# Calibration Cost Functions and Derivatives

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last updated May 2024

## 1 Per-Frequency, Per-Polarization, and Per-Time Calibration

The simplest calibration implementation is not fully polarized. We calibrate the $pp$ and $qq$ visibilities separately and fit the crosspol phase only from the cross-polarized visibilities $pq$ and $qp$. We also exclude the autocorrelations in calibration, assume a single time step, and calibrate each frequency independently.

### 1.1 Cost Function

For a single polarization, time step, and frequency, calibration consists of minimizing the chi-squared quantity

$$\chi^2(\boldsymbol{g}) = \sum_{jk} \frac{1}{\sigma_{jk}^2} \left| m_{jk} - g_j g_k^* v_{jk} \right|^2. \tag{1}$$

### 1.2 Jacobian Calculation

Expanding the chi-squared gives

$$\begin{aligned}
\chi^2(\boldsymbol{g}) &= \sum_{jk} \frac{1}{\sigma_{jk}^2} \left[ |m_{jk}|^2 + |g_j|^2 |g_k|^2 |v_{jk}|^2 - m_{jk}^* g_j g_k^* v_{jk} - m_{jk} g_j^* g_k v_{jk}^* \right] \\
&= \sum_{jk} \frac{1}{\sigma_{jk}^2} \left[ |m_{jk}|^2 + |g_j|^2 |g_k|^2 |v_{jk}|^2 - 2 \operatorname{Re} \left( m_{jk}^* g_j g_k^* v_{jk} \right) \right].
\end{aligned} \tag{2}$$

Now taking derivatives and omitting autocorrelations (assuming $\frac{1}{\sigma_{jk}^2} = 0$ for $j = k$) gives

$$\begin{aligned}
\frac{\partial \chi^2(\boldsymbol{g})}{\partial \operatorname{Re}(g_a)} = \frac{\partial}{\partial \operatorname{Re}(g_a)} \sum_{j \neq a} \Bigg( &\frac{1}{\sigma_{aj}^2} \left[ |m_{aj}|^2 + |g_a|^2 |g_j|^2 |v_{aj}|^2 - 2 \operatorname{Re} \left( m_{aj}^* g_a g_j^* v_{aj} \right) \right] \\
&+ \frac{1}{\sigma_{ja}^2} \left[ |m_{ja}|^2 + |g_j|^2 |g_a|^2 |v_{ja}|^2 - 2 \operatorname{Re} \left( m_{ja}^* g_j g_a^* v_{ja} \right) \right] \Bigg).
\end{aligned} \tag{3}$$

Evaluating the derivative gives

$$\frac{\partial \chi^2(\boldsymbol{g})}{\partial \operatorname{Re}(g_a)} = \sum_{j \neq a} \left( \frac{1}{\sigma_{aj}^2} \left[ 2 \operatorname{Re}(g_a)|g_j|^2|v_{aj}|^2 - 2 \operatorname{Re}\left( m_{aj}^* g_j^* v_{aj} \right) \right] \right.$$
$$\left. + \frac{1}{\sigma_{ja}^2} \left[ 2 \operatorname{Re}(g_a)|g_j|^2|v_{ja}|^2 - 2 \operatorname{Re}\left( m_{ja}^* g_j v_{ja} \right) \right] \right). \tag{4}$$

The derivative with respect to the imaginary component is

$$\frac{\partial \chi^2(\boldsymbol{g})}{\partial \operatorname{Im}(g_a)} = \sum_{j \neq a} \left( \frac{1}{\sigma_{aj}^2} \left[ 2 \operatorname{Im}(g_a)|g_j|^2|v_{aj}|^2 + 2 \operatorname{Im}\left( m_{aj}^* g_j^* v_{aj} \right) \right] \right.$$
$$\left. + \frac{1}{\sigma_{ja}^2} \left[ 2 \operatorname{Im}(g_a)|g_j|^2|v_{ja}|^2 - 2 \operatorname{Im}\left( m_{ja}^* g_j v_{ja} \right) \right] \right). \tag{5}$$

We can rewrite this as

$$\frac{\partial \chi^2(\boldsymbol{g})}{\partial \operatorname{Re}(g_a)} = 2 \operatorname{Re} \sum_{j \neq a} \left( \frac{1}{\sigma_{aj}^2} \left[ g_a|g_j|^2|v_{aj}|^2 - m_{aj} g_j v_{aj}^* \right] \right.$$
$$\left. + \frac{1}{\sigma_{ja}^2} \left[ g_a|g_j|^2|v_{ja}|^2 - m_{ja}^* g_j v_{ja} \right] \right) \tag{6}$$

and

$$\frac{\partial \chi^2(\boldsymbol{g})}{\partial \operatorname{Im}(g_a)} = 2 \operatorname{Im} \sum_{j \neq a} \left( \frac{1}{\sigma_{aj}^2} \left[ g_a|g_j|^2|v_{aj}|^2 - m_{aj} g_j v_{aj}^* \right] \right.$$
$$\left. + \frac{1}{\sigma_{ja}^2} \left[ g_a|g_j|^2|v_{ja}|^2 - m_{ja}^* g_j v_{ja} \right] \right), \tag{7}$$

or alternatively as

$$\frac{\partial \chi^2(\boldsymbol{g})}{\partial \operatorname{Re}(g_a)} = 2 \operatorname{Re} \sum_{j \neq a} \left( \frac{1}{\sigma_{aj}^2} g_j v_{aj}^* \left[ g_a g_j^* v_{aj} - m_{aj} \right] \right.$$
$$\left. + \frac{1}{\sigma_{ja}^2} g_j v_{ja} \left[ g_j^* g_a v_{ja}^* - m_{ja}^* \right] \right) \tag{8}$$

and

$$\frac{\partial \chi^2(\boldsymbol{g})}{\partial \operatorname{Im}(g_a)} = 2 \operatorname{Im} \sum_{j \neq a} \left( \frac{1}{\sigma_{aj}^2} g_j v_{aj}^* \left[ g_a g_j^* v_{aj} - m_{aj} \right] \right.$$
$$\left. + \frac{1}{\sigma_{ja}^2} g_j v_{ja} \left[ g_j^* g_a v_{ja}^* - m_{ja}^* \right] \right). \tag{9}$$

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## 1.3 Hessian Calculation

Taking second derivatives, we get that

$$\frac{\partial^2 \chi^2(\boldsymbol{g})}{\partial \operatorname{Re}^2(g_a)} = \frac{\partial^2 \chi^2(\boldsymbol{g})}{\partial \operatorname{Im}^2(g_a)} = 2 \sum_{j \neq a} \left( \frac{1}{\sigma_{aj}^2} |g_j|^2 |v_{aj}|^2 + \frac{1}{\sigma_{ja}^2} |g_j|^2 |v_{ja}|^2 \right) \tag{10}$$

and

$$\frac{\partial^2 \chi^2(\boldsymbol{g})}{\partial \operatorname{Re}(g_a) \partial \operatorname{Im}(g_a)} = \frac{\partial^2 \chi^2(\boldsymbol{g})}{\partial \operatorname{Im}(g_a) \partial \operatorname{Re}(g_a)} = 0. \tag{11}$$

For $b \neq a$, we get

$$\begin{aligned}
\frac{\partial \chi^2(\boldsymbol{g})}{\partial \operatorname{Re}(g_a) \operatorname{Re}(g_b)} =& 2 \frac{\partial}{\partial \operatorname{Re}(g_b)} \operatorname{Re} \sum_{j \neq a} \left( \frac{1}{\sigma_{aj}^2} \left[ g_a |g_j|^2 |v_{aj}|^2 - m_{aj} g_j v_{aj}^* \right] \right. \\
& \left. + \frac{1}{\sigma_{ja}^2} \left[ g_a |g_j|^2 |v_{ja}|^2 - m_{ja}^* g_j v_{ja} \right] \right) \\
=& 2 \frac{\partial}{\partial \operatorname{Re}(g_b)} \left( \frac{1}{\sigma_{ab}^2} \left[ \operatorname{Re}(g_a) |g_b|^2 |v_{ab}|^2 - \operatorname{Re}(m_{ab} g_b v_{ab}^*) \right] \right. \\
& \left. + \frac{1}{\sigma_{ba}^2} \left[ \operatorname{Re}(g_a) |g_b|^2 |v_{ba}|^2 - \operatorname{Re}(m_{ba}^* g_b v_{ba}) \right] \right) \\
=& 2 \left( \frac{1}{\sigma_{ab}^2} \left[ 2 \operatorname{Re}(g_a) \operatorname{Re}(g_b) |v_{ab}|^2 - \operatorname{Re}(m_{ab} v_{ab}^*) \right] \right. \\
& \left. + \frac{1}{\sigma_{ba}^2} \left[ 2 \operatorname{Re}(g_a) \operatorname{Re}(g_b) |v_{ba}|^2 - \operatorname{Re}(m_{ba}^* v_{ba}) \right] \right),
\end{aligned} \tag{12}$$

$$\begin{aligned}
\frac{\partial \chi^2(\boldsymbol{g})}{\partial \operatorname{Re}(g_a) \operatorname{Im}(g_b)} =& 2 \frac{\partial}{\partial \operatorname{Im}(g_b)} \operatorname{Re} \sum_{j \neq a} \left( \frac{1}{\sigma_{aj}^2} \left[ g_a |g_j|^2 |v_{aj}|^2 - m_{aj} g_j v_{aj}^* \right] \right. \\
& \left. + \frac{1}{\sigma_{ja}^2} \left[ g_a |g_j|^2 |v_{ja}|^2 - m_{ja}^* g_j v_{ja} \right] \right) \\
=& 2 \frac{\partial}{\partial \operatorname{Im}(g_b)} \left( \frac{1}{\sigma_{ab}^2} \left[ \operatorname{Re}(g_a) |g_b|^2 |v_{ab}|^2 - \operatorname{Re}(m_{ab} g_b v_{ab}^*) \right] \right. \\
& \left. + \frac{1}{\sigma_{ba}^2} \left[ \operatorname{Re}(g_a) |g_b|^2 |v_{ba}|^2 - \operatorname{Re}(m_{ba}^* g_b v_{ba}) \right] \right) \\
=& 2 \left( \frac{1}{\sigma_{ab}^2} \left[ 2 \operatorname{Re}(g_a) \operatorname{Im}(g_b) |v_{ab}|^2 + \operatorname{Im}(m_{ab} v_{ab}^*) \right] \right. \\
& \left. + \frac{1}{\sigma_{ba}^2} \left[ 2 \operatorname{Re}(g_a) \operatorname{Im}(g_b) |v_{ba}|^2 + \operatorname{Im}(m_{ba}^* v_{ba}) \right] \right),
\end{aligned} \tag{13}$$

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and

$$
\begin{aligned}
\frac{\partial \chi^2(\boldsymbol{g})}{\partial \operatorname{Im}(g_a) \operatorname{Im}(g_b)} =& 2 \frac{\partial}{\partial \operatorname{Im}(g_b)} \operatorname{Im} \sum_{j \neq a} \left( \frac{1}{\sigma_{aj}^2} \left[ g_a |g_j|^2 |v_{aj}|^2 - m_{aj} g_j v_{aj}^* \right] \right. \\
&+ \frac{1}{\sigma_{ja}^2} \left[ g_a |g_j|^2 |v_{ja}|^2 - m_{ja}^* g_j v_{ja} \right] \bigg) \\
=& 2 \frac{\partial}{\partial \operatorname{Im}(g_b)} \left( \frac{1}{\sigma_{ab}^2} \left[ \operatorname{Im}(g_a) |g_b|^2 |v_{ab}|^2 - \operatorname{Im}(m_{ab} g_b v_{ab}^*) \right] \right. \\
&+ \frac{1}{\sigma_{ba}^2} \left[ \operatorname{Im}(g_a) |g_b|^2 |v_{ba}|^2 - \operatorname{Im}(m_{ba}^* g_b v_{ba}) \right] \bigg) \\
=& 2 \left( \frac{1}{\sigma_{ab}^2} \left[ 2 \operatorname{Im}(g_a) \operatorname{Im}(g_b) |v_{ab}|^2 - \operatorname{Re}(m_{ab} v_{ab}^*) \right] \right. \\
&+ \frac{1}{\sigma_{ba}^2} \left[ 2 \operatorname{Im}(g_a) \operatorname{Im}(g_b) |v_{ba}|^2 - \operatorname{Re}(m_{ba}^* v_{ba}) \right] \bigg),
\end{aligned}
\tag{14}
$$

## 1.4   Phase Regularization

The overall phase of the gains is degenerate. Here we show how to constrain this phase by requiring that the average phase of the gains is zero. This is in contrast to an approach that uses a reference antenna.

We constrain the phase by adding a regularization term to the chi-squared to produce a cost function of the form

$$
L(\boldsymbol{g}) = \sum_t \sum_{jk} \frac{1}{\sigma_{jk}^2(t)} \left| m_{jk}(t) - g_j g_k^* v_{jk}(t) \right|^2 + \lambda \left[ \sum_j \operatorname{Arg}(g_j) \right]^2 . \tag{15}
$$

Taking derivatives gives

$$\frac{\partial}{\partial \operatorname{Re}(g_a)} \lambda \left[ \sum_j \operatorname{Arg}(g_j) \right]^2 \tag{16}$$

$$= 2\lambda \left[ \sum_j \operatorname{Arg}(g_j) \right] \frac{\partial}{\partial \operatorname{Re}(g_a)} \operatorname{Arg}(g_a) \tag{17}$$

$$= 2\lambda \left[ \sum_j \operatorname{Arg}(g_j) \right] \frac{\partial}{\partial \operatorname{Re}(g_a)} \tan^{-1} \left[ \frac{\operatorname{Im}(g_a)}{\operatorname{Re}(g_a)} \right] \tag{18}$$

$$= 2\lambda \left[ \sum_j \operatorname{Arg}(g_j) \right] \frac{1}{1 + \frac{\operatorname{Im}^2(g_a)}{\operatorname{Re}^2(g_a)}} \frac{\partial}{\partial \operatorname{Re}(g_a)} \left[ \frac{\operatorname{Im}(g_a)}{\operatorname{Re}(g_a)} \right] \tag{19}$$

$$= -2\lambda \left[ \sum_j \operatorname{Arg}(g_j) \right] \frac{1}{1 + \frac{\operatorname{Im}^2(g_a)}{\operatorname{Re}^2(g_a)}} \frac{\operatorname{Im}(g_a)}{\operatorname{Re}^2(g_a)} \tag{20}$$

$$= -2\lambda \left[ \sum_j \operatorname{Arg}(g_j) \right] \frac{\operatorname{Im}(g_a)}{|g_a|^2} \tag{21}$$

$$\tag{22}$$

and

$$\frac{\partial}{\partial \operatorname{Im}(g_a)} \lambda \left[ \sum_j \operatorname{Arg}(g_j) \right]^2 \tag{23}$$

$$= 2\lambda \left[ \sum_j \operatorname{Arg}(g_j) \right] \frac{\partial}{\partial \operatorname{Im}(g_a)} \operatorname{Arg}(g_a) \tag{24}$$

$$= 2\lambda \left[ \sum_j \operatorname{Arg}(g_j) \right] \frac{\partial}{\partial \operatorname{Im}(g_a)} \tan^{-1} \left[ \frac{\operatorname{Im}(g_a)}{\operatorname{Re}(g_a)} \right] \tag{25}$$

$$= 2\lambda \left[ \sum_j \operatorname{Arg}(g_j) \right] \frac{1}{1 + \frac{\operatorname{Im}^2(g_a)}{\operatorname{Re}^2(g_a)}} \frac{\partial}{\partial \operatorname{Im}(g_a)} \left[ \frac{\operatorname{Im}(g_a)}{\operatorname{Re}(g_a)} \right] \tag{26}$$

$$= 2\lambda \left[ \sum_j \operatorname{Arg}(g_j) \right] \frac{1}{1 + \frac{\operatorname{Im}^2(g_a)}{\operatorname{Re}^2(g_a)}} \frac{1}{\operatorname{Re}(g_a)} \tag{27}$$

$$= 2\lambda \left[ \sum_j \operatorname{Arg}(g_j) \right] \frac{\operatorname{Re}(g_a)}{|g_a|^2}. \tag{28}$$

$$\tag{29}$$

This can be rewritten as

$$\frac{\partial}{\partial \operatorname{Re}(g_a)} \lambda \left[ \sum_j \operatorname{Arg}(g_j) \right]^2 = 2\lambda \left[ \sum_j \operatorname{Arg}(g_j) \right] \frac{\operatorname{Re}(ig_a)}{|g_a|^2} \tag{30}$$

and

$$\frac{\partial}{\partial \operatorname{Im}(g_a)} \lambda \left[ \sum_j \operatorname{Arg}(g_j) \right]^2 = 2\lambda \left[ \sum_j \operatorname{Arg}(g_j) \right] \frac{\operatorname{Im}(ig_a)}{|g_a|^2}. \tag{31}$$

The second derivatives are given by

$$\frac{\partial}{\partial \operatorname{Re}(g_a) \partial \operatorname{Re}(g_b)} \lambda \left[ \sum_j \operatorname{Arg}(g_j) \right]^2 \tag{32}$$

$$= -2\lambda \frac{\partial}{\partial \operatorname{Re}(g_b)} \left[ \sum_j \operatorname{Arg}(g_j) \right] \frac{\operatorname{Im}(g_a)}{|g_a|^2} \tag{33}$$

$$= -2\lambda \left( \frac{\operatorname{Im}(g_a)}{|g_a|^2} \frac{\partial}{\partial \operatorname{Re}(g_b)} \left[ \sum_j \operatorname{Arg}(g_j) \right] + \left[ \sum_j \operatorname{Arg}(g_j) \right] \frac{\partial}{\partial \operatorname{Re}(g_b)} \frac{\operatorname{Im}(g_a)}{|g_a|^2} \right) \tag{34}$$

$$= -2\lambda \left( \frac{\operatorname{Im}(g_a)}{|g_a|^2} \frac{\partial}{\partial \operatorname{Re}(g_b)} \tan^{-1} \left[ \frac{\operatorname{Im}(g_b)}{\operatorname{Re}(g_b)} \right] + \delta_{ab} \left[ \sum_j \operatorname{Arg}(g_j) \right] \frac{\partial}{\partial \operatorname{Re}(g_a)} \frac{\operatorname{Im}(g_a)}{|g_a|^2} \right) \tag{35}$$

$$= -2\lambda \left( \frac{\operatorname{Im}(g_a)}{|g_a|^2} \frac{1}{1 + \frac{\operatorname{Im}^2(g_b)}{\operatorname{Re}^2(g_b)}} \frac{\partial}{\partial \operatorname{Re}(g_b)} \left[ \frac{\operatorname{Im}(g_b)}{\operatorname{Re}(g_b)} \right] + \delta_{ab} \left[ \sum_j \operatorname{Arg}(g_j) \right] \frac{\partial}{\partial \operatorname{Re}(g_a)} \frac{\operatorname{Im}(g_a)}{|g_a|^2} \right) \tag{36}$$

$$= 2\lambda \left( \frac{\operatorname{Im}(g_a)}{|g_a|^2} \frac{1}{1 + \frac{\operatorname{Im}^2(g_b)}{\operatorname{Re}^2(g_b)}} \left[ \frac{\operatorname{Im}(g_b)}{\operatorname{Re}^2(g_b)} \right] + 2\delta_{ab} \left[ \sum_j \operatorname{Arg}(g_j) \right] \frac{\operatorname{Im}(g_a)\operatorname{Re}(g_a)}{|g_a|^4} \right) \tag{37}$$

$$= 2\lambda \left( \frac{\operatorname{Im}(g_a)\operatorname{Im}(g_b)}{|g_a|^2|g_b|^2} + 2\delta_{ab} \left[ \sum_j \operatorname{Arg}(g_j) \right] \frac{\operatorname{Im}(g_a)\operatorname{Re}(g_a)}{|g_a|^4} \right), \tag{38}$$

$$\tag{39}$$

$$\frac{\partial}{\partial \operatorname{Re}(g_a) \partial \operatorname{Im}(g_b)} \lambda \left[ \sum_j \operatorname{Arg}(g_j) \right]^2 \tag{40}$$

$$= -2\lambda \frac{\partial}{\partial \operatorname{Im}(g_b)} \left[ \sum_j \operatorname{Arg}(g_j) \right] \frac{\operatorname{Im}(g_a)}{|g_a|^2} \tag{41}$$

$$= -2\lambda \left( \frac{\operatorname{Im}(g_a)}{|g_a|^2} \frac{\partial}{\partial \operatorname{Im}(g_b)} \left[ \sum_j \operatorname{Arg}(g_j) \right] + \left[ \sum_j \operatorname{Arg}(g_j) \right] \frac{\partial}{\partial \operatorname{Im}(g_b)} \frac{\operatorname{Im}(g_a)}{|g_a|^2} \right) \tag{42}$$

$$= -2\lambda \left( \frac{\operatorname{Im}(g_a)}{|g_a|^2} \frac{\partial}{\partial \operatorname{Im}(g_b)} \tan^{-1} \left[ \frac{\operatorname{Im}(g_b)}{\operatorname{Re}(g_b)} \right] + \delta_{ab} \left[ \sum_j \operatorname{Arg}(g_j) \right] \frac{\partial}{\partial \operatorname{Im}(g_a)} \frac{\operatorname{Im}(g_a)}{|g_a|^2} \right) \tag{43}$$

$$= -2\lambda \left( \frac{\operatorname{Im}(g_a)}{|g_a|^2} \frac{1}{1 + \frac{\operatorname{Im}^2(g_b)}{\operatorname{Re}^2(g_b)}} \frac{\partial}{\partial \operatorname{Im}(g_b)} \left[ \frac{\operatorname{Im}(g_b)}{\operatorname{Re}(g_b)} \right] + \delta_{ab} \left[ \sum_j \operatorname{Arg}(g_j) \right] \frac{\partial}{\partial \operatorname{Im}(g_a)} \frac{\operatorname{Im}(g_a)}{|g_a|^2} \right) \tag{44}$$

$$= -2\lambda \left( \frac{\operatorname{Im}(g_a)}{|g_a|^2} \frac{1}{1 + \frac{\operatorname{Im}^2(g_b)}{\operatorname{Re}^2(g_b)}} \left[ \frac{1}{\operatorname{Re}(g_b)} \right] + \delta_{ab} \left[ \sum_j \operatorname{Arg}(g_j) \right] \frac{\partial}{\partial \operatorname{Im}(g_a)} \frac{\operatorname{Im}(g_a)}{|g_a|^2} \right) \tag{45}$$

$$= -2\lambda \left( \frac{\operatorname{Im}(g_a) \operatorname{Re}(g_b)}{|g_a|^2 |g_b|^2} + \delta_{ab} \left[ \sum_j \operatorname{Arg}(g_j) \right] \frac{\operatorname{Re}^2(g_a) - \operatorname{Im}^2(g_a)}{|g_a|^4} \right), \tag{46}$$

$$\tag{47}$$

and

$$\frac{\partial}{\partial \operatorname{Im}(g_a)\partial \operatorname{Im}(g_b)}\lambda\left[\sum_j \operatorname{Arg}(g_j)\right]^2 \tag{48}$$

$$= 2\lambda\frac{\partial}{\partial \operatorname{Im}(g_b)}\left[\sum_j \operatorname{Arg}(g_j)\right]\frac{\operatorname{Re}(g_a)}{|g_a|^2} \tag{49}$$

$$= 2\lambda\left(\frac{\operatorname{Re}(g_a)}{|g_a|^2}\frac{\partial}{\partial \operatorname{Im}(g_b)}\left[\sum_j \operatorname{Arg}(g_j)\right] + \left[\sum_j \operatorname{Arg}(g_j)\right]\frac{\partial}{\partial \operatorname{Im}(g_b)}\frac{\operatorname{Re}(g_a)}{|g_a|^2}\right) \tag{50}$$

$$= 2\lambda\left(\frac{\operatorname{Re}(g_a)}{|g_a|^2}\frac{\partial}{\partial \operatorname{Im}(g_b)}\tan^{-1}\left[\frac{\operatorname{Im}(g_b)}{\operatorname{Re}(g_b)}\right] + \delta_{ab}\left[\sum_j \operatorname{Arg}(g_j)\right]\frac{\partial}{\partial \operatorname{Im}(g_a)}\frac{\operatorname{Re}(g_a)}{|g_a|^2}\right) \tag{51}$$

$$= 2\lambda\left(\frac{\operatorname{Re}(g_a)}{|g_a|^2}\frac{1}{1+\frac{\operatorname{Im}^2(g_b)}{\operatorname{Re}^2(g_b)}}\frac{\partial}{\partial \operatorname{Im}(g_b)}\left[\frac{\operatorname{Im}(g_b)}{\operatorname{Re}(g_b)}\right] + \delta_{ab}\left[\sum_j \operatorname{Arg}(g_j)\right]\frac{\partial}{\partial \operatorname{Im}(g_a)}\frac{\operatorname{Re}(g_a)}{|g_a|^2}\right) \tag{52}$$

$$= 2\lambda\left(\frac{\operatorname{Re}(g_a)}{|g_a|^2}\frac{1}{1+\frac{\operatorname{Im}^2(g_b)}{\operatorname{Re}^2(g_b)}}\left[\frac{1}{\operatorname{Re}(g_b)}\right] - 2\delta_{ab}\left[\sum_j \operatorname{Arg}(g_j)\right]\frac{\operatorname{Im}(g_a)\operatorname{Re}(g_a)}{|g_a|^4}\right) \tag{53}$$

$$= 2\lambda\left(\frac{\operatorname{Re}(g_a)\operatorname{Re}(g_b)}{|g_a|^2|g_b|^2} - 2\delta_{ab}\left[\sum_j \operatorname{Arg}(g_j)\right]\frac{\operatorname{Im}(g_a)\operatorname{Re}(g_a)}{|g_a|^4}\right). \tag{54}$$

$$\tag{55}$$

In summary,

$$\frac{\partial}{\partial \operatorname{Re}(g_a)\partial \operatorname{Re}(g_b)}\lambda\left[\sum_j \operatorname{Arg}(g_j)\right]^2 = 2\lambda\left(\frac{\operatorname{Im}(g_a)\operatorname{Im}(g_b)}{|g_a|^2|g_b|^2} + 2\delta_{ab}\left[\sum_j \operatorname{Arg}(g_j)\right]\frac{\operatorname{Im}(g_a)\operatorname{Re}(g_a)}{|g_a|^4}\right), \tag{56}$$

$$\frac{\partial}{\partial \operatorname{Re}(g_a)\partial \operatorname{Im}(g_b)}\lambda\left[\sum_j \operatorname{Arg}(g_j)\right]^2 = 2\lambda\left(-\frac{\operatorname{Im}(g_a)\operatorname{Re}(g_b)}{|g_a|^2|g_b|^2} - \delta_{ab}\left[\sum_j \operatorname{Arg}(g_j)\right]\frac{\operatorname{Re}^2(g_a) - \operatorname{Im}^2(g_a)}{|g_a|^4}\right), \tag{57}$$

$$\tag{58}$$

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and

$$\frac{\partial}{\partial \operatorname{Im}(g_a)\partial \operatorname{Im}(g_b)}\lambda \left[\sum_j \operatorname{Arg}(g_j)\right]^2 = 2\lambda \left(\frac{\operatorname{Re}(g_a)\operatorname{Re}(g_b)}{|g_a|^2|g_b|^2} - 2\delta_{ab}\left[\sum_j \operatorname{Arg}(g_j)\right]\frac{\operatorname{Im}(g_a)\operatorname{Re}(g_a)}{|g_a|^4}\right).$$

(59)

## 1.5   Cross-Polarization Phase

Per-polarization calibration has a degeneracy in the cross-polarization phase, meaning the overall phase between the $p$-polarized and $q$-polarized gains. We can see this by noting that the transformation $g_{jp} \to g_{jp}e^{-i\phi/2}$ and $g_{jq} \to g_{jq}e^{i\phi/2}$ leave $\chi^2$ unchanged, where here $p$ and $q$ denote the two feed polarizations.

We can constrain this phase by matching the cross-polarization data and model. We do this by minimizing this chi-squared:

$$\chi^2(\phi) = \sum_{jk} \frac{1}{\sigma_{jkpq}^2}\left|e^{-i\phi}v_{jkpq}^{\mathrm{cal}} - m_{jkpq}\right|^2 + \sum_{jk} \frac{1}{\sigma_{jkqp}^2}\left|e^{i\phi}v_{jkqp}^{\mathrm{cal}} - m_{jkqp}\right|^2.$$

(60)

Here $v_{jkpq}^{\mathrm{cal}}$ and $v_{jkqp}^{\mathrm{cal}}$ are the cross-polarization visibilities, calibrated up to the cross-polarization phase.

Expanding the chi-squared gives

$$
\begin{aligned}
\chi^2(\phi) =& \sum_{jk} \frac{1}{\sigma_{jkpq}^2} \left[ \left( \mathrm{Re}[e^{-i\phi} v_{jkpq}^{\mathrm{cal}}] - \mathrm{Re}[m_{jkpq}] \right)^2 + \left( \mathrm{Im}[e^{-i\phi} v_{jkpq}^{\mathrm{cal}}] - \mathrm{Im}[m_{jkpq}] \right)^2 \right] \\
&+ \sum_{jk} \frac{1}{\sigma_{jkqp}^2} \left[ \left( \mathrm{Re}[e^{i\phi} v_{jkqp}^{\mathrm{cal}}] - \mathrm{Re}[m_{jkqp}] \right)^2 + \left( \mathrm{Im}[e^{i\phi} v_{jkqp}^{\mathrm{cal}}] - \mathrm{Im}[m_{jkqp}] \right)^2 \right] \\
=& \sum_{jk} \frac{1}{\sigma_{jkpq}^2} \left[ |v_{jkpq}^{\mathrm{cal}}|^2 + |m_{jkpq}|^2 - 2\,\mathrm{Re}[e^{-i\phi} v_{jkpq}^{\mathrm{cal}}]\,\mathrm{Re}[m_{jkpq}] - 2\,\mathrm{Im}[e^{-i\phi} v_{jkpq}^{\mathrm{cal}}]\,\mathrm{Im}[m_{jkpq}] \right] \\
&+ \sum_{jk} \frac{1}{\sigma_{jkqp}^2} \left[ |v_{jkqp}^{\mathrm{cal}}|^2 + |m_{jkqp}|^2 - 2\,\mathrm{Re}[e^{i\phi} v_{jkqp}^{\mathrm{cal}}]\,\mathrm{Re}[m_{jkqp}] - 2\,\mathrm{Im}[e^{i\phi} v_{jkqp}^{\mathrm{cal}}]\,\mathrm{Im}[m_{jkqp}] \right] \\
=& \sum_{jk} \frac{1}{\sigma_{jkpq}^2} \left[ |v_{jkpq}^{\mathrm{cal}}|^2 + |m_{jkpq}|^2 - 2\,\mathrm{Re}[e^{-i\phi} v_{jkpq}^{\mathrm{cal}} m_{jkpq}^*] \right] \\
&+ \sum_{jk} \frac{1}{\sigma_{jkqp}^2} \left[ |v_{jkqp}^{\mathrm{cal}}|^2 + |m_{jkqp}|^2 - 2\,\mathrm{Re}[e^{i\phi} v_{jkqp}^{\mathrm{cal}} m_{jkqp}^*] \right] \\
=& \sum_{jk} \frac{1}{\sigma_{jkpq}^2} \left[ |v_{jkpq}^{\mathrm{cal}}|^2 + |m_{jkpq}|^2 - 2\cos[-\phi]\,\mathrm{Re}[v_{jkpq}^{\mathrm{cal}} m_{jkpq}^*] + 2\sin[-\phi]\,\mathrm{Im}[v_{jkpq}^{\mathrm{cal}} m_{jkpq}^*] \right] \\
&+ \sum_{jk} \frac{1}{\sigma_{jkqp}^2} \left[ |v_{jkqp}^{\mathrm{cal}}|^2 + |m_{jkqp}|^2 - 2\cos\phi\,\mathrm{Re}[v_{jkqp}^{\mathrm{cal}} m_{jkqp}^*] + 2\sin\phi\,\mathrm{Im}[v_{jkqp}^{\mathrm{cal}} m_{jkqp}^*] \right] \\
=& \sum_{jk} \frac{1}{\sigma_{jkpq}^2} \left[ |v_{jkpq}^{\mathrm{cal}}|^2 + |m_{jkpq}|^2 - 2\cos\phi\,\mathrm{Re}[v_{jkpq}^{\mathrm{cal}} m_{jkpq}^*] - 2\sin\phi\,\mathrm{Im}[v_{jkpq}^{\mathrm{cal}} m_{jkpq}^*] \right] \\
&+ \sum_{jk} \frac{1}{\sigma_{jkqp}^2} \left[ |v_{jkqp}^{\mathrm{cal}}|^2 + |m_{jkqp}|^2 - 2\cos\phi\,\mathrm{Re}[v_{jkqp}^{\mathrm{cal}} m_{jkqp}^*] + 2\sin\phi\,\mathrm{Im}[v_{jkqp}^{\mathrm{cal}} m_{jkqp}^*] \right].
\end{aligned}
\tag{61}
$$

Taking the derivative gives

$$
\begin{aligned}
\frac{\partial \chi^2(\phi)}{\partial \phi} =& \sum_{jk} \frac{1}{\sigma_{jkpq}^2} \left[ 2\sin\phi\,\mathrm{Re}[v_{jkpq}^{\mathrm{cal}} m_{jkpq}^*] - 2\cos\phi\,\mathrm{Im}[v_{jkpq}^{\mathrm{cal}} m_{jkpq}^*] \right] \\
&+ \sum_{jk} \frac{1}{\sigma_{jkqp}^2} \left[ 2\sin\phi\,\mathrm{Re}[v_{jkqp}^{\mathrm{cal}} m_{jkqp}^*] + 2\cos\phi\,\mathrm{Im}[v_{jkqp}^{\mathrm{cal}} m_{jkqp}^*] \right].
\end{aligned}
\tag{62}
$$

Using the extremum condition $\left. \frac{\partial \chi^2(\phi)}{\partial \phi} \right|_{\phi=\phi_0} = 0$ gives

$$
\begin{aligned}
\sin\phi_0 &\sum_{jk} \left( \frac{1}{\sigma_{jkpq}^2} \mathrm{Re}[v_{jkpq}^{\mathrm{cal}} m_{jkpq}^*] + \frac{1}{\sigma_{jkqp}^2} \mathrm{Re}[v_{jkqp}^{\mathrm{cal}} m_{jkqp}^*] \right) \\
&= \cos\phi_0 \sum_{jk} \left( \frac{1}{\sigma_{jkpq}^2} \mathrm{Im}[v_{jkpq}^{\mathrm{cal}} m_{jkpq}^*] - \frac{1}{\sigma_{jkqp}^2} \mathrm{Im}[v_{jkqp}^{\mathrm{cal}} m_{jkqp}^*] \right)
\end{aligned}
\tag{63}
$$

or

$$\tan \phi_0 = \frac{\sum_{jk} \left( \frac{1}{\sigma_{jkpq}^2} \operatorname{Im}[v_{jkpq}^{\mathrm{cal}} m_{jkpq}^*] - \frac{1}{\sigma_{jkqp}^2} \operatorname{Im}[v_{jkqp}^{\mathrm{cal}} m_{jkqp}^*] \right)}{\sum_{jk} \left( \frac{1}{\sigma_{jkpq}^2} \operatorname{Re}[v_{jkpq}^{\mathrm{cal}} m_{jkpq}^*] + \frac{1}{\sigma_{jkqp}^2} \operatorname{Re}[v_{jkqp}^{\mathrm{cal}} m_{jkqp}^*] \right)}$$

$$= \frac{\sum_{jk} \left( \frac{1}{\sigma_{jkpq}^2} \operatorname{Im}[v_{jkpq}^{\mathrm{cal}} m_{jkpq}^*] + \frac{1}{\sigma_{jkqp}^2} \operatorname{Im}[(v_{jkqp}^{\mathrm{cal}})^* m_{jkqp}] \right)}{\sum_{jk} \left( \frac{1}{\sigma_{jkpq}^2} \operatorname{Re}[v_{jkpq}^{\mathrm{cal}} m_{jkpq}^*] + \frac{1}{\sigma_{jkqp}^2} \operatorname{Re}[(v_{jkqp}^{\mathrm{cal}})^* m_{jkqp}] \right)} \qquad (64)$$

$$= \frac{\operatorname{Im} \left[ \sum_{jk} \left( \frac{1}{\sigma_{jkpq}^2} v_{jkpq}^{\mathrm{cal}} m_{jkpq}^* + \frac{1}{\sigma_{jkqp}^2} (v_{jkqp}^{\mathrm{cal}})^* m_{jkqp} \right) \right]}{\operatorname{Re} \left[ \sum_{jk} \left( \frac{1}{\sigma_{jkpq}^2} v_{jkpq}^{\mathrm{cal}} m_{jkpq}^* + \frac{1}{\sigma_{jkqp}^2} (v_{jkqp}^{\mathrm{cal}})^* m_{jkqp} \right) \right]}.$$

It follows that

$$\phi_0 = \operatorname{Arg} \left[ \sum_{jk} \left( \frac{1}{\sigma_{jkpq}^2} v_{jkpq}^{\mathrm{cal}} m_{jkpq}^* + \frac{1}{\sigma_{jkqp}^2} (v_{jkqp}^{\mathrm{cal}})^* m_{jkqp} \right) \right], \qquad (65)$$

where Arg denotes the complex argument.

## 2 Per-Frequency, Per-Polarization Including the Time Axis

Extending the simple implementation to include time steps is trivial. We assume that the gains do not vary on the timescale of the observation, so only the data and model visibilities are time-dependent. We then get a chi-squared in the form

$$\chi^2(\boldsymbol{g}) = \sum_t \sum_{jk} \frac{1}{\sigma_{jk}^2(t)} \left| m_{jk}(t) - g_j g_k^* v_{jk}(t) \right|^2. \qquad (66)$$

The derivatives calculated in §1.2 then become

$$\frac{\partial \chi^2(\boldsymbol{g})}{\partial \operatorname{Re}(g_a)} = 2 \operatorname{Re} \sum_t \sum_{j \neq a} \left( \frac{1}{\sigma_{aj}^2(t)} \left[ g_a |g_j|^2 |v_{aj}(t)|^2 - m_{aj}(t) g_j v_{aj}^*(t) \right] \right.$$
$$\left. + \frac{1}{\sigma_{ja}^2(t)} \left[ g_a |g_j|^2 |v_{ja}(t)|^2 - m_{ja}^*(t) g_j v_{ja}(t) \right] \right). \qquad (67)$$

and

$$\frac{\partial \chi^2(\boldsymbol{g})}{\partial \operatorname{Im}(g_a)} = 2 \operatorname{Im} \sum_t \sum_{j \neq a} \left( \frac{1}{\sigma_{aj}^2(t)} \left[ g_a |g_j|^2 |v_{aj}(t)|^2 - m_{aj}(t) g_j v_{aj}^*(t) \right] \right.$$
$$\left. + \frac{1}{\sigma_{ja}^2(t)} \left[ g_a |g_j|^2 |v_{ja}(t)|^2 - m_{ja}^*(t) g_j v_{ja}(t) \right] \right). \tag{68}$$

The Hessian quantities in §1.3 become

$$\frac{\partial^2 \chi^2(\boldsymbol{g})}{\partial \operatorname{Re}^2(g_a)} = \frac{\partial^2 \chi^2(\boldsymbol{g})}{\partial \operatorname{Im}^2(g_a)} = 2 \sum_t \sum_{j \neq a} \left( \frac{1}{\sigma_{aj}^2(t)} |g_j|^2 |v_{aj}(t)|^2 + \frac{1}{\sigma_{ja}^2(t)} |g_j|^2 |v_{ja}(t)|^2 \right). \tag{69}$$

Once again,

$$\frac{\partial^2 \chi^2(\boldsymbol{g})}{\partial \operatorname{Re}(g_a) \partial \operatorname{Im}(g_a)} = \frac{\partial^2 \chi^2(\boldsymbol{g})}{\partial \operatorname{Im}(g_a) \partial \operatorname{Re}(g_a)} = 0. \tag{70}$$

For $b \neq a$, we get

$$\frac{\partial^2 \chi^2(\boldsymbol{g})}{\partial \operatorname{Re}(g_a) \operatorname{Re}(g_b)} = 2 \sum_t \left( \frac{1}{\sigma_{ab}^2(t)} \left[ 2 \operatorname{Re}(g_a) \operatorname{Re}(g_b) |v_{ab}(t)|^2 - \operatorname{Re}[m_{ab}(t) v_{ab}^*(t)] \right] \right.$$
$$\left. + \frac{1}{\sigma_{ba}^2(t)} \left[ 2 \operatorname{Re}(g_a) \operatorname{Re}(g_b) |v_{ba}(t)|^2 - \operatorname{Re}[m_{ba}^*(t) v_{ba}(t)] \right] \right), \tag{71}$$

$$\frac{\partial^2 \chi^2(\boldsymbol{g})}{\partial \operatorname{Re}(g_a) \operatorname{Im}(g_b)} = 2 \sum_t \left( \frac{1}{\sigma_{ab}^2(t)} \left[ 2 \operatorname{Re}(g_a) \operatorname{Im}(g_b) |v_{ab}(t)|^2 + \operatorname{Im}[m_{ab}(t) v_{ab}^*(t)] \right] \right.$$
$$\left. + \frac{1}{\sigma_{ba}^2(t)} \left[ 2 \operatorname{Re}(g_a) \operatorname{Im}(g_b) |v_{ba}(t)|^2 + \operatorname{Im}[m_{ba}^*(t) v_{ba}(t)] \right] \right), \tag{72}$$

and

$$\frac{\partial^2 \chi^2(\boldsymbol{g})}{\partial \operatorname{Im}(g_a) \operatorname{Im}(g_b)} = 2 \sum_t \left( \frac{1}{\sigma_{ab}^2(t)} \left[ 2 \operatorname{Im}(g_a) \operatorname{Im}(g_b) |v_{ab}(t)|^2 - \operatorname{Re}[m_{ab}(t) v_{ab}^*(t)] \right] \right.$$
$$\left. + \frac{1}{\sigma_{ba}^2(t)} \left[ 2 \operatorname{Im}(g_a) \operatorname{Im}(g_b) |v_{ba}(t)|^2 - \operatorname{Re}[m_{ba}^*(t) v_{ba}(t)] \right] \right). \tag{73}$$

The phase regularization term does not depend on either the data or model visibilities, so it is time-independent. We can therefore use the quantities derived in §1.4.

# 3 Absolute Calibration

## 3.1 Cost Function

The absolute calibration chi-squared for each frequency and time is

$$\chi^2(A, \Delta_x, \Delta_y) = \sum_{jk} \frac{1}{\sigma_{jk}^2} \left| A^2 e^{i(\Delta_x x_{jk} + \Delta_y y_{jk})} v_{jk}^{\text{rel}} - m_{jk} \right|^2. \tag{74}$$

Here $v_{jk}^{\text{rel}}$ are the relatively calibrated visibilities and $x_{jk}$ and $y_{jk}$ are the coordinates of baseline $\{j, k\}$.

## 3.2 Jacobian Calculation

Expanding the cost function from above gives

$$
\begin{aligned}
\chi^2(A, \Delta_x, \Delta_y) &= \sum_{jk} \frac{1}{\sigma_{jk}^2} \left| A^2 e^{i(\Delta_x x_{jk} + \Delta_y y_{jk})} v_{jk}^{\text{rel}} - m_{jk} \right|^2 \\
&= \sum_{jk} \frac{1}{\sigma_{jk}^2} \left[ \left( A^2 \operatorname{Re}[e^{i(\Delta_x x_{jk} + \Delta_y y_{jk})} v_{jk}^{\text{rel}}] - \operatorname{Re}[m_{jk}] \right)^2 \right. \\
&\quad \left. + \left( A^2 \operatorname{Im}[e^{i(\Delta_x x_{jk} + \Delta_y y_{jk})} v_{jk}^{\text{rel}}] - \operatorname{Im}[m_{jk}] \right)^2 \right] \\
&= \sum_{jk} \frac{1}{\sigma_{jk}^2} \left( A^4 |v_{jk}^{\text{rel}}|^2 + |m_{jk}|^2 \right. \\
&\quad - 2A^2 \operatorname{Re}[e^{i(\Delta_x x_{jk} + \Delta_y y_{jk})} v_{jk}^{\text{rel}}] \operatorname{Re}[m_{jk}] \\
&\quad \left. - 2A^2 \operatorname{Im}[e^{i(\Delta_x x_{jk} + \Delta_y y_{jk})} v_{jk}^{\text{rel}}] \operatorname{Im}[m_{jk}] \right) \\
&= \sum_{jk} \frac{1}{\sigma_{jk}^2} \left( A^4 |v_{jk}^{\text{rel}}|^2 + |m_{jk}|^2 \right. \\
&\quad \left. - 2A^2 \operatorname{Re}[e^{i(\Delta_x x_{jk} + \Delta_y y_{jk})} v_{jk}^{\text{rel}} m_{jk}^*] \right) \\
&= \sum_{jk} \frac{1}{\sigma_{jk}^2} \left( A^4 |v_{jk}^{\text{rel}}|^2 + |m_{jk}|^2 \right. \\
&\quad - 2A^2 \cos(\Delta_x x_{jk} + \Delta_y y_{jk}) \operatorname{Re}[v_{jk}^{\text{rel}} m_{jk}^*] \\
&\quad \left. + 2A^2 \sin(\Delta_x x_{jk} + \Delta_y y_{jk}) \operatorname{Im}[v_{jk}^{\text{rel}} m_{jk}^*] \right).
\end{aligned}
\tag{75}
$$

Taking the derivative with respect to the overall amplitude $A$ gives

$$\frac{\partial \chi^2(A, \Delta_x, \Delta_y)}{\partial A} = \sum_{jk} \frac{1}{\sigma_{jk}^2} \left( 4A^3 |v_{jk}^{\text{rel}}|^2 - 4A \operatorname{Re}[e^{i(\Delta_x x_{jk} + \Delta_y y_{jk})} v_{jk}^{\text{rel}} m_{jk}^*] \right). \tag{76}$$

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Taking the derivative with respect to the phase gradient term $\Delta_x$ gives

$$\frac{\partial \chi^2(A, \Delta_x, \Delta_y)}{\partial \Delta_x} = \sum_{jk} \frac{1}{\sigma_{jk}^2} \left( 2A^2 x_{jk} \sin(\Delta_x x_{jk} + \Delta_y y_{jk}) \operatorname{Re}[v_{jk}^{\text{rel}} m_{jk}^*] \right.$$
$$\left. + 2A^2 x_{jk} \cos(\Delta_x x_{jk} + \Delta_y y_{jk}) \operatorname{Im}[v_{jk}^{\text{rel}} m_{jk}^*] \right) \tag{77}$$
$$= 2A^2 \sum_{jk} \frac{1}{\sigma_{jk}^2} x_{jk} \operatorname{Im}[e^{i(\Delta_x x_{jk} + \Delta_y y_{jk})} v_{jk}^{\text{rel}} m_{jk}^*].$$

It follows that

$$\frac{\partial \chi^2(A, \Delta_x, \Delta_y)}{\partial \Delta_y} = 2A^2 \sum_{jk} \frac{1}{\sigma_{jk}^2} y_{jk} \operatorname{Im}[e^{i(\Delta_x x_{jk} + \Delta_y y_{jk})} v_{jk}^{\text{rel}} m_{jk}^*]. \tag{78}$$

## 3.3 Hessian Calculation

Taking the second derivatives gives

$$\frac{\partial^2 \chi^2(A, \Delta_x, \Delta_y)}{\partial A^2} = \sum_{jk} \frac{1}{\sigma_{jk}^2} \left( 12A^2 |v_{jk}^{\text{rel}}|^2 - 4 \operatorname{Re}[e^{i(\Delta_x x_{jk} + \Delta_y y_{jk})} v_{jk}^{\text{rel}} m_{jk}^*] \right),$$
$$\tag{79}$$

$$\frac{\partial^2 \chi^2(A, \Delta_x, \Delta_x)}{\partial A \partial \Delta_y} = 4A \sum_{jk} \frac{1}{\sigma_{jk}^2} x_{jk} \operatorname{Im}[e^{i(\Delta_x x_{jk} + \Delta_y y_{jk})} v_{jk}^{\text{rel}} m_{jk}^*], \tag{80}$$

and

$$\frac{\partial^2 \chi^2(A, \Delta_x, \Delta_y)}{\partial A \partial \Delta_y} = 4A \sum_{jk} \frac{1}{\sigma_{jk}^2} y_{jk} \operatorname{Im}[e^{i(\Delta_x x_{jk} + \Delta_y y_{jk})} v_{jk}^{\text{rel}} m_{jk}^*]. \tag{81}$$

The second derivatives with respect to the phase gradients are

$$\frac{\partial^2 \chi^2(A, \Delta_x, \Delta_y)}{\partial \Delta_x^2} = \sum_{jk} \frac{1}{\sigma_{jk}^2} \left( 2A^2 x_{jk}^2 \cos(\Delta_x x_{jk} + \Delta_y y_{jk}) \operatorname{Re}[v_{jk}^{\text{rel}} m_{jk}^*] \right.$$
$$\left. - 2A^2 x_{jk}^2 \sin(\Delta_x x_{jk} + \Delta_y y_{jk}) \operatorname{Im}[v_{jk}^{\text{rel}} m_{jk}^*] \right) \tag{82}$$
$$= 2A^2 \sum_{jk} \frac{1}{\sigma_{jk}^2} x_{jk}^2 \operatorname{Re}[e^{i(\Delta_x x_{jk} + \Delta_y y_{jk})} v_{jk}^{\text{rel}} m_{jk}^*].$$

It follows that

$$\frac{\partial^2 \chi^2(A, \Delta_x, \Delta_y)}{\partial \Delta_y^2} = 2A^2 \sum_{jk} \frac{1}{\sigma_{jk}^2} y_{jk}^2 \operatorname{Re}[e^{i(\Delta_x x_{jk} + \Delta_y y_{jk})} v_{jk}^{\text{rel}} m_{jk}^*] \tag{83}$$

and

$$\frac{\partial^2 \chi^2(A, \Delta_x, \Delta_y)}{\partial \Delta_x \Delta_y} = 2A^2 \sum_{jk} \frac{1}{\sigma_{jk}^2} x_{jk} y_{jk} \operatorname{Re}[e^{i(\Delta_x x_{jk} + \Delta_y y_{jk})} v_{jk}^{\text{rel}} m_{jk}^*]. \tag{84}$$

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## 3.4 Applying Abscal Solutions

For per-polarization abscal, the outputs are parameters $A_p$, $\Delta_{xp}$, $\Delta_{yp}$, $A_q$, $\Delta_{xq}$, and $\Delta_{yq}$. Applying these solutions to the single-polarization visibilities is simple:

$$v_{jk\,pp}^{\text{cal}} = A_p^2 e^{i(\Delta_{xp}x_{jk}+\Delta_{yp}y_{jk})} v_{jk\,pp}^{\text{rel}} \tag{85}$$

and

$$v_{jk\,qq}^{\text{cal}} = A_q^2 e^{i(\Delta_{xq}x_{jk}+\Delta_{yq}y_{jk})} v_{jk\,qq}^{\text{rel}}, \tag{86}$$

where $v_{jk\,pp}^{\text{cal}}$ and $v_{jk\,qq}^{\text{cal}}$ are the calibrated visibilities.

However, calibrating the crosspol visibilities is slightly more complicated. Here we get that

$$v_{jk\,pq}^{\text{cal}} = A_p A_q e^{i(\Delta_{xp}x_j-\Delta_{xq}x_k+\Delta_{yp}y_j-\Delta_{yq}y_k)} v_{jk\,pq}^{\text{rel}} \tag{87}$$

and

$$v_{jk\,qp}^{\text{cal}} = A_p A_q e^{i(\Delta_{xq}x_j-\Delta_{xp}x_k+\Delta_{yq}y_j-\Delta_{yp}y_k)} v_{jk\,qp}^{\text{rel}}. \tag{88}$$

Here $\{x_j, y_j\}$ and $\{x_k, y_k\}$ are the coordinates of antennas $j$ and $k$, respectively. Note that we require calibrating with the antenna positions, not with the baseline locations.