cancel{\sum_i x_i^2} + 2n\bar{x}\mu_1 - n\mu_1^2\right)\right\}$$

$$= \exp\left\{-\frac{1}{2\sigma^2}\left(2n\bar{x}(\mu_1-\mu_0) + n(\mu_0^2-\mu_1^2)\right)\right\}$$

$$= \exp\left\{-\frac{1}{2\sigma^2}\left(2n\bar{x}(\mu_1-\mu_0) + n(\mu_0-\mu_1)(\mu_0+\mu_1)\right)\right\}$$

$$= \exp\left\{\frac{n(\mu_0-\mu_1)}{\sigma^2}\left(\bar{x} - \frac{\mu_0+\mu_1}{2}\right)\right\}$$

Likelihood ratio test

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We want to to test:

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Against the ***simple*** alternative hypothesis:

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The **likelihood ratio test** rejects H_0 when

Likelihood ratio = $\dfrac{f_{\theta_0}(x_1, \ldots, x_n)}{f_{\theta_1}(x_1, \ldots, x_n)} < c_\alpha$

Some number that depends on the significance level α

Likelihood ratio example

Assume that we have some IID data $X_1, \ldots, X_n$ from a $N(\mu, \sigma^2)$ distribution. We want to test

$$H_0 : \mu = \mu_0 \qquad H_1 : \mu = \mu_1$$

The likelihood ratio is $$\frac{lik(\mu_0)}{lik(\mu_1)} = \exp\left\{ \frac{n(\mu_0 - \mu_1)}{\sigma^2} \left(\bar{x} - \frac{\mu_0 + \mu_1}{2} \right) \right\}$$

So the likelihood ratio test is reject H_0 when $$\exp\left\{ \frac{n(\mu_0 - \mu_1)}{\sigma^2} \left(\bar{x} - \frac{\mu_0 + \mu_1}{2} \right) \right\} < c_\alpha$$

Which can be re-arranged to be $$\bar{x} < \frac{\log(c_\alpha)\sigma^2}{n(\mu_0 - \mu_1)} + \frac{\mu_0 - \mu_1}{2} \qquad \text{(If } \mu_1 < \mu_0\text{)}$$

$$\bar{x} > \frac{\log(c_\alpha)\sigma^2}{n(\mu_0 - \mu_1)} + \frac{\mu_0 - \mu_1}{2} \qquad \text{(If } \mu_1 > \mu_0\text{)}$$

Likelihood ratio example

Assume that we have some IID data $X_1, \ldots, X_n$ from a $N(\mu, \sigma^2)$ distribution. We want to test

$$H_0 : \mu = \mu_0 \qquad H_1 : \mu = \mu_1$$

The likelihood ratio is $$\frac{lik(\mu_0)}{lik(\mu_1)} = \exp\left\{ \frac{n(\mu_0 - \mu_1)}{\sigma^2} \left(\bar{x} - \frac{\mu_0 + \mu_1}{2} \right) \right\}$$

So the likelihood ratio test is reject H_0 when $$\exp\left\{ \frac{n(\mu_0 - \mu_1)}{\sigma^2} \left(\bar{x} - \frac{\mu_0 + \mu_1}{2} \right) \right\} < c_\alpha$$

Which can be re-arranged to be $\bar{x} < \tilde{c}_\alpha$ (If $\mu_1 < \mu_0$)

$\bar{x} > \tilde{c}_\alpha$ (If $\mu_1 > \mu_0$)

Where $$\tilde{c}_\alpha = \frac{\log(c_\alpha)\sigma^2}{n(\mu_0 - \mu_1)} + \frac{\mu_0 - \mu_1}{2}$$

But we don't care that much about this expression… just treat it as a constant

Likelihood ratio example continued

Assume that we have some IID data $X_1, \ldots, X_n$ from a $N(\mu, \sigma^2)$ distribution. We want to test

$$H_0 : \mu = \mu_0 \qquad H_1 : \mu = \mu_1$$

The LRT has format: reject H_0 when $\bar{x} > \tilde{c}_\alpha$ (If $\mu_1 > \mu_0$)

What is $\tilde{c}_\alpha$?

To achieve significance level α, we want $\tilde{c}_\alpha$ to satisfy:

$$\alpha = P(\bar{X} > \tilde{c}_\alpha \,|\, H_0) \quad = P\left(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} > \frac{\tilde{c}_\alpha - \mu_0}{\sigma/\sqrt{n}} \,\middle|\, H_0 \right) = 1 - \Phi\left(\frac{\tilde{c}_\alpha - \mu_0}{\sigma/\sqrt{n}} \right)$$

But, we also have the following definition: $\alpha = 1 - \Phi(z_\alpha)$ So $\tilde{c}_\alpha = \mu_0 + z_\alpha \sigma/\sqrt{n}$

$$\boxed{\text{The LRT has format: reject } H_0 \text{ when } \quad \bar{x} > \mu_0 + z_\alpha \sigma/\sqrt{n}}$$

Neyman Pearson lemma

Neyman-Pearson lemma

Neyman-Pearson lemma:

Suppose that we are testing the **simple** hypotheses $H_0 : \theta = \theta_0$ and $H_1 : \theta = \theta_1$.

Suppose that the likelihood ratio test that rejects H_0 when $\dfrac{f_{\theta_0}(X_1, \ldots, X_n)}{f_{\theta_1}(X_1, \ldots, X_n)} < c(\alpha)$ has

significance level α.

<u>Then any other simple test with significance level $\alpha' \leq \alpha$ has power less than or equal to that of the LRT</u>

Conclusion: if we can design a likelihood ratio test with significance level α, then it is the most powerful (i.e., best) test at this significance level (among tests with simple hypotheses)!

Note there is no such result for *composite* hypotheses such as $H_1 : \theta > \theta_0$

Strategy for finding the best test for simple hypotheses

Observe $X_1, \ldots, X_n$ *IID*

$H_0 : \mu = \mu_0$ and $H_1 : \mu = \mu_1$.

How to find the best test for this problem?

1. Compute the likelihood ratio test

   Goal: Find the values for which LR is small

2. In practice, LR is complicated, find a simpler statistic, T, that determines when LR is small

3. Determine the rejection region for T by finding the cutoff point that guarantees level α