Reduction to Hessenberg or Tridiagonal Form

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1 Research Background and Significance

Eigenvalues and eigenvectors tell us by how much the linear transformation scales up or down the sides of certain parallelograms. Eigenvectors and eigenvalues have a wide range of applications in science and engineering, such as stability analysis, facial recognition, and matrix diagonalization. A significant application is performing singular value decomposition resulting in diagonalization of matrix with left and right orthogonal vectors. Additionally, eigenvalues are used in calculating conditions numbers which is important in accessing the accuracy of solutions to linear systems. The project aims at tackling eigenvalue problem by implementing the algorithms specified in Lecture 26 of [1]. This involves reducing matrix A to a Hessenberg or Tridiagonal form (for Hermitian matrices).

2 Mathematical Formulation

Given a matrix $A \in \mathbb{C}^{m\times m}$, an eigenvalue problem encompasses finding a subspace S^m where the action of A on S mimics scalar multiplication of S as shown in equation (1). Here, S, x, and λ are referred to as the eigenspace, eigenvector and eigenvalue respectively. As a result, we can see that the domain and the range space of A is the same. The set of all eigenvalues of matrix A is the spectrum of A, subset of S denoted by $\Lambda(A)$.

$$Ax = \lambda x, \quad x \ \in S \tag{1}$$

There are several eigenvalue yielding factorizations. A notable one is Schur factorization equation (2), where given an n by n matrix A with complex entries, A can be expressed as:

$$A = QTQ^*, \tag{2}$$

where Q is a unitary matrix whereby Q^{-1} is also Q^*, and T is an upper triangular matrix. As can be seen matrix A and T are similar, hence the eigenvalues of A are the entries in the diagonal of T. Inspired by Householder triangulation, the task was to use the householder reflection to find the sequence of Qs that transforms A to upper-triangular. The traditional first householder reflector is the form Q_1^*A which introduces zeros to first columns except diagonal entry. For Schur decomposition, the form is $Q_1^*AQ_1$, thus right-multiplying Q_1^*A by Q_1 refills the zeros that had previously been taken out by the Householder reflection. An alternative method was proposed in the [1] (applying householder reflector to produce a sub diagonal upper-triangular referred as Hessenberg form).

$$QAQ^* = H \tag{3}$$

3 Proposed Research Plan

In finding eigenvalues, two phases are involved; phase 1 is where matrix A is reduced to a structured form and phase 2 entails converging the structured form to eigenvalues using algorithms like power iteration. The implementable steps for our projects are listed below:

1. Implementing Householder Reduction algorithm for Hessenberg form in programming language of choice.

2. Comparing out implementation with in-built methods for computing Hessenberg form.

3. Performing stability and flops analysis of Householder Reduction algorithm for Hessenberg form.

4. Time permit: Implementing an algorithm to converge Hessenberg form to a diagonal matrix with corresponding eigenvalues and compare results with in-built methods.

References

[1] L. N. Trefethen and D. Bau III, *Numerical linear algebra*, vol. 50. Siam, 1997.