

# **9.S918: Statistical Inference in Brain and Cognitive Sciences**

**Week 2 Day 1: Introduction to causal inference**

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# Introductory causal inference

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- You have probably had previous exposure to both probability and statistics
- You are less likely to have had exposure to **causal inference**
- Causal inference uses probability and statistics, but it is something separate from the traditional construal of those two fields
- You can think of causal inference as being a framework extending more traditional statistics by:
  - Adding new probability-based mathematical constructs; and,
  - Developing a set of practice for statistical inference based on those constructs
- Two causal inference frameworks:
  - The **potential outcomes** framework
  - The **causal graphical models** framework

# The potential-outcomes framework

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- In epidemiology and many other areas of statistics, causal inference was developed out of the idea of **potential outcomes** (Neyman 1923, Rubin 1974)
- Consider an outcome,  $Y$ , and a potential **treatment**  $A$
- **Example:**  
 $Y$ : an individual survives to the end of the year (0: no, 1: yes)  
 $A$ : an individual with heart disease receives a heart transplant (0: no, 1: yes)

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- The **potential outcomes**, or **counterfactual outcomes**, are random variables for  $Y$  for each potential value of  $A$

$Y^{a=0}$

The value that  $Y$  would take if  $A$  were 0

$Y^{a=1}$

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The value that  $Y$  would take if  $A$  were 1

- **Counterfactual risk** is the **expected value** of each counterfactual-outcome random variable:

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$$E[X] = \sum_x xP(X = x)$$



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- So we are interested in (and likewise for  $Y^{a=1}$ ):

$$E[Y^{a=0}] = \sum_y yP(Y^{a=0} = y) = 0 \times P(Y^{a=0} = 0) + 1 \times P(Y^{a=0} = 1) = P(Y^{a=0} = 1)$$

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# Counterfactual data and causal effects

(Hernan & Robins, 2020, Table 1.1)

	$Y^{a=0}$	$Y^{a=1}$
Rheia	0	1
Kronos	1	0
Demeter	0	0
Hades	0	0
Hestia	0	0
Poseidon	1	0
Hera	0	0
Zeus	0	1
Artemis	1	1
Apollo	1	0
Leto	0	1
Ares	1	1
Athena	1	1
Hephaestus	0	1
Aphrodite	0	1
Cyclope	0	1
Persephone	1	1
Hermes	1	0
Hebe	1	0
Dionysus	1	0
$P(Y^{a=*}) = 1$	0.5	0.5

# Counterfactual data and causal effects

- Suppose we knew **what would happen** for each individual in the population under each value of the treatment

(Hernan & Robins, 2020, Table 1.1)

	$Y^{a=0}$	$Y^{a=1}$
Rheia	0	1
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$$E[Y^{a=1}] = 0.5$$

- The **average causal effect** of treatment  $A$  is defined as the difference of counterfactual risks:

$$E[Y^{a=1}] - E[Y^{a=0}] = 0$$

(Hernan & Robins, 2020, Table 1.1)

	$Y^{a=0}$	$Y^{a=1}$
Rheia	0	1
Kronos	1	0
Demeter	0	0
Hades	0	0
Hestia	0	0
Poseidon	1	0
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$P(Y^{a=*}) = 1$	0.5	0.5

# Counterfactual data and causal effects

- Suppose we knew **what would happen** for each individual in the population under each value of the treatment
- Then we could compute the counterfactual risks:

$$E[Y^{a=0}] = 0.5 \qquad E[Y^{a=1}] = 0.5$$

- The **average causal effect** of treatment  $A$  is defined as the difference of counterfactual risks:

$$E[Y^{a=1}] - E[Y^{a=0}] = 0$$

- Here, treatment is **ineffective**

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$P(Y^{a=*}) = 1$	0.5	0.5

# Estimating causal effects

Remember,  $E[Y^{a=i}] = P(Y^{a=i} = 1)$

	$L$	$A$	$Y$	$Y^0$	$Y^1$
Rheia	0	0	0	0	?
Kronos	0	0	1	1	?
Demeter	0	0	0	0	?
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Poseidon	0	1	0	?	0
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- Naively, we might estimate the counterfactual risks  $P(Y^{a=i} = 1)$  directly from observed  $A$  and  $Y$ :

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*Crucial step;  
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$$= \frac{\text{Count}(Y^{a=1} = 1 \wedge A = i)}{\text{Count}(A = i)} = \hat{P}_{MLE}(Y^{a=i} = 1 | A = i)$$

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- But under what circumstances  $\hat{P}_{MLE}(Y | A = i) = \hat{P}_{MLE}(Y^{a=i} = 1)$ ?

- The following is certainly true:

$$\hat{P}_{MLE}(Y = 1 | A = i) = \frac{\text{Count}(Y = 1 \wedge A = i)}{\text{Count}(A = i)}$$

**CONSISTENCY:** when  
 $A = i, Y = Y^{a=i}$

$$= \frac{\text{Count}(Y^{a=1} = 1 \wedge A = i)}{\text{Count}(A = i)}$$

$$= \hat{P}_{MLE}(Y^{a=i} = 1 | A = i)$$

*Crucial step;  
make sure you  
understand it!*

- So, the following condition suffices:

$$P(Y^{a=i} = 1 | A = i) = P(Y^{a=i} = 1)$$

# Estimating causal effects

Remember,  $E[Y^{a=i}] = P(Y^{a=i} = 1)$

	$L$	$A$	$Y$	$Y^0$	$Y^1$
Rheia	0	0	0	0	?
Kronos	0	0	1	1	?
Demeter	0	0	0	0	?
Hades	0	0	0	0	?
Hestia	0	1	0	?	0
Poseidon	0	1	0	?	0
Hera	0	1	0	?	0
Zeus	0	1	1	?	1
Artemis	1	0	1	1	?
Apollo	1	0	1	1	?
Leto	1	0	0	0	?
Ares	1	1	1	?	1
Athena	1	1	1	?	1
Hephaestus	1	1	1	?	1
Aphrodite	1	1	1	?	1
Polyphemos	1	1	1	?	1
Persephone	1	1	1	?	1
Hermes	1	1	0	?	0
Hebe	1	1	0	?	0
Dionysus	1	1	0	?	0

- Naively, we might estimate the counterfactual risks  $P(Y^{a=i} = 1)$  directly from observed  $A$  and  $Y$ :

$$\hat{P}_{MLE}(Y = 1 | A = 0) = \frac{3}{7} \quad \hat{P}_{MLE}(Y = 1 | A = 1) = \frac{7}{13}$$

- But under what circumstances  $\hat{P}_{MLE}(Y | A = i) = \hat{P}_{MLE}(Y^{a=i} = 1)$ ?
- The following is certainly true:

$$\hat{P}_{MLE}(Y = 1 | A = i) = \frac{\text{Count}(Y = 1 \wedge A = i)}{\text{Count}(A = i)}$$

**CONSISTENCY:** when  
 $A = i, Y = Y^{a=i}$

$$= \frac{\text{Count}(Y^{a=1} = 1 \wedge A = i)}{\text{Count}(A = i)}$$

$$= \hat{P}_{MLE}(Y^{a=i} = 1 | A = i)$$

*Crucial step;  
make sure you  
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- So, the following condition suffices:
- This is called **EXCHANGEABILITY**:

$$Y^a \perp A | \{ \}$$

# Exchangeability and randomization

**Goal:**  $\hat{P}(Y^a = 1)$

	$Y^{a=0}$	$Y^{a=1}$
Rheia	0	1
Kronos	1	0
Demeter	0	0
Hades	0	0
Hestia	0	0
Poseidon	1	0
Hera	0	0
Zeus	0	1
Artemis	1	1
Apollo	1	0
Leto	0	1
Ares	1	1
Athena	1	1
Hephaestus	0	1
Aphrodite	0	1
Polyphemus	0	1
Persephone	1	1
Hermes	1	0
Hebe	1	0
Dionysus	1	0

# Exchangeability and randomization

**Goal:**  $\hat{P}(Y^a = 1)$

- Why is a randomized experiment so powerful?

	$Y^{a=0}$	$Y^{a=1}$
Rheia	0	1
Kronos	1	0
Demeter	0	0
Hades	0	0
Hestia	0	0
Poseidon	1	0
Hera	0	0
Zeus	0	1
Artemis	1	1
Apollo	1	0
Leto	0	1
Ares	1	1
Athena	1	1
Hephaestus	0	1
Aphrodite	0	1
Polyphemus	0	1
Persephone	1	1
Hermes	1	0
Hebe	1	0
Dionysus	1	0

# Exchangeability and randomization

**Goal:**  $\hat{P}(Y^a = 1)$

- Why is a randomized experiment so powerful?
- Recap of exchangeability criterion:

$$Y^a \perp A \mid \{\}$$

	$Y^{a=0}$	$Y^{a=1}$
Rheia	0	1
Kronos	1	0
Demeter	0	0
Hades	0	0
Hestia	0	0
Poseidon	1	0
Hera	0	0
Zeus	0	1
Artemis	1	1
Apollo	1	0
Leto	0	1
Ares	1	1
Athena	1	1
Hephaestus	0	1
Aphrodite	0	1
Polyphemus	0	1
Persephone	1	1
Hermes	1	0
Hebe	1	0
Dionysus	1	0



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- If we ourselves determine  $A$  in a way that is *truly blind* to  $Y^a$ , it **imposes** exchangeability!

	$Y^{a=0}$	$Y^{a=1}$
Rheia	0	1
Kronos	1	0
Demeter	0	0
Hades	0	0
Hestia	0	0
Poseidon	1	0
Hera	0	0
Zeus	0	1
Artemis	1	1
Apollo	1	0
Leto	0	1
Ares	1	1
Athena	1	1
Hephaestus	0	1
Aphrodite	0	1
Polyphemus	0	1
Persephone	1	1
Hermes	1	0
Hebe	1	0
Dionysus	1	0

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---

---

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Kronos  
Demeter  
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	$A$
Rheia	0
Kronos	0
Demeter	0
Hades	0
Hestia	1
Poseidon	1
Hera	1
Zeus	1
Artemis	0
Apollo	0
Leto	0
Ares	1
Athena	1
Hephaestus	1
Aphrodite	1
Polyphemos	1
Persephone	1
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Hebe	1
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	$A$	$Y$
Rheia	0	0
Kronos	0	1
Demeter	0	0
Hades	0	0
Hestia	1	0
Poseidon	1	0
Hera	1	0
Zeus	1	1
Artemis	0	1
Apollo	0	1
Leto	0	0
Ares	1	1
Athena	1	1
Hephaestus	1	1
Aphrodite	1	1
Polyphemos	1	1
Persephone	1	1
Hermes	1	0
Hebe	1	0
Dionysus	1	0

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	$Y^{a=0}$	$Y^{a=1}$	$A$	$Y$
Rheia	0	1	0	0
Kronos	1	0	0	1
Demeter	0	0	0	0
Hades	0	0	0	0
Hestia	0	0	1	0
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- We can now go ahead and estimate

$$\hat{P}(Y^{a=i} = 1) = \hat{P}(Y = 1 \mid A = i)$$

	$Y^{a=0}$	$Y^{a=1}$	$A$	$Y$
Rheia	0	1	0	0
Kronos	1	0	0	1
Demeter	0	0	0	0
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$$\hat{P}(Y^{a=i} = 1) = \hat{P}(Y = 1 \mid A = i)$$

- Hooray!!!

	$Y^{a=0}$	$Y^{a=1}$	$A$	$Y$
Rheia	0	1	0	0
Kronos	1	0	0	1
Demeter	0	0	0	0
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	$L$
Rheia	0
Kronos	0
Demeter	0
Hades	0
Hestia	0
Poseidon	0
Hera	0
Zeus	0
Artemis	1
Apollo	1
Leto	1
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Hera	0	1	0	0	0
Zeus	0	1	0	1	1
Artemis	1	0	1	1	1
Apollo	1	0	1	0	1
Leto	1	0	0	1	0
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Aphrodite	1	1	0	1	1
Polyphemus	1	1	0	1	1
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Hera	0	1	0	0	0
Zeus	0	1	0	1	1
Artemis	1	0	1	1	1
Apollo	1	0	1	0	1
Leto	1	0	0	1	0
Ares	1	1	1	1	1
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Hephaestus	1	1	0	1	1
Aphrodite	1	1	0	1	1
Polyphemus	1	1	0	1	1
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- In general,  $L$  will be related to  $Y^a$ 
  - E.g., in this example, patients in critical condition are surely more likely to die overall!




	$L$	$A$	$Y^{a=0}$	$Y^{a=1}$	$Y$
Rheia	0	0	0	1	0
Kronos	0	0	1	0	1
Demeter	0	0	0	0	0
Hades	0	0	0	0	0
Hestia	0	1	0	0	0
Poseidon	0	1	1	0	0
Hera	0	1	0	0	0
Zeus	0	1	0	1	1
Artemis	1	0	1	1	1
Apollo	1	0	1	0	1
Leto	1	0	0	1	0
Ares	1	1	1	1	1
Athena	1	1	1	1	1
Hephaestus	1	1	0	1	1
Aphrodite	1	1	0	1	1
Polyphemus	1	1	0	1	1
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Hermes	1	1	1	0	0
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$$A \perp Y^a \mid \{ \}$$




	$L$	$A$	$Y^{a=0}$	$Y^{a=1}$	$Y$
Rheia	0	0	0	1	0
Kronos	0	0	1	0	1
Demeter	0	0	0	0	0
Hades	0	0	0	0	0
Hestia	0	1	0	0	0
Poseidon	0	1	1	0	0
Hera	0	1	0	0	0
Zeus	0	1	0	1	1
Artemis	1	0	1	1	1
Apollo	1	0	1	0	1
Leto	1	0	0	1	0
Ares	1	1	1	1	1
Athena	1	1	1	1	1
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Aphrodite	1	1	0	1	1
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$A \perp L \mid \{\}$

	$L$	$A$	$Y^{a=0}$	$Y^{a=1}$	$Y$
Rheia	0	0	0	1	0
Kronos	0	0	1	0	1
Demeter	0	0	0	0	0
Hades	0	0	0	0	0
Hestia	0	1	0	0	0
Poseidon	0	1	1	0	0
Hera	0	1	0	0	0
Zeus	0	1	0	1	1
Artemis	1	0	1	1	1
Apollo	1	0	1	0	1
Leto	1	0	0	1	0
Ares	1	1	1	1	1
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Hephaestus	1	1	0	1	1
Aphrodite	1	1	0	1	1
Polyphemus	1	1	0	1	1
Persephone	1	1	1	1	1
Hermes	1	1	1	0	0
Hebe	1	1	1	0	0
Dionysus	1	1	1	0	0

# Conditional exchangeability

---

	$L$	$A$	$Y^{a=0}$	$Y^{a=1}$	$Y$
Rheia	0	0	0	1	0
Kronos	0	0	1	0	1
Demeter	0	0	0	0	0
Hades	0	0	0	0	0
Hestia	0	1	0	0	0
Poseidon	0	1	1	0	0
Hera	0	1	0	0	0
Zeus	0	1	0	1	1
Artemis	1	0	1	1	1
Apollo	1	0	1	0	1
Leto	1	0	0	1	0
Ares	1	1	1	1	1
Athena	1	1	1	1	1
Hephaestus	1	1	0	1	1
Aphrodite	1	1	0	1	1
Polyphemus	1	1	0	1	1
Persephone	1	1	1	1	1
Hermes	1	1	1	0	0
Hebe	1	1	1	0	0
Dionysus	1	1	1	0	0

# Conditional exchangeability

- But now suppose we have observed (i.e., it's in our dataset) the factor  $L$  that affected whether the treatment  $A$  was applied

	$L$	$A$	$Y^{a=0}$	$Y^{a=1}$	$Y$
Rheia	0	0	0	1	0
Kronos	0	0	1	0	1
Demeter	0	0	0	0	0
Hades	0	0	0	0	0
Hestia	0	1	0	0	0
Poseidon	0	1	1	0	0
Hera	0	1	0	0	0
Zeus	0	1	0	1	1
Artemis	1	0	1	1	1
Apollo	1	0	1	0	1
Leto	1	0	0	1	0
Ares	1	1	1	1	1
Athena	1	1	1	1	1
Hephaestus	1	1	0	1	1
Aphrodite	1	1	0	1	1
Polyphemus	1	1	0	1	1
Persephone	1	1	1	1	1
Hermes	1	1	1	0	0
Hebe	1	1	1	0	0
Dionysus	1	1	1	0	0



# Conditional exchangeability

- But now suppose we have observed (i.e., it's in our dataset) the factor  $L$  that affected whether the treatment  $A$  was applied
- If the following condition holds, it can help us estimate the counterfactual risks  $P(Y^a)$ :

$$A \perp Y^a \mid L$$

	$L$	$A$	$Y^{a=0}$	$Y^{a=1}$	$Y$
Rheia	0	0	0	1	0
Kronos	0	0	1	0	1
Demeter	0	0	0	0	0
Hades	0	0	0	0	0
Hestia	0	1	0	0	0
Poseidon	0	1	1	0	0
Hera	0	1	0	0	0
Zeus	0	1	0	1	1
Artemis	1	0	1	1	1
Apollo	1	0	1	0	1
Leto	1	0	0	1	0
Ares	1	1	1	1	1
Athena	1	1	1	1	1
Hephaestus	1	1	0	1	1
Aphrodite	1	1	0	1	1
Polyphemus	1	1	0	1	1
Persephone	1	1	1	1	1
Hermes	1	1	1	0	0
Hebe	1	1	1	0	0
Dionysus	1	1	1	0	0

# Conditional exchangeability

- But now suppose we have observed (i.e., it's in our dataset) the factor  $L$  that affected whether the treatment  $A$  was applied
- If the following condition holds, it can help us estimate the counterfactual risks  $P(Y^a)$ :

$$A \perp Y^a \mid L$$

- That is,  $L$  captures all the information available in  $A$  that is relevant to all  $Y^a$

	$L$	$A$	$Y^{a=0}$	$Y^{a=1}$	$Y$
Rheia	0	0	0	1	0
Kronos	0	0	1	0	1
Demeter	0	0	0	0	0
Hades	0	0	0	0	0
Hestia	0	1	0	0	0
Poseidon	0	1	1	0	0
Hera	0	1	0	0	0
Zeus	0	1	0	1	1
Artemis	1	0	1	1	1
Apollo	1	0	1	0	1
Leto	1	0	0	1	0
Ares	1	1	1	1	1
Athena	1	1	1	1	1
Hephaestus	1	1	0	1	1
Aphrodite	1	1	0	1	1
Polyphemus	1	1	0	1	1
Persephone	1	1	1	1	1
Hermes	1	1	1	0	0
Hebe	1	1	1	0	0
Dionysus	1	1	1	0	0

# Conditional exchangeability

- But now suppose we have observed (i.e., it's in our dataset) the factor  $L$  that affected whether the treatment  $A$  was applied
- If the following condition holds, it can help us estimate the counterfactual risks  $P(Y^a)$ :

$$A \perp Y^a \mid L$$

- That is,  $L$  captures all the information available in  $A$  that is relevant to all  $Y^a$
- This is called **CONDITIONAL EXCHANGEABILITY**

	$L$	$A$	$Y^{a=0}$	$Y^{a=1}$	$Y$
Rheia	0	0	0	1	0
Kronos	0	0	1	0	1
Demeter	0	0	0	0	0
Hades	0	0	0	0	0
Hestia	0	1	0	0	0
Poseidon	0	1	1	0	0
Hera	0	1	0	0	0
Zeus	0	1	0	1	1
Artemis	1	0	1	1	1
Apollo	1	0	1	0	1
Leto	1	0	0	1	0
Ares	1	1	1	1	1
Athena	1	1	1	1	1
Hephaestus	1	1	0	1	1
Aphrodite	1	1	0	1	1
Polyphemus	1	1	0	1	1
Persephone	1	1	1	1	1
Hermes	1	1	1	0	0
Hebe	1	1	1	0	0
Dionysus	1	1	1	0	0



# Using conditional exchangeability

---

	$L$	$Y$
Rheia	0	0
Kronos	0	1
Demeter	0	0
Hades	0	0
Hestia	0	0
Poseidon	0	0
Hera	0	0
Zeus	0	1
Artemis	1	1
Apollo	1	1
Leto	1	0
Ares	1	1
Athena	1	1
Hephaestus	1	1
Aphrodite	1	1
Polyphemus	1	1
Persephone	1	1
Hermes	1	0
Hebe	1	0
Dionysus	1	0

# Using conditional exchangeability

---

- Can we estimate  $P(Y^{a=i} = 1 | L)$ ?

	$L$	$Y$
Rheia	0	0
Kronos	0	1
Demeter	0	0
Hades	0	0
Hestia	0	0
Poseidon	0	0
Hera	0	0
Zeus	0	1
Artemis	1	1
Apollo	1	1
Leto	1	0
Ares	1	1
Athena	1	1
Hephaestus	1	1
Aphrodite	1	1
Polyphemus	1	1
Persephone	1	1
Hermes	1	0
Hebe	1	0
Dionysus	1	0

# Using conditional exchangeability

---

- Can we estimate  $P(Y^{a=i} = 1 | L)$ ?
- It turns out we can!

	$L$	$Y$
Rheia	0	0
Kronos	0	1
Demeter	0	0
Hades	0	0
Hestia	0	0
Poseidon	0	0
Hera	0	0
Zeus	0	1
Artemis	1	1
Apollo	1	1
Leto	1	0
Ares	1	1
Athena	1	1
Hephaestus	1	1
Aphrodite	1	1
Polyphemus	1	1
Persephone	1	1
Hermes	1	0
Hebe	1	0
Dionysus	1	0

# Using conditional exchangeability

- Can we estimate  $P(Y^{a=i} = 1 | L)$ ?
- It turns out we can!

$$P(Y^{a=i} = 1 | L) = P(Y^{a=i} = 1 | L, A)$$

CONDITIONAL  
EXCHANGEABILITY

	$L$	$Y$
Rheia	0	0
Kronos	0	1
Demeter	0	0
Hades	0	0
Hestia	0	0
Poseidon	0	0
Hera	0	0
Zeus	0	1
Artemis	1	1
Apollo	1	1
Leto	1	0
Ares	1	1
Athena	1	1
Hephaestus	1	1
Aphrodite	1	1
Polyphemus	1	1
Persephone	1	1
Hermes	1	0
Hebe	1	0
Dionysus	1	0

# Using conditional exchangeability

- Can we estimate  $P(Y^{a=i} = 1 | L)$ ?
- It turns out we can!

$$P(Y^{a=i} = 1 | L) = P(Y^{a=i} = 1 | L, A)$$

CONDITIONAL  
EXCHANGEABILITY

	$L$	$A$	$Y$
Rheia	0	0	0
Kronos	0	0	1
Demeter	0	0	0
Hades	0	0	0
Hestia	0	1	0
Poseidon	0	1	0
Hera	0	1	0
Zeus	0	1	1
Artemis	1	0	1
Apollo	1	0	1
Leto	1	0	0
Ares	1	1	1
Athena	1	1	1
Hephaestus	1	1	1
Aphrodite	1	1	1
Polyphemus	1	1	1
Persephone	1	1	1
Hermes	1	1	0
Hebe	1	1	0
Dionysus	1	1	0

# Using conditional exchangeability

- Can we estimate  $P(Y^{a=i} = 1 | L)$ ?
- It turns out we can!

$$P(Y^{a=i} = 1 | L) = P(Y^{a=i} = 1 | L, A)$$

CONDITIONAL  
EXCHANGEABILITY

	$L$	$A$	$Y$	$Y^0$	$Y^1$
Rheia	0	0	0	0	?
Kronos	0	0	1	1	?
Demeter	0	0	0	0	?
Hades	0	0	0	0	?
Hestia	0	1	0	?	0
Poseidon	0	1	0	?	0
Hera	0	1	0	?	0
Zeus	0	1	1	?	1
Artemis	1	0	1	1	?
Apollo	1	0	1	1	?
Leto	1	0	0	0	?
Ares	1	1	1	?	1
Athena	1	1	1	?	1
Hephaestus	1	1	1	?	1
Aphrodite	1	1	1	?	1
Polyphemus	1	1	1	?	1
Persephone	1	1	1	?	1
Hermes	1	1	0	?	0
Hebe	1	1	0	?	0
Dionysus	1	1	0	?	0

# Using conditional exchangeability

- Can we estimate  $P(Y^{a=i} = 1 | L)$ ?
- It turns out we can!

$$P(Y^{a=i} = 1 | L) = P(Y^{a=i} = 1 | L, A) \quad \text{CONDITIONAL EXCHANGEABILITY}$$

$$\hat{P}_{\text{MLE}}(Y^{a=i} = 1 | L = j, A = k) = \frac{\text{Count}(Y^{a=i} = 1, L = j, A = k)}{\text{Count}(L = j, A = k)}$$

	$L$	$A$	$Y$	$Y^0$	$Y^1$
Rheia	0	0	0	0	?
Kronos	0	0	1	1	?
Demeter	0	0	0	0	?
Hades	0	0	0	0	?
Hestia	0	1	0	?	0
Poseidon	0	1	0	?	0
Hera	0	1	0	?	0
Zeus	0	1	1	?	1
Artemis	1	0	1	1	?
Apollo	1	0	1	1	?
Leto	1	0	0	0	?
Ares	1	1	1	?	1
Athena	1	1	1	?	1
Hephaestus	1	1	1	?	1
Aphrodite	1	1	1	?	1
Polyphemus	1	1	1	?	1
Persephone	1	1	1	?	1
Hermes	1	1	0	?	0
Hebe	1	1	0	?	0
Dionysus	1	1	0	?	0



# Using conditional exchangeability

- Can we estimate  $P(Y^{a=i} = 1 | L)$ ?
- It turns out we can!

$$P(Y^{a=i} = 1 | L) = P(Y^{a=i} = 1 | L, A) \quad \text{CONDITIONAL EXCHANGEABILITY}$$

$$\hat{P}_{\text{MLE}}(Y^{a=i} = 1 | L = j, A = k) = \frac{\text{Count}(Y^{a=i} = 1, L = j, A = k)}{\text{Count}(L = j, A = k)}$$

- In estimating this condprob:

	$L$	$A$	$Y$	$Y^0$	$Y^1$
Rheia	0	0	0	0	?
Kronos	0	0	1	1	?
Demeter	0	0	0	0	?
Hades	0	0	0	0	?
Hestia	0	1	0	?	0
Poseidon	0	1	0	?	0
Hera	0	1	0	?	0
Zeus	0	1	1	?	1
Artemis	1	0	1	1	?
Apollo	1	0	1	1	?
Leto	1	0	0	0	?
Ares	1	1	1	?	1
Athena	1	1	1	?	1
Hephaestus	1	1	1	?	1
Aphrodite	1	1	1	?	1
Polyphemus	1	1	1	?	1
Persephone	1	1	1	?	1
Hermes	1	1	0	?	0
Hebe	1	1	0	?	0
Dionysus	1	1	0	?	0

# Using conditional exchangeability

- Can we estimate  $P(Y^{a=i} = 1 | L)$ ?

- It turns out we can!

$$P(Y^{a=i} = 1 | L) = P(Y^{a=i} = 1 | L, A) \quad \text{CONDITIONAL EXCHANGEABILITY}$$

$$\hat{P}_{\text{MLE}}(Y^{a=i} = 1 | L = j, A = k) = \frac{\text{Count}(Y^{a=i} = 1, L = j, A = k)}{\text{Count}(L = j, A = k)}$$

- In estimating this condprob:

- when  $i = k$  we use  
CONSISTENCY

CONSISTENCY: when  
 $A = i, Y = Y^{a=i}$

	$L$	$A$	$Y$	$Y^0$	$Y^1$
Rheia	0	0	0	0	?
Kronos	0	0	1	1	?
Demeter	0	0	0	0	?
Hades	0	0	0	0	?
Hestia	0	1	0	?	0
Poseidon	0	1	0	?	0
Hera	0	1	0	?	0
Zeus	0	1	1	?	1
Artemis	1	0	1	1	?
Apollo	1	0	1	1	?
Leto	1	0	0	0	?
Ares	1	1	1	?	1
Athena	1	1	1	?	1
Hephaestus	1	1	1	?	1
Aphrodite	1	1	1	?	1
Polyphemus	1	1	1	?	1
Persephone	1	1	1	?	1
Hermes	1	1	0	?	0
Hebe	1	1	0	?	0
Dionysus	1	1	0	?	0

# Using conditional exchangeability

- Can we estimate  $P(Y^{a=i} = 1 | L)$ ?

- It turns out we can!

$$P(Y^{a=i} = 1 | L) = P(Y^{a=i} = 1 | L, A) \quad \text{CONDITIONAL EXCHANGEABILITY}$$

$$\hat{P}_{\text{MLE}}(Y^{a=i} = 1 | L = j, A = k) = \frac{\text{Count}(Y^{a=i} = 1, L = j, A = k)}{\text{Count}(L = j, A = k)}$$

- In estimating this condprob:
  - when  $i = k$  we use CONSISTENCY
  - WHEN  $i \neq k$  we have "missing data", so ignore those instances

$L$     $A$     $\text{Count}(L, A)$

	$L$	$A$	$Y$	$Y^0$	$Y^1$
Rheia	0	0	0	0	?
Kronos	0	0	1	1	?
Demeter	0	0	0	0	?
Hades	0	0	0	0	?
Hestia	0	1	0	?	0
Poseidon	0	1	0	?	0
Hera	0	1	0	?	0
Zeus	0	1	1	?	1
Artemis	1	0	1	1	?
Apollo	1	0	1	1	?
Leto	1	0	0	0	?
Ares	1	1	1	?	1
Athena	1	1	1	?	1
Hephaestus	1	1	1	?	1
Aphrodite	1	1	1	?	1
Polyphemus	1	1	1	?	1
Persephone	1	1	1	?	1
Hermes	1	1	0	?	0
Hebe	1	1	0	?	0
Dionysus	1	1	0	?	0

# Using conditional exchangeability

- Can we estimate  $P(Y^{a=i} = 1 | L)$ ?

- It turns out we can!

$$P(Y^{a=i} = 1 | L) = P(Y^{a=i} = 1 | L, A) \quad \text{CONDITIONAL EXCHANGEABILITY}$$

$$\hat{P}_{\text{MLE}}(Y^{a=i} = 1 | L = j, A = k) = \frac{\text{Count}(Y^{a=i} = 1, L = j, A = k)}{\text{Count}(L = j, A = k)}$$

- In estimating this condprob:
  - when  $i = k$  we use CONSISTENCY
  - WHEN  $i \neq k$  we have "missing data", so ignore those instances

$L$     $A$     $\text{Count}(L, A)$

0   0

0   1

1   0

1   1

	$L$	$A$	$Y$	$Y^0$	$Y^1$
Rheia	0	0	0	0	?
Kronos	0	0	1	1	?
Demeter	0	0	0	0	?
Hades	0	0	0	0	?
Hestia	0	1	0	?	0
Poseidon	0	1	0	?	0
Hera	0	1	0	?	0
Zeus	0	1	1	?	1
Artemis	1	0	1	1	?
Apollo	1	0	1	1	?
Leto	1	0	0	0	?
Ares	1	1	1	?	1
Athena	1	1	1	?	1
Hephaestus	1	1	1	?	1
Aphrodite	1	1	1	?	1
Polyphemus	1	1	1	?	1
Persephone	1	1	1	?	1
Hermes	1	1	0	?	0
Hebe	1	1	0	?	0
Dionysus	1	1	0	?	0

# Using conditional exchangeability

- Can we estimate  $P(Y^{a=i} = 1 | L)$ ?

- It turns out we can!

$$P(Y^{a=i} = 1 | L) = P(Y^{a=i} = 1 | L, A) \quad \text{CONDITIONAL EXCHANGEABILITY}$$

$$\hat{P}_{\text{MLE}}(Y^{a=i} = 1 | L = j, A = k) = \frac{\text{Count}(Y^{a=i} = 1, L = j, A = k)}{\text{Count}(L = j, A = k)}$$

- In estimating this condprob:
  - when  $i = k$  we use CONSISTENCY
  - WHEN  $i \neq k$  we have "missing data", so ignore those instances

$L$	$A$	Count( $L, A$ )
0	0	4
0	1	4
1	0	3
1	1	9

	$L$	$A$	$Y$	$Y^0$	$Y^1$
Rheia	0	0	0	0	?
Kronos	0	0	1	1	?
Demeter	0	0	0	0	?
Hades	0	0	0	0	?
Hestia	0	1	0	?	0
Poseidon	0	1	0	?	0
Hera	0	1	0	?	0
Zeus	0	1	1	?	1
Artemis	1	0	1	1	?
Apollo	1	0	1	1	?
Leto	1	0	0	0	?
Ares	1	1	1	?	1
Athena	1	1	1	?	1
Hephaestus	1	1	1	?	1
Aphrodite	1	1	1	?	1
Polyphemus	1	1	1	?	1
Persephone	1	1	1	?	1
Hermes	1	1	0	?	0
Hebe	1	1	0	?	0
Dionysus	1	1	0	?	0



# Using conditional exchangeability

- Can we estimate  $P(Y^{a=i} = 1 | L)$ ?

- It turns out we can!

$$P(Y^{a=i} = 1 | L) = P(Y^{a=i} = 1 | L, A) \quad \text{CONDITIONAL EXCHANGEABILITY}$$

$$\hat{P}_{\text{MLE}}(Y^{a=i} = 1 | L = j, A = k) = \frac{\text{Count}(Y^{a=i} = 1, L = j, A = k)}{\text{Count}(L = j, A = k)}$$

- In estimating this condprob:
  - when  $i = k$  we use CONSISTENCY
  - WHEN  $i \neq k$  we have "missing data", so ignore those instances

CONSISTENCY: when  
 $A = i, Y = Y^{a=i}$

	$L$	$A$	$Y$	$Y^0$	$Y^1$
Rheia	0	0	0	0	?
Kronos	0	0	1	1	?
Demeter	0	0	0	0	?
Hades	0	0	0	0	?
Hestia	0	1	0	?	0
Poseidon	0	1	0	?	0
Hera	0	1	0	?	0
Zeus	0	1	1	?	1
Artemis	1	0	1	1	?
Apollo	1	0	1	1	?
Leto	1	0	0	0	?
Ares	1	1	1	?	1
Athena	1	1	1	?	1
Hephaestus	1	1	1	?	1
Aphrodite	1	1	1	?	1
Polyphemus	1	1	1	?	1
Persephone	1	1	1	?	1
Hermes	1	1	0	?	0
Hebe	1	1	0	?	0
Dionysus	1	1	0	?	0

$L$	$A$	$\text{Count}(L, A)$	$\text{Count}(Y^{a=0} = 1, L, A)$	$\text{Count}(Y^{a=1} = 1, L, A)$
0	0	4		
0	1	4		
1	0	3		
1	1	9		

# Using conditional exchangeability

- Can we estimate  $P(Y^{a=i} = 1 | L)$ ?

- It turns out we can!

$$P(Y^{a=i} = 1 | L) = P(Y^{a=i} = 1 | L, A) \quad \text{CONDITIONAL EXCHANGEABILITY}$$

$$\hat{P}_{\text{MLE}}(Y^{a=i} = 1 | L = j, A = k) = \frac{\text{Count}(Y^{a=i} = 1, L = j, A = k)}{\text{Count}(L = j, A = k)}$$

- In estimating this condprob:

- when  $i = k$  we use CONSISTENCY
- WHEN  $i \neq k$  we have "missing data", so ignore those instances

CONSISTENCY: when  
 $A = i, Y = Y^{a=i}$

	$L$	$A$	$Y$	$Y^0$	$Y^1$
Rheia	0	0	0	0	?
Kronos	0	0	1	1	?
Demeter	0	0	0	0	?
Hades	0	0	0	0	?
Hestia	0	1	0	?	0
Poseidon	0	1	0	?	0
Hera	0	1	0	?	0
Zeus	0	1	1	?	1
Artemis	1	0	1	1	?
Apollo	1	0	1	1	?
Leto	1	0	0	0	?
Ares	1	1	1	?	1
Athena	1	1	1	?	1
Hephaestus	1	1	1	?	1
Aphrodite	1	1	1	?	1
Polyphemus	1	1	1	?	1
Persephone	1	1	1	?	1
Hermes	1	1	0	?	0
Hebe	1	1	0	?	0
Dionysus	1	1	0	?	0

$L$	$A$	$\text{Count}(L, A)$	$\text{Count}(Y^{a=0} = 1, L, A)$	$\text{Count}(Y^{a=1} = 1, L, A)$
0	0	4	1	?
0	1	4		
1	0	3		
1	1	9		



# Using conditional exchangeability

- Can we estimate  $P(Y^{a=i} = 1 | L)$ ?

- It turns out we can!

$$P(Y^{a=i} = 1 | L) = P(Y^{a=i} = 1 | L, A) \quad \text{CONDITIONAL EXCHANGEABILITY}$$

$$\hat{P}_{\text{MLE}}(Y^{a=i} = 1 | L = j, A = k) = \frac{\text{Count}(Y^{a=i} = 1, L = j, A = k)}{\text{Count}(L = j, A = k)}$$

- In estimating this condprob:

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- WHEN  $i \neq k$  we have "missing data", so ignore those instances

CONSISTENCY: when  
 $A = i, Y = Y^{a=i}$

	$L$	$A$	$Y$	$Y^0$	$Y^1$
Rheia	0	0	0	0	?
Kronos	0	0	1	1	?
Demeter	0	0	0	0	?
Hades	0	0	0	0	?
Hestia	0	1	0	?	0
Poseidon	0	1	0	?	0
Hera	0	1	0	?	0
Zeus	0	1	1	?	1
Artemis	1	0	1	1	?
Apollo	1	0	1	1	?
Leto	1	0	0	0	?
Ares	1	1	1	?	1
Athena	1	1	1	?	1
Hephaestus	1	1	1	?	1
Aphrodite	1	1	1	?	1
Polyphemus	1	1	1	?	1
Persephone	1	1	1	?	1
Hermes	1	1	0	?	0
Hebe	1	1	0	?	0
Dionysus	1	1	0	?	0

$L$	$A$	$\text{Count}(L, A)$	$\text{Count}(Y^{a=0} = 1, L, A)$	$\text{Count}(Y^{a=1} = 1, L, A)$
0	0	4	1	?
0	1	4	?	1
1	0	3		
1	1	9		

# Using conditional exchangeability

- Can we estimate  $P(Y^{a=i} = 1 | L)$ ?

- It turns out we can!

$$P(Y^{a=i} = 1 | L) = P(Y^{a=i} = 1 | L, A) \quad \text{CONDITIONAL EXCHANGEABILITY}$$

$$\hat{P}_{MLE}(Y^{a=i} = 1 | L = j, A = k) = \frac{\text{Count}(Y^{a=i} = 1, L = j, A = k)}{\text{Count}(L = j, A = k)}$$

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Leto	1	0	0	0	?
Ares	1	1	1	?	1
Athena	1	1	1	?	1
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Persephone	1	1	1	?	1
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0	0	4	1	?
0	1	4	?	1
1	0	3	2	?
1	1	9		

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$$\hat{P}_{MLE}(Y^{a=i} = 1 | L = j, A = k) = \frac{\text{Count}(Y^{a=i} = 1, L = j, A = k)}{\text{Count}(L = j, A = k)}$$

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0	0	4	1	?
0	1	4	?	1
1	0	3	2	?
1	1	9	?	6

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$L$	$A$	$\text{Count}(L, A)$	$\text{Count}(Y^{a=0} = 1, L, A)$	$\text{Count}(Y^{a=1} = 1, L, A)$
-----	-----	----------------------	-----------------------------------	-----------------------------------

0	0	4	1	?
---	---	---	---	---

0	1	4	?	1
---	---	---	---	---

1	0	3	2	?
---	---	---	---	---

1	1	9	?	6
---	---	---	---	---

$$\hat{P}_{\text{MLE}}(Y^{a=i} = 1 | L)$$

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$L$	$A$	$\text{Count}(L, A)$	$\text{Count}(Y^{a=0} = 1, L, A)$	$\text{Count}(Y^{a=1} = 1, L, A)$	$\hat{P}_{MLE}(Y^{a=i} = 1   L)$
0	0	4	1	?	1/4
0	1	4	?	1	1/4
1	0	3	2	?	2/3
1	1	9	?	6	2/3



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-----	-----	----------------------	-----------------------------------	-----------------------------------

0	0	4	1	?
---	---	---	---	---

0	1	4	?	1
---	---	---	---	---

1	0	3	2	?
---	---	---	---	---

1	1	9	?	6
---	---	---	---	---

$\hat{P}_{\text{MLE}}(Y^{a=i} = 1 | L)$

1/4

1/4

2/3

2/3

*This is just like  
estimating  
 $P(Y | L, A)$ !*



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- Often, this may be all you need or want
  - If the causal effect of  $A$  depends on  $L$ , then "summarizing out"  $L$  discards information!
- But there are situations where the basic counterfactual risk  $E[Y^{a=i}]$  may be of interest
  - e.g., "how many lives would it save if everyone who came to the hospital with heart disease received a heart transplant?"
- But we can recover the basic counterfactual risks through **standardization** (or the mathematically equivalent **inverse probability weighting**)

# Standardization

$L$  = whether the patient was in critical condition (1=yes, 0=no)

$L$	$A$	$\hat{P}_{\text{MLE}}(Y^{a=i}   L)$
0	0	1/4
0	1	1/4
1	0	2/3
1	1	2/3

	$L$	$A$	$Y$	$Y^0$	$Y^1$
Rheia	0	0	0	0	?
Kronos	0	0	1	1	?
Demeter	0	0	0	0	?
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Hebe	1	1	0	?	0
Dionysus	1	1	0	?	0

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- By the law of total probability,

$$P(Y^{a=i}) = \sum_j P(Y^{a=i} | L) P(L)$$

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$$\hat{P}_{MLE}(L = 1) = \frac{12}{20} = \frac{3}{5}$$

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$$P(Y^{a=i} = 1) = P(Y^{a=i} = 1 | L = 0)P(L = 0) + P(Y^{a=i} = 1 | L = 1)P(L = 1)$$

	$L$	$A$	$Y$	$Y^0$	$Y^1$
Rheia	0	0	0	0	?
Kronos	0	0	1	1	?
Demeter	0	0	0	0	?
Hades	0	0	0	0	?
Hestia	0	1	0	?	0
Poseidon	0	1	0	?	0
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Ares	1	1	1	?	1
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$$\hat{P}_{MLE}(L = 1) = \frac{12}{20} = \frac{3}{5}$$

# Standardization

$L$  = whether the patient was in critical condition (1=yes, 0=no)

$L$	$A$	$\hat{P}_{MLE}(Y^{a=i}   L)$
0	0	1/4
0	1	1/4
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*We have just estimated this*

$$P(Y^{a=i}) = \sum_j P(Y^{a=i} | L) P(L)$$

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(Because  $\hat{P}_{MLE}(Y^{a=i} | L)$  are the same for  $a = 0$  and  $a = 1$ , this work gives us the result for both counterfactual treatments, and the risk ratio is 1)

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*Not an issue of sample size—no amount of data would help!*

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- If all three criteria hold, we can estimate causal effects



# Summary of intro to potential outcomes

- The potential outcomes framework formalizes causal effects (or risks) through **counterfactual outcome** (also called **potential outcome**) variables
- At most one counterfactual outcome is observable in each datum, so causal effects cannot in general be naively estimated from data

	$L$	$A$	$Y$	$Y^0$	$Y^1$
Rheia	0	0	0	0	?
Kronos	0	0	1	1	?
Demeter	0	0	0	0	?
Hades	0	0	0	0	?
Hestia	0	1	0	?	0
Poseidon	0	1	0	?	0

- However, if the three following conditions hold, the data can be viewed as a **conditionally randomized experiment** and causal effects can be estimated

**Consistency**

$$Y = Y^{a=i} \text{ whenever } A = i$$

**Conditional Exchangeability**

$$\exists Z. \forall i. Z \text{ is observed} \\ \wedge Y^{a=i} \perp A | Z$$

**Positivity**

$$\forall i. \forall Z. P(A = i | Z) > 0$$

- This analysis also sheds light on the power of randomized experiments: they offer **unconditional exchangeability**

# Why causal inference is challenging

---

(Hernan & Robins, 2020, Table 1.1)

	$Y^{a=0}$	$Y^{a=1}$
Rheia	0	1
Kronos	1	0
Demeter	0	0
Hades	0	0
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Poseidon	1	0
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Zeus	0	1
Artemis	1	1
Apollo	1	0
Leto	0	1
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- But, we don't get to see individual-specific counterfactual RV values!
- At most we see one **actual** outcome
- Let us bundle up everything we don't know about the process by which actual  $A$  and  $Y$  were determined into the RV  $U$

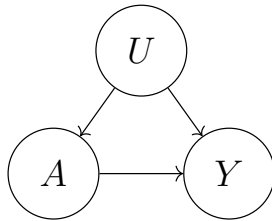
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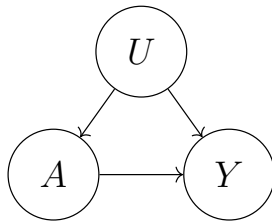
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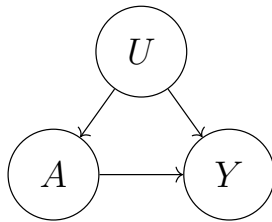
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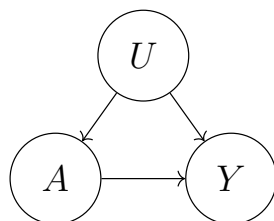
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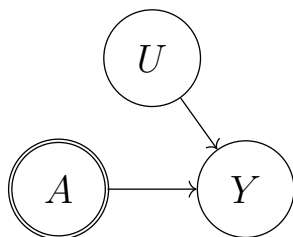
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Artemis	1	1
Apollo	1	0
Leto	0	1
Ares	1	1
Athena	1	1
Hephaestus	0	1
Aphrodite	0	1
Cyclope	0	1
Persephone	1	1
Hermes	1	0
Hebe	1	0
Dionysus	1	0

Typical observed  $Y$

# Pearl's functional causal models

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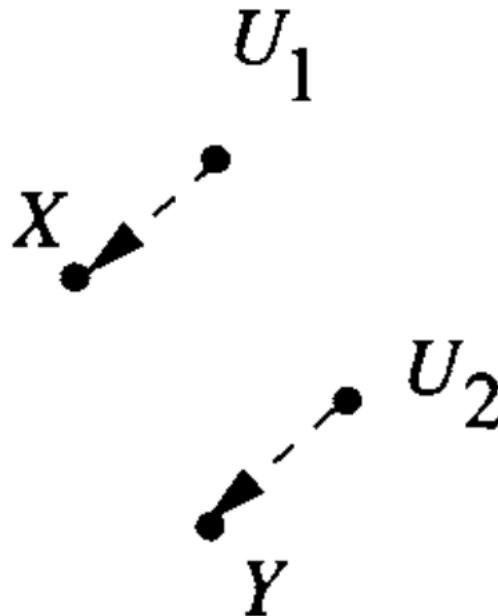
- where  $\text{pa}_i$  is the subset of  $V$  that are parents of  $X_i$

# Bayes Nets & functional causal models

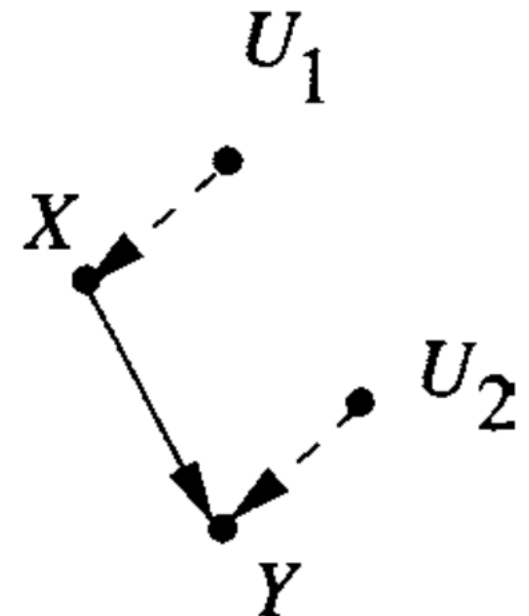
Bayes Net



Corresponding  
functional  
causal model

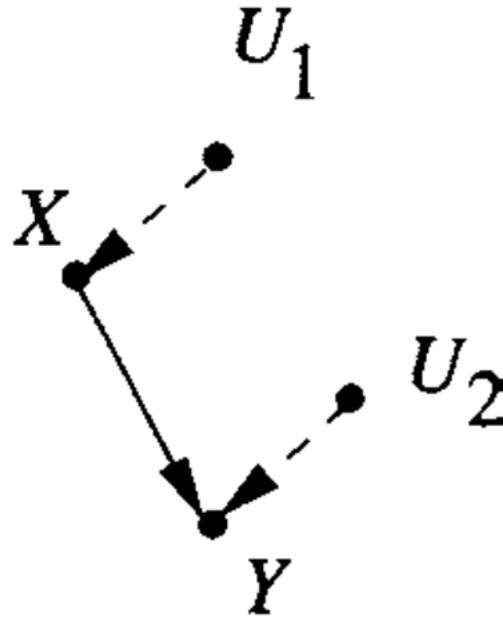


Functional  
causal model  
with  $X$  causally  
upstream of  $Y$



# Example model & structural equations

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$$x = f_1(u_1)$$

$$y = f_2(x, u_2)$$

- An example structural equation:

$$y = a + bx + u_2$$

(Could be linear regression!)