Confidence intervals, hypothesis testing, Monte Carlo, and generalized linear models

Roger Levy

9.S918: Quantitative inference in brain and cognitive sciences

19 February 2025

Credible intervals & confidence intervals

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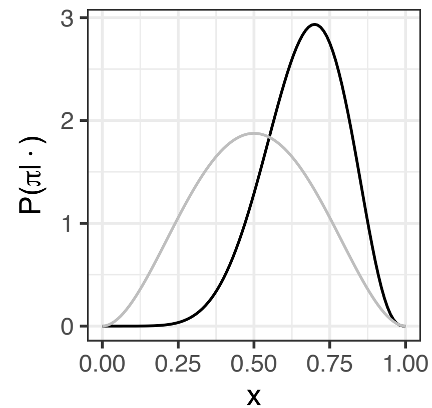

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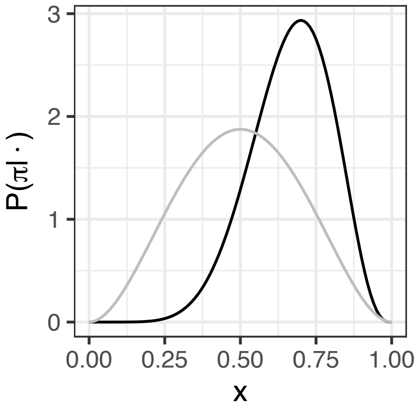

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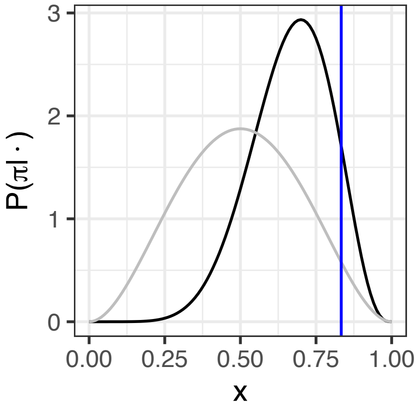

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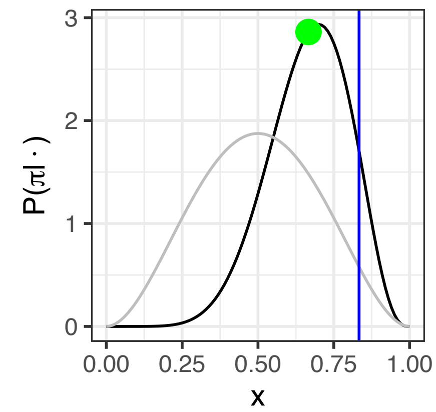

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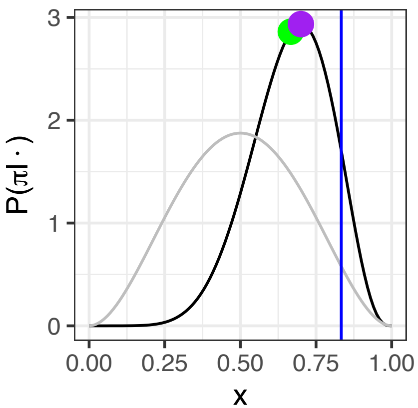

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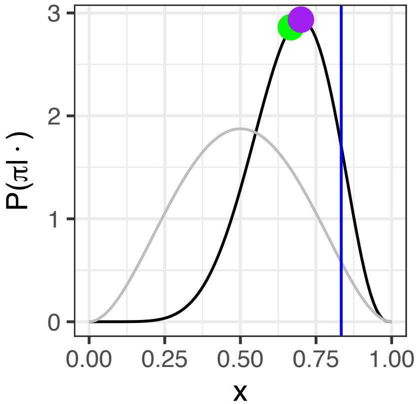

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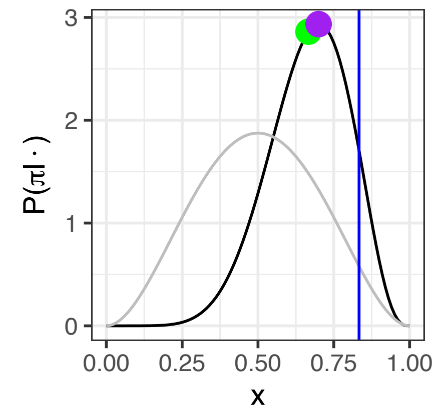

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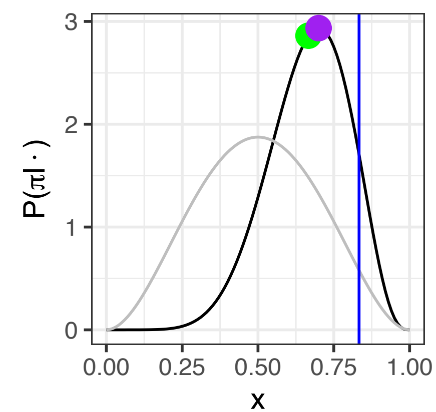

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- Credible intervals (Bayesian) and confidence intervals (frequentist) provide a bit more information about this uncertainty

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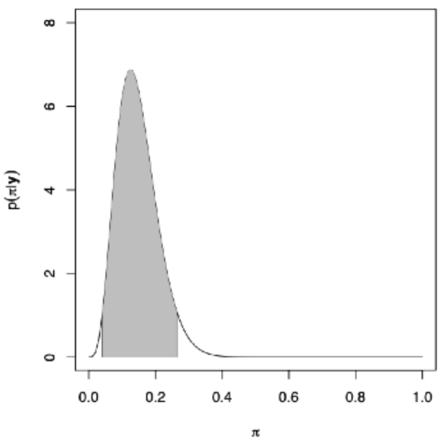

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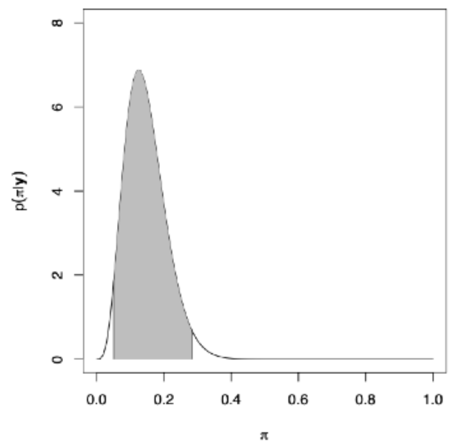

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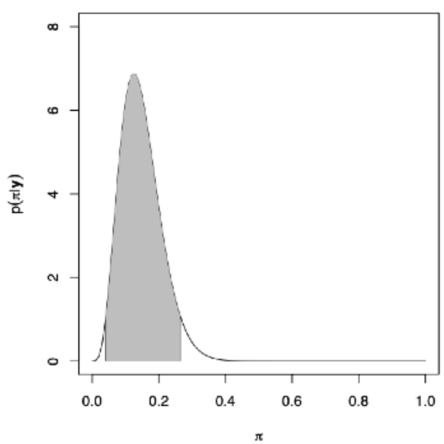

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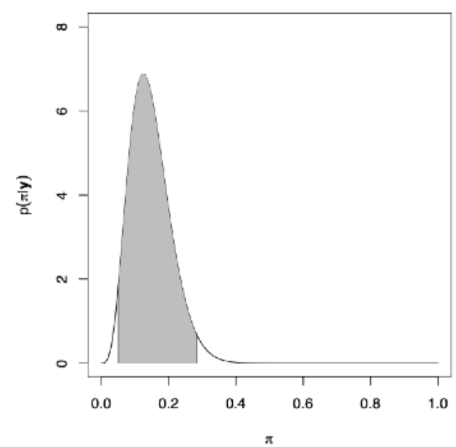

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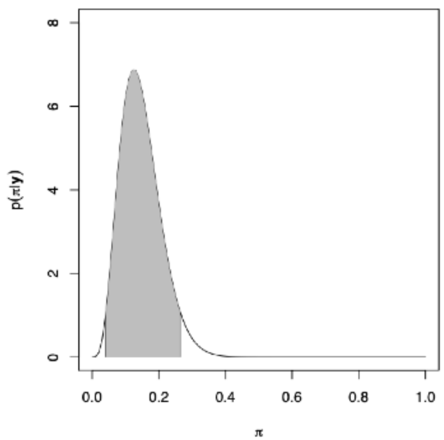

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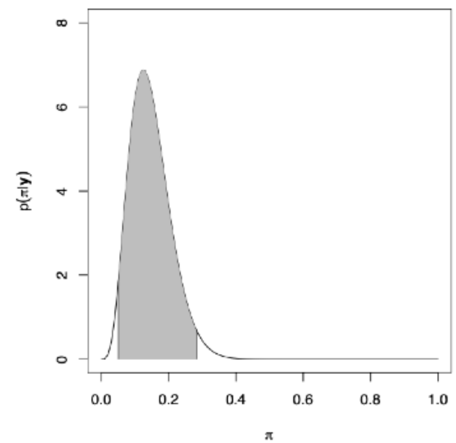

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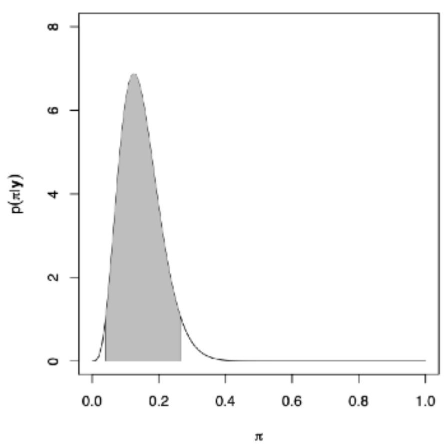

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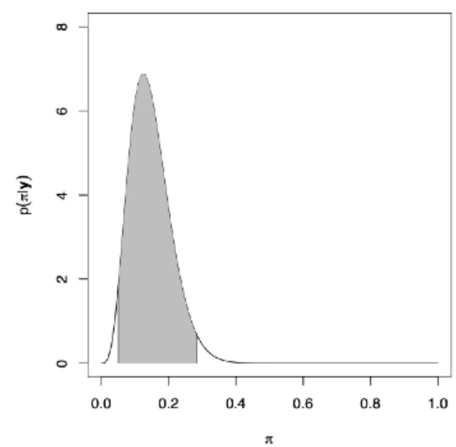

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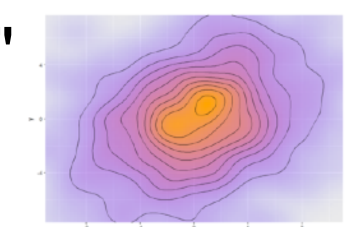

https://www.martinmodrak.cz/2018/02/19/taming-divergences-in-stan-models/

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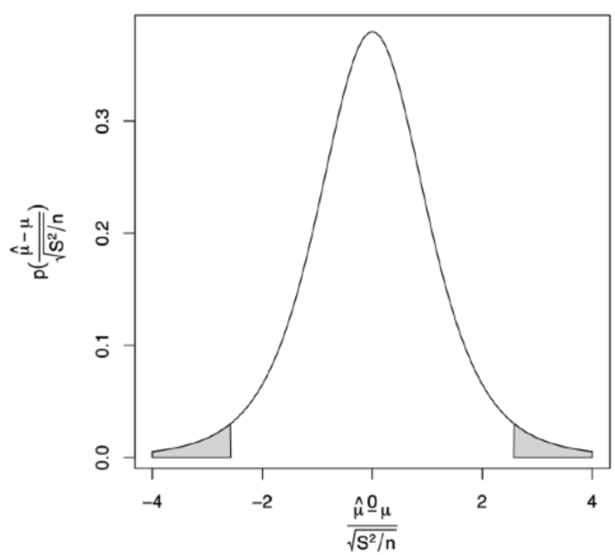

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Bayes Factor: $\dfrac{P(\boldsymbol{y}|H)}{P(\boldsymbol{y}|H')}$

Interpreting Bayes Factors

$$K = \frac{P(\boldsymbol{y}|H)}{P(\boldsymbol{y}|H')}$$

$\log_{10} K$	K	Strength of evidence
0 to 1/2	1 to 3.2	Not worth more than a bare mention
1/2 to 1	3.2 to 10	Substantial
1 to 2	10 to 100	Strong
> 2	> 100	Decisive

(Kass & Raftery, 1995; table from https://en.wikipedia.org/wiki/Bayes_factor)

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$$= 1.64$$

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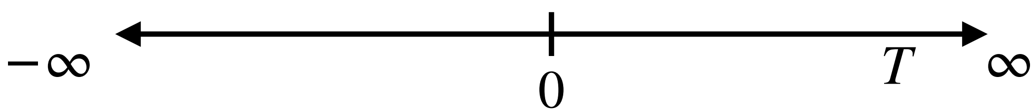

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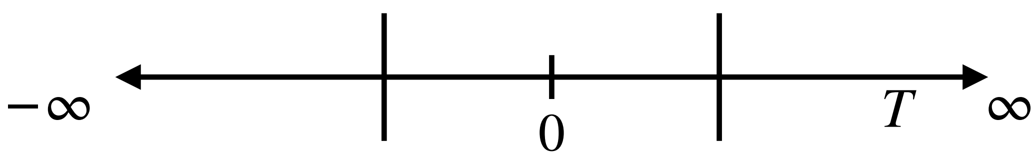

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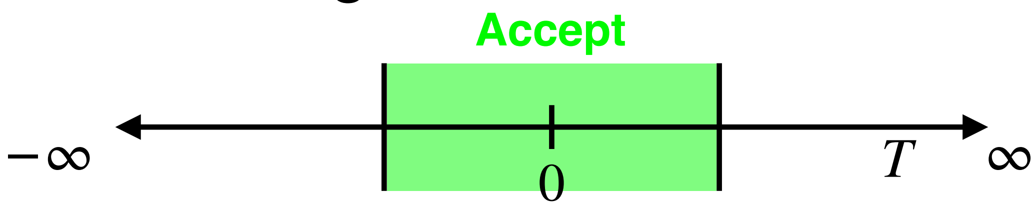

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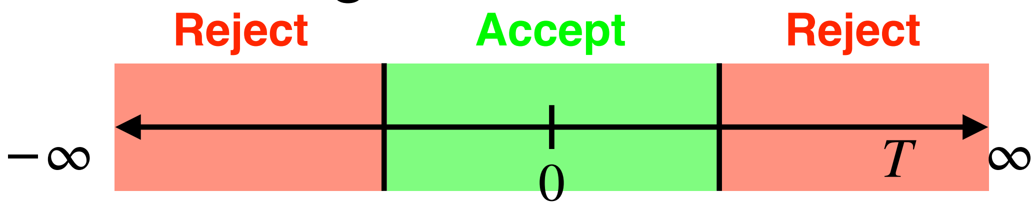

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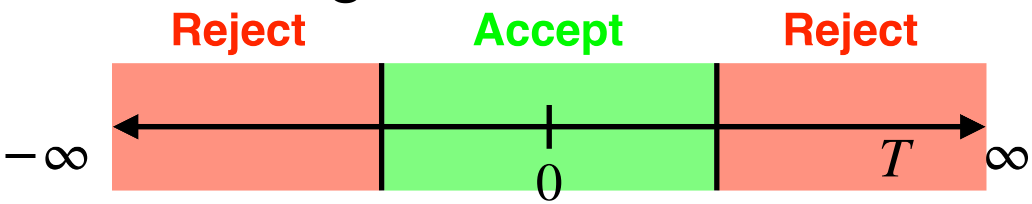

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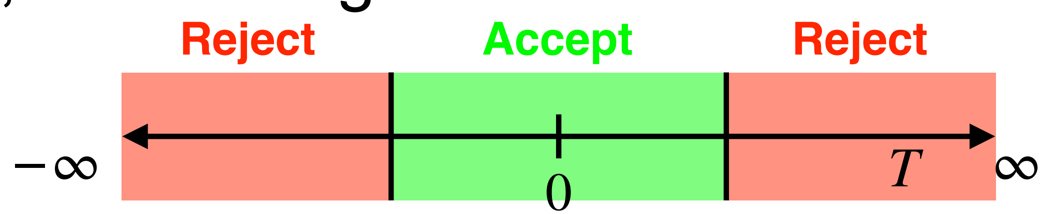

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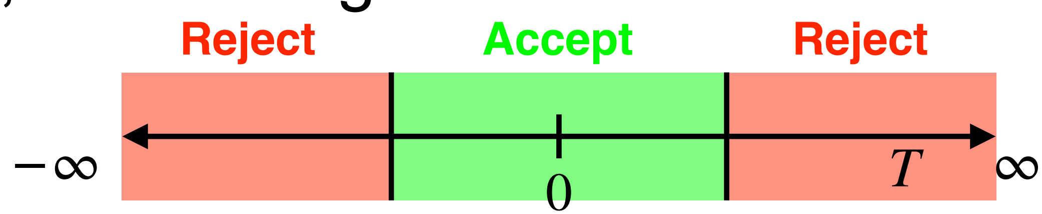

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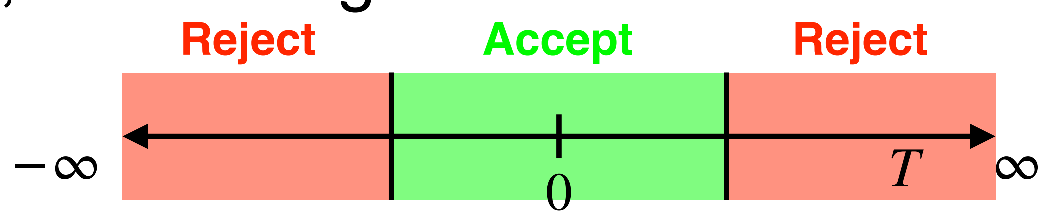

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Power

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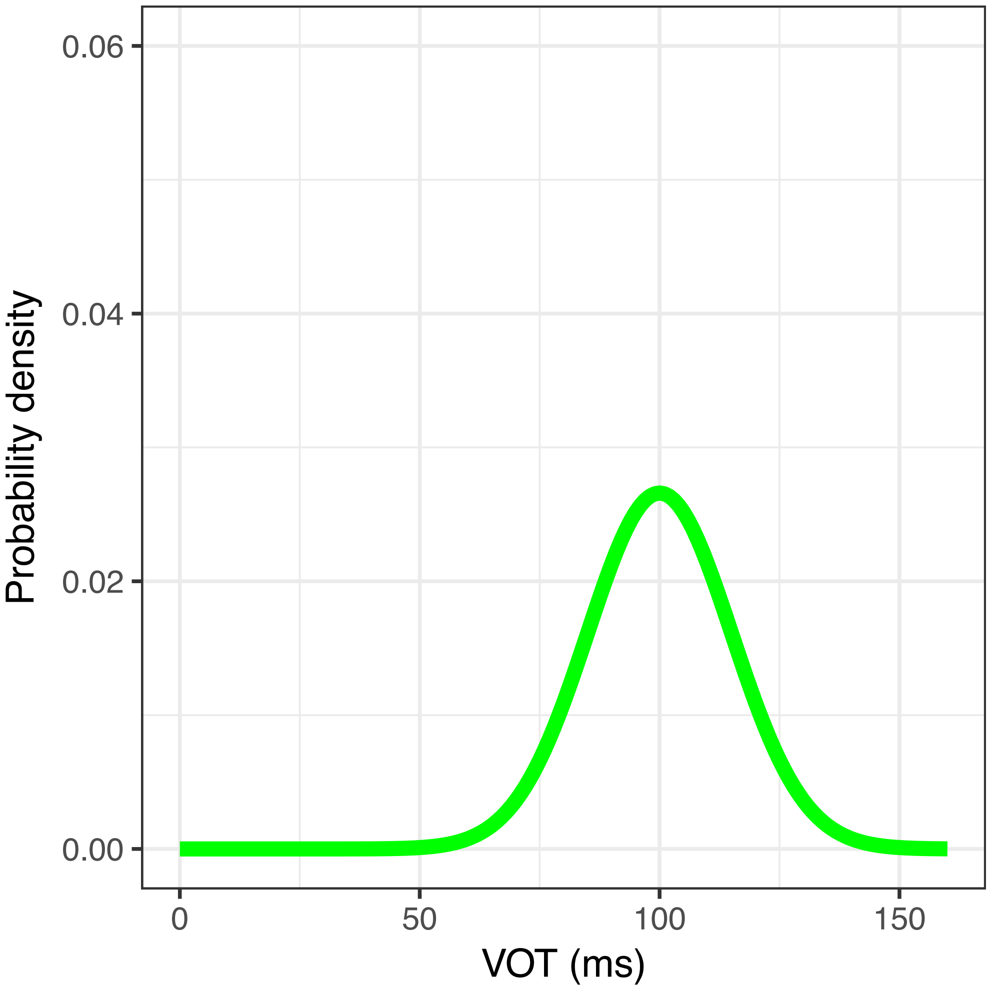

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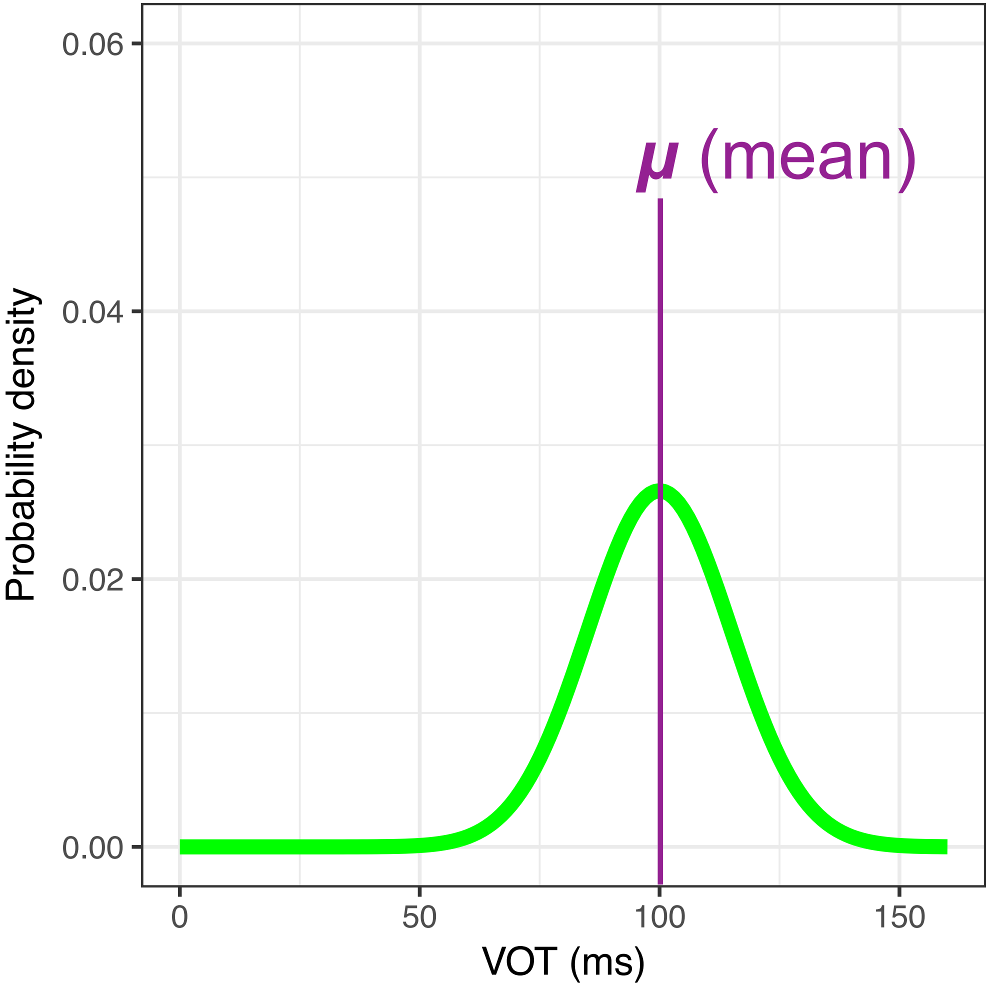

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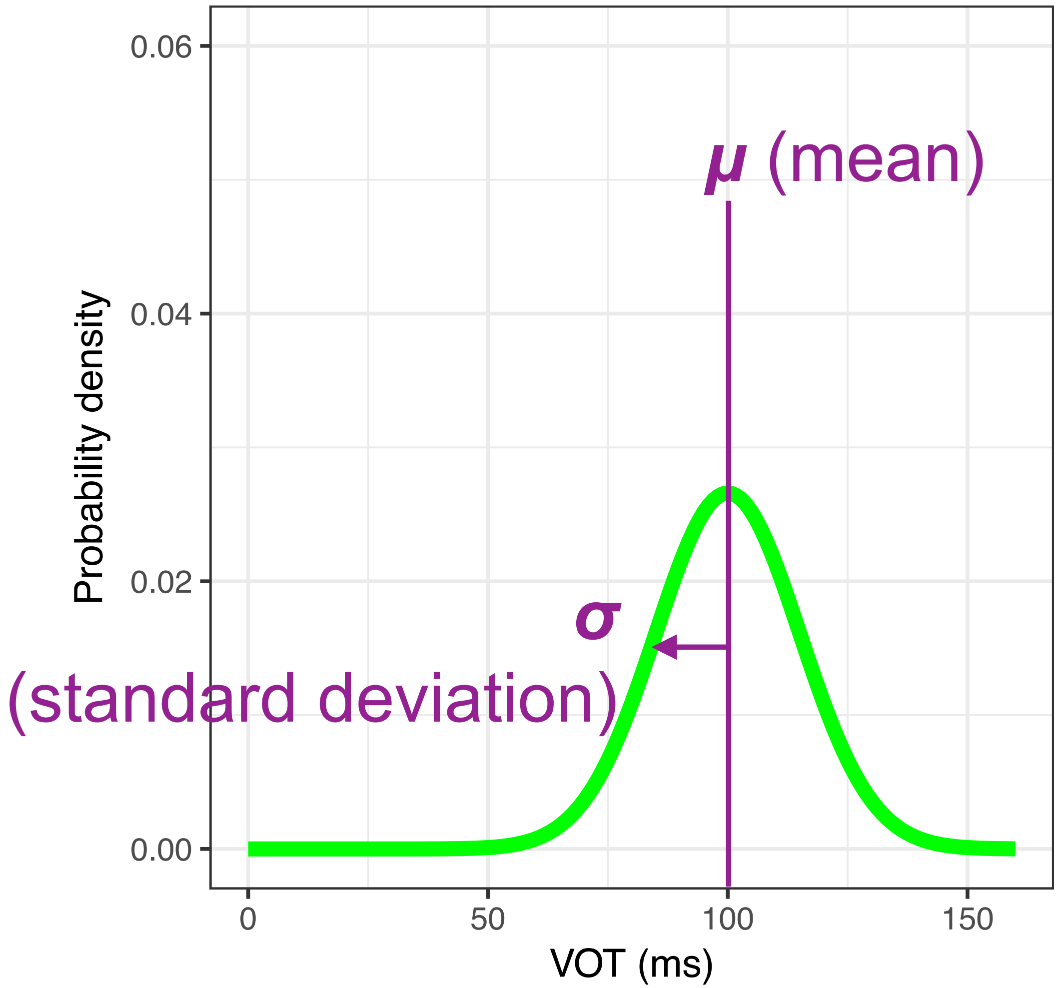

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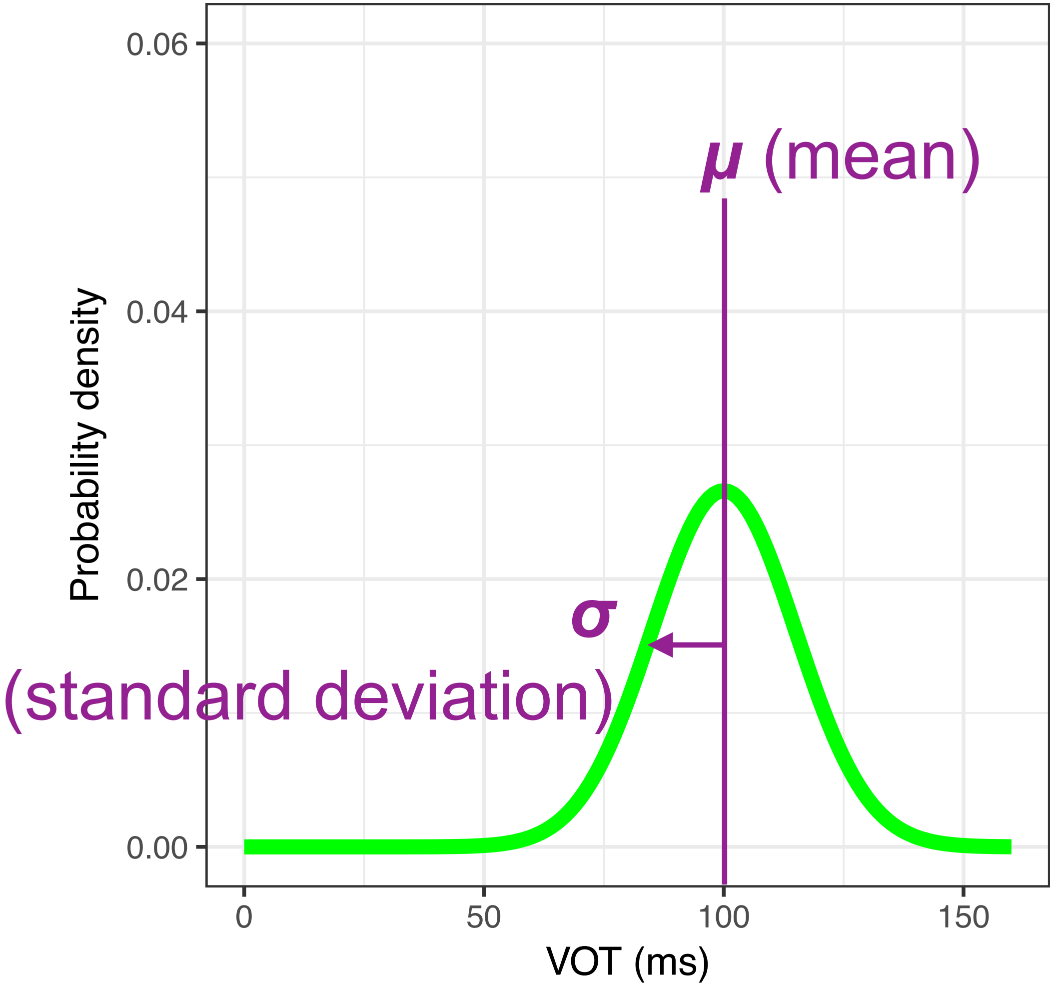

$$p(x\,|\,\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]$$

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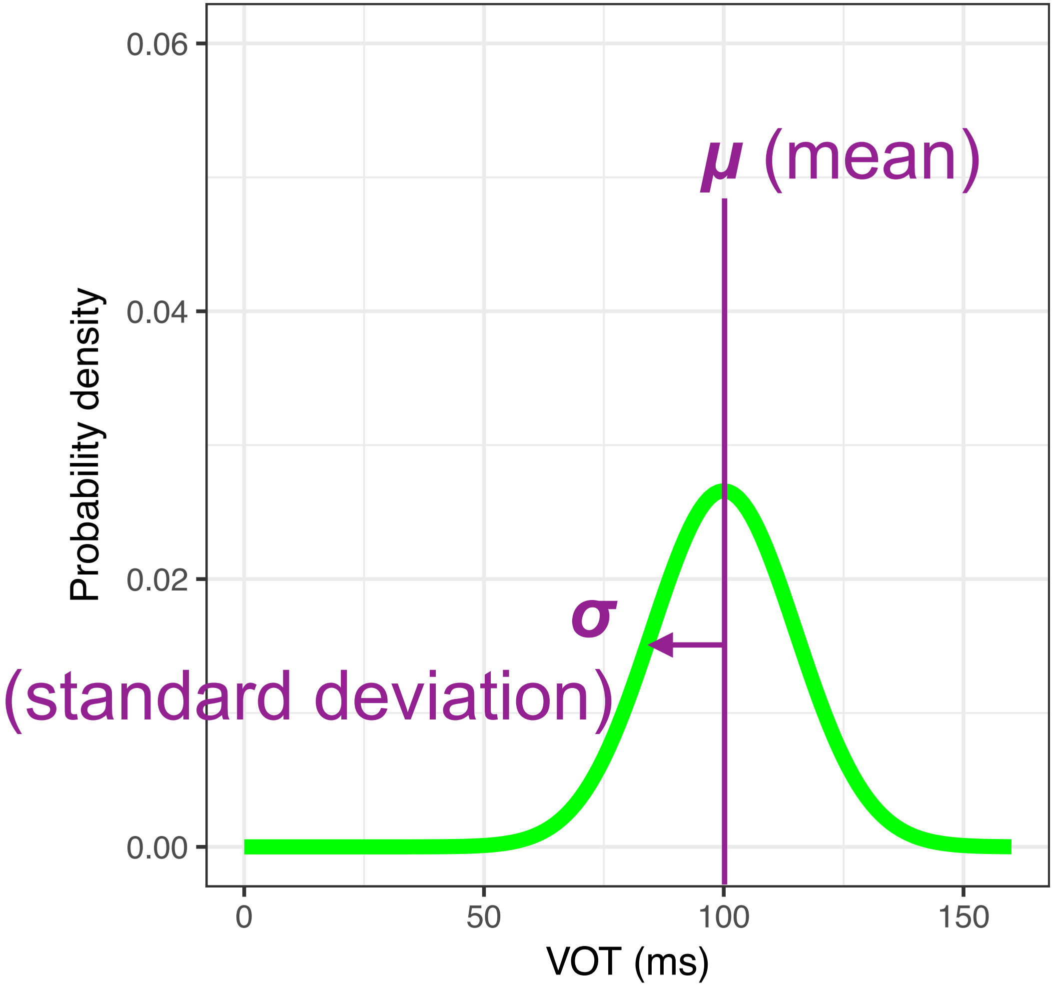

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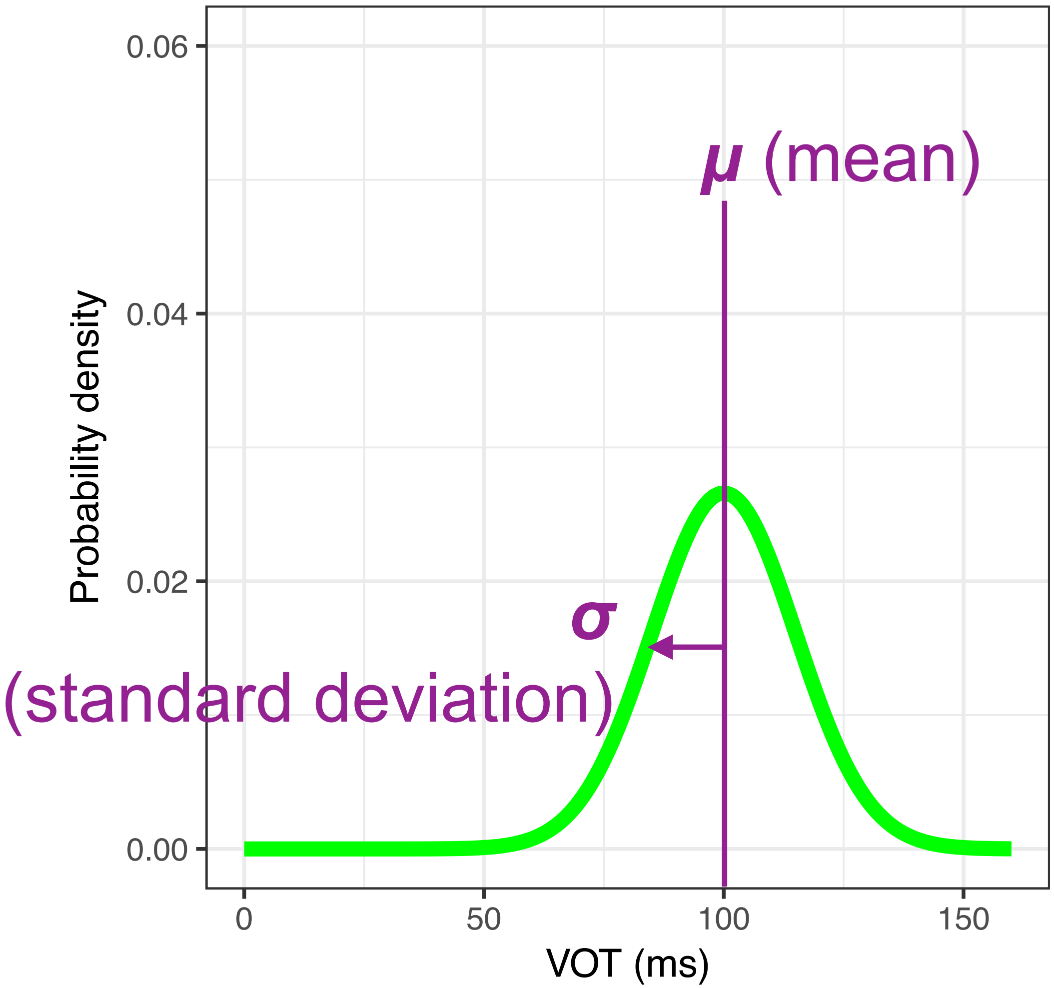

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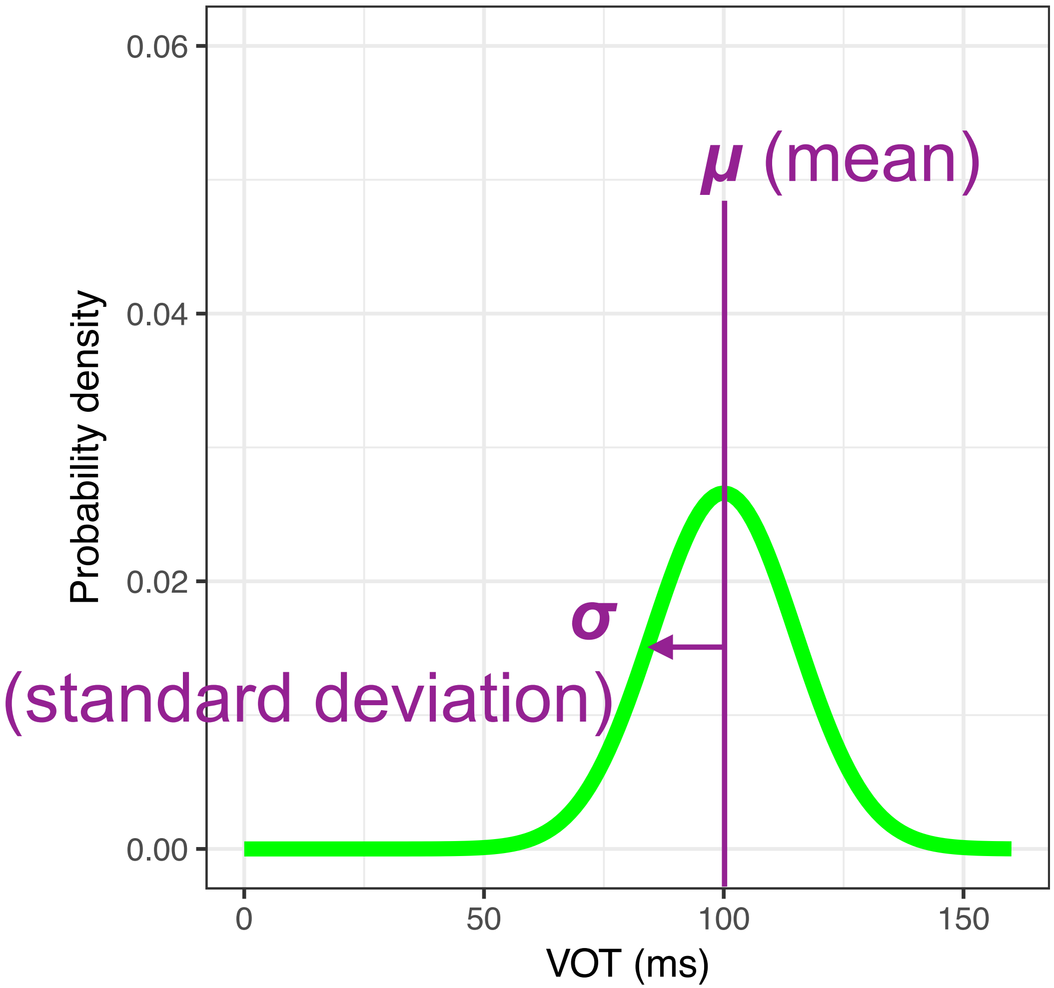

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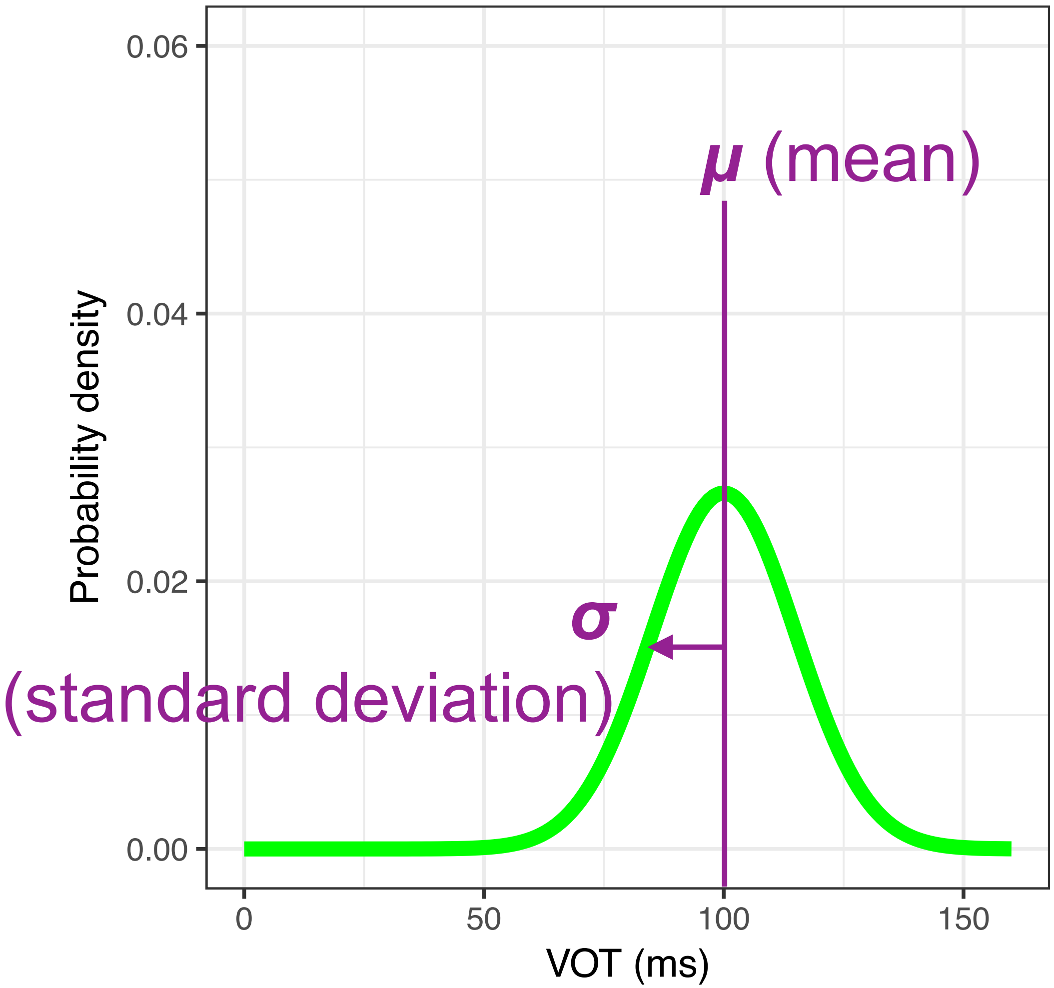

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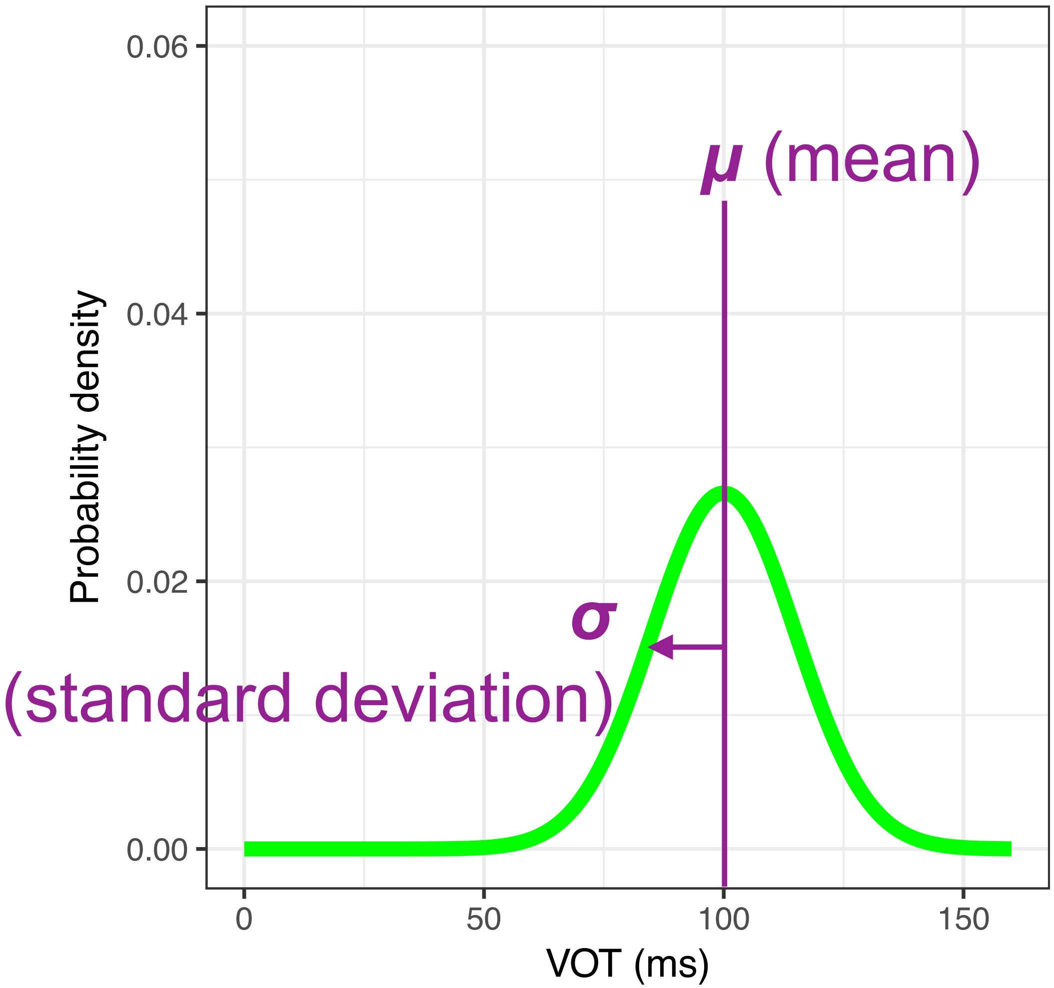

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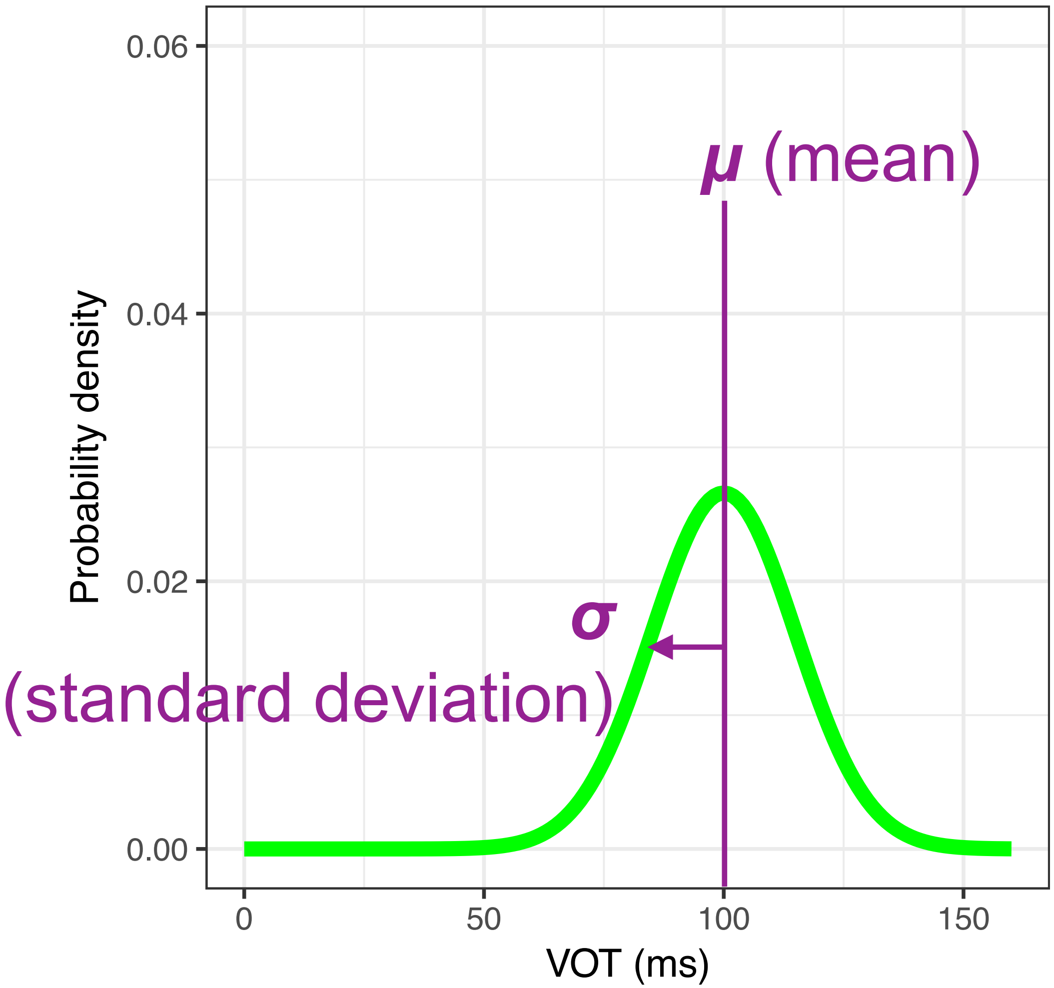

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The t-test: three variants

- **One sample (Student's) test:** Does the underlying population mean of a sample differ from zero?
- **Two-sample test (unpaired):** do the underlying population means of two samples differ from one another?
- **Two-sample test (paired):** You have a sample of individuals from the population and take measurements from each member of the sample in two different conditions. Do the underlying population means in the two conditions differ from one another?

William Sealy Gosset, a.k.a. Student

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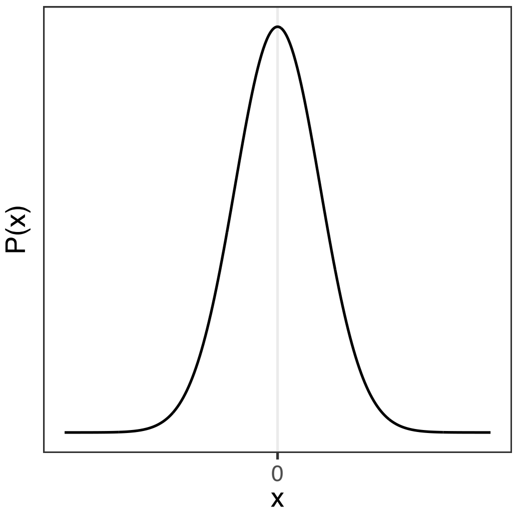

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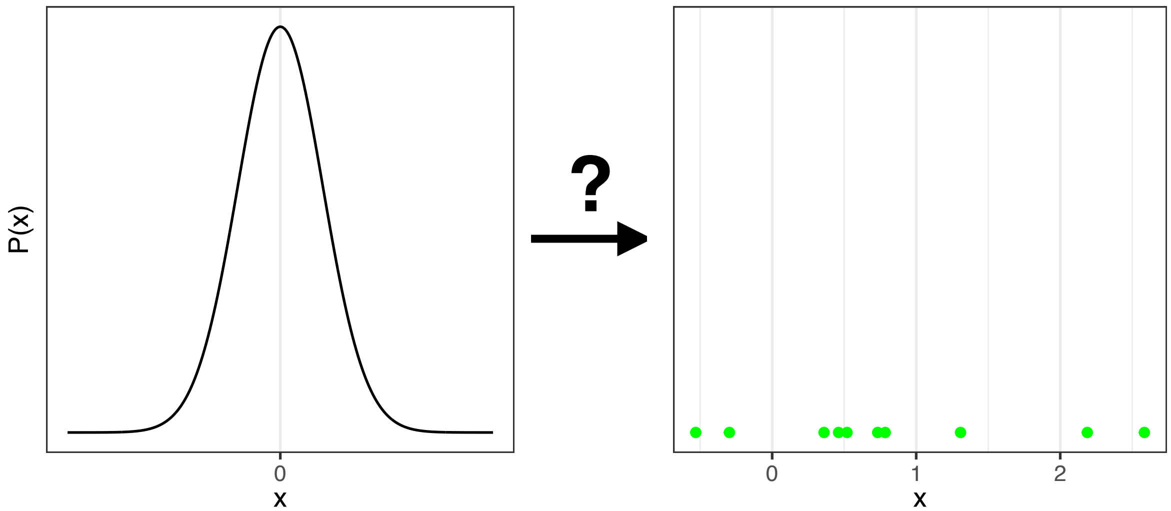

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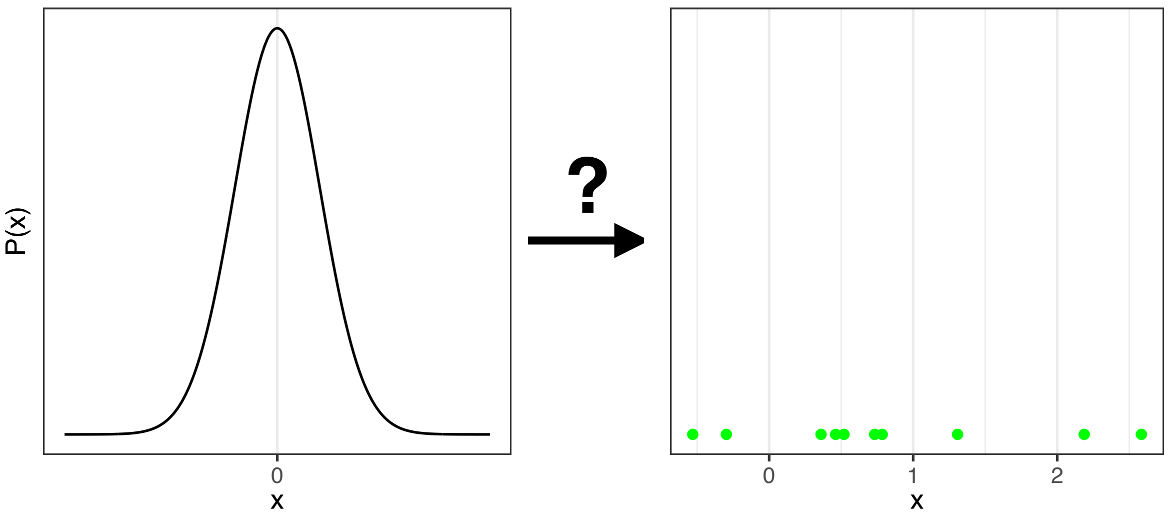

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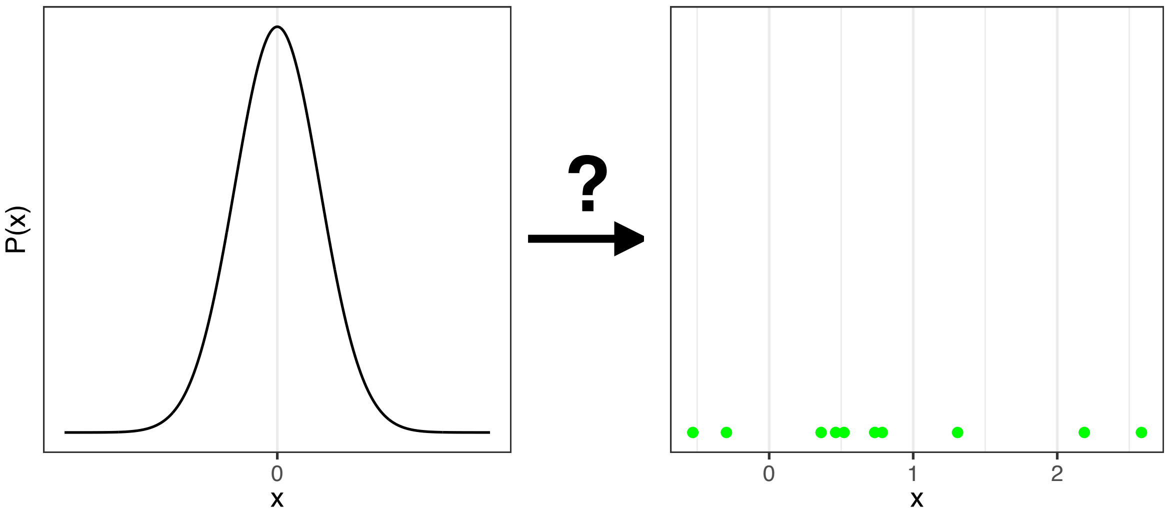

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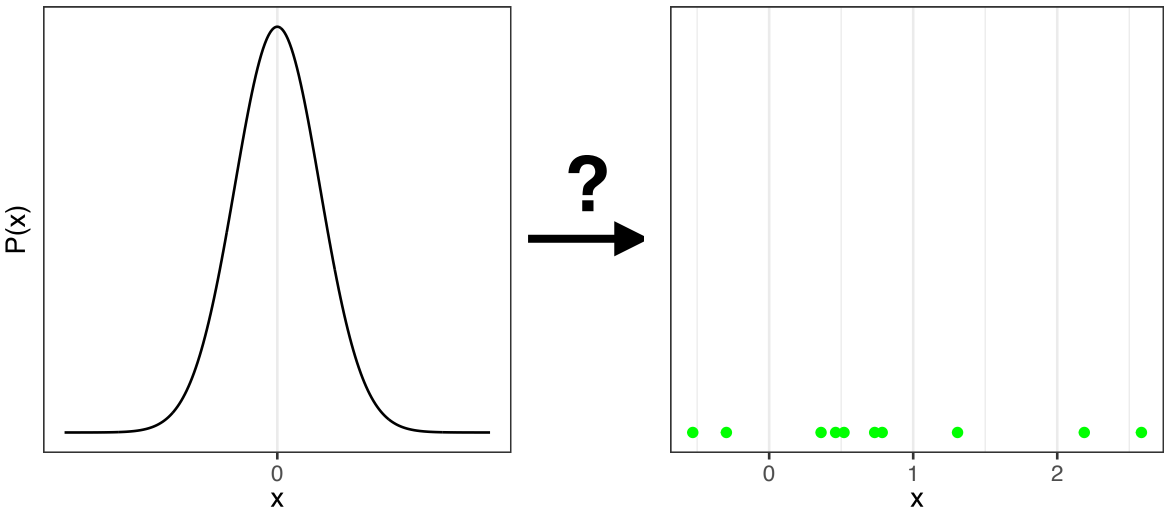

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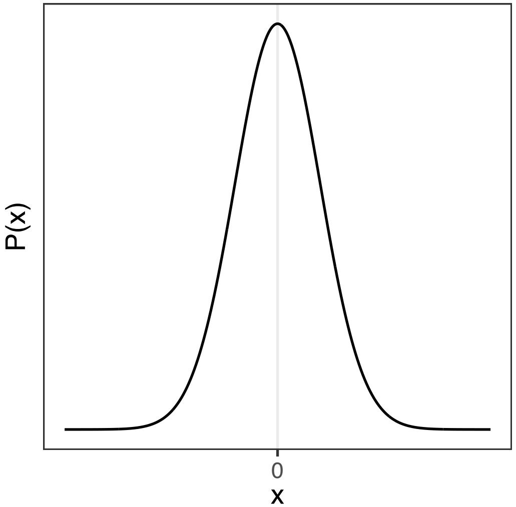

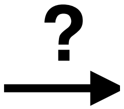

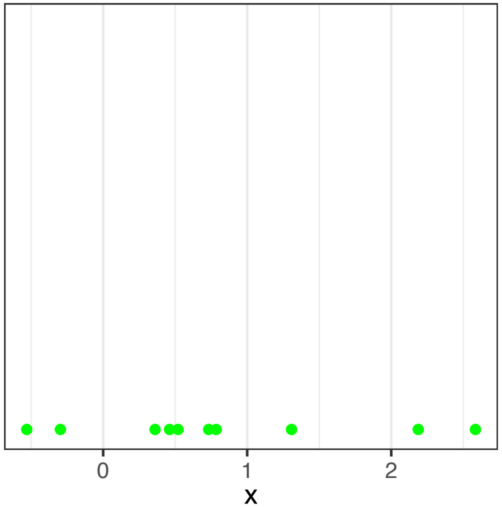

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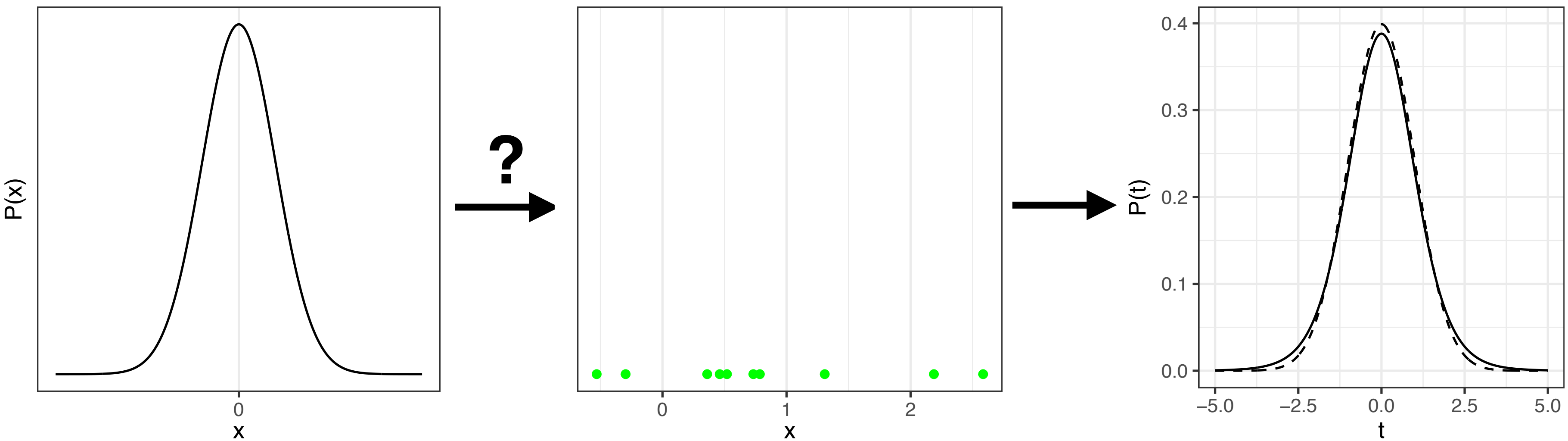

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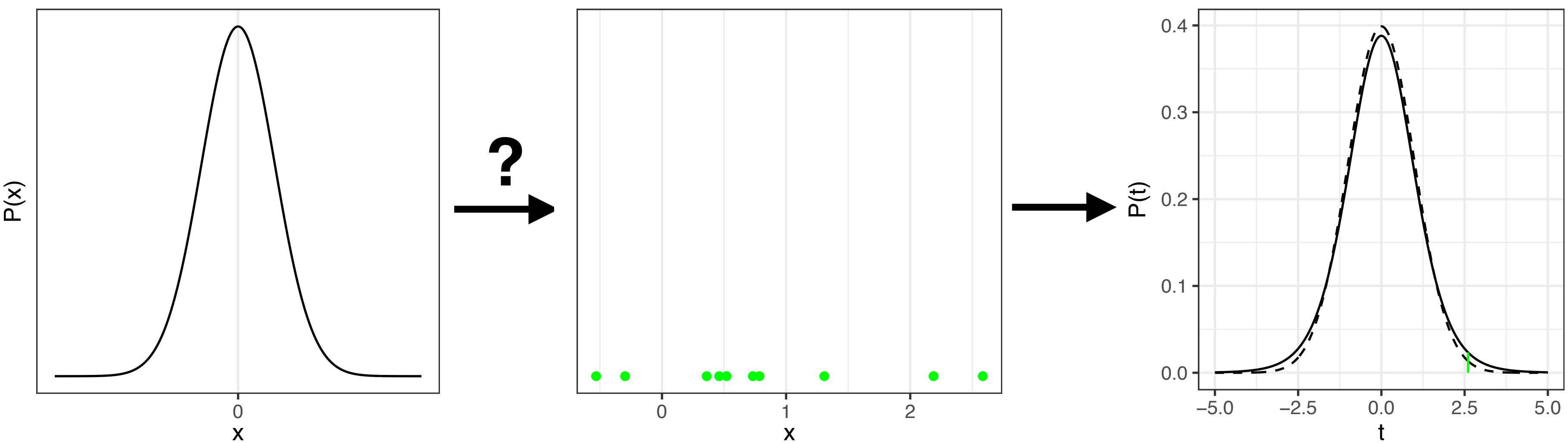

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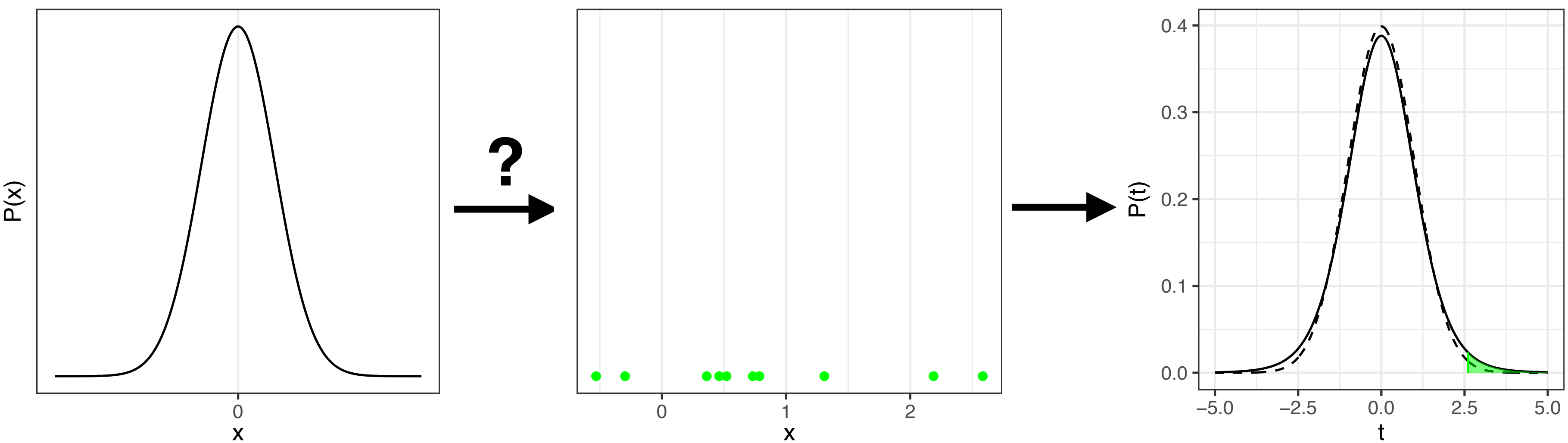

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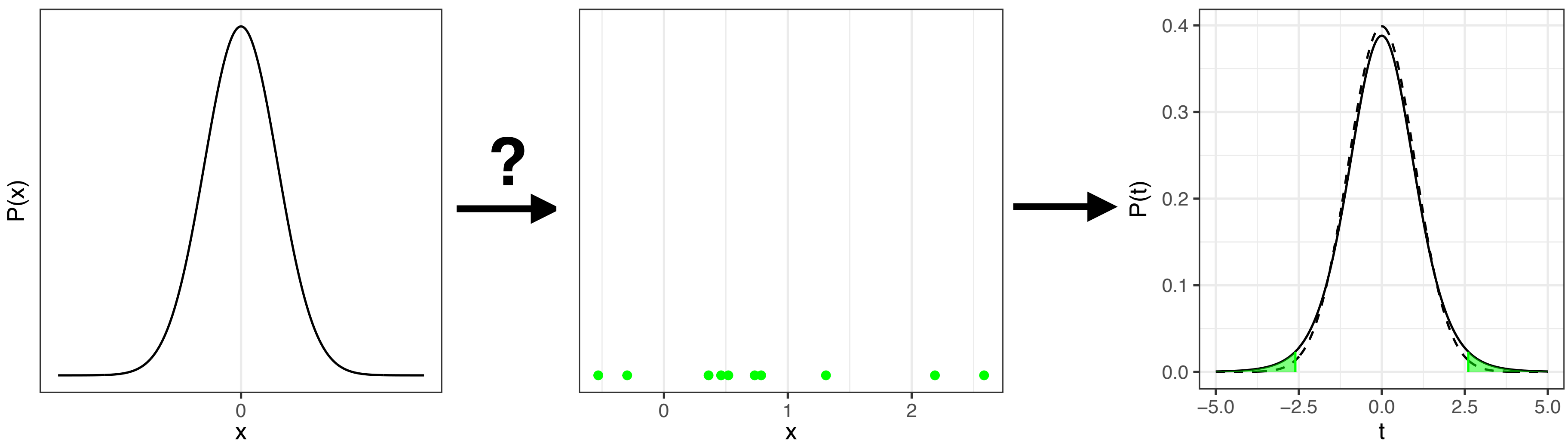

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t-distributed with a complex number of degrees of freedom whose formula can easily be looked up

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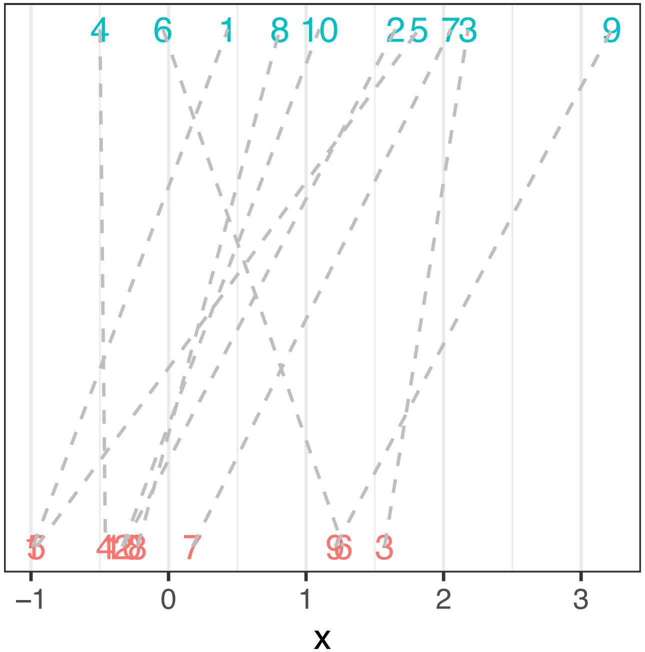

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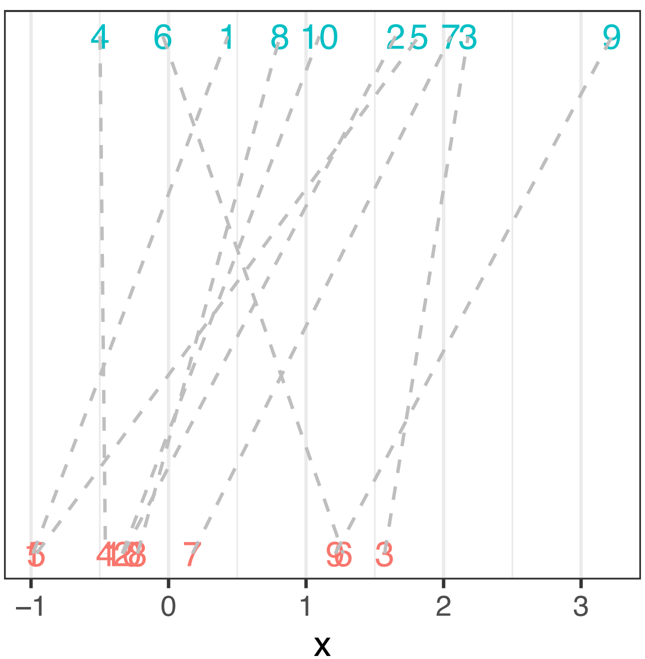

Paired two-sample t-test

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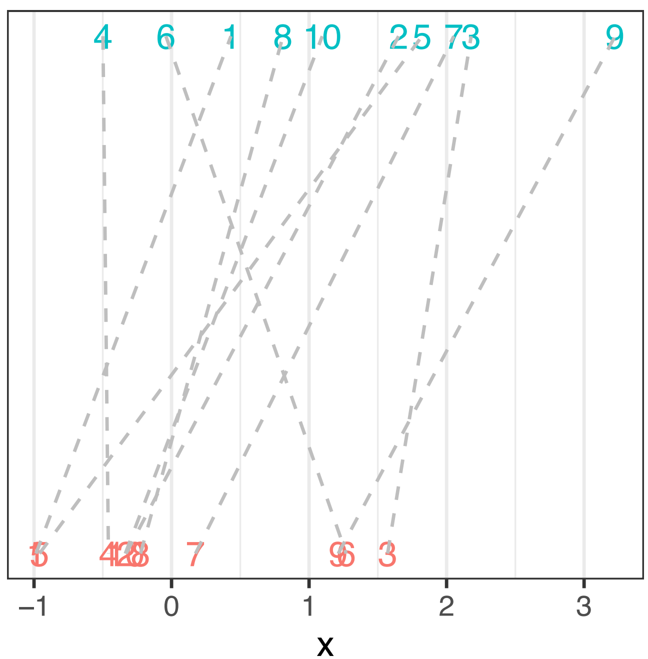

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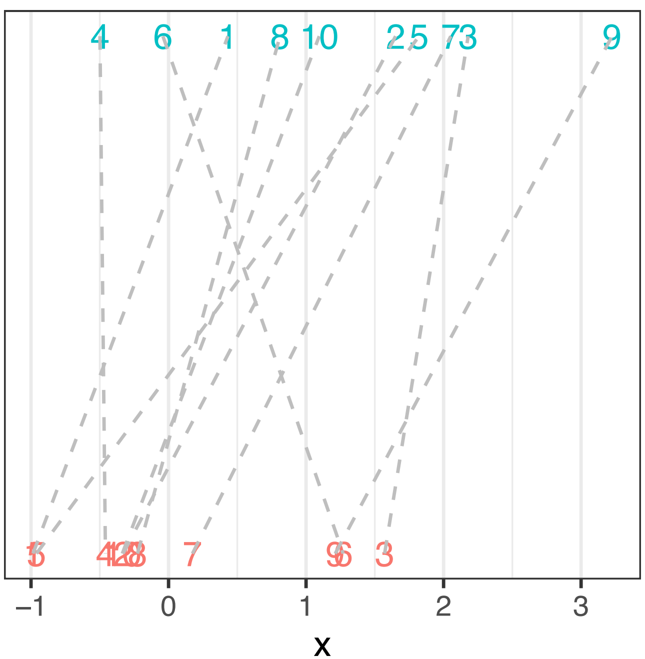

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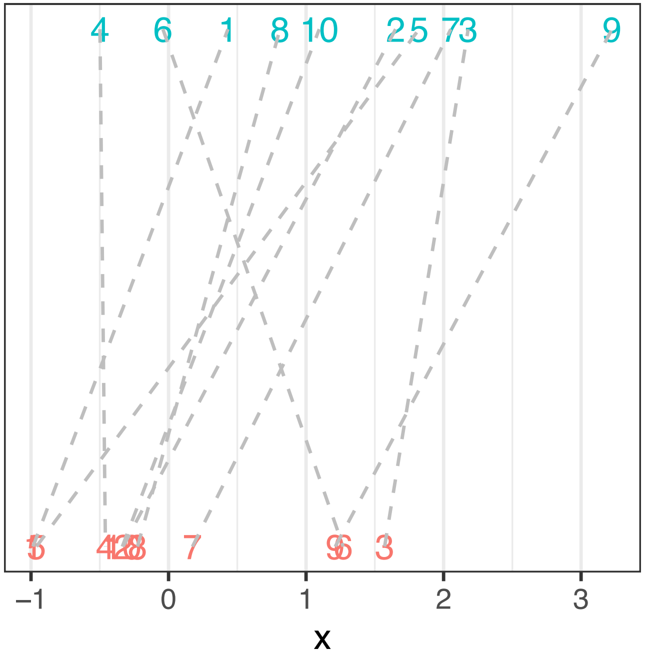

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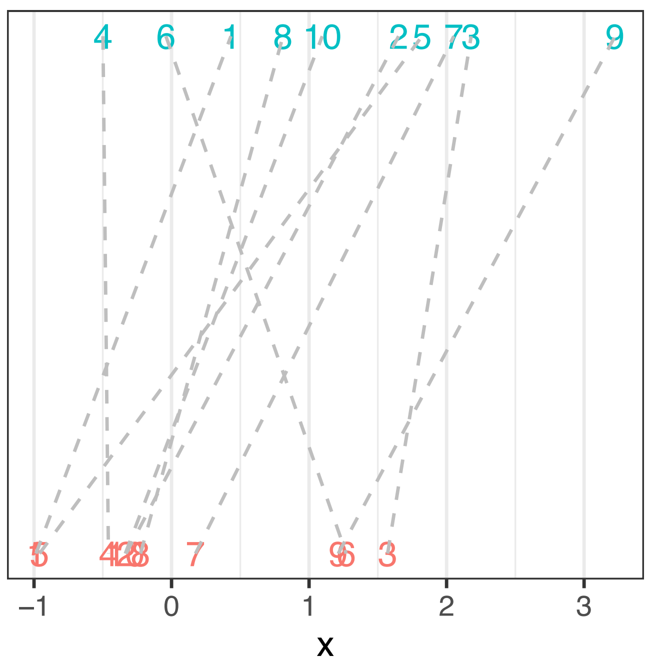

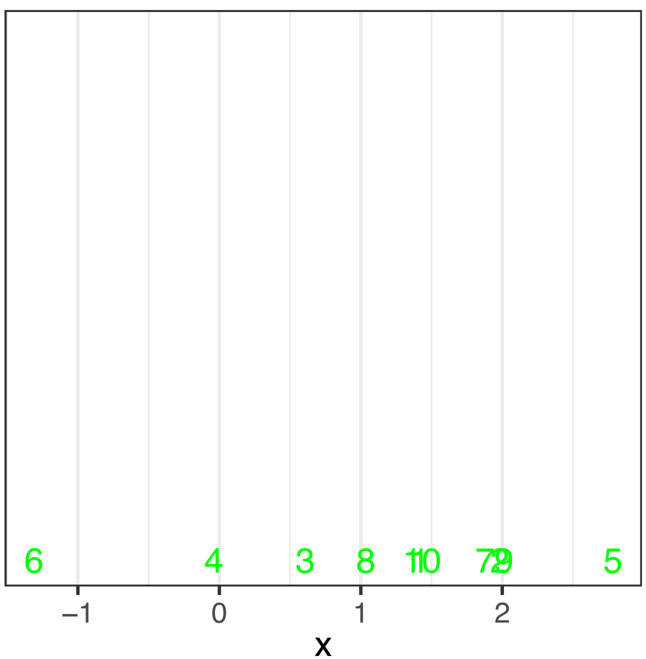

The likelihood ratio test

The likelihood ratio test

- The **likelihood ratio**:

$$\Lambda^* = \frac{\max \mathrm{Lik}_{H_0}(\boldsymbol{y})}{\max \mathrm{Lik}_{H_A}(\boldsymbol{y})}$$

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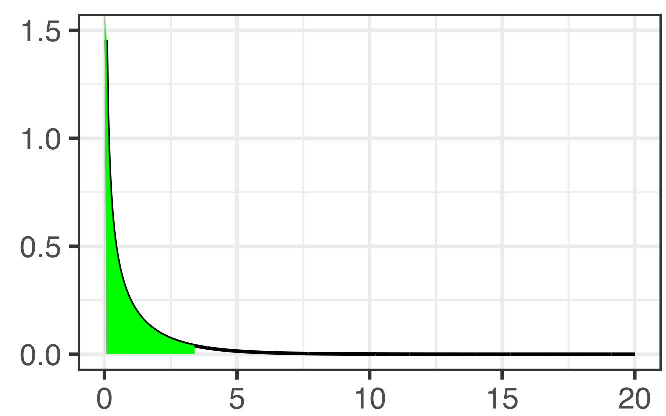

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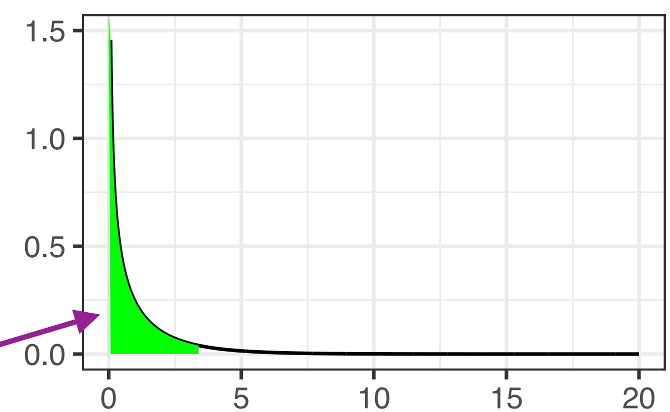


93.5% of probability mass under $G^2 \rightarrow p = 0.065$

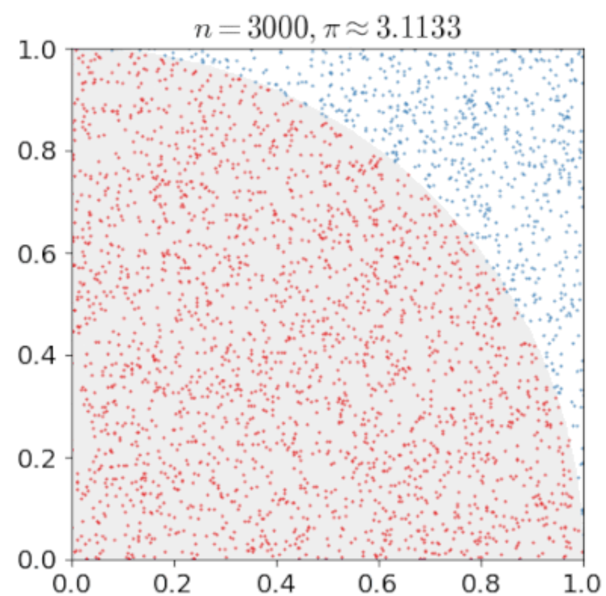
Simulation and approximate computation

Simulation and approximate computation

- All the statistical analysis that I've shown you so far has involved **exact** computation using **analytic expressions**
- This is facilitated by:
  - **Strong assumptions** regarding the data generating process (e.g., iid normal data for t-tests); and/or
  - **Conjugate priors** for Bayesian inference (e.g., Beta prior for Bernoulli/binomial data)
- But often, exact computation is not possible
- Solution: use more computationally intensive methods that don't rely on these strong assumptions. Examples:
  - Bootstrapped confidence intervals
  - Nonparametric statistical tests
  - Monte Carlo methods **(today)**

Monte Carlo methods, or "probabilistic simulation"

- Generally speaking:
  1. Define a domain of possible inputs
  2. Generate n iid random inputs from a probability distribution on the domain
  3. Perform a deterministic computation on each randomly generated input
  4. Aggregate the results of the deterministic computation
- As n grows larger, the simulated result approaches the true value



https://en.wikipedia.org/wiki/Monte_Carlo_method

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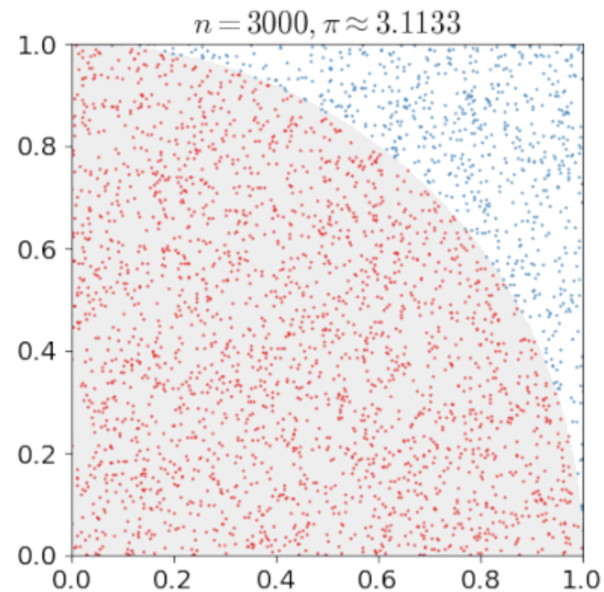

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Simple example of Monte Carlo

- Suppose I want to do a two-sample *t*-test but my data aren't normally distributed


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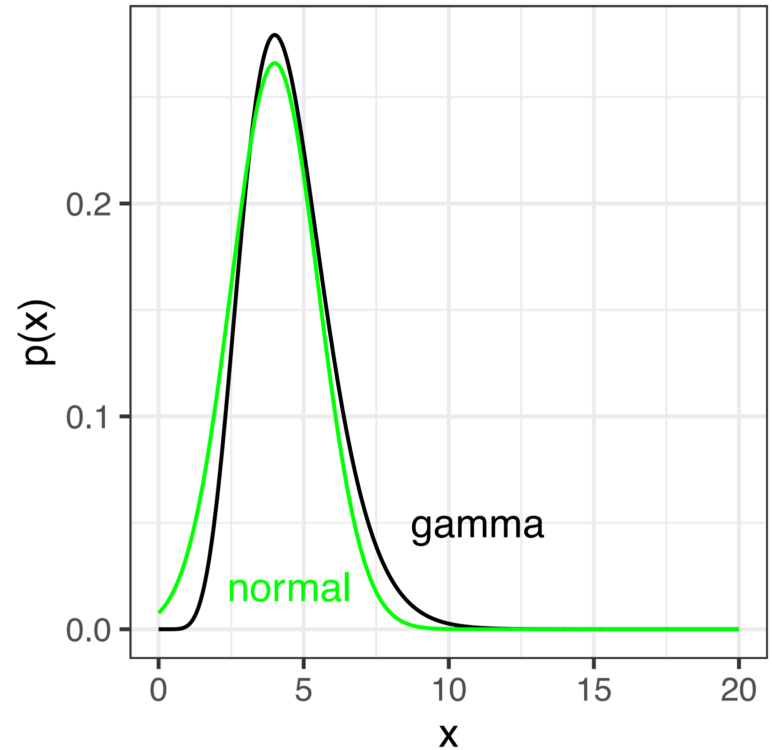
Student's *t*-test

From Wikipedia, the free encyclopedia

The ***t*-test** is any statistical hypothesis test in which the test statistic follows a Student's *t*-distribution under the null hypothesis.

A *t*-test is the most commonly applied when the test statistic would follow a normal distribution if the value of a scaling term in the test statistic were known. When the scaling term is unknown and is replaced by an estimate based on the data, the test statistics (under certain conditions) follow a Student's *t* distribution. The *t*-test can be used, for example, to determine if the means of two sets of data are significantly different from each other.



- **How bad will this be for my *t*-test????**

Monte Carlo, in action

```r
library(ggplot2)
library(tidyverse)

# Manually compute Student t-statistic
f <- function(seed,N=100,shape=9,scale=0.5) {
  set.seed(seed)
  y1 <- rgamma(N,shape=shape,scale=scale)
  y2 <- rgamma(N,shape=shape,scale=scale)
  s_p <- sqrt( (var(y1) + var(y2)) / 2 )
  t_statistic <- ( mean(y1) - mean(y2) ) / ( s_p*sqrt(2/N ) )
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N <- 100
Ts <- sapply(1:10000,f)

t_reference <- tibble(x=seq(-4,4,by=0.01),t=dt(x,df=2*(N-1)))

ggplot(data=tibble(t=Ts),aes(x=t)) +
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Compare against Student's *t* distribution

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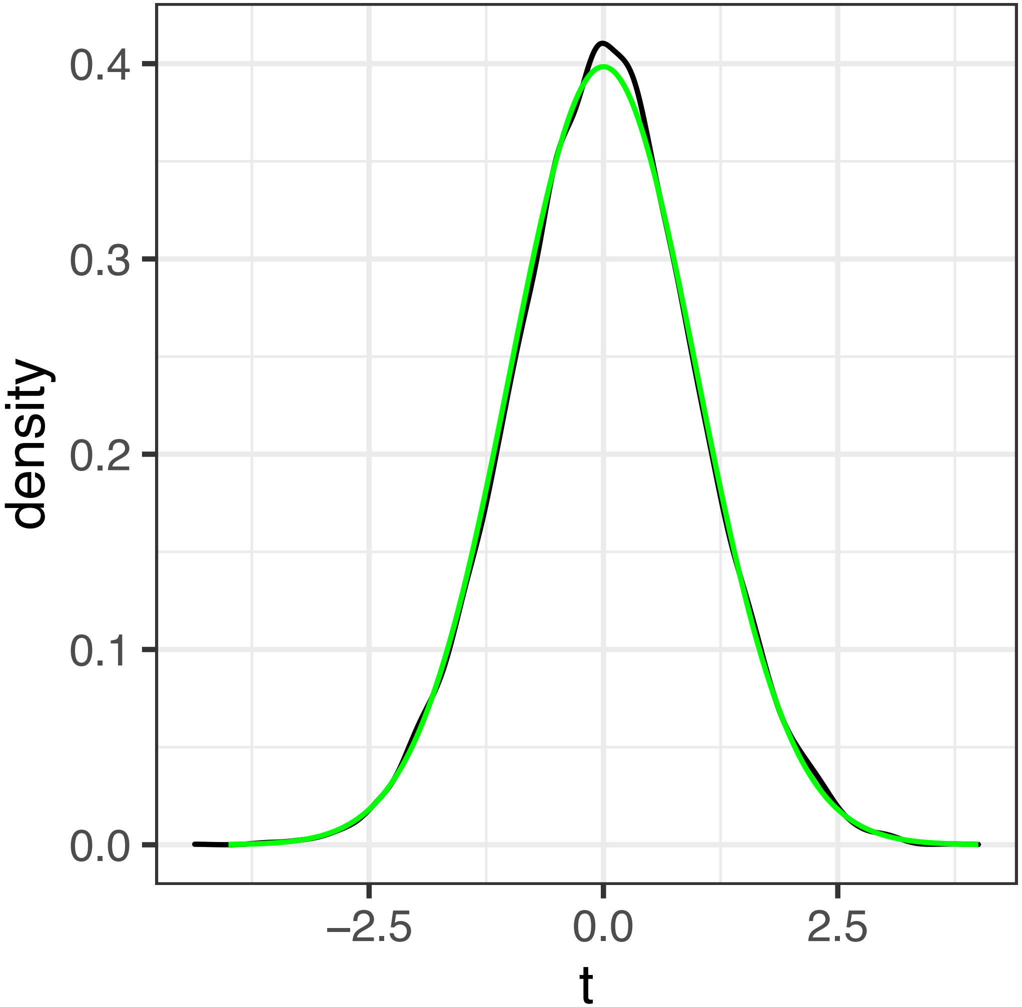

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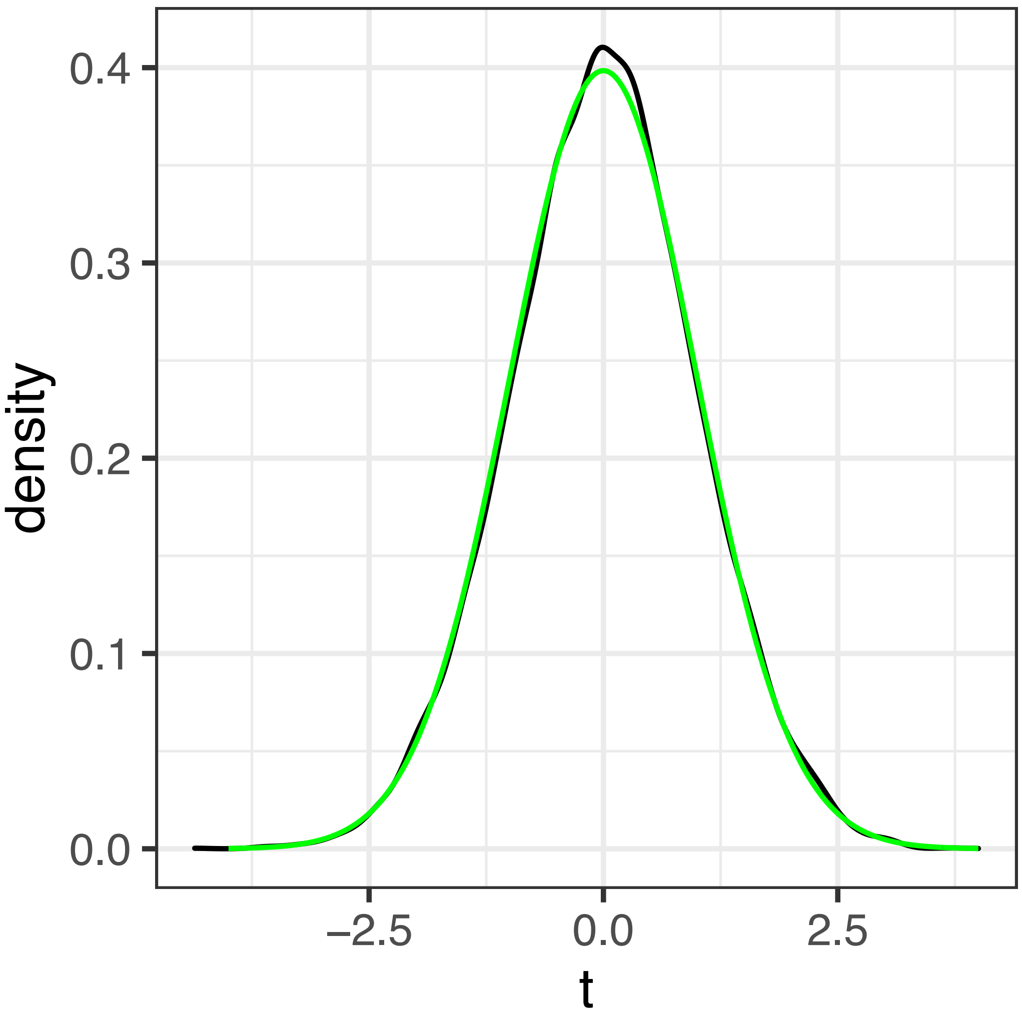

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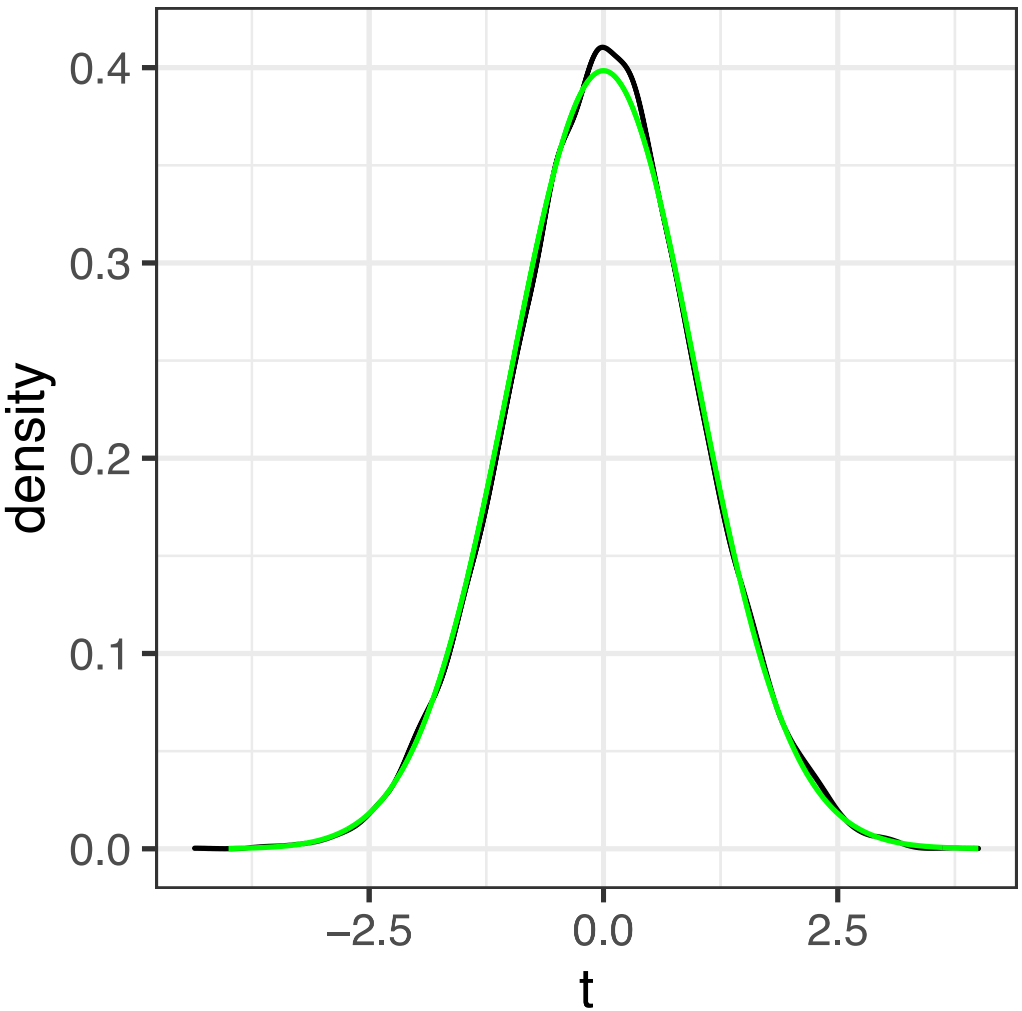


- The *t* distribution is still a pretty good approximation of the distribution of the *t* statistic, even when the underlying distribution is gamma!
- This exemplifies what is meant when people say that the *t* test is **robust to deviations from normality**

Unnormalizable posteriors

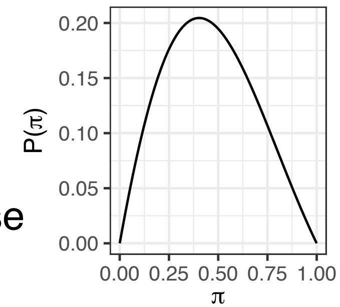
- Our motivation: Bayesian posterior inference

$$P(\theta \mid \mathbf{y}, I) = \frac{P(\mathbf{y} \mid \theta, I) P(\theta \mid I)}{P(\mathbf{y} \mid I)}$$

- Sometimes $P(\mathbf{y} \mid I)$ can't be calculated exactly. Example

Bernoulli data with non-conjugate prior:

$$P(\pi) \propto \begin{cases} \pi(1-\pi)e^{-\pi^2} & \pi \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$



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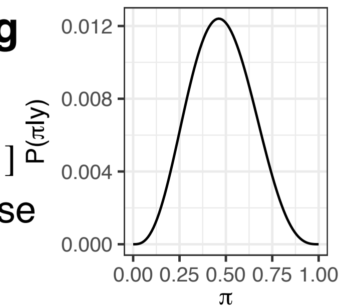
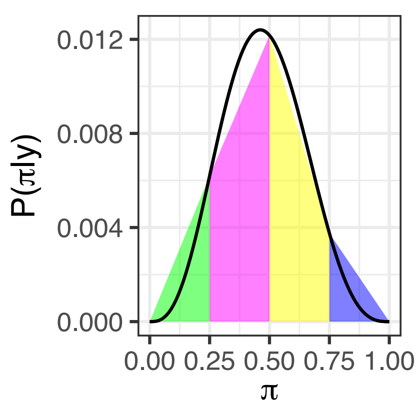


- In simple cases like this, we can numerically approximate the integral:



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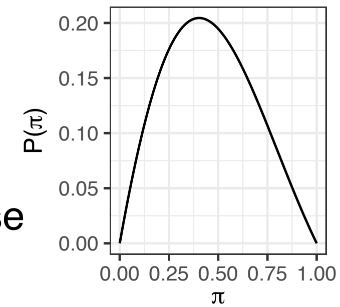

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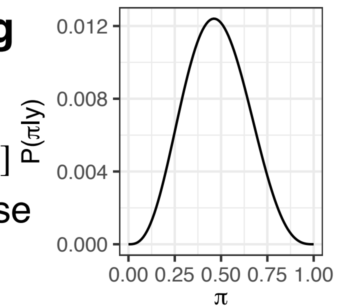

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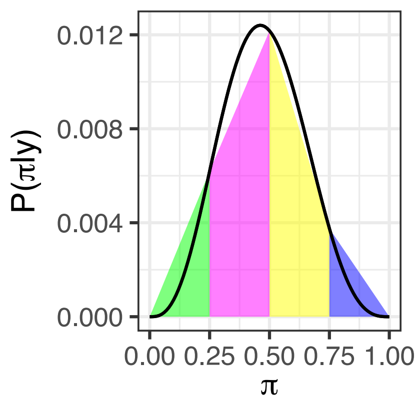

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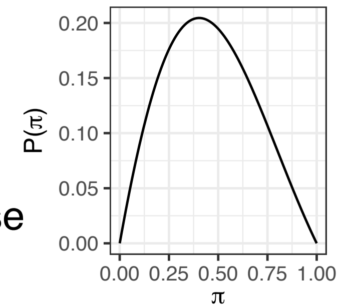

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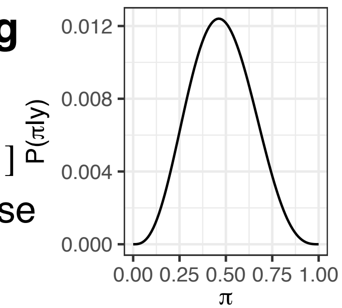

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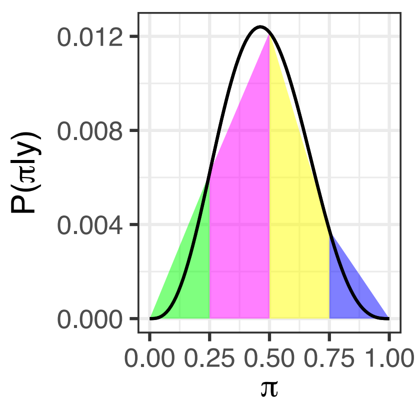

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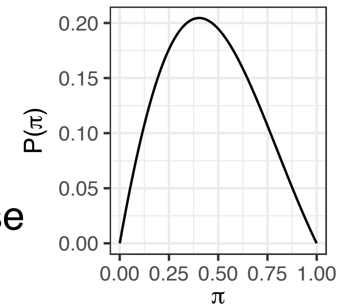

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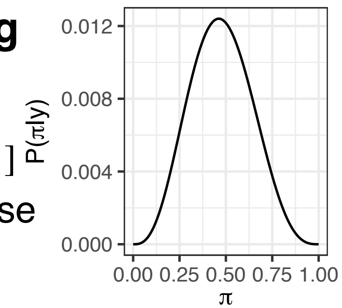

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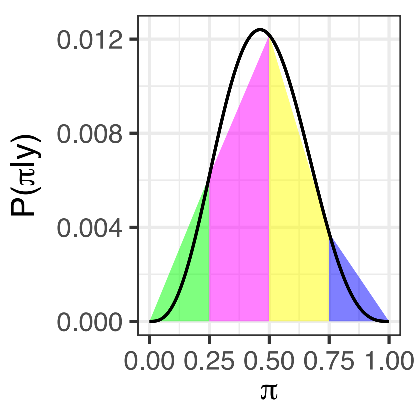

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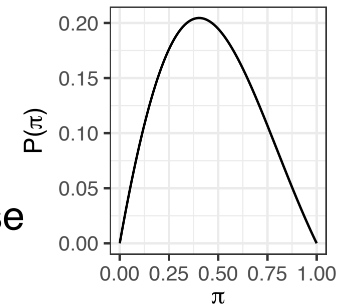

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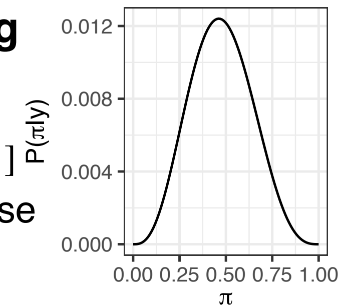
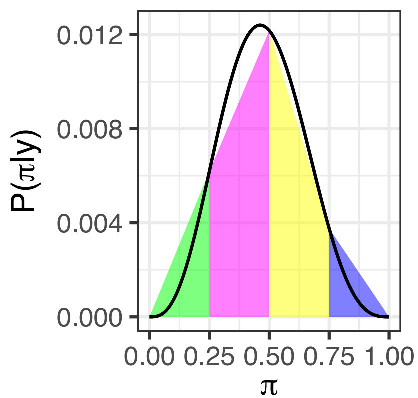
No closed form!



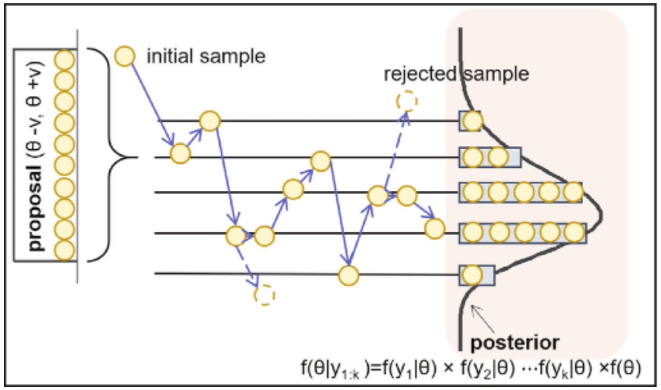
- In simple cases like this, we can numerically approximate the integral:



- But in high dimension and/or unbounded ranges, difficult or even impossible!

Markov chain Monte Carlo

- However, we can often **take samples from the posterior** even when we can't compute normalized probabilities
- One general and widely used approach: **Markov chain Monte Carlo (MCMC)**
- MCMC is a mathematically principled random walk on a non-negative function, directed toward regions where the function takes on a larger value



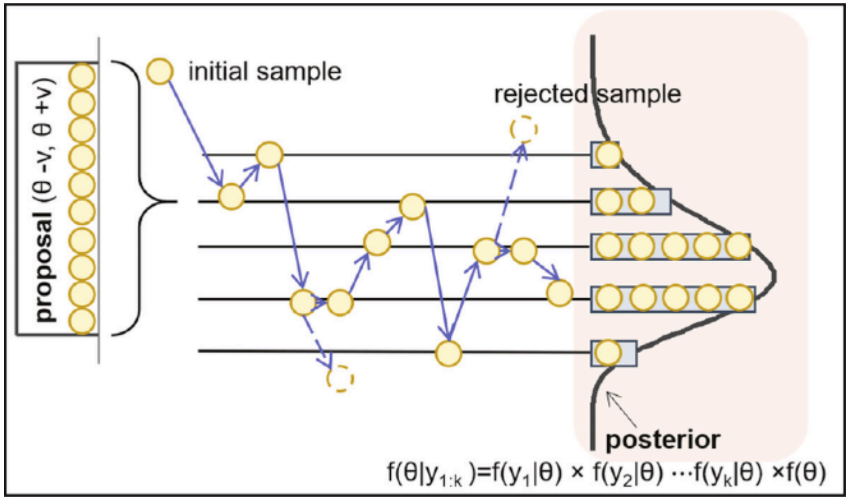
- Asymptotically, the random walk gives us samples from in proportion to the height of the function

Image credit: Dong et al., 2019, http://dx.doi.org/10.5772/intechopen.82781

MCMC for posterior sampling

- We use the *unnormalized* form of the posterior:

$$P(\theta \mid \mathbf{y}, I) \propto P(\mathbf{y} \mid \theta, I) P(\theta \mid I)$$

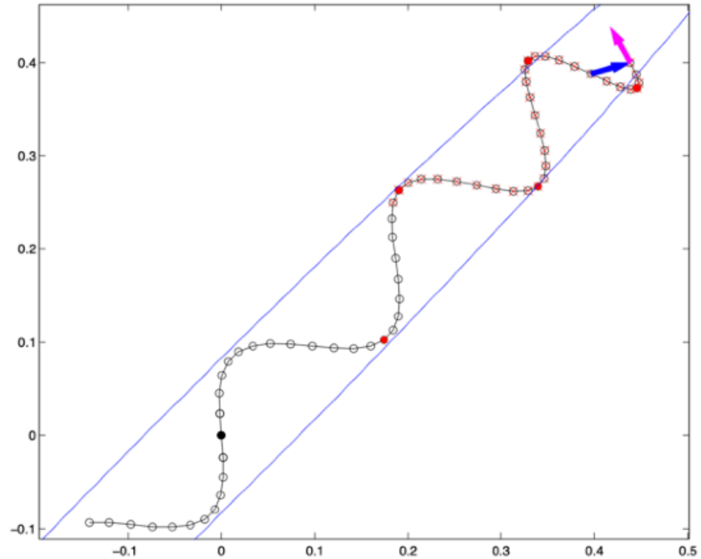


- We run MCMC and then treat the chain of values as samples from the posterior
- The full set of samples is not iid (nearby values on the chain are correlated), but methods exist for estimating "effectively" how many independent samples we have

Image credit: Dong et al., 2019, http://dx.doi.org/10.5772/intechopen.82781

Stan, HMC, and NUTS

- There are many different MCMC algorithms (e.g., Metropolis, Gibbs Sampling)
- We will use the probabilistic programming language **Stan** for Bayesian inference about model parameters
- Stan uses an algorithm called **Hamiltonian Monte Carlo** (**HMC**) with the **No U-Turn Sampler (NUTS)**



(Hoffman & Gelman, 2014)

- This algorithm tends to be particularly efficient for many problems we'll face

Bayesian posterior inference with Stan

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4. Provide the data Y you want to analyze, and ask Stan to sample from the posterior $P(\theta \mid Y, I) \propto P(Y \mid \theta) P(\theta \mid I)$ (often written as $P(\theta \mid Y) \propto P(Y \mid \theta) P(\theta)$, i.e. eliding I)

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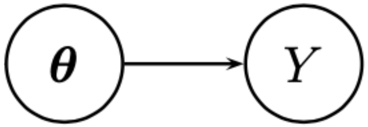

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$$I \rightarrow \boldsymbol{\theta} \rightarrow Y$$