Approximate computation: Monte Carlo

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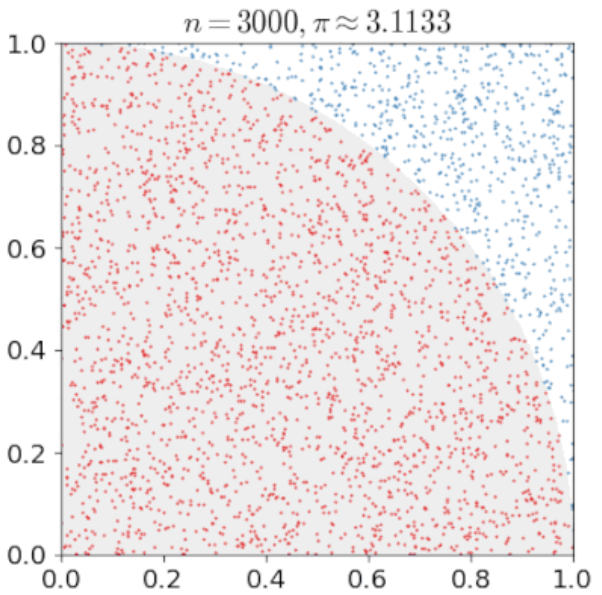
9.S918: Quantitative inference in brain and cognitive sciences

24 February 2025

Monte Carlo methods, or "probabilistic simulation"

- Generally speaking:
  1. Define a domain of possible inputs
  2. Generate n iid random inputs from a probability distribution on the domain
  3. Perform a deterministic computation on each randomly generated input
  4. Aggregate the results of the deterministic computation
- As n grows larger, the simulated result approaches the true value



https://en.wikipedia.org/wiki/Monte_Carlo_method

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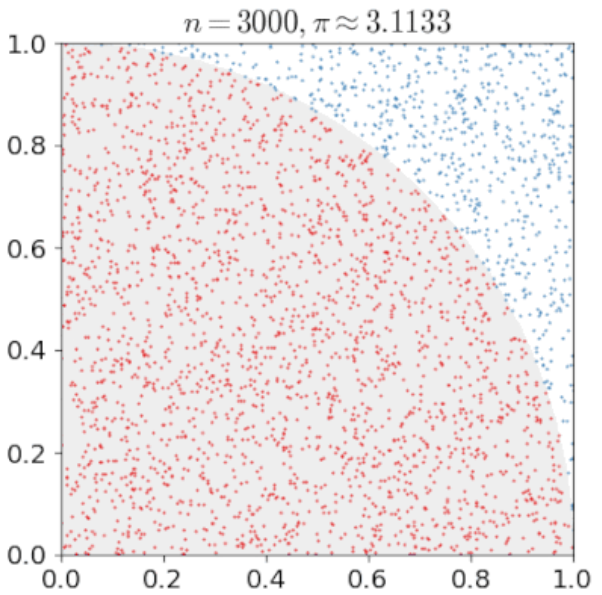

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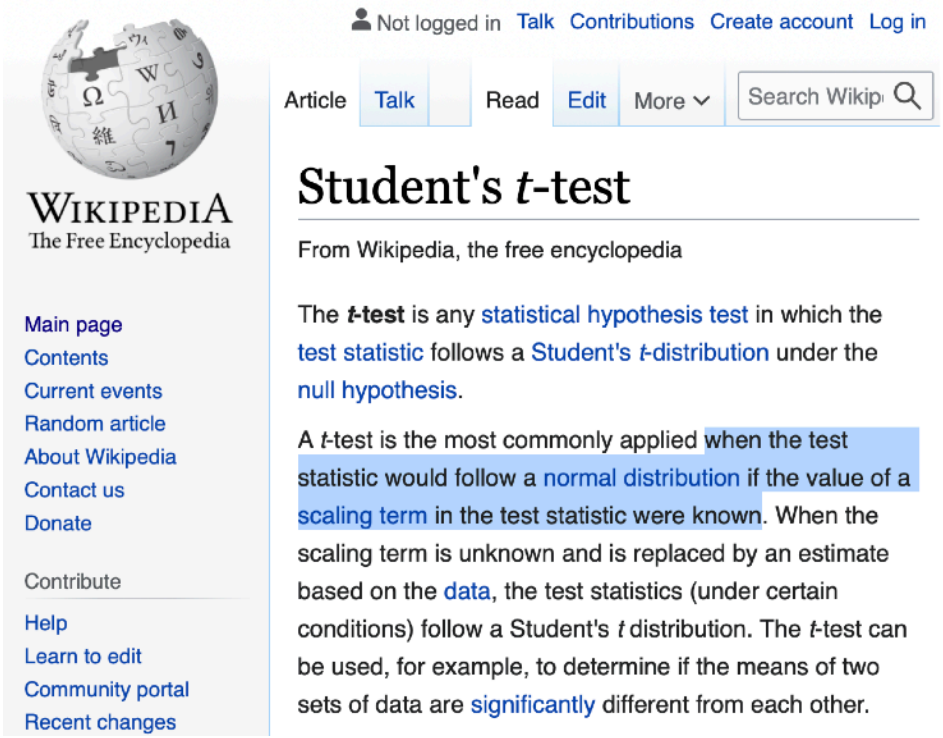
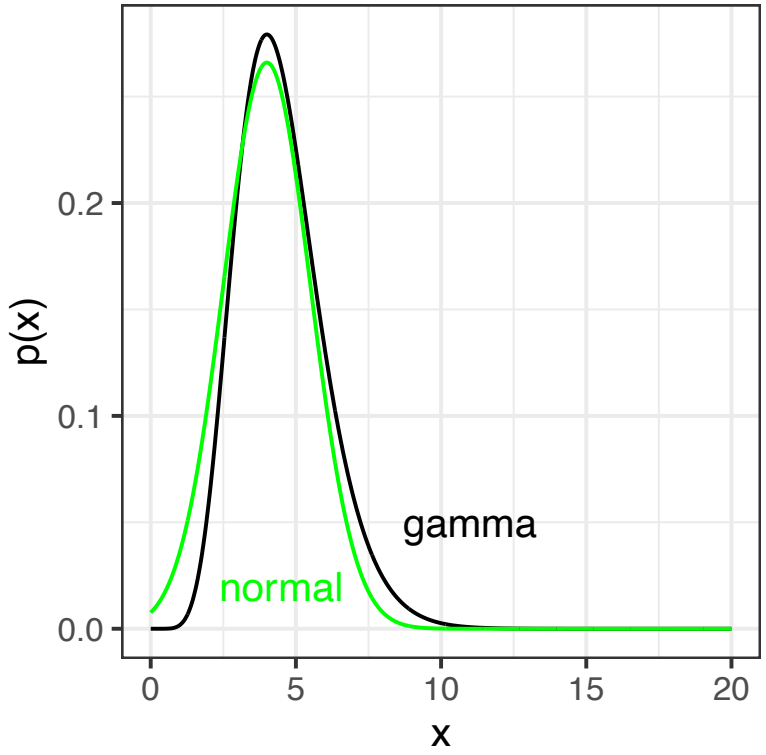
Simple example of Monte Carlo

- Suppose I want to do a two-sample *t*-test but my data aren't normally distributed





- **How bad will this be for my *t*-test????**

Monte Carlo, in action

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library(ggplot2)
library(tidyverse)

# Manually compute Student t-statistic
f <- function(seed,N=100,shape=9,scale=0.5) {
  set.seed(seed)
  y1 <- rgamma(N,shape=shape,scale=scale)
  y2 <- rgamma(N,shape=shape,scale=scale)
  s_p <- sqrt( (var(y1) + var(y2)) / 2 )
  t_statistic <- ( mean(y1) - mean(y2) ) / ( s_p*sqrt(2/N ) )
  return(t_statistic)
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N <- 100
Ts <- sapply(1:10000,f)

t_reference <- tibble(x=seq(-4,4,by=0.01),t=dt(x,df=2*(N-1)))

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Compare against Student's *t* distribution

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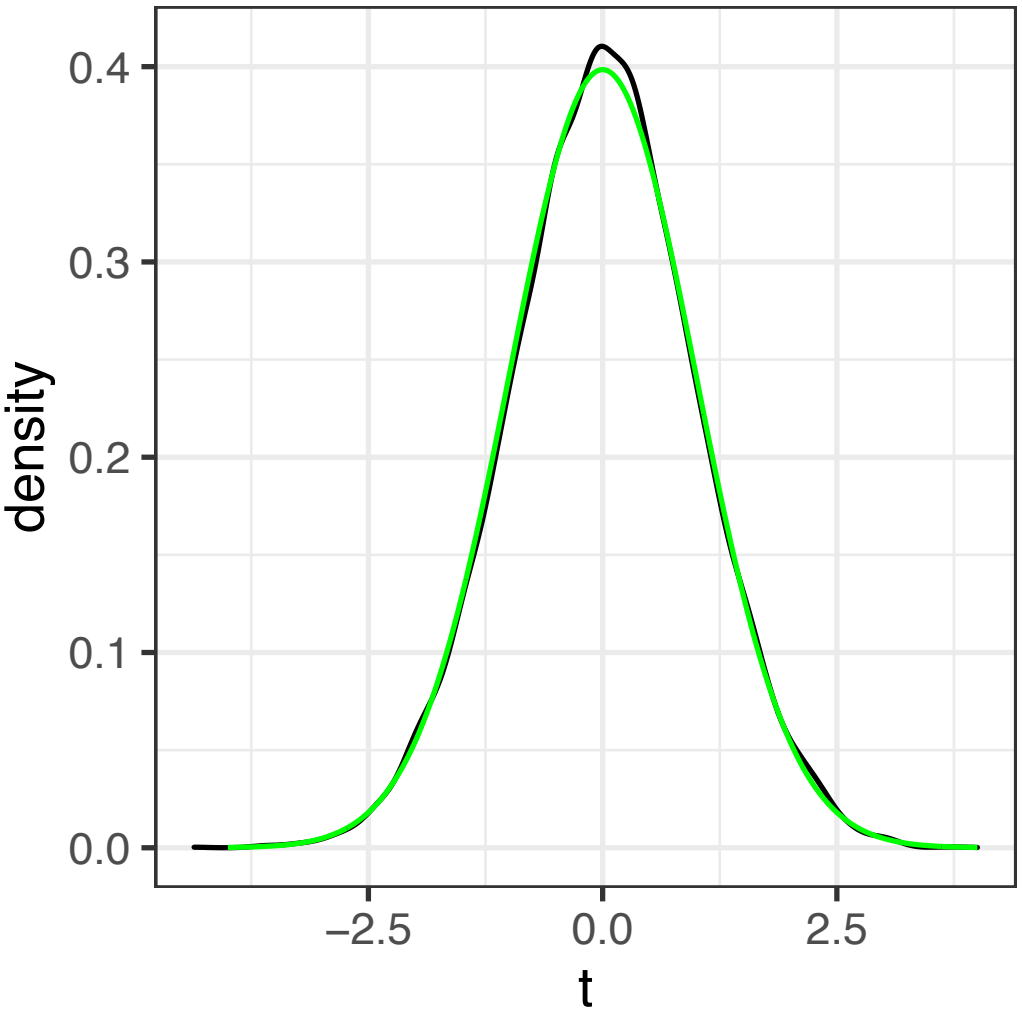

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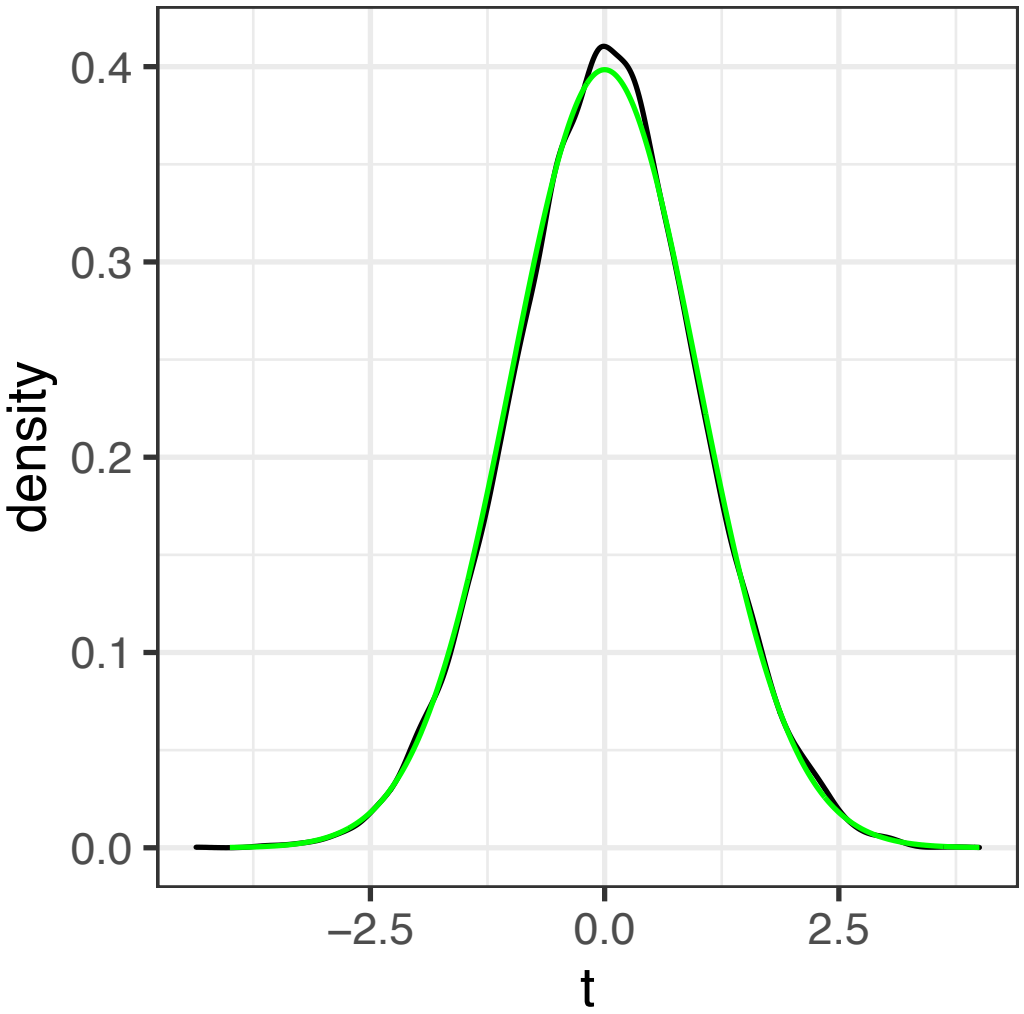

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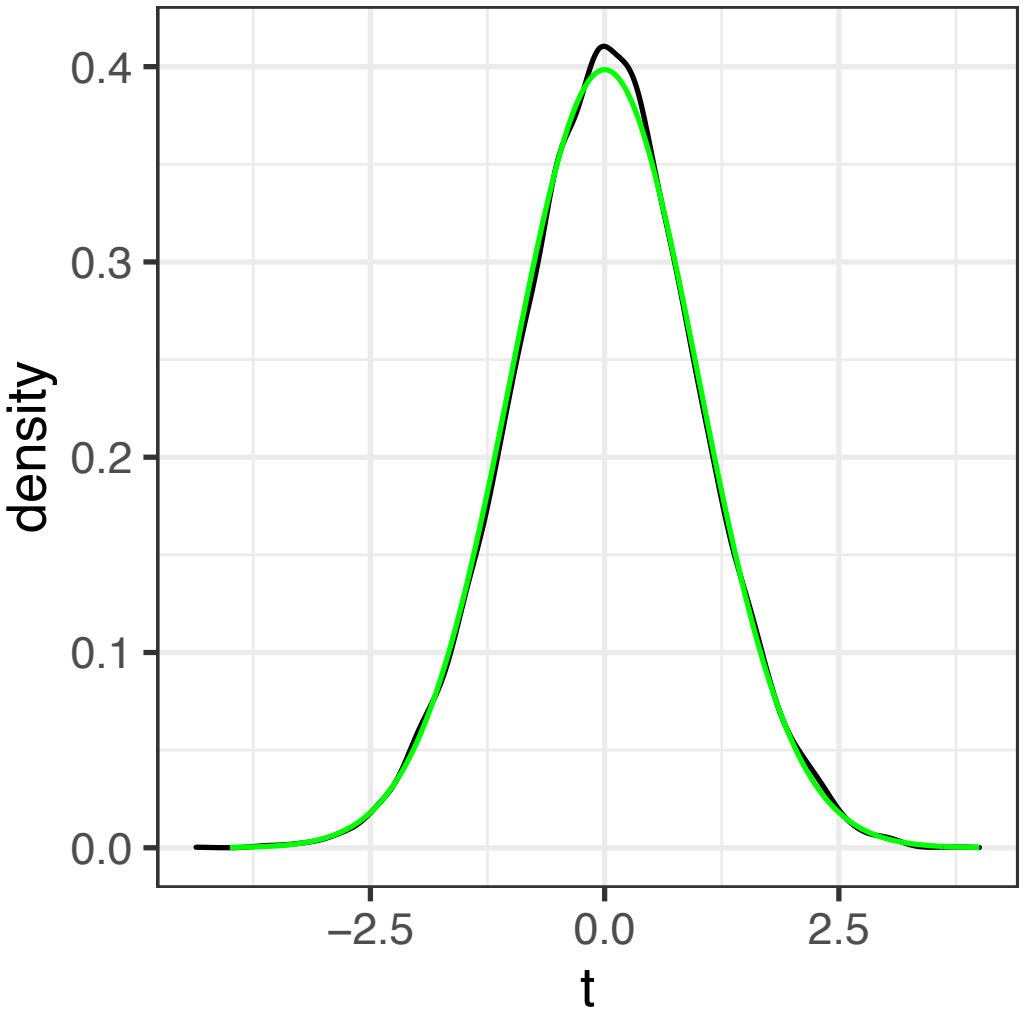

- The *t* distribution is still a pretty good approximation of the distribution of the *t* statistic, even when the underlying distribution is gamma!
- This exemplifies what is meant when people say that the *t* test is **robust to deviations from normality**

Unnormalizable posteriors

- Our motivation: Bayesian posterior inference

$$P(\theta \mid \mathbf{y}, I) = \frac{P(\mathbf{y} \mid \theta, I) P(\theta \mid I)}{P(\mathbf{y} \mid I)}$$

- Sometimes $P(\mathbf{y} \mid I)$ can't be calculated exactly. Example

Bernoulli data with non-conjugate prior:

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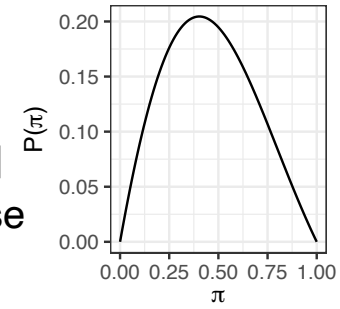

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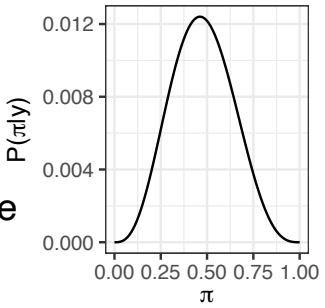

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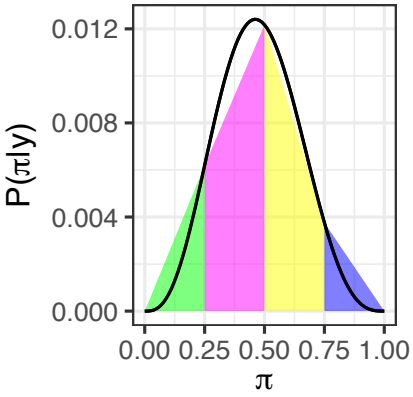

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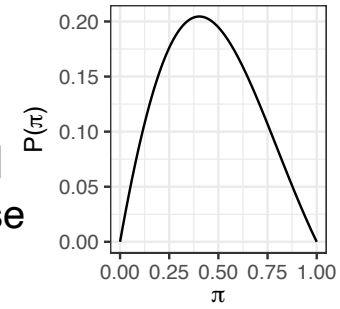

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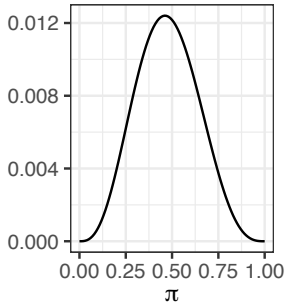

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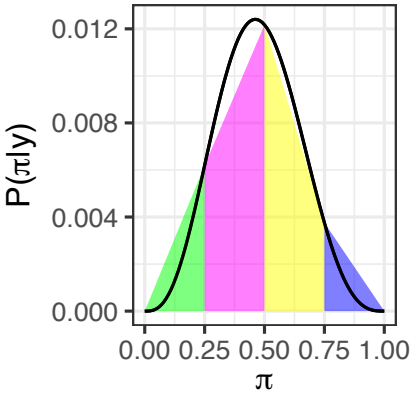

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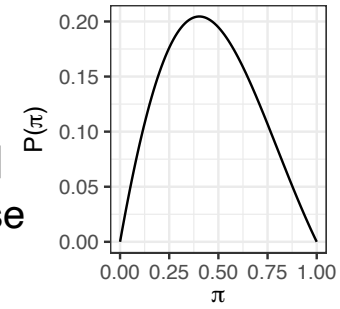

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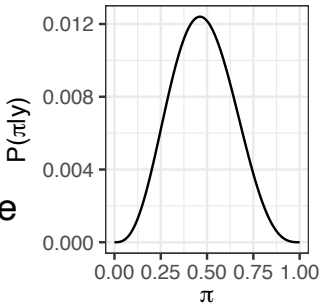

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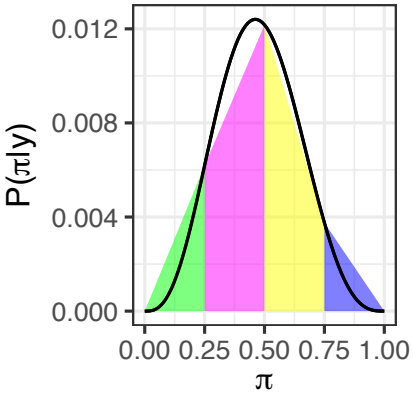

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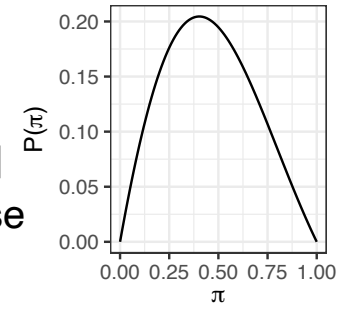

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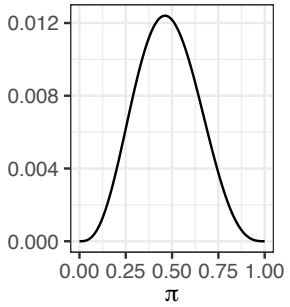

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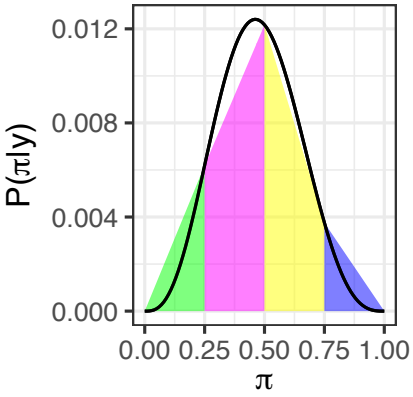

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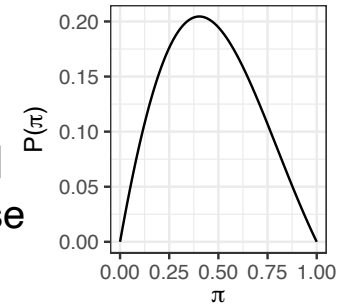

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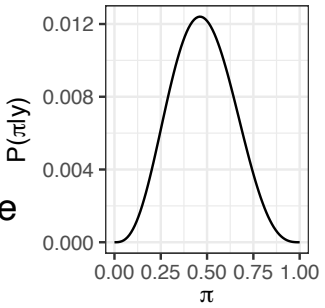
No closed form!



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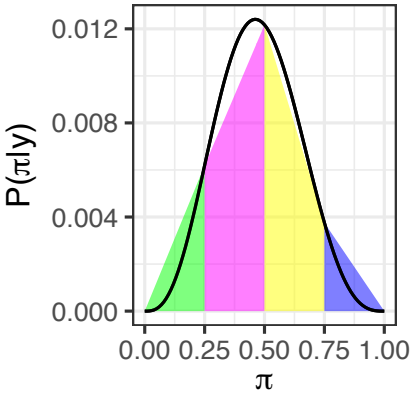

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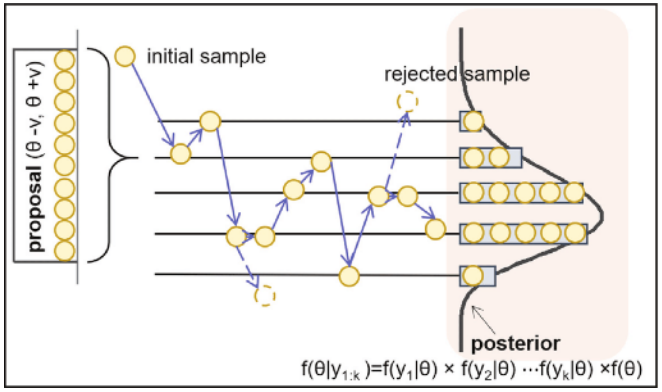
Markov chain Monte Carlo

- However, we can often **take samples from the posterior** even when we can't compute normalized probabilities
- One general and widely used approach: **Markov chain Monte Carlo (MCMC)**
- MCMC is a mathematically principled random walk on a non-negative function, directed toward regions where the function takes on a larger value



- Asymptotically, the random walk gives us samples from in proportion to the height of the function

Image credit: Dong et al., 2019, http://dx.doi.org/10.5772/intechopen.82781

MCMC for posterior sampling

- We use the *unnormalized* form of the posterior:

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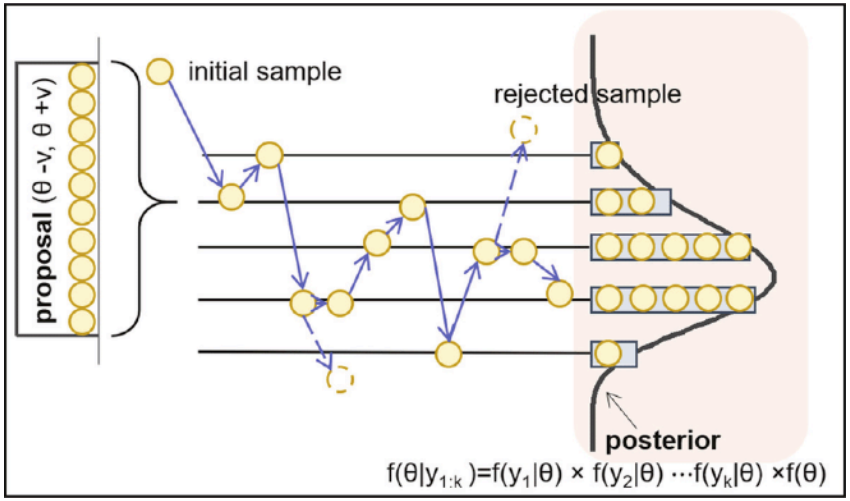

- We run MCMC and then treat the chain of values as samples from the posterior
- The full set of samples is not iid (nearby values on the chain are correlated), but methods exist for estimating "effectively" how many independent samples we have

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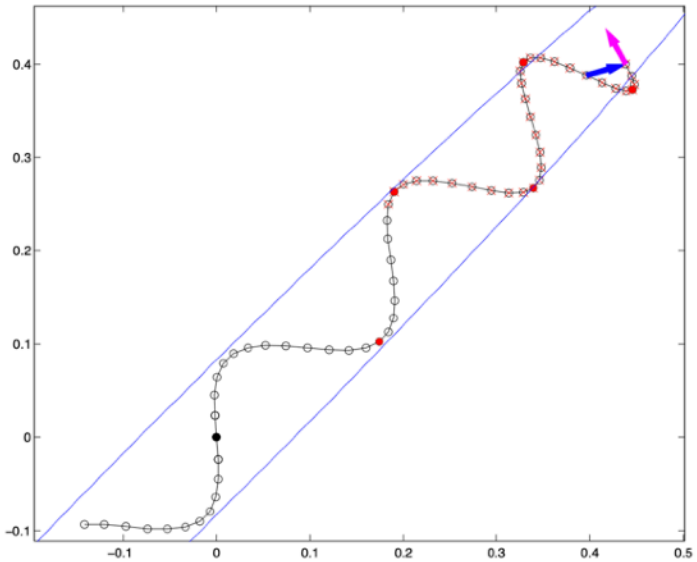
Stan, HMC, and NUTS

- There are many different MCMC algorithms (e.g., Metropolis, Gibbs Sampling)
- We will use the probabilistic programming language **Stan** for Bayesian inference about model parameters
- Stan uses an algorithm called **Hamiltonian Monte Carlo** (**HMC**) with the **No U-Turn Sampler (NUTS)**



(Hoffman & Gelman, 2014)

- This algorithm tends to be particularly efficient for many problems we'll face

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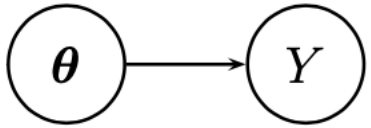

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