Generalized linear models, linear regression, parameter inference, the F test, credit assignment

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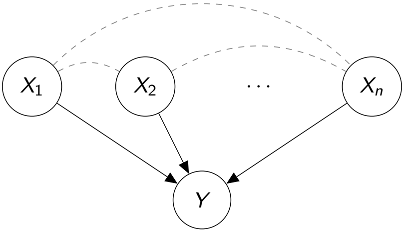

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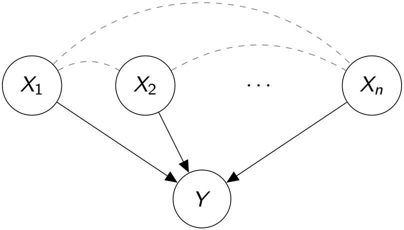

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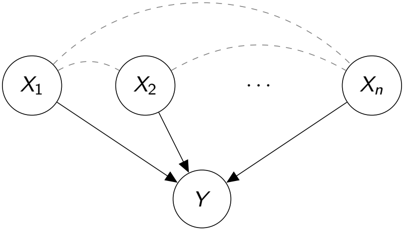

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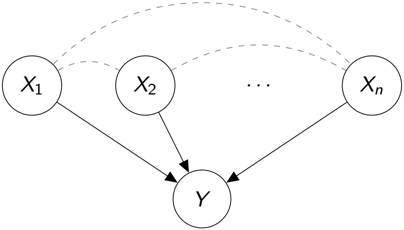

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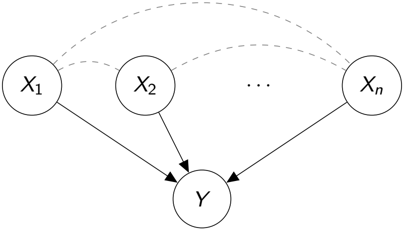

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  - What is the *shape* of the predictive relationship between the X's and Y?

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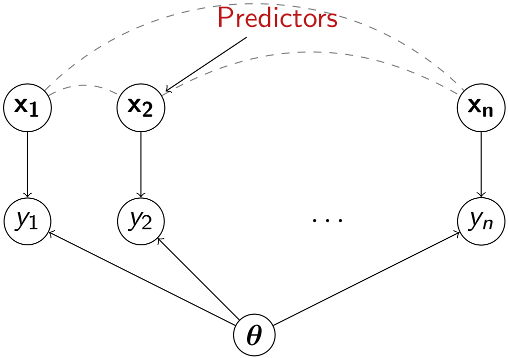

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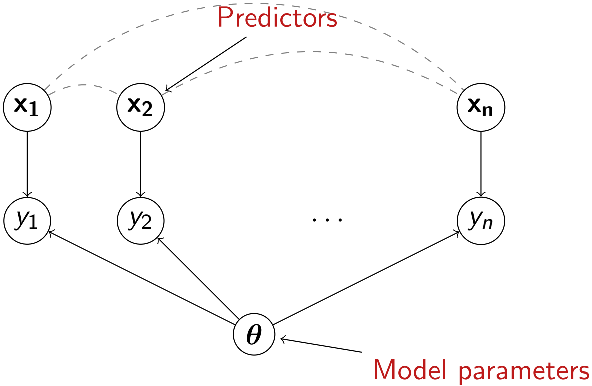

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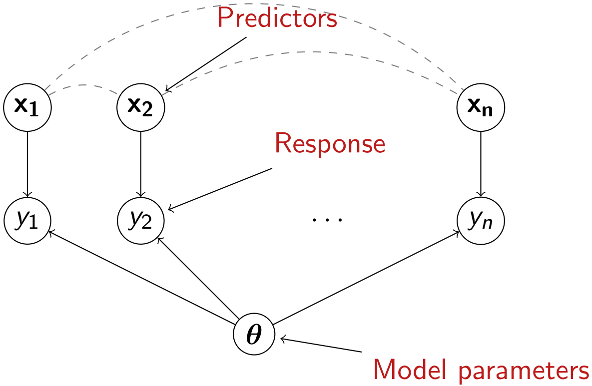

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4. There is some noise distribution of Y around the predicted mean μ of Y:

$$P(Y = y; \mu)$$

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- This gives us the traditional linear regression equation:

$$Y = \overbrace{\alpha + \beta_1 X_1 + \cdots + \beta_n X_n}^{\text{Predicted Mean } \mu = \eta} + \overbrace{\epsilon}^{\text{Noise}\sim N(0,\sigma)}$$

GLMs IV

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- e.g., "Does neighborhood density affects RT?"$\rightarrow$ is β reliably non-zero?

GLMs VI

- We'll use length-4 nonword data from (Bicknell et al., 2010) (thanks!), such as:

Few neighbors			*Many neighbors*		
`gaty`	`peme`	`rixy`	`lish`	`pait`	`yine`

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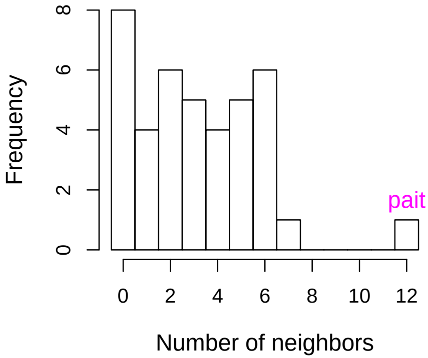
- There's a wide range of neighborhood density:



GLMs VII: maximum-likelihood model fitting

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- Here's a translation of our simple model into `R`:

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Gaussian noise, implicit intercept

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GLMs: maximum-likelihood fitting VIII

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GLMs: maximum-likelihood fitting VIII

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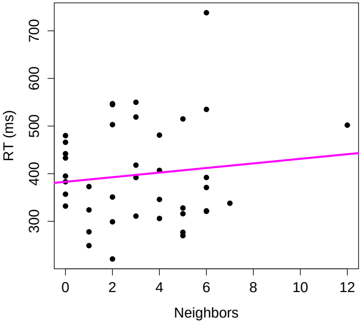

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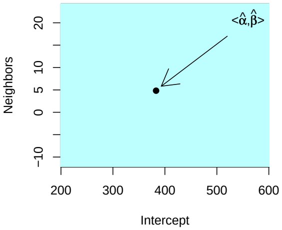

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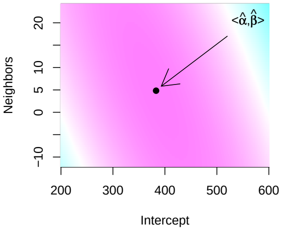

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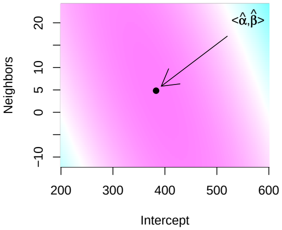

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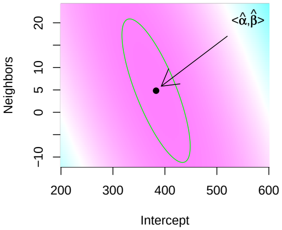

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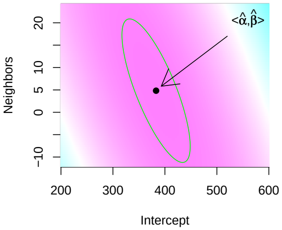

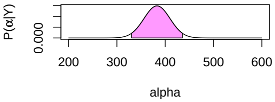

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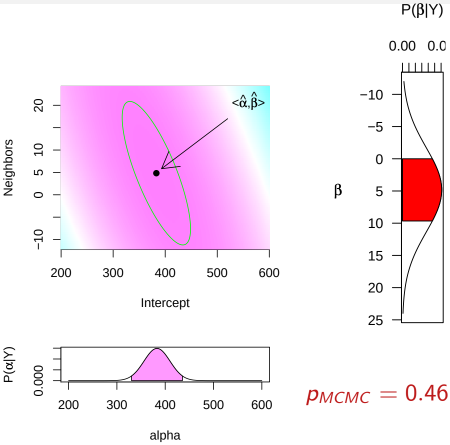

$p_{MCMC} = 0.46$

- p_{MCMC} (Baayen et al., 2008) is 1 minus the largest possible symmetric confidence interval wholly on one side of 0

Linear regression

$$Y = \overbrace{\alpha + \beta_1 X_1 + \cdots + \beta_n X_n}^{\text{Predicted Mean}} + \overbrace{\epsilon}^{\text{Noise}\sim N(0,\sigma)}$$

- More compact representation with matrices is very useful: for m predictors and n observations,

Data vector (length n)	Model matrix (dims $n \times (m+1)$)	Coefficients (length $m+1$)	Error vector (length n)

$$\mathbf{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad \mathbf{X} = \begin{bmatrix} 1 & x_{11} & x_{12} & \dots & x_{1m} \\ 1 & x_{21} & x_{22} & \dots & x_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & x_{n2} & \dots & x_{nm} \end{bmatrix} \quad \boldsymbol{\beta} = \begin{bmatrix} \alpha \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix} \quad \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}$$

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- The linear regression equation is then specified as

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

A little linear algebra

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If $\mathbf{X}$ is an $L \times M$ matrix and $\mathbf{Y}$ is an $M \times N$ matrix, then $\mathbf{X}$ and $\mathbf{Y}$ can be multiplied together; the resulting matrix $\mathbf{XY}$ is an $L \times M$ matrix. If $\mathbf{Z} = \mathbf{XY}$, the i, j-th entry of $\mathbf{Z}$ is:

$$z_{ij} = \sum_{k=1}^{M} x_{ik} y_{kj}$$

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Thus for our linear regression equation (note that $M = m + 1$):

$$\mathbf{X}\boldsymbol{\beta} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

A little linear algebra

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A little linear algebra

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Linear regression

- So we have our regression equation

$$\mathbf{Y} = \mathbf{X}\beta + \epsilon$$

Linear regression

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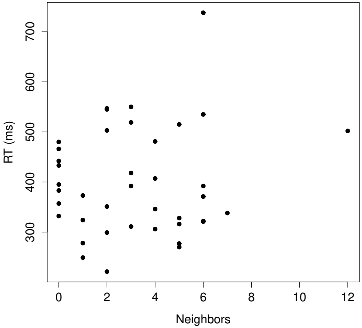

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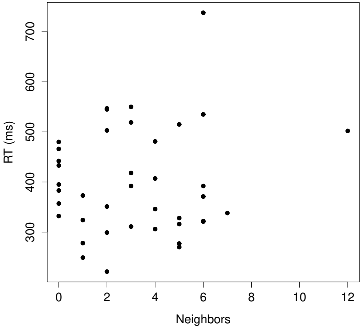

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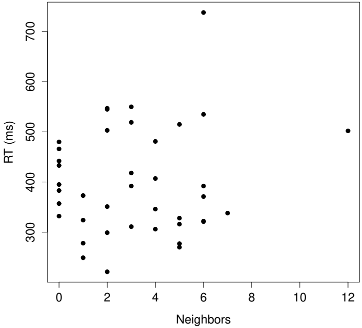

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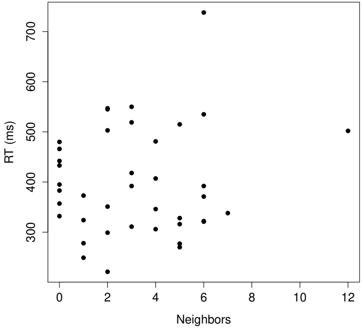

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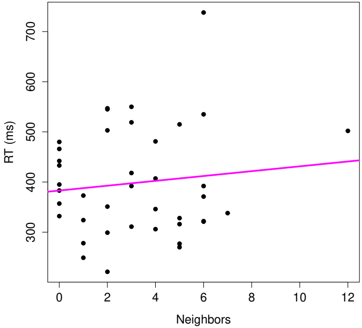

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- The quantity $s^2 = RSS/(n - m - 1)$ is an unbiased estimator of the the error variance σ^2

Frequentist confidence regions for linear regression

$$\boxed{\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}}$$

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- In our example,

$$(\mathbf{X}^\mathsf{T}\mathbf{X})^{-1} = \begin{bmatrix} 0.06265 & -0.01186 \\ -0.01186 & 0.003734 \end{bmatrix}$$

hence the correlation between $\widehat{\alpha}$ and $\widehat{\beta}$ is $-0.78 (= \frac{-0.01186}{\sqrt{0.06265 * 0.003734}})$

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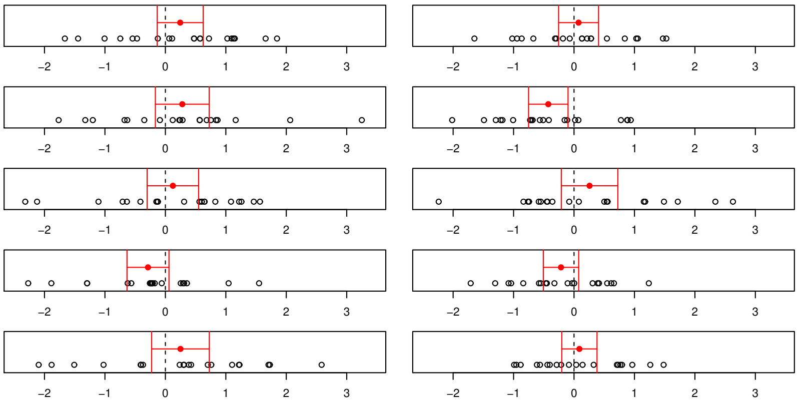
- Recall that a $1-p$ frequentist confidence interval I for a parameter θ is one that, if the same procedure is used to construct intervals from many different randomly generated datasets, contain θ with probability $1-p$



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  - It will always be an *ellipsoid* whose shape is determined by $X^T X$ and whose size is determined by p (the size of the region) and s^2 (the estimate of the error variance)

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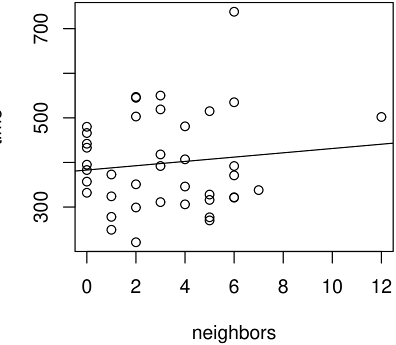
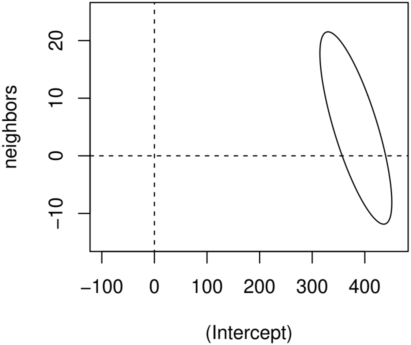
Our original example:





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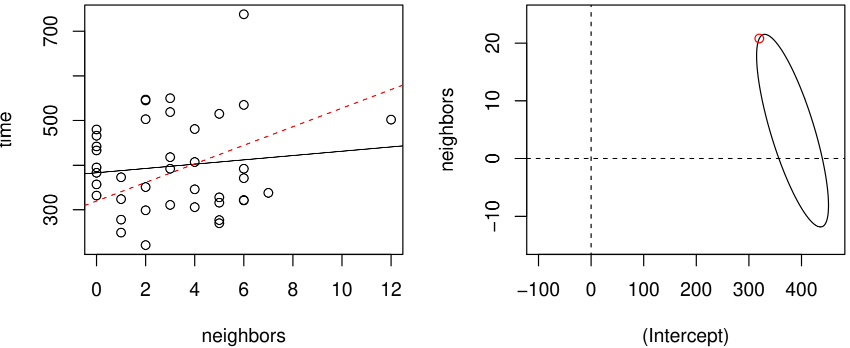

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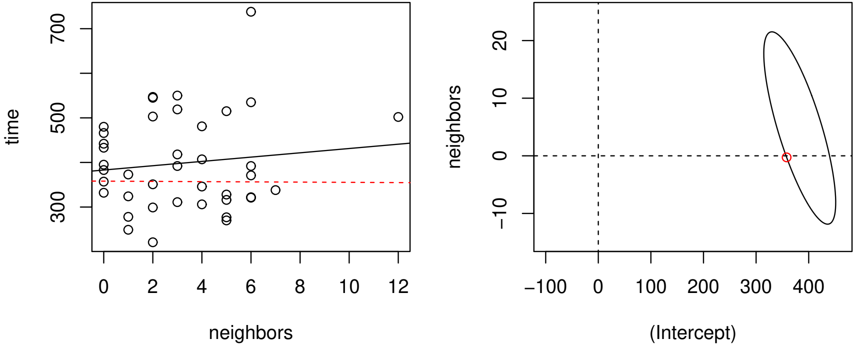

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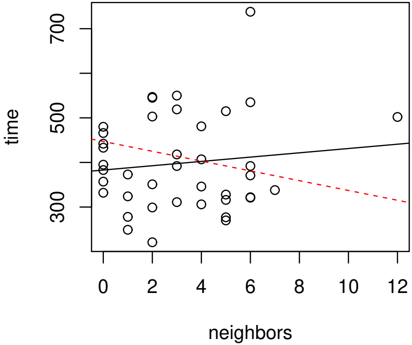

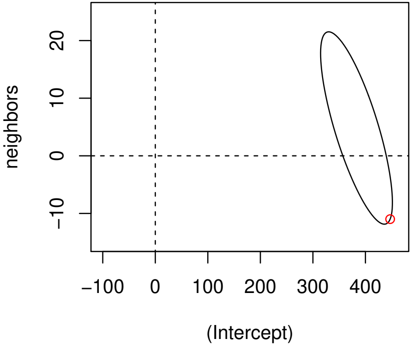

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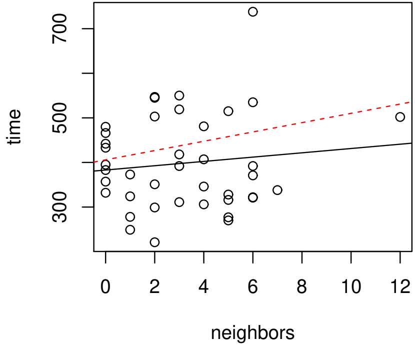

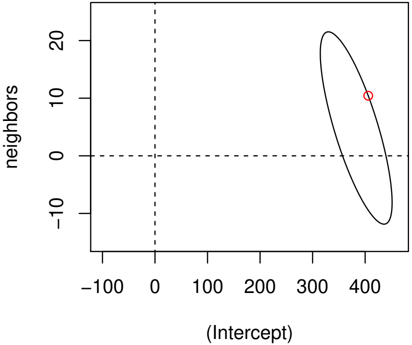

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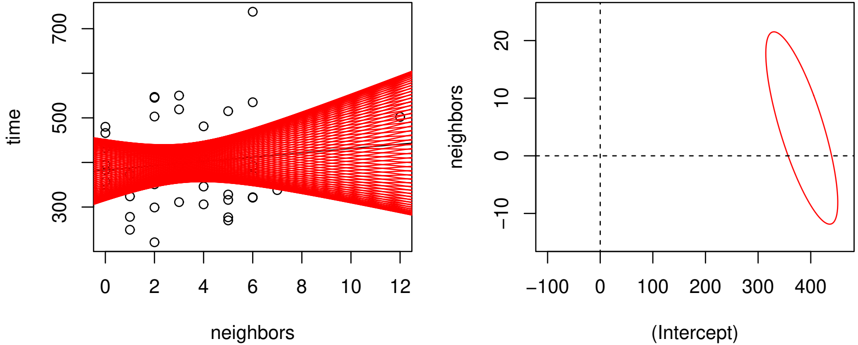

Null hypothesis significance testing with the t-statistic

- Recall: general confidence region is built on the fact that

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- The quantity $1/\frac{(X^T X)_{ii}}{s^2}$ is often called the standard error of the estimate $\widehat{\beta}_i$

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- **Example:** in our case, $\widehat{\beta}_{RT} = 4.8$; $SE_{RT} = 6.6$, so the t-statistic of the estimate is 0.74. This is statistically **insignificant**

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- Additionally, we'll look at the possible predictive value of two other predictors: **word frequency** and **word length**

The interpretation of regression coefficients

```
summary(lm(exp(RTnaming) ~ Ncount + LengthInLetters + WrittenFrequency,dat.engl

##
## Call:
## lm(formula = exp(RTnaming) ~ Ncount + LengthInLetters + WrittenFrequency,
##     data = dat.english)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -57.274 -13.143  -0.197  13.256  63.203
##
## Coefficients:
##                  Estimate Std. Error t value Pr(>|t|)
## (Intercept)      469.1812     9.0089  52.080  < 2e-16 ***
## Ncount            -0.7519     0.2297  -3.274 0.001130 **
## LengthInLetters    5.2222     1.4588   3.580 0.000375 ***
## WrittenFrequency  -3.4998     0.8959  -3.906 0.000106 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 18.61 on 535 degrees of freedom
## Multiple R-squared:  0.125, Adjusted R-squared:  0.1201
## F-statistic: 25.47 on 3 and 535 DF,  p-value: 2.054e-15
```

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- **-0.75ms per additional neighbor**
- **5.22ms per additional character of length**
- **-3.50ms per doubling of word frequency**

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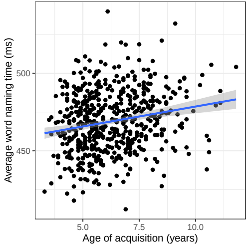

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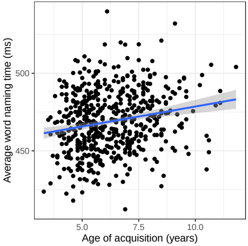

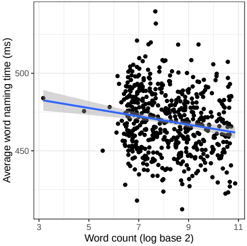

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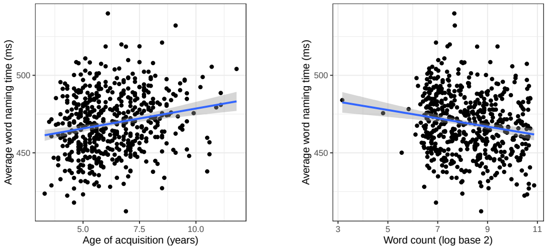

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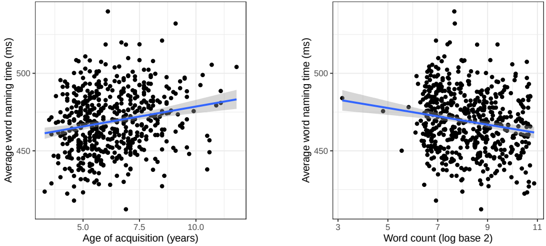

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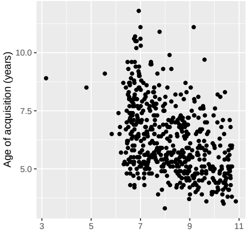

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- Remember, we can read off the correlations among the parameter estimates from $(\mathbf{X}^{\mathsf{T}}\mathbf{X})^{-1}$, which in this case is

$$(\mathbf{X}^{\mathsf{T}}\mathbf{X})^{-1} = \begin{bmatrix} 0.183 & -0.0143 & -0.0102 \\ -0.0143 & 0.00137 & 0.000499 \\ -0.0102 & 0.000499 & 0.000966 \end{bmatrix}$$

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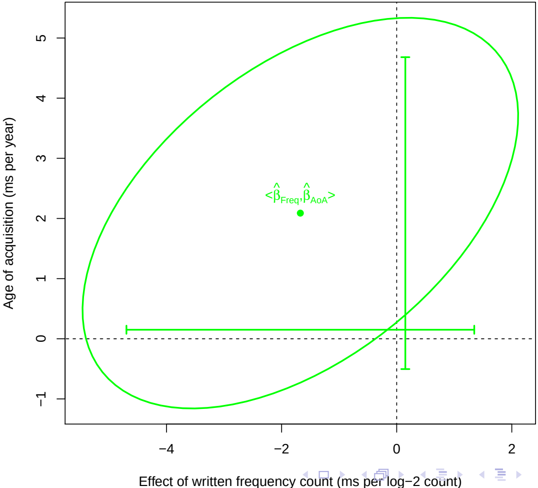
- Thus the correlation between $\widehat{\beta}_{\mathsf{Freq}}$ and $\widehat{\beta}_{\mathsf{AoA}}$ is $\frac{0.000499}{\sqrt{0.00137 \times 0.000966}} = 0.4341883$

A problem of credit assignment

A 95% confidence region for a random subset of 100 words from the English Lexicon Project (Spieler and Balota, 1997):



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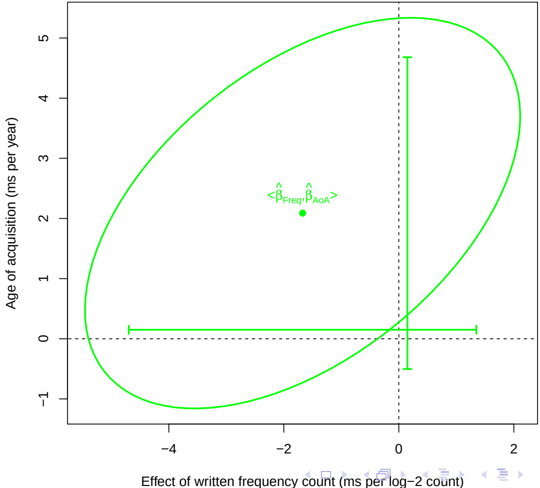

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We see this in R output as well:

```
summary(m)

##
## Call:
## lm(formula = exp(RTnaming) ~ WrittenFrequencyLog2 + AoA, data = dat.english.subset)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -44.429 -12.719   0.439  11.026  41.035
##
## Coefficients:
##                      Estimate Std. Error t value Pr(>|t|)
## (Intercept)           471.958     17.893  26.376   <2e-16 ***
## WrittenFrequencyLog2   -1.674      1.523  -1.100    0.274
## AoA                     2.089      1.306   1.599    0.113
## ---
## Signif. codes:  0 `***' 0.001 `**' 0.01 `*' 0.05 `.' 0.1 ` ' 1
##
## Residual standard error: 17.12 on 97 degrees of freedom
## Multiple R-squared:  0.07014, Adjusted R-squared:  0.05097
## F-statistic: 3.659 on 2 and 97 DF,  p-value: 0.02939
```

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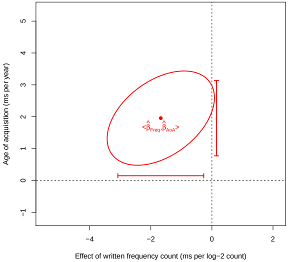

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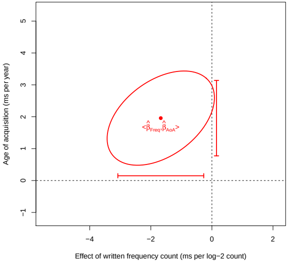

- But suppose we *didn't* have more data: can we test the hypothesis that *some* combination of frequency and age of acquisition predicts naming time?

Nesting of models

- Sometimes, one model will be a more restricted version of another

$$\begin{aligned} M_1 &: & y &= \alpha + \beta_{\mathsf{Freq}}\mathsf{Freq} + \epsilon \\ M_2 &: & y &= \alpha + \beta_{\mathsf{Freq}}\mathsf{Freq} + \beta_{\mathsf{AoA}}\mathsf{AoA} + \epsilon \end{aligned}$$

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- Under such situations, we say that M_1 is nested inside M_2
- Sometimes, it will be less apparent that two models are in a nesting relation—but it can be checked automatically!

The decomposition of variance

- Beautiful property of linear models: decomposition of variance. If model M produces fitted values $\{\hat{y}_i\}$, then the overall variance $\mathrm{Var}(y)$ can be decomposed:

$$\begin{aligned}\mathrm{Var}(y) &= \sum_j (y_j - \bar{y})^2 \\ &= \overbrace{\sum_j (y_j - \hat{y}_j)^2}^{\text{unexplained}} + \overbrace{\sum_j (\hat{y}_j - \bar{y})^2}^{\mathrm{Var}_M(y)}\end{aligned}$$

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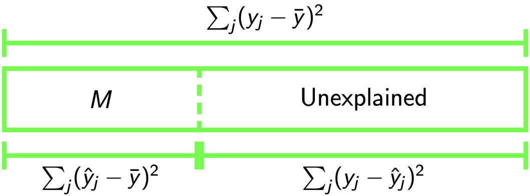

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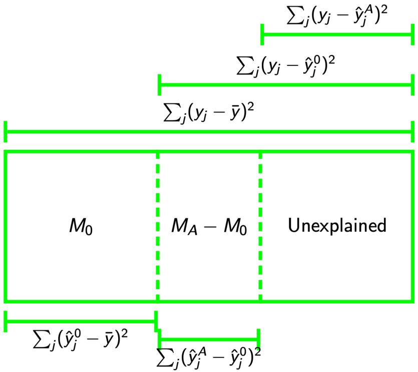
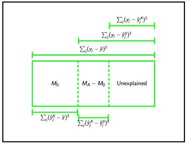
- Then the variance can be decomposed into three pieces:

$$\overbrace{\sum_j (y_j - \bar{y})^2}^{\mathrm{Var}(y)} = \overbrace{\sum_j (\hat{y}_j^0 - \bar{y})^2}^{\mathrm{Var}_0(y)} + \overbrace{\sum_j (\hat{y}_j^A - \hat{y}_j^0)^2}^{\mathrm{Var}_A(y) - \mathrm{Var}_0(y)} + \overbrace{\sum_j (y_j - \hat{y}_j^A)^2}^{\text{unexplained}}$$

The decomposition of variance—graphically represented



ANOVA: NHST model comparison with the F statistic



$\sum_j (y_j - \hat{y}_j^0)^2 \quad RSS_0$

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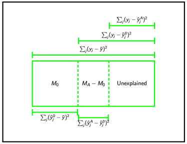

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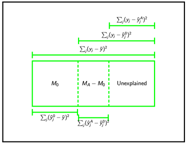

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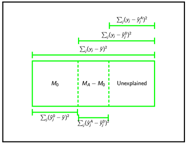

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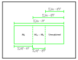
- This hypothesis test is the Analysis of Variance (ANOVA)!

Analysis of Variance: an example

- In our example, formulate the hypothesis: does *some combination of frequency and age of acquisition* meaningfully predict naming time?



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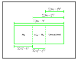

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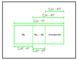

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- For us (note that $n = 100$):

$$
\frac{(RSS_0 - RSS_A)/(m_A - m_0)}{RSS_A/(n - m_A - 1)} = \frac{(30589 - 28443)/2}{28443/97}
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= 3.66
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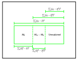

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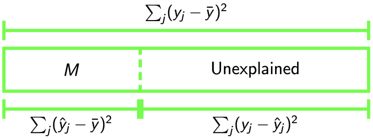
- Consulting the cumulative distribution function for $F_{2,97}$, we get a p-value of $0.0294 \rightarrow$yes, there is good evidence that some combination of frequency and AoA predicts RTs!

How much of the variance does your model explain?



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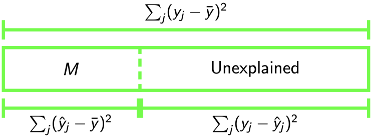

- The coefficient of determination, R^2, is often used to quantify overall model fit in a scale-independent way

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&= 1 - \frac{\sum_j (y_j - \hat{y}_j)^2}{\sum_j (y_j - \bar{y})^2}
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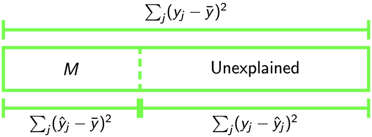

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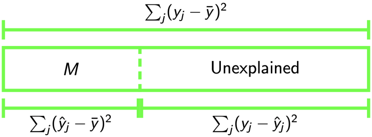

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  has $R^2 = 0.0495$
- $\rightarrow$Although both are useful in predicting RTs, only a small fraction of variance across words in average RT is explained!

References I

Baayen, R. H., Davidson, D. J., and Bates, D. M. (2008). Mixed-effects modeling with crossed random effects for subjects and items. *Journal of Memory and Language*, 59(4):390–412.

Bicknell, K., Elman, J. L., Hare, M., McRae, K., and Kutas, M. (2010). Effects of event knowledge in processing verbal arguments. *Journal of Memory and Language*, 63:489–505.

Spieler, D. H. and Balota, D. A. (1997). Bringing computational models of word naming down to the item level. *Psychological Science*, 6:411–416.

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- Note that so-called "p_{MCMC}" is NOT a p-value in the Neyman-Pearson sense!
- Weakness, both in practice and in principle: the alternative hypothesis is never actually used (except indirectly in determining optimal acceptance and rejection regions)

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- Alternative: Bayesian hypothesis testing, which is symmetric:

$$\frac{P(H_0|D)}{P(H_1|D)} = \frac{P(D|H_0)}{P(D|H_1)} \frac{P(H_0)}{P(H_1)}$$

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- Technically, such a measure doesn't need to be a true Neyman-Pearson p-value (p_{MCMC} falls into this category)