

Confidence intervals, hypothesis testing, Monte Carlo, and generalized linear models

Roger Levy

9.S916: Statistical data analysis for scientific inference in
cognitive science

16 April 2024

Credible intervals & confidence intervals

Credible intervals & confidence intervals

- A **point estimate** of a model parameter is one example of a **statistic**

Credible intervals & confidence intervals

- A **point estimate** of a model parameter is one example of a **statistic**

(Wikipedia: "A **statistic**...or **sample statistic** is any quantity computed from values in a sample which is considered for a statistical purpose")

Credible intervals & confidence intervals

- A **point estimate** of a model parameter is one example of a **statistic**

(Wikipedia: "A **statistic**...or **sample statistic** is any quantity computed from values in a sample which is considered for a statistical purpose")

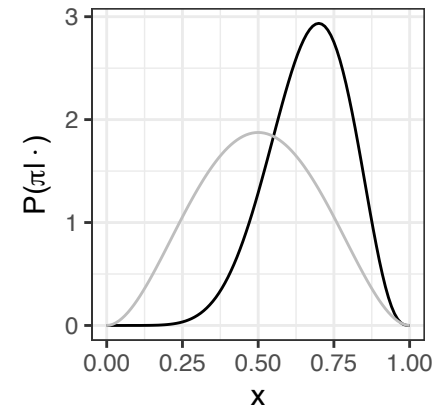
Beta-binomial

$$\alpha_1 = 3$$

$$\alpha_2 = 3$$

$$r = 5$$

$$n = 6$$



Credible intervals & confidence intervals

- A **point estimate** of a model parameter is one example of a **statistic**

(Wikipedia: "A **statistic**...or **sample statistic** is any quantity computed from values in a sample which is considered for a statistical purpose")

- Point estimates we have seen thus far:

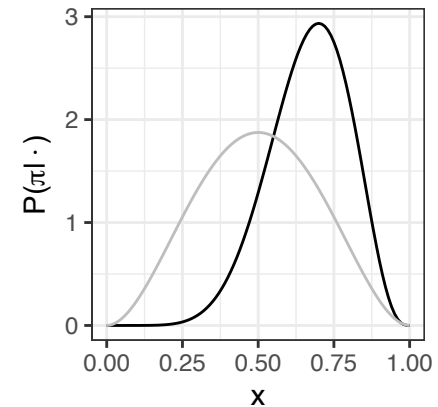
Beta-binomial

$$\alpha_1 = 3$$

$$\alpha_2 = 3$$

$$r = 5$$

$$n = 6$$



Credible intervals & confidence intervals

- A **point estimate** of a model parameter is one example of a **statistic**

(Wikipedia: "A **statistic**...or **sample statistic** is any quantity computed from values in a sample which is considered for a statistical purpose")

- Point estimates we have seen thus far:

- **Maximum likelihood estimate**

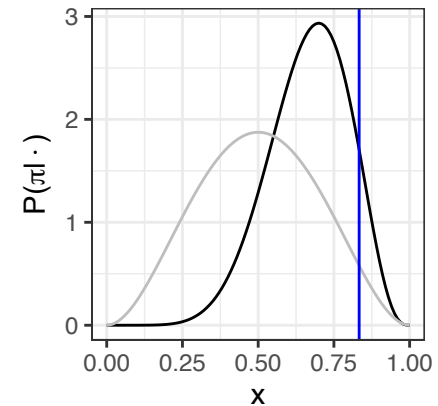
Beta-binomial

$$\alpha_1 = 3$$

$$\alpha_2 = 3$$

$$r = 5$$

$$n = 6$$



Credible intervals & confidence intervals

- A **point estimate** of a model parameter is one example of a **statistic**

(Wikipedia: "A **statistic**...or **sample statistic** is any quantity computed from values in a sample which is considered for a statistical purpose")

- Point estimates we have seen thus far:

- Maximum likelihood estimate
- Bayesian posterior mean

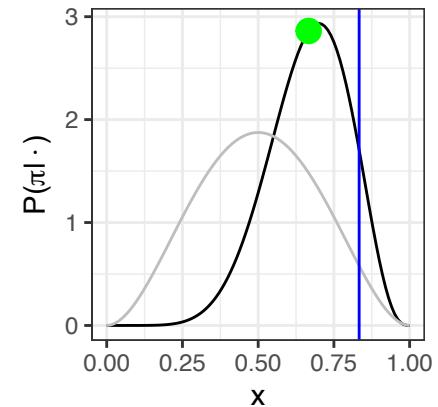
Beta-binomial

$$\alpha_1 = 3$$

$$\alpha_2 = 3$$

$$r = 5$$

$$n = 6$$



Credible intervals & confidence intervals

- A **point estimate** of a model parameter is one example of a **statistic**

(Wikipedia: "A **statistic**...or **sample statistic** is any quantity computed from values in a sample which is considered for a statistical purpose")

- Point estimates we have seen thus far:

- Maximum likelihood estimate
- Bayesian posterior mean
- Bayesian posterior mode

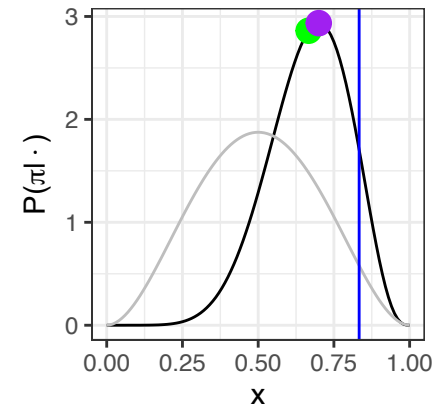
Beta-binomial

$$\alpha_1 = 3$$

$$\alpha_2 = 3$$

$$r = 5$$

$$n = 6$$



Credible intervals & confidence intervals

- A **point estimate** of a model parameter is one example of a **statistic**

(Wikipedia: "A **statistic**...or **sample statistic** is any quantity computed from values in a sample which is considered for a statistical purpose")

- Point estimates we have seen thus far:

- Maximum likelihood estimate
- Bayesian posterior mean
- Bayesian posterior mode

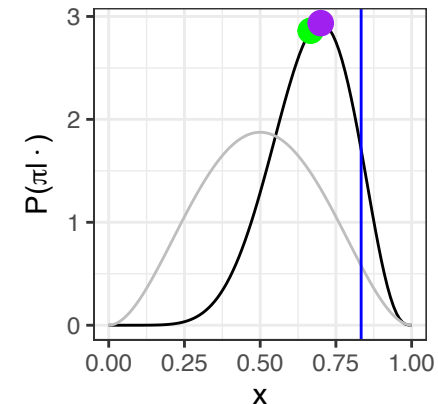
Beta-binomial

$$\alpha_1 = 3$$

$$\alpha_2 = 3$$

$$r = 5$$

$$n = 6$$



- All of these point estimates discard a lot of information about the shape of the curve that they come from!

Credible intervals & confidence intervals

- A **point estimate** of a model parameter is one example of a **statistic**

(Wikipedia: "A **statistic**...or **sample statistic** is any quantity computed from values in a sample which is considered for a statistical purpose")

- Point estimates we have seen thus far:

- Maximum likelihood estimate
- Bayesian posterior mean
- Bayesian posterior mode

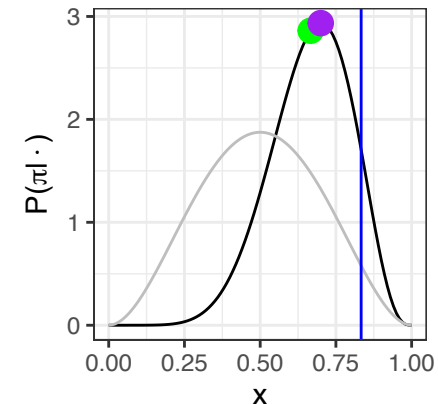
Beta-binomial

$$\alpha_1 = 3$$

$$\alpha_2 = 3$$

$$r = 5$$

$$n = 6$$



- All of these point estimates discard a lot of information about the shape of the curve that they come from!
 - Curve shape captures **uncertainty** about parameter

Credible intervals & confidence intervals

- A **point estimate** of a model parameter is one example of a **statistic**

(Wikipedia: "A **statistic**...or **sample statistic** is any quantity computed from values in a sample which is considered for a statistical purpose")

- Point estimates we have seen thus far:

- Maximum likelihood estimate
- Bayesian posterior mean
- Bayesian posterior mode

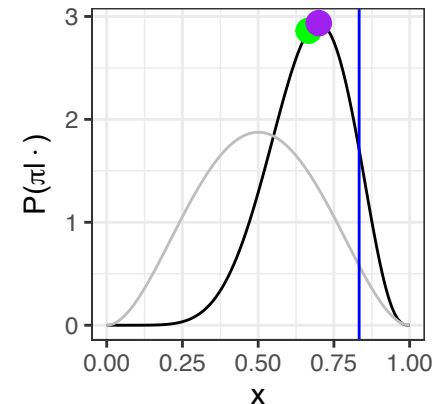
Beta-binomial

$$\alpha_1 = 3$$

$$\alpha_2 = 3$$

$$r = 5$$

$$n = 6$$



- All of these point estimates discard a lot of information about the shape of the curve that they come from!
 - Curve shape captures **uncertainty** about parameter
- Credible intervals (Bayesian) and confidence intervals (frequentist) provide a bit more information about this uncertainty

Bayesian credible intervals

$$P(\theta|\mathbf{y}) = \frac{P(\mathbf{y}|\theta)P(\theta)}{P(\mathbf{y})}$$

Bayesian credible intervals

$$P(\theta|\mathbf{y}) = \frac{P(\mathbf{y}|\theta)P(\theta)}{P(\mathbf{y})}$$

- A $(1 - \alpha)$ Bayesian credible interval (CI) on parameter π is an interval containing $(1 - \alpha)$ of the posterior mass

Bayesian credible intervals

$$P(\theta|\mathbf{y}) = \frac{P(\mathbf{y}|\theta)P(\theta)}{P(\mathbf{y})}$$

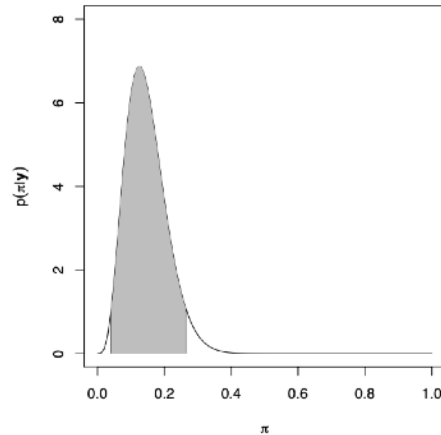
- A $(1 - \alpha)$ Bayesian credible interval (CI) on parameter π is an interval containing $(1 - \alpha)$ of the posterior mass
- Two common standards for Bayesian CI construction:

Bayesian credible intervals

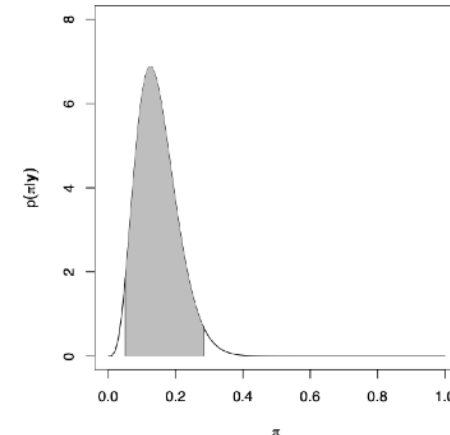
$$P(\theta|\mathbf{y}) = \frac{P(\mathbf{y}|\theta)P(\theta)}{P(\mathbf{y})}$$

- A $(1 - \alpha)$ Bayesian credible interval (CI) on parameter π is an interval containing $(1 - \alpha)$ of the posterior mass
- Two common standards for Bayesian CI construction:

Highest posterior density



Symmetric

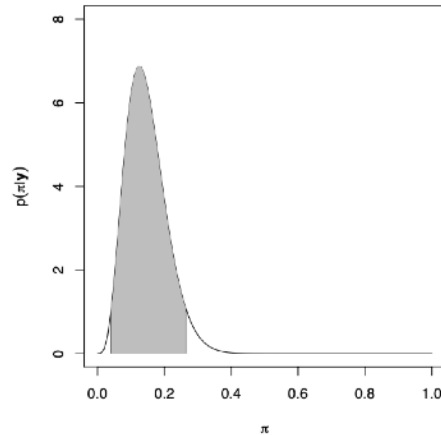


Bayesian credible intervals

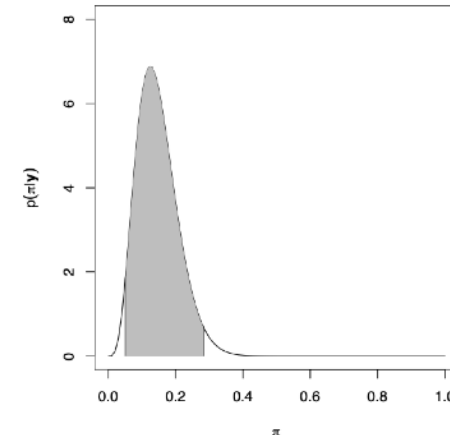
$$P(\theta|\mathbf{y}) = \frac{P(\mathbf{y}|\theta)P(\theta)}{P(\mathbf{y})}$$

- A $(1 - \alpha)$ Bayesian credible interval (CI) on parameter π is an interval containing $(1 - \alpha)$ of the posterior mass
- Two common standards for Bayesian CI construction:

Highest posterior density



Symmetric



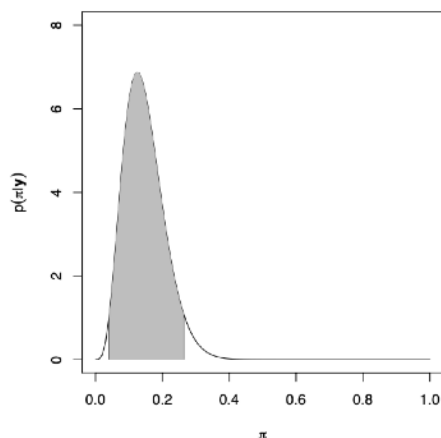
- Older term: "**Bayesian confidence interval**"

Bayesian credible intervals

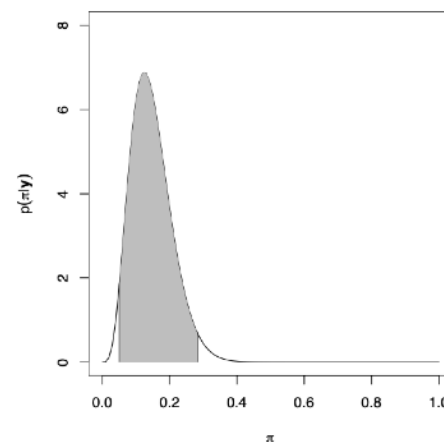
$$P(\theta|\mathbf{y}) = \frac{P(\mathbf{y}|\theta)P(\theta)}{P(\mathbf{y})}$$

- A $(1 - \alpha)$ Bayesian credible interval (CI) on parameter π is an interval containing $(1 - \alpha)$ of the posterior mass
- Two common standards for Bayesian CI construction:

Highest posterior density



Symmetric



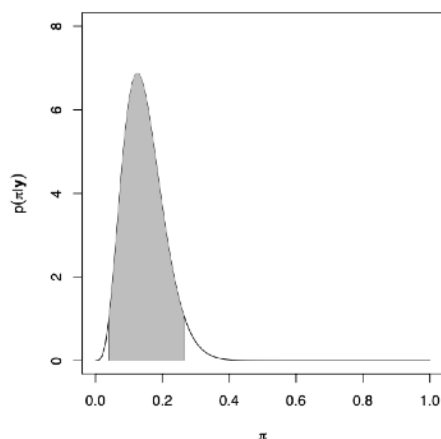
- Older term: "**Bayesian confidence interval**"
- Multivariate generalization: interval→region

Bayesian credible intervals

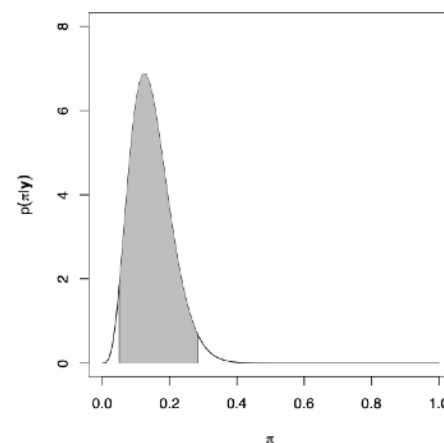
$$P(\theta|\mathbf{y}) = \frac{P(\mathbf{y}|\theta)P(\theta)}{P(\mathbf{y})}$$

- A $(1 - \alpha)$ Bayesian credible interval (CI) on parameter π is an interval containing $(1 - \alpha)$ of the posterior mass
- Two common standards for Bayesian CI construction:

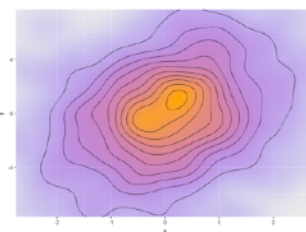
Highest posterior density



Symmetric



- Older term: "**Bayesian confidence interval**"
- Multivariate generalization: interval→region



Frequentist confidence intervals

Frequentist confidence intervals

- For model parameter θ , define a procedure for constructing from data \mathbf{y} an interval I for possible θ

Frequentist confidence intervals

- For model parameter θ , define a procedure for constructing from data \mathbf{y} an interval I for possible θ

$$\text{Proc}(\mathbf{y}) = I$$

Frequentist confidence intervals

- For model parameter θ , define a procedure for constructing from data \mathbf{y} an interval I for possible θ
$$\text{Proc}(\mathbf{y}) = I$$
- Suppose I repeat my experiment over and over again, each time collecting data \mathbf{y} and constructing $I = \text{Proc}(\mathbf{y})$

Frequentist confidence intervals

- For model parameter θ , define a procedure for constructing from data \mathbf{y} an interval I for possible θ
$$\text{Proc}(\mathbf{y}) = I$$
- Suppose I repeat my experiment over and over again, each time collecting data \mathbf{y} and constructing $I = \text{Proc}(\mathbf{y})$
- If $(1 - \alpha)$ of these intervals contain the **true value of θ** , then Proc is a method for constructing a $(1 - \alpha)$ frequentist confidence interval

Frequentist confidence intervals

- For model parameter θ , define a procedure for constructing from data \mathbf{y} an interval I for possible θ
$$\text{Proc}(\mathbf{y}) = I$$
- Suppose I repeat my experiment over and over again, each time collecting data \mathbf{y} and constructing $I = \text{Proc}(\mathbf{y})$
- If $(1 - \alpha)$ of these intervals contain the **true value of θ** , then Proc is a method for constructing a $(1 - \alpha)$ frequentist confidence interval

Confidence interval for mean
 μ of a normal distribution

$$\frac{\hat{\mu} - \mu}{\sqrt{S^2/n}} \sim t_{n-1}$$

Frequentist confidence intervals

- For model parameter θ , define a procedure for constructing from data \mathbf{y} an interval I for possible θ
$$\text{Proc}(\mathbf{y}) = I$$
- Suppose I repeat my experiment over and over again, each time collecting data \mathbf{y} and constructing $I = \text{Proc}(\mathbf{y})$
- If $(1 - \alpha)$ of these intervals contain the **true value of θ** , then Proc is a method for constructing a $(1 - \alpha)$ frequentist confidence interval

Confidence interval for mean
 μ of a normal distribution

Sample mean

$$\frac{\hat{\mu} - \mu}{\sqrt{S^2/n}} \sim t_{n-1}$$

Frequentist confidence intervals

- For model parameter θ , define a procedure for constructing from data \mathbf{y} an interval I for possible θ
$$\text{Proc}(\mathbf{y}) = I$$
- Suppose I repeat my experiment over and over again, each time collecting data \mathbf{y} and constructing $I = \text{Proc}(\mathbf{y})$
- If $(1 - \alpha)$ of these intervals contain the **true value of θ** , then Proc is a method for constructing a $(1 - \alpha)$ frequentist confidence interval

Confidence interval for mean
 μ of a normal distribution

Sample mean

$$\frac{\hat{\mu} - \mu}{\sqrt{S^2/n}} \sim t_{n-1}$$

Standard error (of the mean)

Frequentist confidence intervals

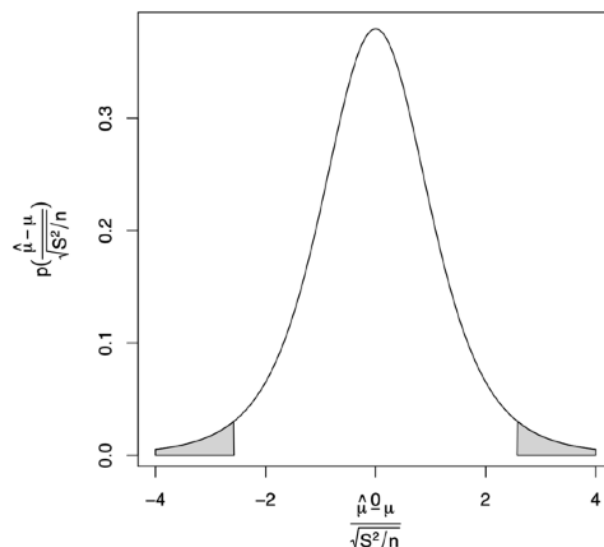
- For model parameter θ , define a procedure for constructing from data \mathbf{y} an interval I for possible θ
$$\text{Proc}(\mathbf{y}) = I$$
- Suppose I repeat my experiment over and over again, each time collecting data \mathbf{y} and constructing $I = \text{Proc}(\mathbf{y})$
- If $(1 - \alpha)$ of these intervals contain the **true value of θ** , then Proc is a method for constructing a $(1 - \alpha)$ frequentist confidence interval

Confidence interval for mean
 μ of a normal distribution

Sample mean

$$\frac{\hat{\mu} - \mu}{\sqrt{S^2/n}} \sim t_{n-1}$$

Standard error (of the mean)



Bayesian hypothesis testing

Bayesian hypothesis testing

- **Hypothesis:** a candidate theory/model for the generative process by which data \mathbf{y} come into the world

Bayesian hypothesis testing

- **Hypothesis:** a candidate theory/model for the generative process by which data \mathbf{y} come into the world
- To compare hypotheses $\{H_i\}$: simply Bayesian inference!

$$P(H_i|\mathbf{y}) = \frac{P(\mathbf{y}|H_i)P(H_i)}{P(\mathbf{y})}$$

Bayesian hypothesis testing

- **Hypothesis:** a candidate theory/model for the generative process by which data \mathbf{y} come into the world
- To compare hypotheses $\{H_i\}$: simply Bayesian inference!

$$P(H_i|\mathbf{y}) = \frac{P(\mathbf{y}|H_i)P(H_i)}{P(\mathbf{y})}$$

Normalizing constant, not of great
interest for present purposes

Bayesian hypothesis testing

- **Hypothesis:** a candidate theory/model for the generative process by which data \mathbf{y} come into the world
- To compare hypotheses $\{H_i\}$: simply Bayesian inference!

$$P(H_i|\mathbf{y}) = \frac{P(\mathbf{y}|H_i)P(H_i)}{P(\mathbf{y})}$$

Normalizing constant, not of great interest for present purposes

$$P(\mathbf{y}) = \sum_{j=1}^n P(\mathbf{y}|H_j)P(H_j)$$

Bayesian hypothesis testing

- **Hypothesis:** a candidate theory/model for the generative process by which data \mathbf{y} come into the world
- To compare hypotheses $\{H_i\}$: simply Bayesian inference!

$$P(H_i|\mathbf{y}) = \frac{P(\mathbf{y}|H_i)P(H_i)}{P(\mathbf{y})}$$

Normalizing constant, not of great interest for present purposes

$$P(\mathbf{y}) = \sum_{j=1}^n P(\mathbf{y}|H_j)P(H_j)$$

- Focus on contribution of data to posterior: **Bayes factor**

Bayesian hypothesis testing

- **Hypothesis:** a candidate theory/model for the generative process by which data \mathbf{y} come into the world
- To compare hypotheses $\{H_i\}$: simply Bayesian inference!

$$P(H_i|\mathbf{y}) = \frac{P(\mathbf{y}|H_i)P(H_i)}{P(\mathbf{y})}$$

Normalizing constant, not of great interest for present purposes

$$P(\mathbf{y}) = \sum_{j=1}^n P(\mathbf{y}|H_j)P(H_j)$$

- Focus on contribution of data to posterior: **Bayes factor**

$$\overbrace{\frac{P(H|\mathbf{y})}{P(H'|\mathbf{y})}}^{\text{Posterior odds}} = \overbrace{\frac{P(\mathbf{y}|H)}{P(\mathbf{y}|H')}}^{\text{Likelihood ratio}} \overbrace{\frac{P(H)}{P(H')}}^{\text{Prior odds}}$$

Bayesian hypothesis testing

- **Hypothesis:** a candidate theory/model for the generative process by which data \mathbf{y} come into the world
- To compare hypotheses $\{H_i\}$: simply Bayesian inference!

$$P(H_i|\mathbf{y}) = \frac{P(\mathbf{y}|H_i)P(H_i)}{P(\mathbf{y})}$$

Normalizing constant, not of great interest for present purposes

$$P(\mathbf{y}) = \sum_{j=1}^n P(\mathbf{y}|H_j)P(H_j)$$

- Focus on contribution of data to posterior: **Bayes factor**

$$\overbrace{\frac{P(H|\mathbf{y})}{P(H'|\mathbf{y})}}^{\text{Posterior odds}} = \overbrace{\frac{P(\mathbf{y}|H)}{P(\mathbf{y}|H')}}^{\text{Likelihood ratio}} \overbrace{\frac{P(H)}{P(H')}}^{\text{Prior odds}}$$

Bayesian hypothesis testing

- **Hypothesis:** a candidate theory/model for the generative process by which data \mathbf{y} come into the world
- To compare hypotheses $\{H_i\}$: simply Bayesian inference!

$$P(H_i|\mathbf{y}) = \frac{P(\mathbf{y}|H_i)P(H_i)}{P(\mathbf{y})}$$

Normalizing constant, not of great interest for present purposes $\longrightarrow P(\mathbf{y}) = \sum_{j=1}^n P(\mathbf{y}|H_j)P(H_j)$

- Focus on contribution of data to posterior: **Bayes factor**

$$\overbrace{\frac{P(H|\mathbf{y})}{P(H'|\mathbf{y})}}^{\text{Posterior odds}} = \overbrace{\frac{P(\mathbf{y}|H)}{P(\mathbf{y}|H')}}^{\text{Likelihood ratio}} \overbrace{\frac{P(H)}{P(H')}}^{\text{Prior odds}}$$

Bayes Factor: $\frac{P(\mathbf{y}|H)}{P(\mathbf{y}|H')}$

Interpreting Bayes Factors

$$K = \frac{P(\mathbf{y}|H)}{P(\mathbf{y}|H')}$$

$\log_{10} K$	K	Strength of evidence
0 to 1/2	1 to 3.2	Not worth more than a bare mention
1/2 to 1	3.2 to 10	Substantial
1 to 2	10 to 100	Strong
> 2	> 100	Decisive

Example of Bayesian hypothesis testing

Example of Bayesian hypothesis testing

- Once again the case of the bent coin

Example of Bayesian hypothesis testing

- Once again the case of the bent coin

$$H_1 : P(\pi|H_1) = \begin{cases} 1 & \pi = 0.5 \\ 0 & \pi \neq 0.5 \end{cases}$$

Example of Bayesian hypothesis testing

- Once again the case of the bent coin

$$H_1 : P(\pi|H_1) = \begin{cases} 1 & \pi = 0.5 \\ 0 & \pi \neq 0.5 \end{cases} \quad \text{"The coin is fair"}$$

Example of Bayesian hypothesis testing

- Once again the case of the bent coin

$$H_1 : P(\pi|H_1) = \begin{cases} 1 & \pi = 0.5 \\ 0 & \pi \neq 0.5 \end{cases} \quad \text{"The coin is fair"}$$

$$H_3 : P(\pi|H_3) = 1 \quad 0 \leq \pi \leq 1$$

Example of Bayesian hypothesis testing

- Once again the case of the bent coin

$$H_1 : P(\pi|H_1) = \begin{cases} 1 & \pi = 0.5 \\ 0 & \pi \neq 0.5 \end{cases} \quad \text{"The coin is fair"}$$

$$H_3 : P(\pi|H_3) = 1 \quad 0 \leq \pi \leq 1 \quad \text{"The coin is not fair"}^*$$

Example of Bayesian hypothesis testing

- Once again the case of the bent coin

$$H_1 : P(\pi|H_1) = \begin{cases} 1 & \pi = 0.5 \\ 0 & \pi \neq 0.5 \end{cases} \quad \text{"The coin is fair"}$$

$$H_3 : P(\pi|H_3) = 1 \quad 0 \leq \pi \leq 1 \quad \text{"The coin is not fair"}^*$$

- I flip a coin six times, and it comes up heads four times

Example of Bayesian hypothesis testing

- Once again the case of the bent coin

$$H_1 : P(\pi|H_1) = \begin{cases} 1 & \pi = 0.5 \\ 0 & \pi \neq 0.5 \end{cases} \quad \text{"The coin is fair"}$$

$$H_3 : P(\pi|H_3) = 1 \quad 0 \leq \pi \leq 1 \quad \text{"The coin is not fair"}^*$$

- I flip a coin six times, and it comes up heads four times

$$P(\mathbf{y}|H_1) = \binom{6}{4} \pi^4 (1 - \pi)^2 = \binom{6}{4} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^2 = 0.23$$

Example of Bayesian hypothesis testing

- Once again the case of the bent coin

$$H_1 : P(\pi|H_1) = \begin{cases} 1 & \pi = 0.5 \\ 0 & \pi \neq 0.5 \end{cases} \quad \text{"The coin is fair"}$$

$$H_3 : P(\pi|H_3) = 1 \quad 0 \leq \pi \leq 1 \quad \text{"The coin is not fair"}^*$$

- I flip a coin six times, and it comes up heads four times

$$P(\mathbf{y}|H_1) = \binom{6}{4} \pi^4 (1 - \pi)^2 = \binom{6}{4} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^2 = 0.23$$

$$P(\mathbf{y}|H_3) = \int_{\pi} P(\mathbf{y}|\pi) P(\pi|H_3) d\pi = \int_0^1 \overbrace{\binom{6}{4} \pi^4 (1 - \pi)^2}^{P(\mathbf{y}|\pi)} \underbrace{1}_{P(\pi|H_3)} d\pi = \binom{6}{4} B(5, 3) = 0.14$$

Example of Bayesian hypothesis testing

- Once again the case of the bent coin

$$H_1 : P(\pi|H_1) = \begin{cases} 1 & \pi = 0.5 \\ 0 & \pi \neq 0.5 \end{cases} \quad \text{"The coin is fair"}$$

$$H_3 : P(\pi|H_3) = 1 \quad 0 \leq \pi \leq 1 \quad \text{"The coin is not fair"}^*$$

- I flip a coin six times, and it comes up heads four times

$$P(\mathbf{y}|H_1) = \binom{6}{4} \pi^4 (1 - \pi)^2 = \binom{6}{4} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^2 = 0.23$$

$$P(\mathbf{y}|H_3) = \int_{\pi} P(\mathbf{y}|\pi) P(\pi|H_3) d\pi = \int_0^1 \overbrace{\binom{6}{4} \pi^4 (1 - \pi)^2}^{P(\mathbf{y}|\pi)} \underbrace{1}_{P(\pi|H_3)} d\pi = \binom{6}{4} B(5, 3) = 0.14$$

$$\begin{aligned} \frac{P(\mathbf{y}|H_1)}{P(\mathbf{y}|H_3)} &= \frac{0.23}{0.14} \\ &= 1.64 \end{aligned}$$

Frequentist hypothesis testing

Frequentist hypothesis testing

- The **Neyman–Pearson paradigm**: Formulate two hypotheses about generative process underlying the data

Frequentist hypothesis testing

- The **Neyman–Pearson paradigm**: Formulate two hypotheses about generative process underlying the data

NULL HYPOTHESIS H_0

Frequentist hypothesis testing

- The **Neyman–Pearson paradigm**: Formulate two hypotheses about generative process underlying the data

NULL HYPOTHESIS H_0

ALTERNATIVE HYPOTHESIS H_A within which H_0 is **nested**

Frequentist hypothesis testing

- The **Neyman–Pearson paradigm**: Formulate two hypotheses about generative process underlying the data

NULL HYPOTHESIS H_0

ALTERNATIVE HYPOTHESIS H_A within which H_0 is **nested**

- Choose a TEST STATISTIC T that you'll compute from data

Frequentist hypothesis testing

- The **Neyman–Pearson paradigm**: Formulate two hypotheses about generative process underlying the data

NULL HYPOTHESIS H_0

ALTERNATIVE HYPOTHESIS H_A within which H_0 is **nested**

- Choose a TEST STATISTIC T that you'll compute from data
- Pre-data, divide range of T into ACCEPT/REJECT regions

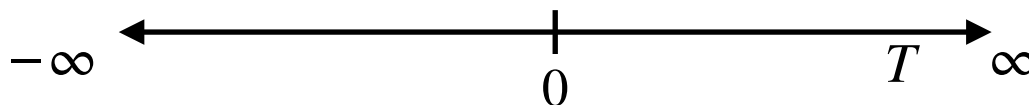
Frequentist hypothesis testing

- The **Neyman–Pearson paradigm**: Formulate two hypotheses about generative process underlying the data

NULL HYPOTHESIS H_0

ALTERNATIVE HYPOTHESIS H_A within which H_0 is **nested**

- Choose a TEST STATISTIC T that you'll compute from data
- Pre-data, divide range of T into ACCEPT/REJECT regions



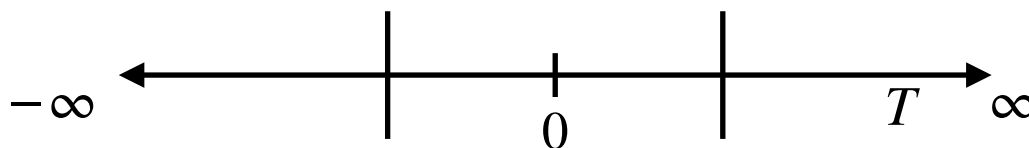
Frequentist hypothesis testing

- The **Neyman–Pearson paradigm**: Formulate two hypotheses about generative process underlying the data

NULL HYPOTHESIS H_0

ALTERNATIVE HYPOTHESIS H_A within which H_0 is **nested**

- Choose a TEST STATISTIC T that you'll compute from data
- Pre-data, divide range of T into ACCEPT/REJECT regions



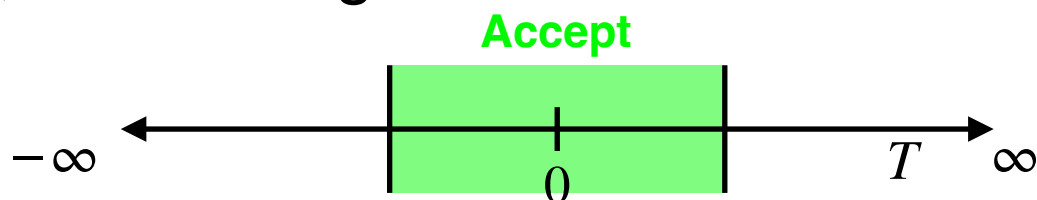
Frequentist hypothesis testing

- The **Neyman–Pearson paradigm**: Formulate two hypotheses about generative process underlying the data

NULL HYPOTHESIS H_0

ALTERNATIVE HYPOTHESIS H_A within which H_0 is **nested**

- Choose a TEST STATISTIC T that you'll compute from data
- Pre-data, divide range of T into ACCEPT/REJECT regions



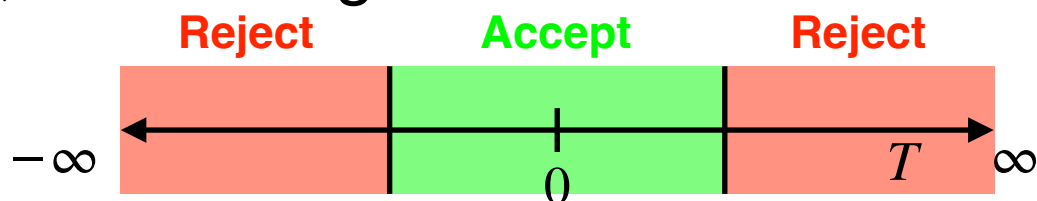
Frequentist hypothesis testing

- The **Neyman–Pearson paradigm**: Formulate two hypotheses about generative process underlying the data

NULL HYPOTHESIS H_0

ALTERNATIVE HYPOTHESIS H_A within which H_0 is **nested**

- Choose a TEST STATISTIC T that you'll compute from data
- Pre-data, divide range of T into ACCEPT/REJECT regions



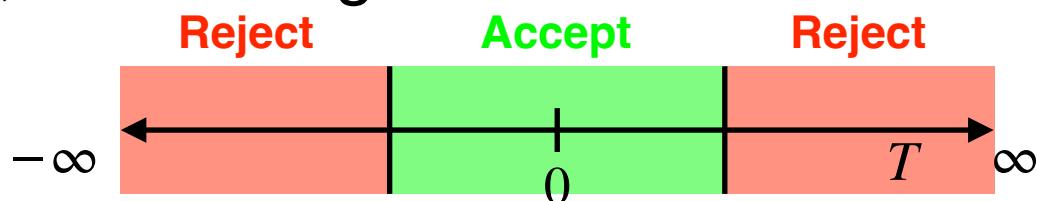
Frequentist hypothesis testing

- The **Neyman–Pearson paradigm**: Formulate two hypotheses about generative process underlying the data

NULL HYPOTHESIS H_0

ALTERNATIVE HYPOTHESIS H_A within which H_0 is **nested**

- Choose a TEST STATISTIC T that you'll compute from data
- Pre-data, divide range of T into ACCEPT/REJECT regions



- Collect data, compute T , see where it falls!

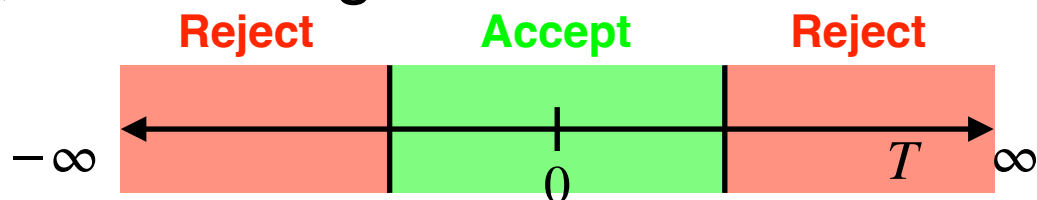
Frequentist hypothesis testing

- The **Neyman–Pearson paradigm**: Formulate two hypotheses about generative process underlying the data

NULL HYPOTHESIS H_0

ALTERNATIVE HYPOTHESIS H_A within which H_0 is **nested**

- Choose a TEST STATISTIC T that you'll compute from data
- Pre-data, divide range of T into ACCEPT/REJECT regions



- Collect data, compute T , see where it falls!

		Accept H_0	Reject H_0
H_0 is...	True	Correct decision (prob. $1 - \alpha$)	Type I error (prob. α)
	False	Type II error (prob. β)	Correct decision (prob. $1 - \beta$)

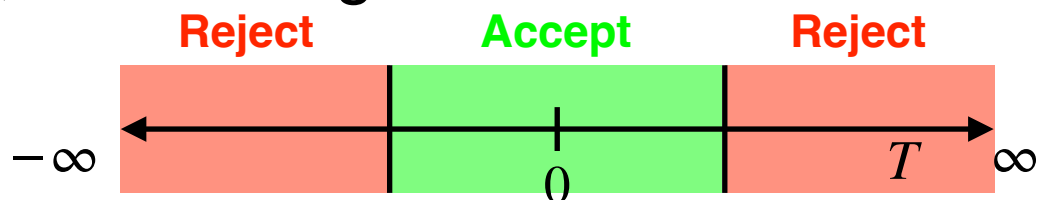
Frequentist hypothesis testing

- The **Neyman–Pearson paradigm**: Formulate two hypotheses about generative process underlying the data

NULL HYPOTHESIS H_0

ALTERNATIVE HYPOTHESIS H_A within which H_0 is **nested**

- Choose a TEST STATISTIC T that you'll compute from data
- Pre-data, divide range of T into ACCEPT/REJECT regions



- Collect data, compute T , see where it falls!

		Accept H_0	Reject H_0	Significance level
H_0 is...	True	Correct decision (prob. $1 - \alpha$)	Type I error (prob. α)	
	False	Type II error (prob. β)	Correct decision (prob. $1 - \beta$)	

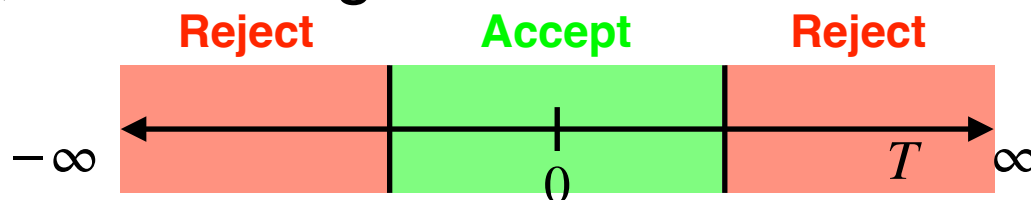
Frequentist hypothesis testing

- The **Neyman–Pearson paradigm**: Formulate two hypotheses about generative process underlying the data

NULL HYPOTHESIS H_0

ALTERNATIVE HYPOTHESIS H_A within which H_0 is **nested**

- Choose a TEST STATISTIC T that you'll compute from data
- Pre-data, divide range of T into ACCEPT/REJECT regions

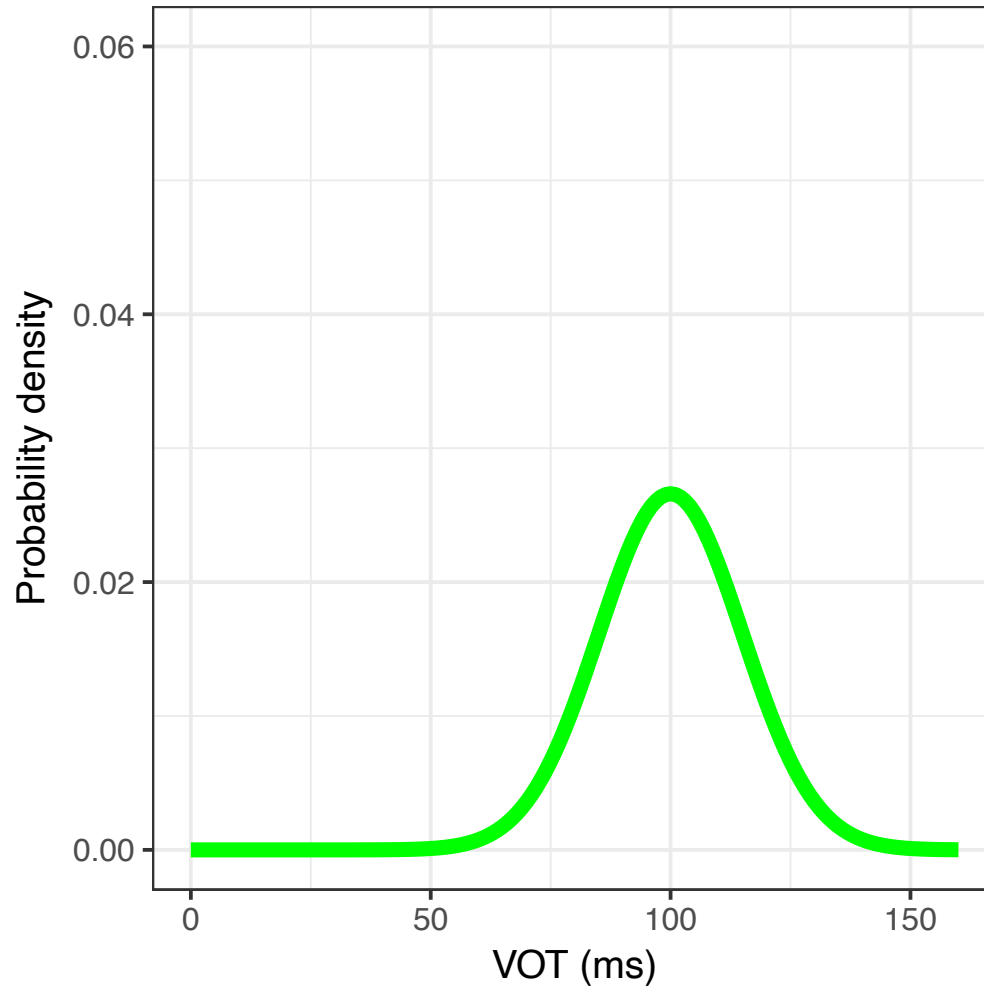


- Collect data, compute T , see where it falls!

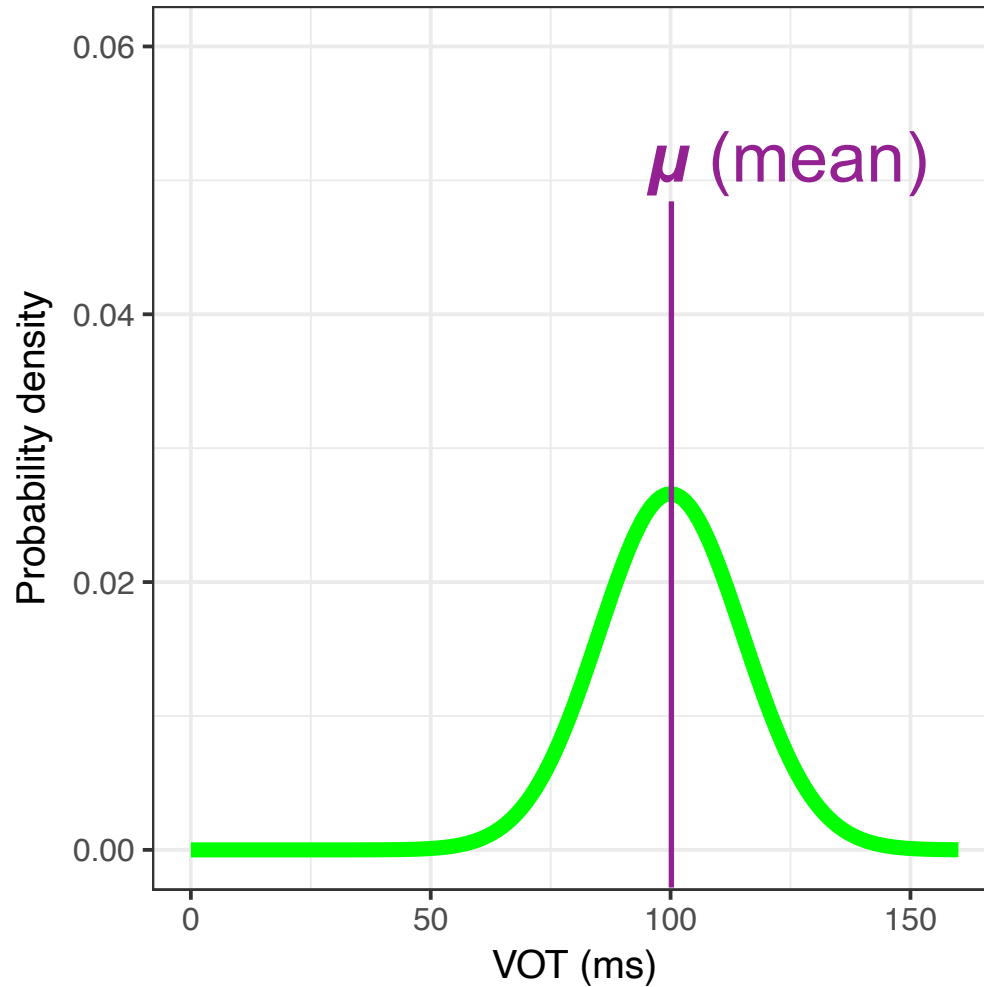
		Accept H_0	Reject H_0	Significance level
H_0 is...	True	Correct decision (prob. $1 - \alpha$)	Type I error (prob. α)	
	False	Type II error (prob. β)	Correct decision (prob. $1 - \beta$)	Power

The Gaussian, or normal, distribution

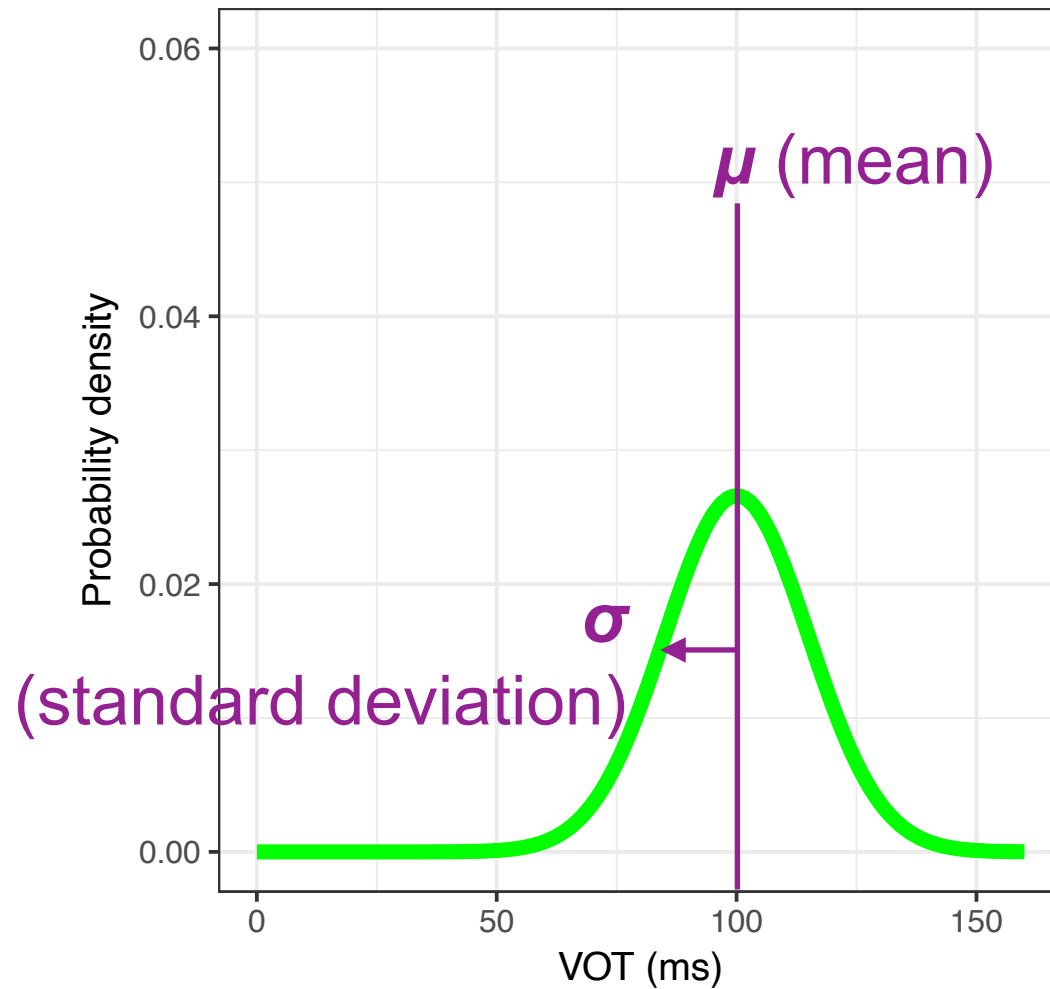
The Gaussian, or normal, distribution



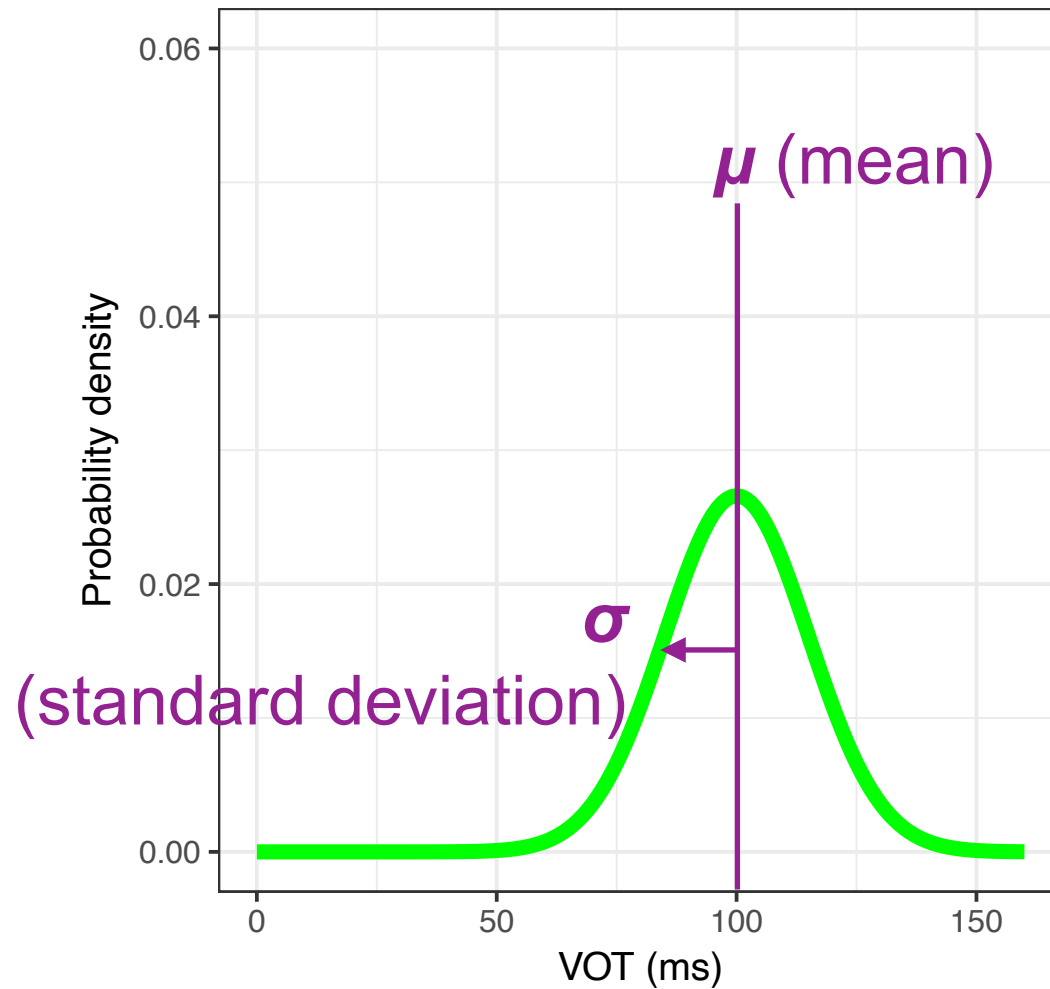
The Gaussian, or normal, distribution



The Gaussian, or normal, distribution

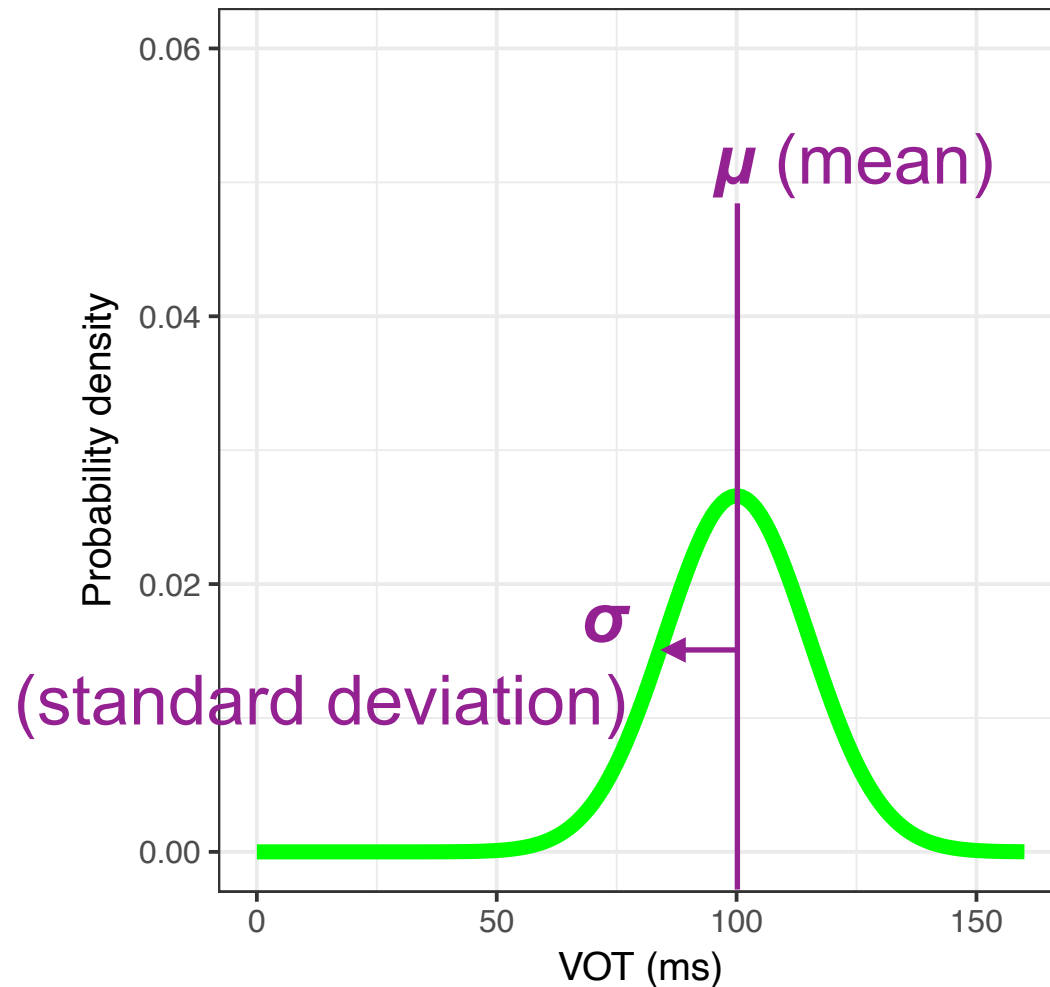


The Gaussian, or normal, distribution



$$p(x | \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[\frac{-(x - \mu)^2}{2\sigma^2} \right]$$

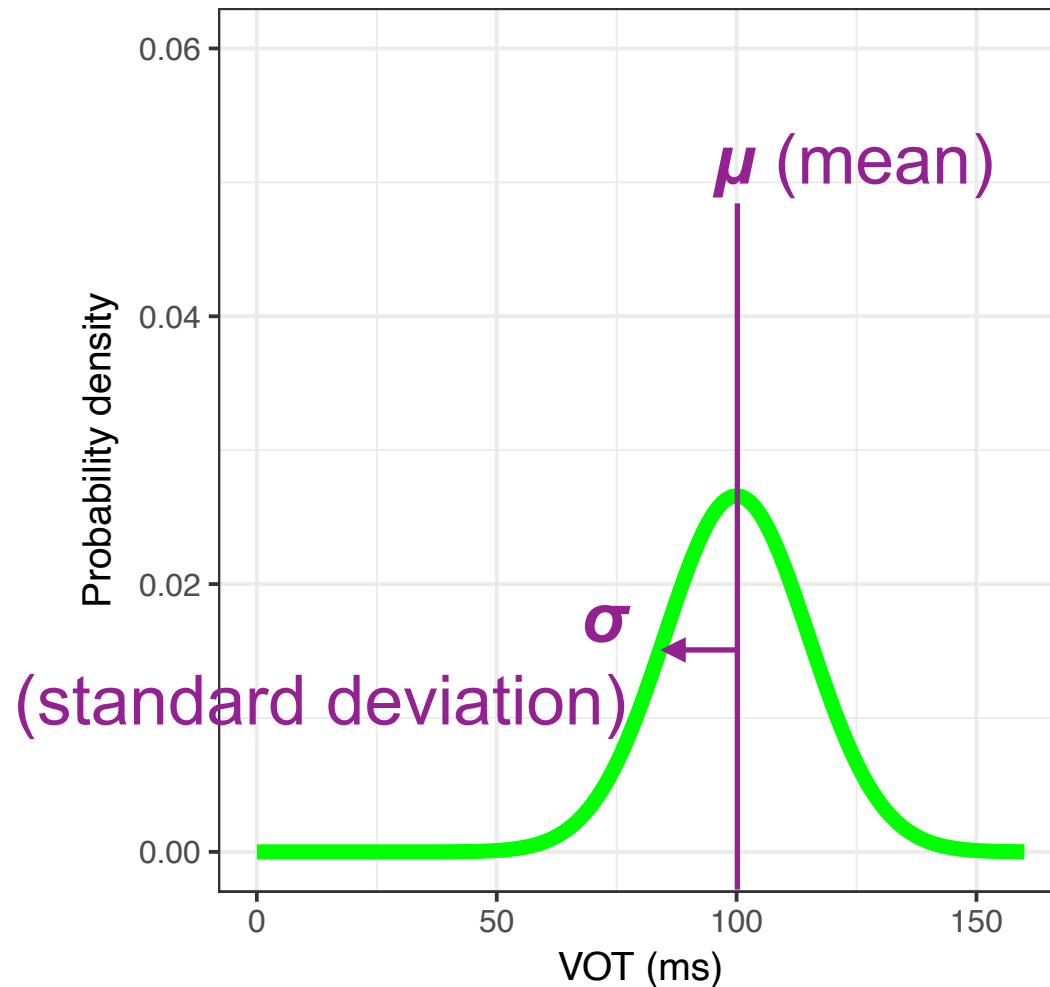
The Gaussian, or normal, distribution



*Squared deviation
from mean*

$$p(x | \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[\frac{-(x - \mu)^2}{2\sigma^2} \right]$$

The Gaussian, or normal, distribution

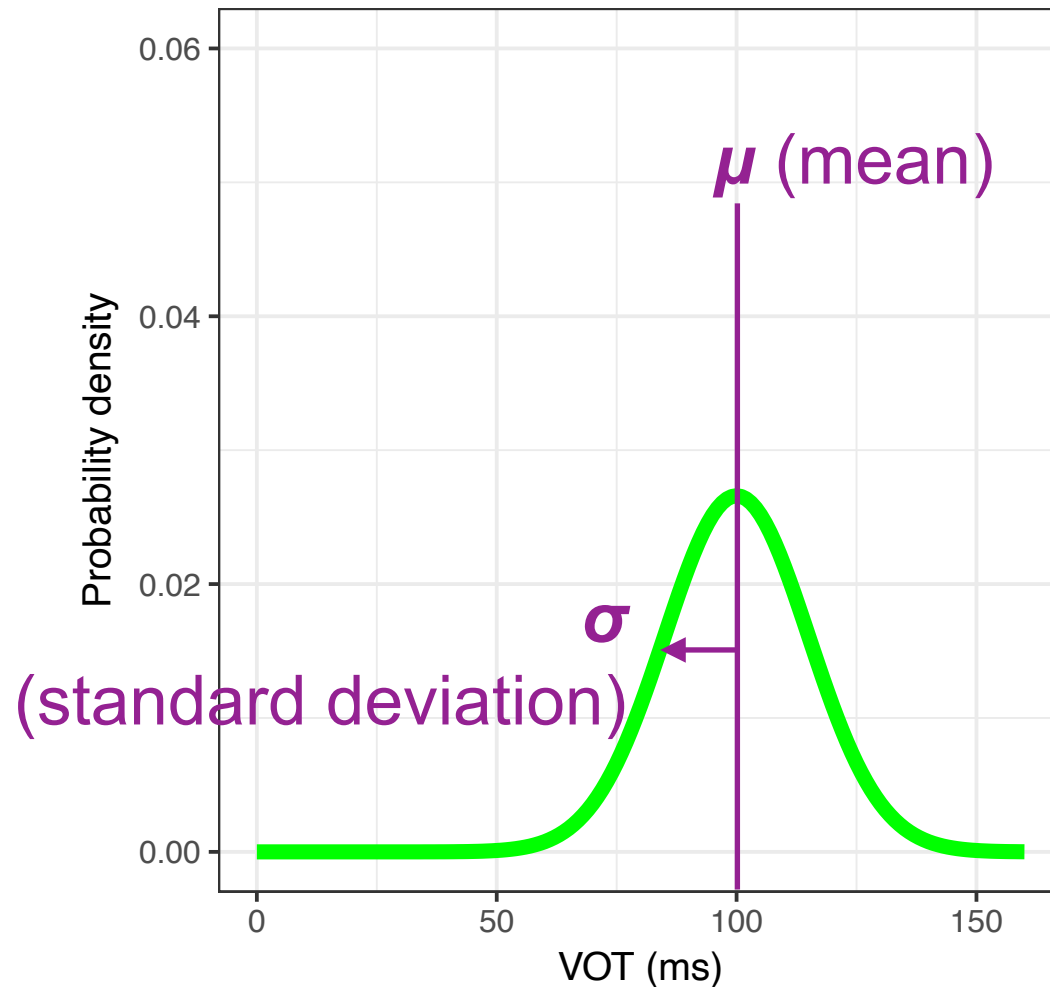


Squared deviation from mean

$$p(x | \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[\frac{-(x - \mu)^2}{2\sigma^2} \right]$$

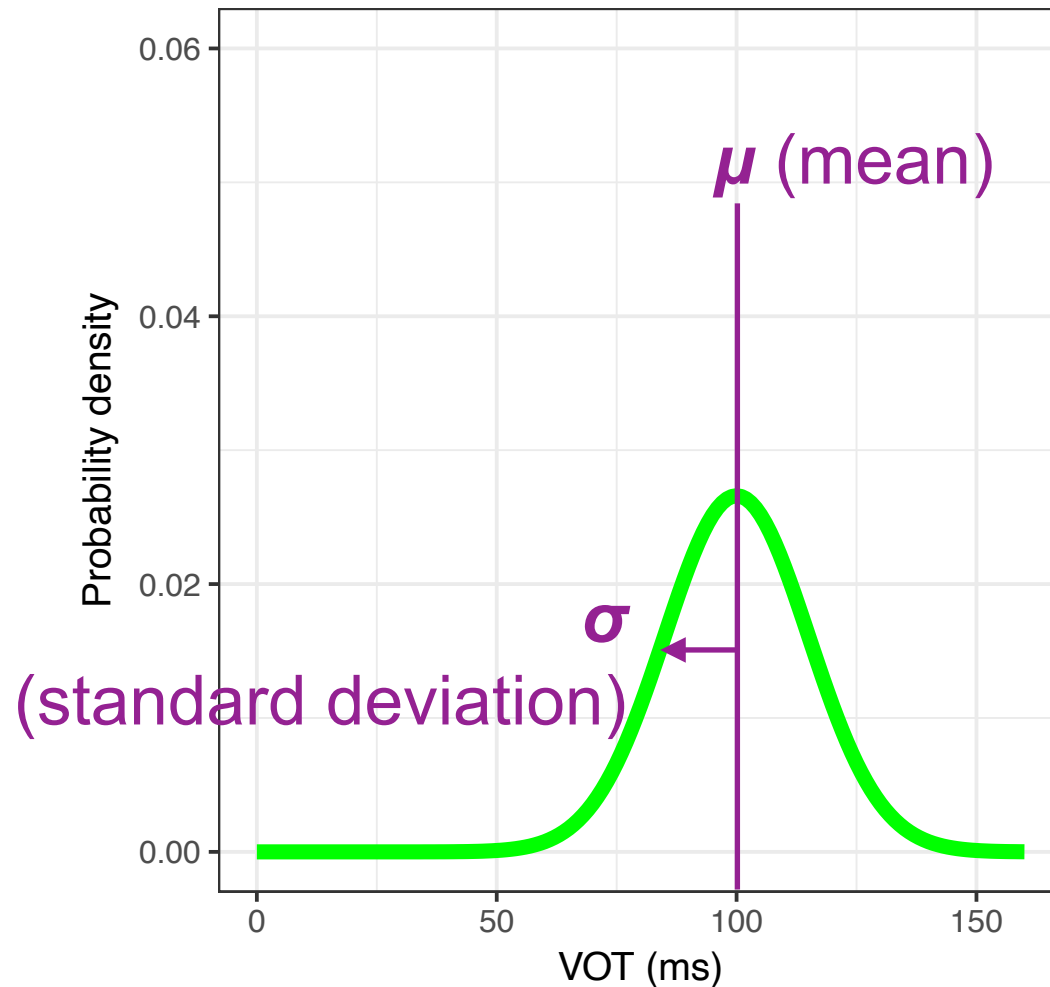
Scaled by variance

The Gaussian, or normal, distribution



$$p(x | \mu, \sigma) = \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}}}_{\text{(normalizing constant)}} \exp \left[\frac{\underbrace{-(x - \mu)^2}_{\text{Squared deviation from mean}}}{\underbrace{2\sigma^2}_{\text{Scaled by variance}}} \right]$$

The Gaussian, or normal, distribution



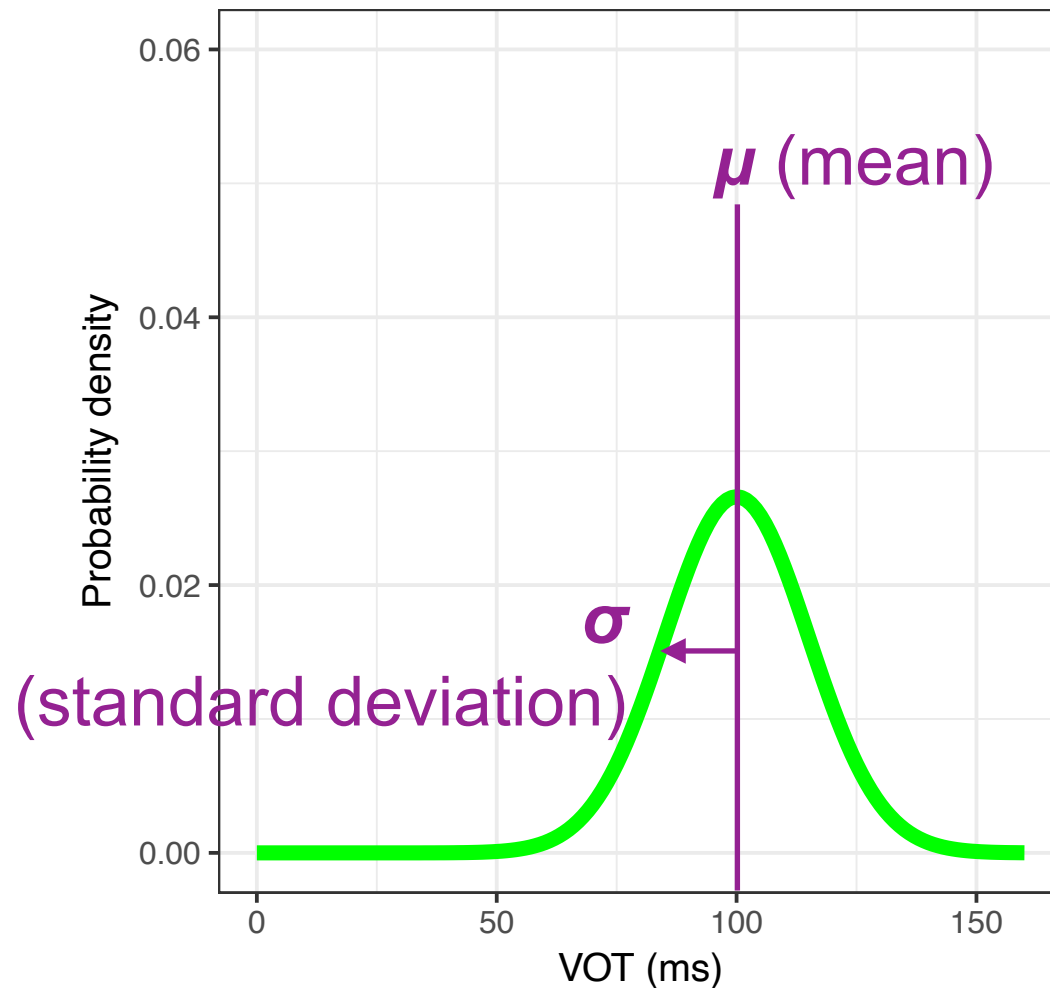
$$p(x | \mu, \sigma) = \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}}}_{\text{(normalizing constant)}} \exp \left[\underbrace{\frac{-(x - \mu)^2}{2\sigma^2}}_{\text{Squared deviation from mean Scaled by variance}} \right]$$

- Unbiased parameter estimates from a size- N sample:

$$\hat{\mu} = \bar{x}$$

$$\hat{\sigma} = \sqrt{\frac{1}{N-1} \sum_{i=1}^N (x_i - \bar{x})^2} \triangleq s$$

The Gaussian, or normal, distribution



Squared deviation from mean

$$p(x | \mu, \sigma) = \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}}}_{\text{(normalizing constant)}} \exp \left[\frac{\underbrace{-(x - \mu)^2}_{\text{Squared deviation from mean}}}{\underbrace{2\sigma^2}_{\text{Scaled by variance}}} \right]$$

(normalizing constant)

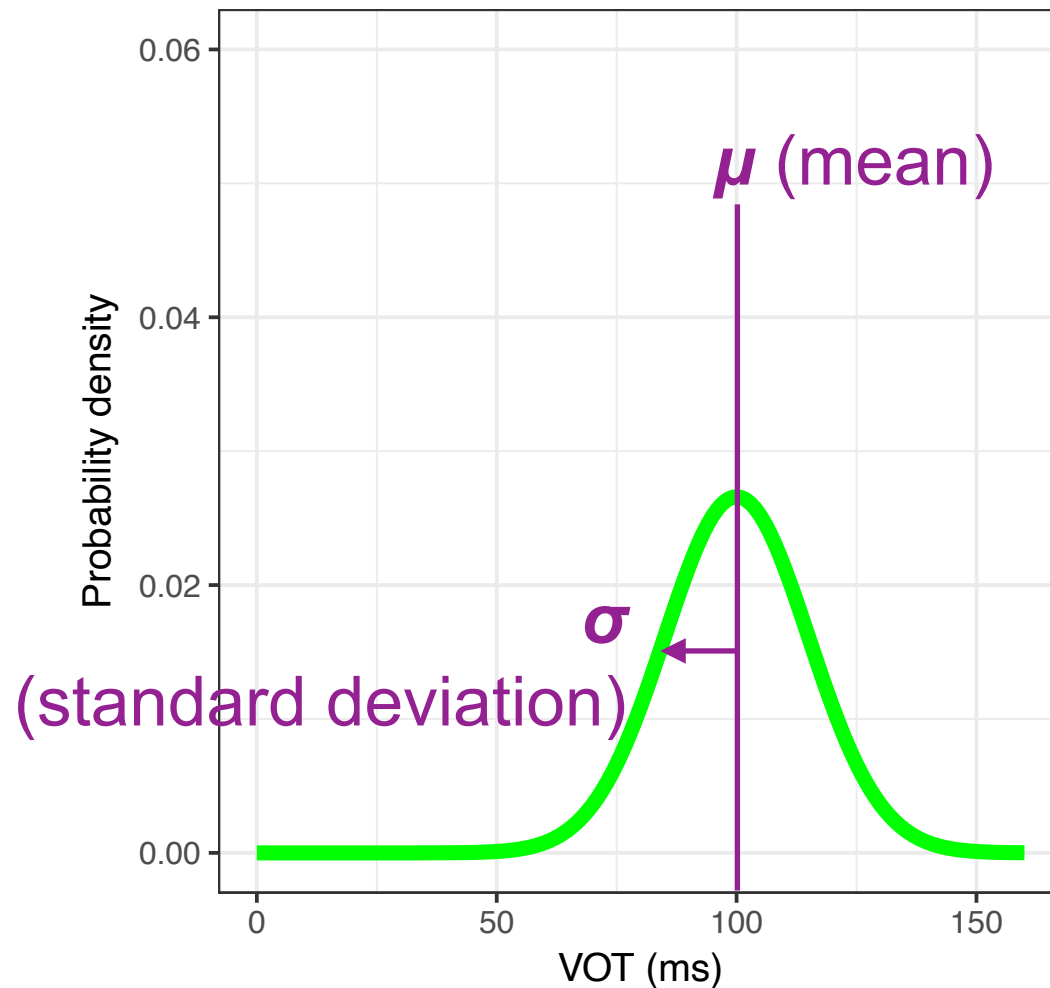
Scaled by variance

- Unbiased parameter estimates from a size- N sample:

$$\hat{\mu} = \bar{x} \quad \text{Sample mean}$$

$$\hat{\sigma} = \sqrt{\frac{1}{N-1} \sum_{i=1}^N (x_i - \bar{x})^2} \triangleq s$$

The Gaussian, or normal, distribution



$$p(x | \mu, \sigma) = \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}}}_{\text{(normalizing constant)}} \exp \left[\underbrace{\frac{-(x - \mu)^2}{2\sigma^2}}_{\text{Squared deviation from mean Scaled by variance}} \right]$$

- Unbiased parameter estimates from a size- N sample:

$$\hat{\mu} = \bar{x} \quad \text{Sample mean} \qquad \hat{\sigma} = \sqrt{\frac{1}{N-1} \sum_{i=1}^N (x_i - \bar{x})^2} \triangleq s \quad \text{Sample standard deviation}$$

The t -test: three variants

- **One sample (Student's) test:** Does the underlying population mean of a sample differ from zero?
- **Two-sample test (unpaired):** do the underlying population means of two samples differ from one another?
- **Two-sample test (paired):** You have a sample of individuals from the population and take measurements from each member of the sample in two different conditions. Do the underlying population means in the two conditions differ from one another?



*William Sealy
Gosset, a.k.a.
Student*

One-sample t -test

One-sample t -test

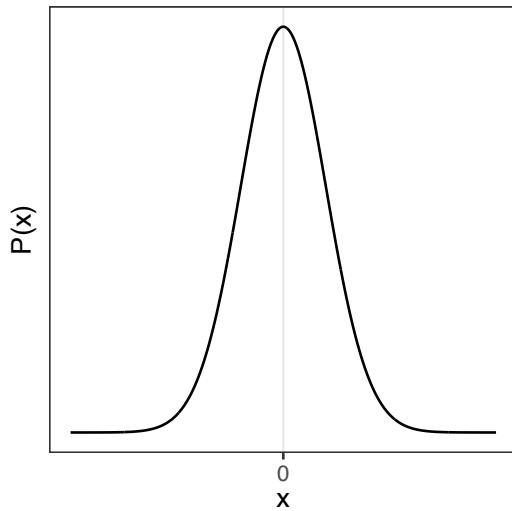
- **Null hypothesis H_0 :** the mean of the normally-distributed population underlying a sample is taken is $\mu = 0$

One-sample t -test

- **Null hypothesis H_0 :** the mean of the normally-distributed population underlying a sample is taken is $\mu = 0$
- **Alternative hypothesis H_1 :** $\mu \neq 0$ (two tailed; generally preferred) or $\mu > 0$ (one tailed; generally dispreferred)

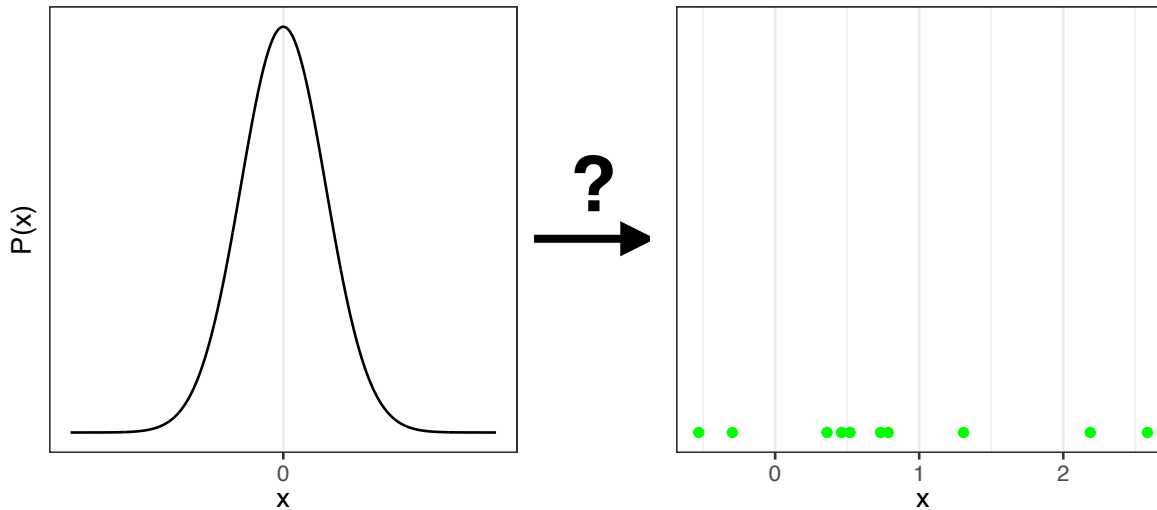
One-sample t -test

- **Null hypothesis H_0 :** the mean of the normally-distributed population underlying a sample is taken is $\mu = 0$
- **Alternative hypothesis H_1 :** $\mu \neq 0$ (two tailed; generally preferred) or $\mu > 0$ (one tailed; generally dispreferred)



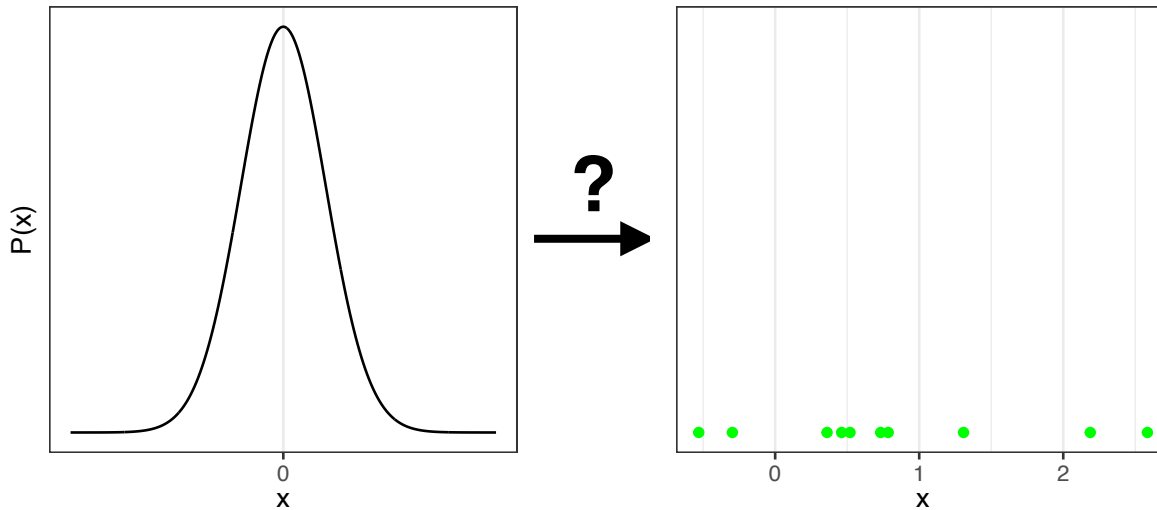
One-sample t -test

- **Null hypothesis H_0 :** the mean of the normally-distributed population underlying a sample is taken is $\mu = 0$
- **Alternative hypothesis H_1 :** $\mu \neq 0$ (two tailed; generally preferred) or $\mu > 0$ (one tailed; generally dispreferred)



One-sample t -test

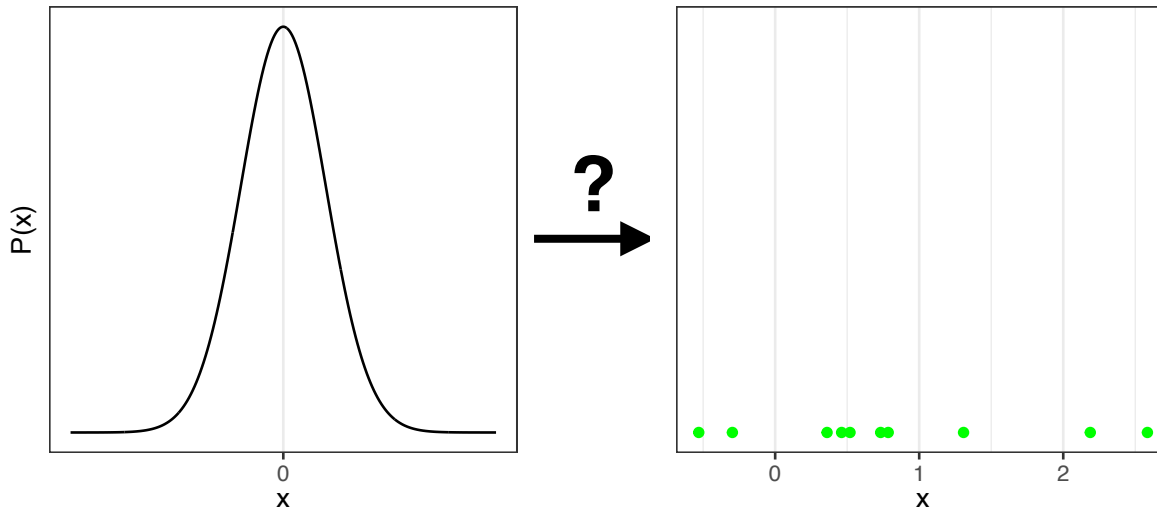
- **Null hypothesis H_0 :** the mean of the normally-distributed population underlying a sample is taken is $\mu = 0$
- **Alternative hypothesis H_1 :** $\mu \neq 0$ (two tailed; generally preferred) or $\mu > 0$ (one tailed; generally dispreferred)



Test statistic: $t = \frac{\bar{x}}{s/\sqrt{n}}$

One-sample t -test

- **Null hypothesis H_0 :** the mean of the normally-distributed population underlying a sample is taken is $\mu = 0$
- **Alternative hypothesis H_1 :** $\mu \neq 0$ (two tailed; generally preferred) or $\mu > 0$ (one tailed; generally dispreferred)

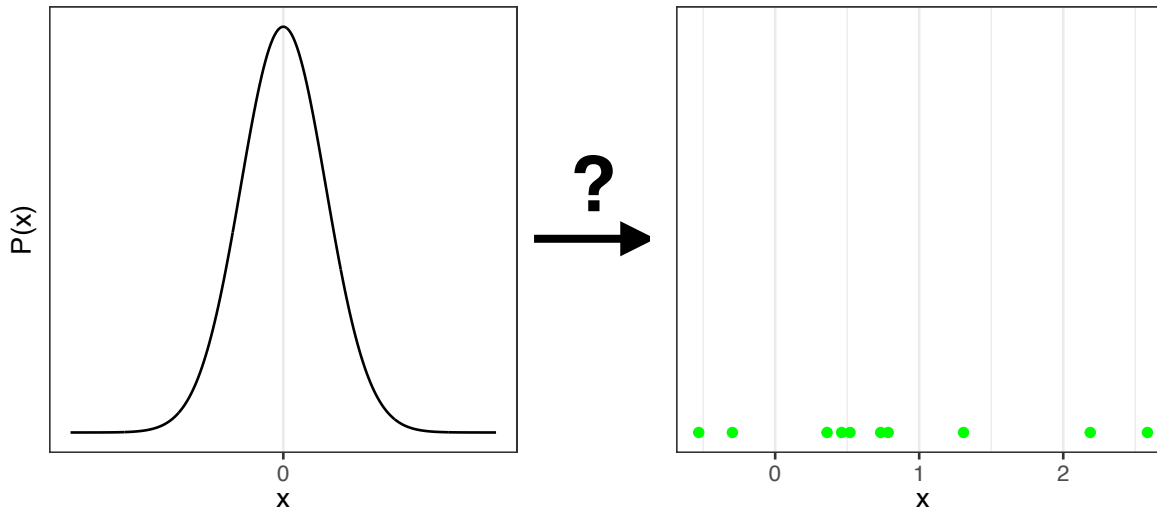


Test statistic: $t = \frac{\bar{x}}{s/\sqrt{n}}$

Remember: Sample mean

One-sample t -test

- **Null hypothesis H_0 :** the mean of the normally-distributed population underlying a sample is taken is $\mu = 0$
- **Alternative hypothesis H_1 :** $\mu \neq 0$ (two tailed; generally preferred) or $\mu > 0$ (one tailed; generally dispreferred)



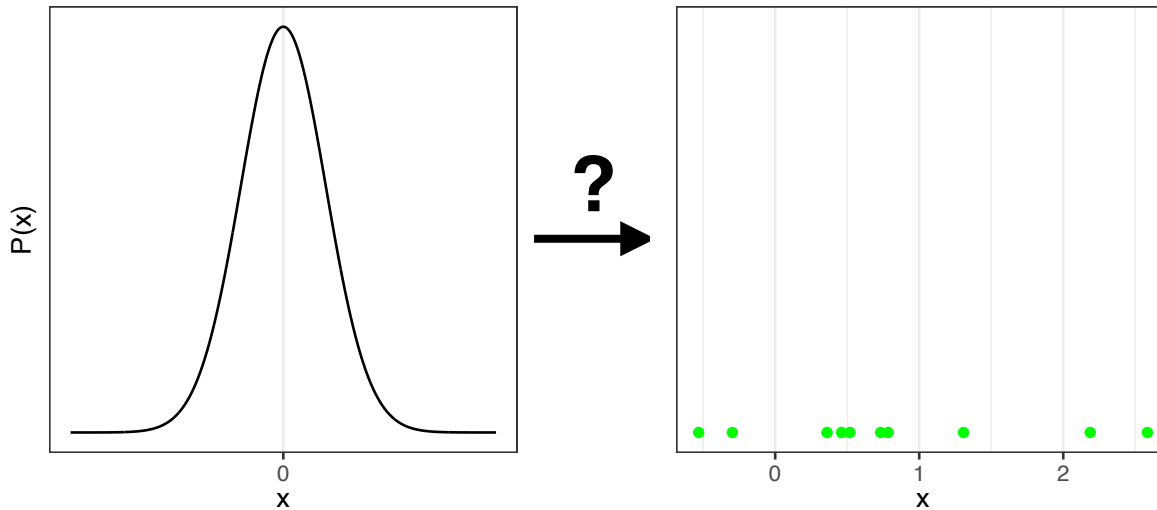
Test statistic: $t = \frac{\bar{x}}{s/\sqrt{n}}$

Remember: Sample mean

Remember: Sample standard deviation

One-sample t -test

- **Null hypothesis H_0 :** the mean of the normally-distributed population underlying a sample is taken is $\mu = 0$
- **Alternative hypothesis H_1 :** $\mu \neq 0$ (two tailed; generally preferred) or $\mu > 0$ (one tailed; generally dispreferred)



t-distributed with $n-1$
degrees of freedom

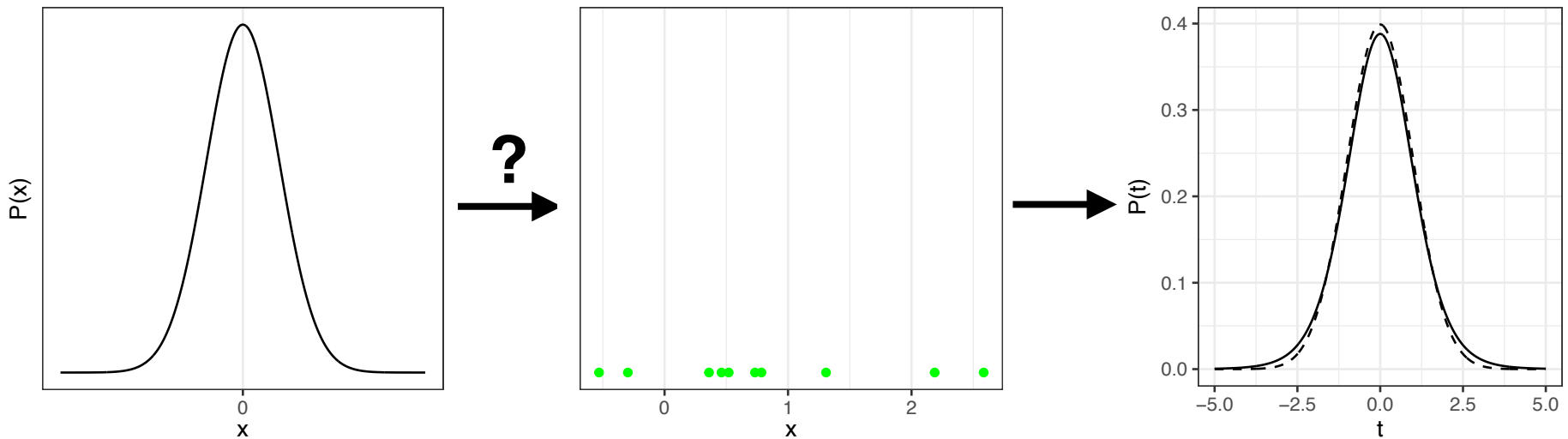
Remember:
Sample mean

Test statistic: $t = \frac{\bar{x}}{s/\sqrt{n}}$

Remember: Sample
standard deviation

One-sample t -test

- **Null hypothesis H_0 :** the mean of the normally-distributed population underlying a sample is taken is $\mu = 0$
- **Alternative hypothesis H_1 :** $\mu \neq 0$ (two tailed; generally preferred) or $\mu > 0$ (one tailed; generally dispreferred)



*t -distributed with $n-1$
degrees of freedom*

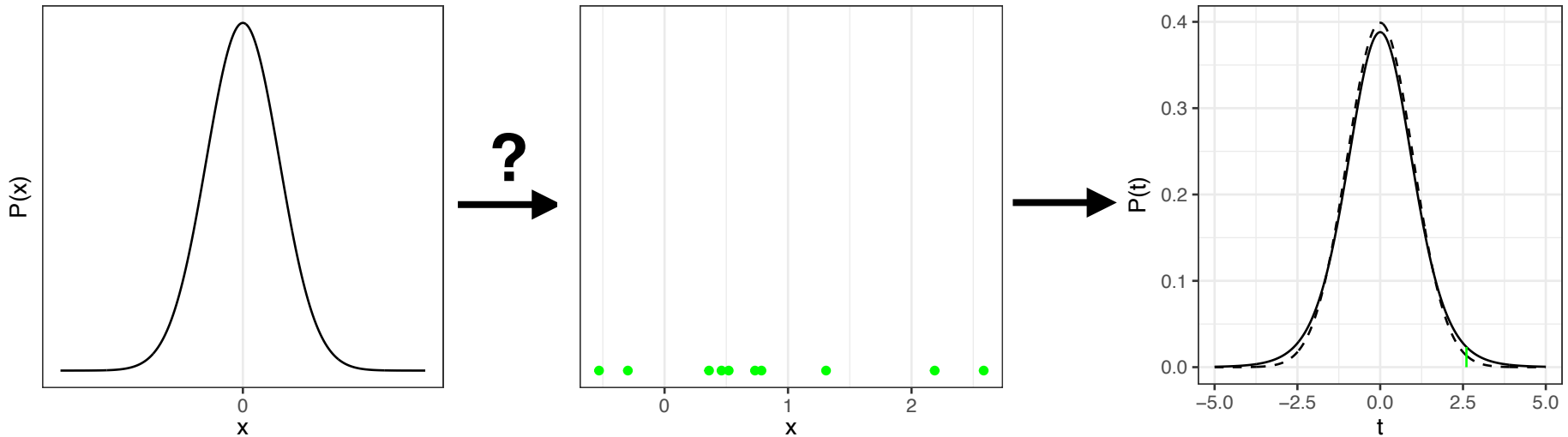
*Remember:
Sample mean*

Test statistic: $t = \frac{\bar{x}}{s/\sqrt{n}}$

Remember: Sample standard deviation

One-sample t -test

- **Null hypothesis H_0 :** the mean of the normally-distributed population underlying a sample is taken is $\mu = 0$
- **Alternative hypothesis H_1 :** $\mu \neq 0$ (two tailed; generally preferred) or $\mu > 0$ (one tailed; generally dispreferred)



*t-distributed with $n-1$
degrees of freedom*

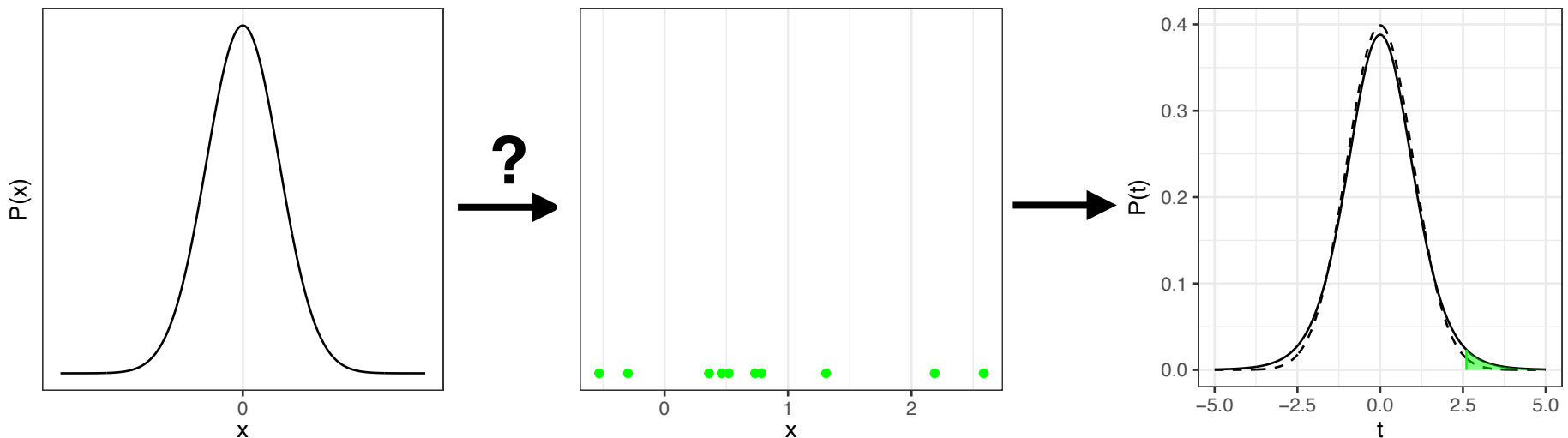
*Remember:
Sample mean*

Test statistic: $t = \frac{\bar{x}}{s/\sqrt{n}}$

*Remember: Sample
standard deviation*

One-sample t -test

- **Null hypothesis H_0 :** the mean of the normally-distributed population underlying a sample is taken is $\mu = 0$
- **Alternative hypothesis H_1 :** $\mu \neq 0$ (two tailed; generally preferred) or $\mu > 0$ (one tailed; generally dispreferred)



*t -distributed with $n-1$
degrees of freedom*

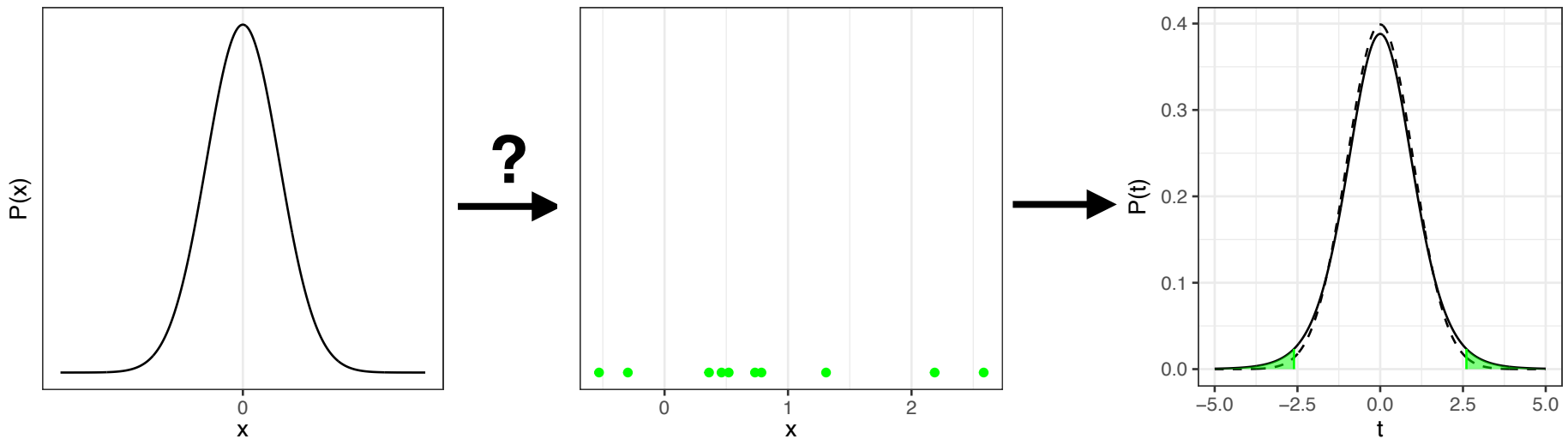
*Remember:
Sample mean*

Test statistic: $t = \frac{\bar{x}}{s/\sqrt{n}}$

*Remember: Sample
standard deviation*

One-sample t -test

- **Null hypothesis H_0 :** the mean of the normally-distributed population underlying a sample is taken is $\mu = 0$
- **Alternative hypothesis H_1 :** $\mu \neq 0$ (two tailed; generally preferred) or $\mu > 0$ (one tailed; generally dispreferred)



t-distributed with $n-1$
degrees of freedom

Remember:
Sample mean

Test statistic: $t = \frac{\bar{x}}{s/\sqrt{n}}$

Remember: Sample standard deviation

Two-sample t -test (unpaired)

Two-sample t -test (unpaired)

- **Assumptions:** samples 1 and 2 are each iid normal

Two-sample t -test (unpaired)

- **Assumptions:** samples 1 and 2 are each iid normal
- **Null hypothesis H_0 :** $\mu_1 = \mu_2$

Two-sample t -test (unpaired)

- **Assumptions:** samples 1 and 2 are each iid normal
- **Null hypothesis H_0 :** $\mu_1 = \mu_2$
- **Alternative hypothesis H_1 :** $\mu_1 \neq \mu_2$ (two-tailed); $\mu_1 > \mu_2$ (one-tailed; generally dispreferred)

Two-sample t -test (unpaired)

- **Assumptions:** samples 1 and 2 are each iid normal
- **Null hypothesis H_0 :** $\mu_1 = \mu_2$
- **Alternative hypothesis H_1 :** $\mu_1 \neq \mu_2$ (two-tailed); $\mu_1 > \mu_2$ (one-tailed; generally dispreferred)
- If we assume that the two underlying populations have **equal variance** ("Student's" t -test):

Two-sample t -test (unpaired)

- **Assumptions:** samples 1 and 2 are each iid normal
- **Null hypothesis H_0 :** $\mu_1 = \mu_2$
- **Alternative hypothesis H_1 :** $\mu_1 \neq \mu_2$ (two-tailed); $\mu_1 > \mu_2$ (one-tailed; generally dispreferred)
- If we assume that the two underlying populations have **equal variance** ("Student's" t -test):

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s_p \sqrt{1/n_1 + 1/n_2}} \quad \text{where} \quad s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

Two-sample t -test (unpaired)

- **Assumptions:** samples 1 and 2 are each iid normal
- **Null hypothesis H_0 :** $\mu_1 = \mu_2$
- **Alternative hypothesis H_1 :** $\mu_1 \neq \mu_2$ (two-tailed); $\mu_1 > \mu_2$ (one-tailed; generally dispreferred)
- If we assume that the two underlying populations have **equal variance** ("Student's" t -test):

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s_p \sqrt{1/n_1 + 1/n_2}} \quad \text{where } \boxed{s_p} = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

Pooled sample standard deviation

Two-sample t -test (unpaired)

- **Assumptions:** samples 1 and 2 are each iid normal
- **Null hypothesis H_0 :** $\mu_1 = \mu_2$
- **Alternative hypothesis H_1 :** $\mu_1 \neq \mu_2$ (two-tailed); $\mu_1 > \mu_2$ (one-tailed; generally dispreferred)
- If we assume that the two underlying populations have **equal variance** ("Student's" t -test):

$$\underbrace{t}_{\substack{\text{t-distributed w/} \\ n_1 + n_2 - 2 \text{ degrees of} \\ \text{freedom}}} = \frac{\bar{x}_1 - \bar{x}_2}{s_p \sqrt{1/n_1 + 1/n_2}} \quad \text{where} \quad \underbrace{s_p}_{\substack{\text{Pooled sample} \\ \text{standard deviation}}} = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

Two-sample t -test (unpaired)

- **Assumptions:** samples 1 and 2 are each iid normal
- **Null hypothesis H_0 :** $\mu_1 = \mu_2$
- **Alternative hypothesis H_1 :** $\mu_1 \neq \mu_2$ (two-tailed); $\mu_1 > \mu_2$ (one-tailed; generally dispreferred)
- If we assume that the two underlying populations have **equal variance** ("Student's" t -test):

$$\underbrace{t}_{\substack{\text{t-distributed w/} \\ n_1 + n_2 - 2 \text{ degrees of} \\ \text{freedom}}} = \frac{\bar{x}_1 - \bar{x}_2}{s_p \sqrt{1/n_1 + 1/n_2}} \quad \text{where} \quad \underbrace{s_p}_{\substack{\text{Pooled sample} \\ \text{standard deviation}}} = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

- If we **do not** assume that the two underlying populations have equal variance ("Welch's" t -test):

Two-sample t -test (unpaired)

- **Assumptions:** samples 1 and 2 are each iid normal
- **Null hypothesis H_0 :** $\mu_1 = \mu_2$
- **Alternative hypothesis H_1 :** $\mu_1 \neq \mu_2$ (two-tailed); $\mu_1 > \mu_2$ (one-tailed; generally dispreferred)
- If we assume that the two underlying populations have **equal variance** ("Student's" t -test):

$$\underset{\substack{\text{t-distributed w/} \\ n_1 + n_2 - 2 \text{ degrees of} \\ \text{freedom}}}{t} = \frac{\bar{x}_1 - \bar{x}_2}{s_p \sqrt{1/n_1 + 1/n_2}} \quad \text{where} \quad \underset{\substack{\text{Pooled sample} \\ \text{standard deviation}}}{s_p} = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

- If we **do not** assume that the two underlying populations have equal variance ("Welch's" t -test):

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{s_1^2/n_1 + s_2^2/n_2}} \quad \text{t-distributed with a complex number of degrees of freedom whose formula can easily be looked up}$$

Paired two-sample t -test

Paired two-sample t -test

- Assumptions:

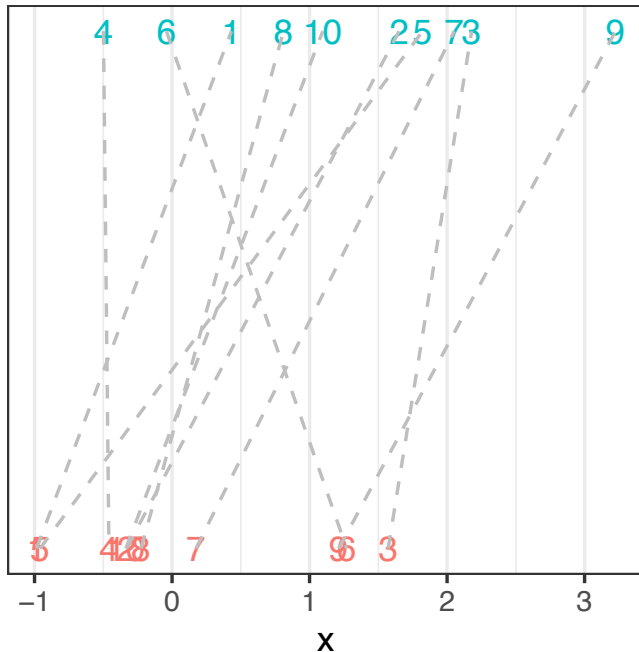
Paired two-sample t -test

- **Assumptions:**
 - In a sample of **units** from a population; for each unit we have two **measurements** $\langle x_1, x_2 \rangle$ on the same scale

Paired two-sample t -test

- **Assumptions:**

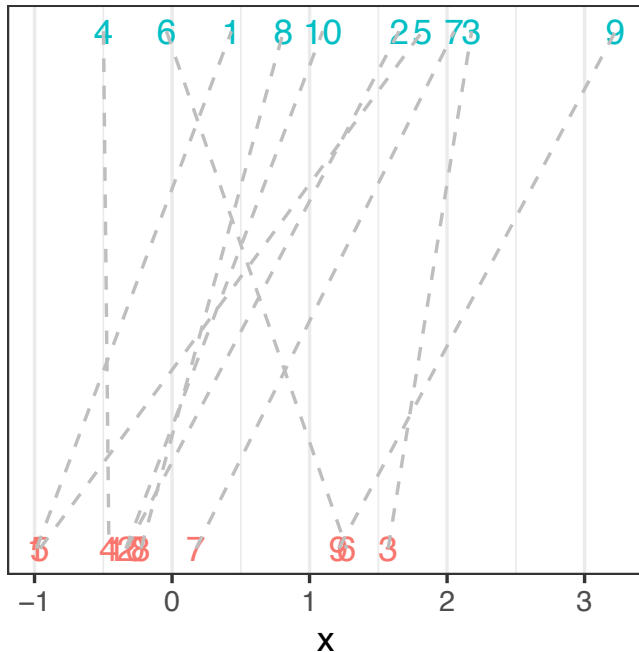
- In a sample of **units** from a population; for each unit we have two **measurements** $\langle x_1, x_2 \rangle$ on the same scale



Paired two-sample t -test

- **Assumptions:**

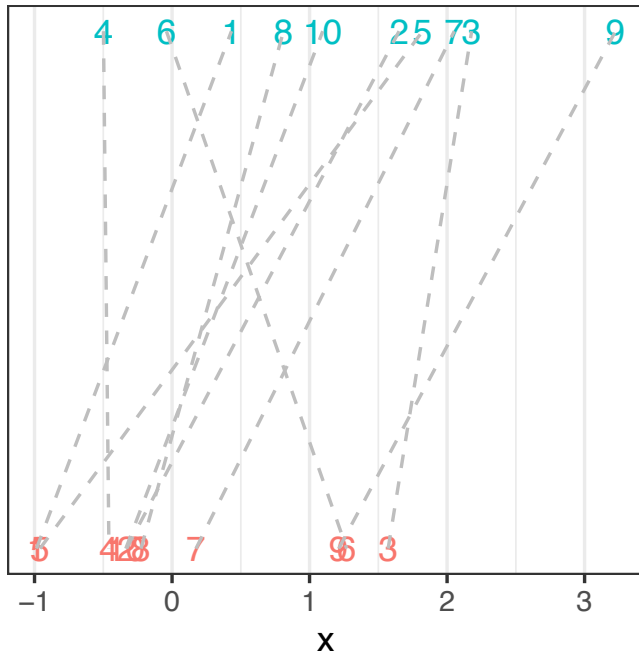
- In a sample of **units** from a population; for each unit we have two **measurements** $\langle x_1, x_2 \rangle$ on the same scale
- The difference between measurements is iid normal



Paired two-sample t -test

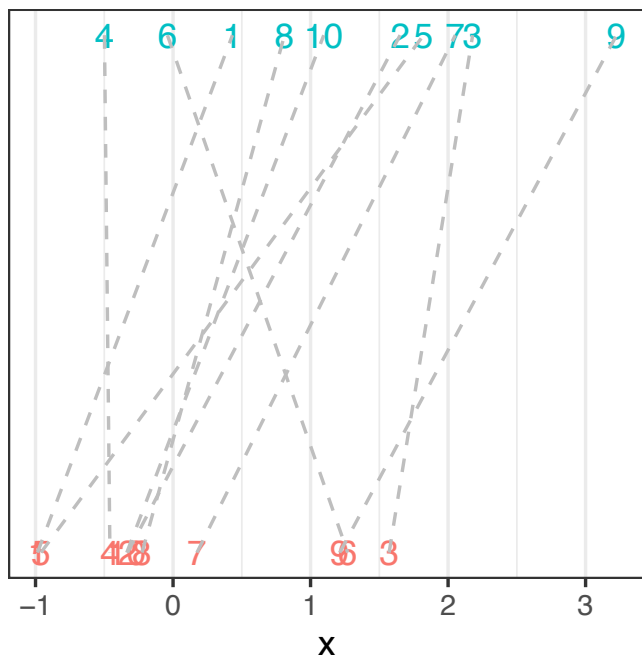
- **Assumptions:**

- In a sample of **units** from a population; for each unit we have two **measurements** $\langle x_1, x_2 \rangle$ on the same scale
- The difference between measurements is iid normal
- (Sufficient condition: paired measurements are **bivariate normal** – a distr. we haven't yet covered)



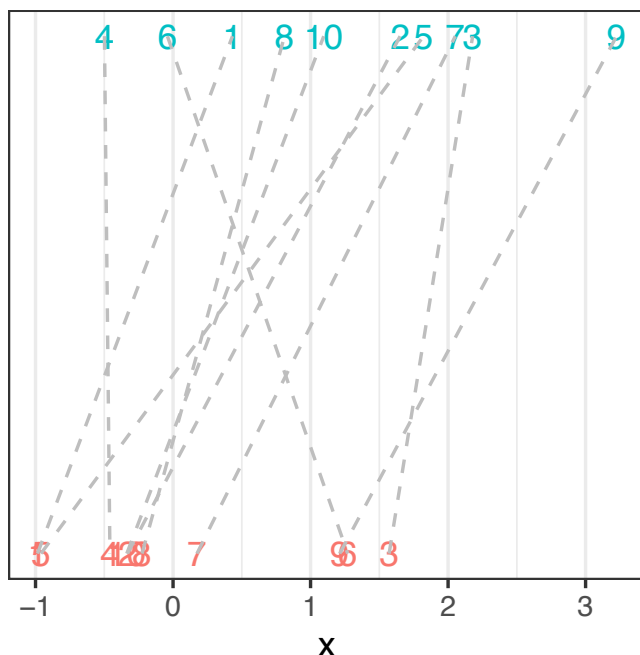
Paired two-sample t -test

- **Assumptions:**
 - In a sample of **units** from a population; for each unit we have two **measurements** $\langle x_1, x_2 \rangle$ on the same scale
 - The difference between measurements is iid normal
 - (Sufficient condition: paired measurements are **bivariate normal** – a distr. we haven't yet covered)
- **H₀:** $\mu_1 = \mu_2$; **H₁:** $\mu_1 \neq \mu_2$ (2-tailed) or $\mu_1 > \mu_2$ (1-tailed; generally dispreferred)



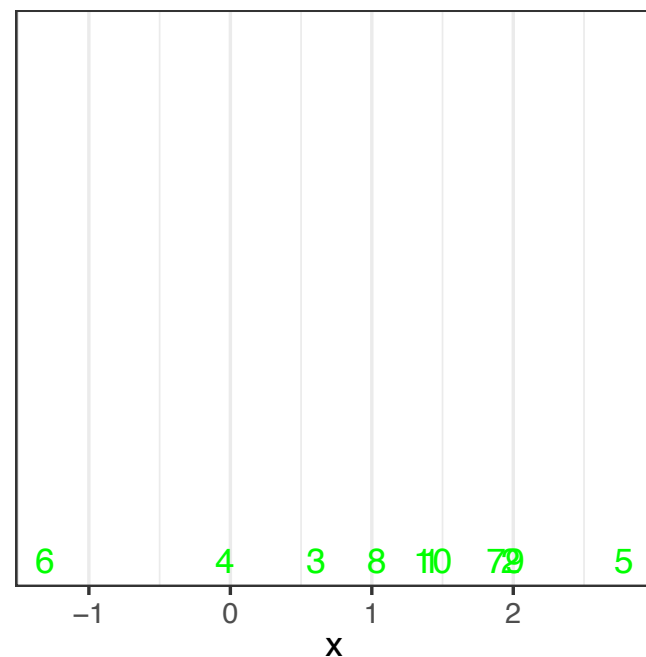
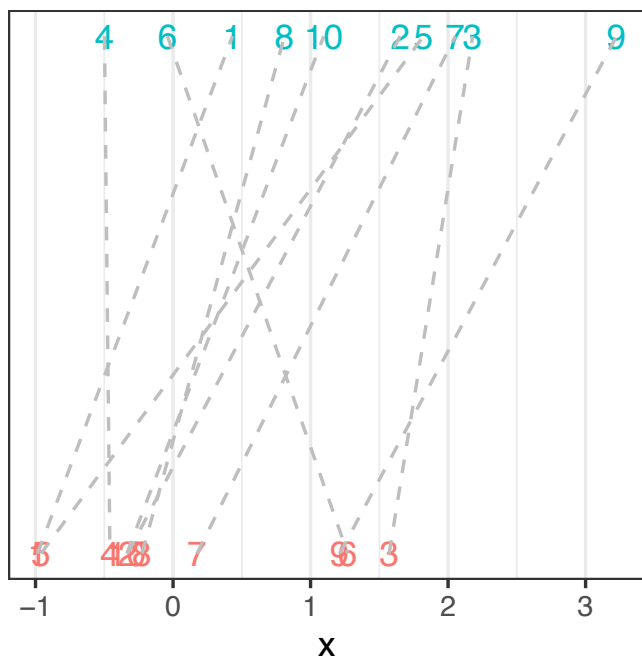
Paired two-sample t -test

- **Assumptions:**
 - In a sample of **units** from a population; for each unit we have two **measurements** $\langle x_1, x_2 \rangle$ on the same scale
 - The difference between measurements is iid normal
 - (Sufficient condition: paired measurements are **bivariate normal** – a distr. we haven't yet covered)
- **H₀:** $\mu_1 = \mu_2$; **H₁:** $\mu_1 \neq \mu_2$ (2-tailed) or $\mu_1 > \mu_2$ (1-tailed; generally dispreferred)
- **Strategy:** take within-unit difference scores and apply a 1-sample t -test!



Paired two-sample t -test

- **Assumptions:**
 - In a sample of **units** from a population; for each unit we have two **measurements** $\langle x_1, x_2 \rangle$ on the same scale
 - The difference between measurements is iid normal
 - (Sufficient condition: paired measurements are **bivariate normal** – a distr. we haven't yet covered)
- **H₀:** $\mu_1 = \mu_2$; **H₁:** $\mu_1 \neq \mu_2$ (2-tailed) or $\mu_1 > \mu_2$ (1-tailed; generally dispreferred)
- **Strategy:** take within-unit difference scores and apply a 1-sample t -test!



The likelihood ratio test

The likelihood ratio test

- The **likelihood ratio**:

$$\Lambda^* = \frac{\max \text{Lik}_{H_0}(\mathbf{y})}{\max \text{Lik}_{H_A}(\mathbf{y})}$$

The likelihood ratio test

- The **likelihood ratio**:

$$\Lambda^* = \frac{\max \text{Lik}_{H_0}(\mathbf{y})}{\max \text{Lik}_{H_A}(\mathbf{y})}$$

Data likelihood under MLE of H_0

The likelihood ratio test

- The **likelihood ratio**:

$$\Lambda^* = \frac{\max \text{Lik}_{H_0}(\mathbf{y})}{\max \text{Lik}_{H_A}(\mathbf{y})}$$

Data likelihood under MLE of H_0

Data likelihood under MLE of H_A

The likelihood ratio test

- The **likelihood ratio**:

$$\Lambda^* = \frac{\max \text{Lik}_{H_0}(\mathbf{y})}{\max \text{Lik}_{H_A}(\mathbf{y})}$$

Data likelihood under MLE of H_0

Data likelihood under MLE of H_A

- The **deviance** is (asymptotically) χ^2 -distributed with **degrees of freedom** equal to diff. in # model parameters

The likelihood ratio test

- The **likelihood ratio**:

$$\Lambda^* = \frac{\max \text{Lik}_{H_0}(\mathbf{y})}{\max \text{Lik}_{H_A}(\mathbf{y})}$$

Data likelihood under MLE of H_0

Data likelihood under MLE of H_A

- The **deviance** is (asymptotically) χ^2 -distributed with **degrees of freedom** equal to diff. in # model parameters

$$G^2 \stackrel{\text{def}}{=} -2 \log \Lambda^*$$

The likelihood ratio test

- The **likelihood ratio**:

$$\Lambda^* = \frac{\max \text{Lik}_{H_0}(\mathbf{y})}{\max \text{Lik}_{H_A}(\mathbf{y})}$$

Data likelihood under MLE of H_0

Data likelihood under MLE of H_A

- The **deviance** is (asymptotically) χ^2 -distributed with **degrees of freedom** equal to diff. in # model parameters

$$G^2 \stackrel{\text{def}}{=} -2 \log \Lambda^*$$

- **Example**: is a coin flipped 30 times, 20H 10T, fair?

The likelihood ratio test

- The **likelihood ratio**:

$$\Lambda^* = \frac{\max \text{Lik}_{H_0}(\mathbf{y})}{\max \text{Lik}_{H_A}(\mathbf{y})}$$

Data likelihood under MLE of H_0

Data likelihood under MLE of H_A

- The **deviance** is (asymptotically) χ^2 -distributed with **degrees of freedom** equal to diff. in # model parameters

$$G^2 \stackrel{\text{def}}{=} -2 \log \Lambda^*$$

- **Example**: is a coin flipped 30 times, 20H 10T, fair?

$$G^2 = -2 \log \frac{\left(\frac{1}{2}\right)^{30}}{\left(\frac{2}{3}\right)^{20} \left(\frac{1}{3}\right)^{10}} \approx 3.4$$

The likelihood ratio test

- The **likelihood ratio**:

$$\Lambda^* = \frac{\max \text{Lik}_{H_0}(\mathbf{y})}{\max \text{Lik}_{H_A}(\mathbf{y})}$$

Data likelihood under MLE of H_0

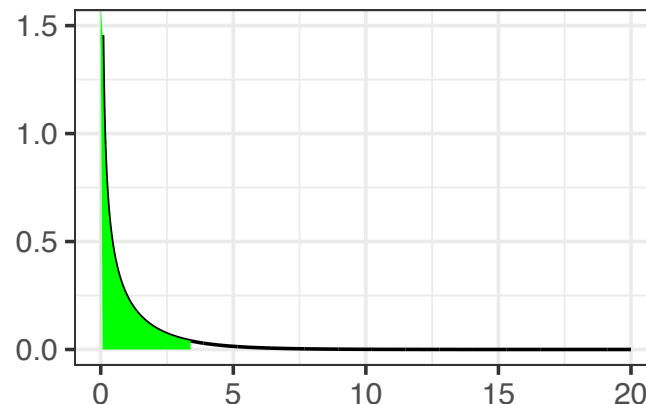
Data likelihood under MLE of H_A

- The **deviance** is (asymptotically) χ^2 -distributed with **degrees of freedom** equal to diff. in # model parameters

$$G^2 \stackrel{\text{def}}{=} -2 \log \Lambda^*$$

- **Example:** is a coin flipped 30 times, 20H 10T, fair?

$$G^2 = -2 \log \frac{\left(\frac{1}{2}\right)^{30}}{\left(\frac{2}{3}\right)^{20} \left(\frac{1}{3}\right)^{10}} \approx 3.4$$



The likelihood ratio test

- The **likelihood ratio**:

$$\Lambda^* = \frac{\max \text{Lik}_{H_0}(\mathbf{y})}{\max \text{Lik}_{H_A}(\mathbf{y})}$$

Data likelihood under MLE of H_0

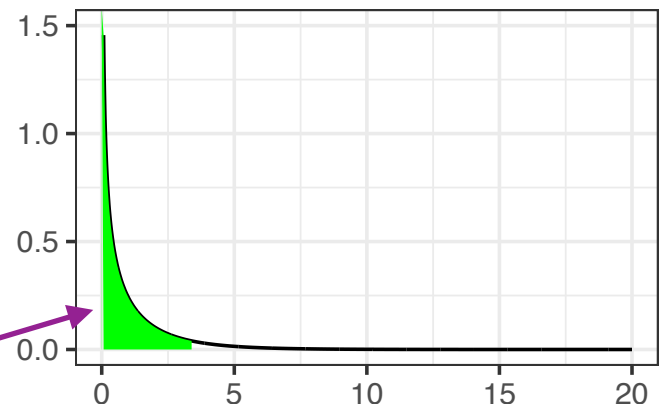
Data likelihood under MLE of H_A

- The **deviance** is (asymptotically) χ^2 -distributed with **degrees of freedom** equal to diff. in # model parameters

$$G^2 \stackrel{\text{def}}{=} -2 \log \Lambda^*$$

- **Example:** is a coin flipped 30 times, 20H 10T, fair?

$$G^2 = -2 \log \frac{\left(\frac{1}{2}\right)^{30}}{\left(\frac{2}{3}\right)^{20} \left(\frac{1}{3}\right)^{10}} \approx 3.4$$



93.5% of probability mass under $G^2 \rightarrow p = 0.065$

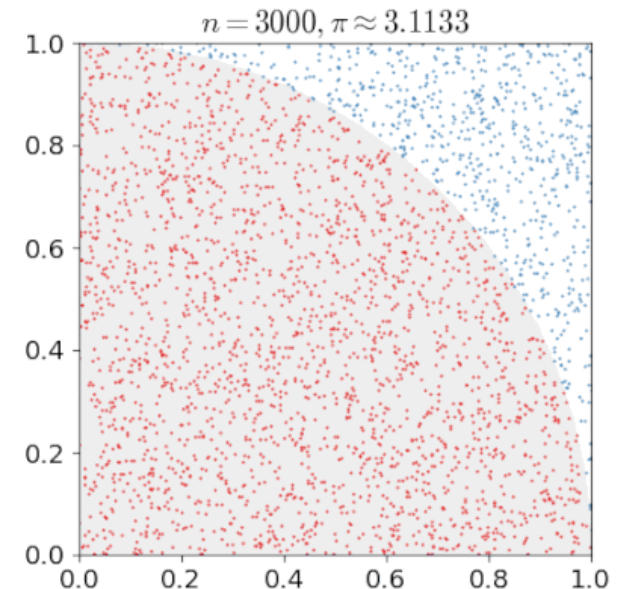
Simulation and approximate computation

Simulation and approximate computation

- All the statistical analysis that I've shown you so far has involved **exact** computation using **analytic expressions**
- This is facilitated by:
 - **Strong assumptions** regarding the data generating process (e.g., iid normal data for t -tests); and/or
 - **Conjugate priors** for Bayesian inference (e.g., Beta prior for Bernoulli/binomial data)
- But often, exact computation is not possible
- Solution: use more computationally intensive methods that don't rely on these strong assumptions. Examples:
 - Bootstrapped confidence intervals
 - Nonparametric statistical tests
 - Monte Carlo methods (**today**)

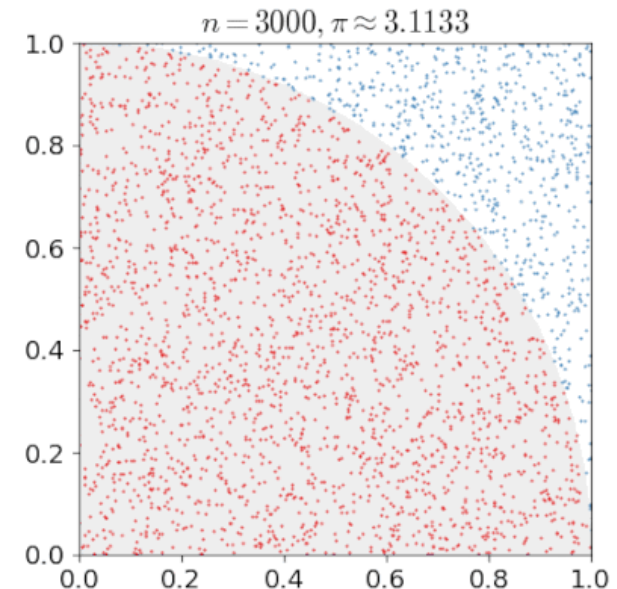
Monte Carlo methods, or "probabilistic simulation"

- Generally speaking:
 1. Define a domain of possible inputs
 2. Generate n iid random inputs from a probability distribution on the domain
 3. Perform a deterministic computation on each randomly generated input
 4. Aggregate the results of the deterministic computation
- As n grows larger, the simulated result approaches the true value



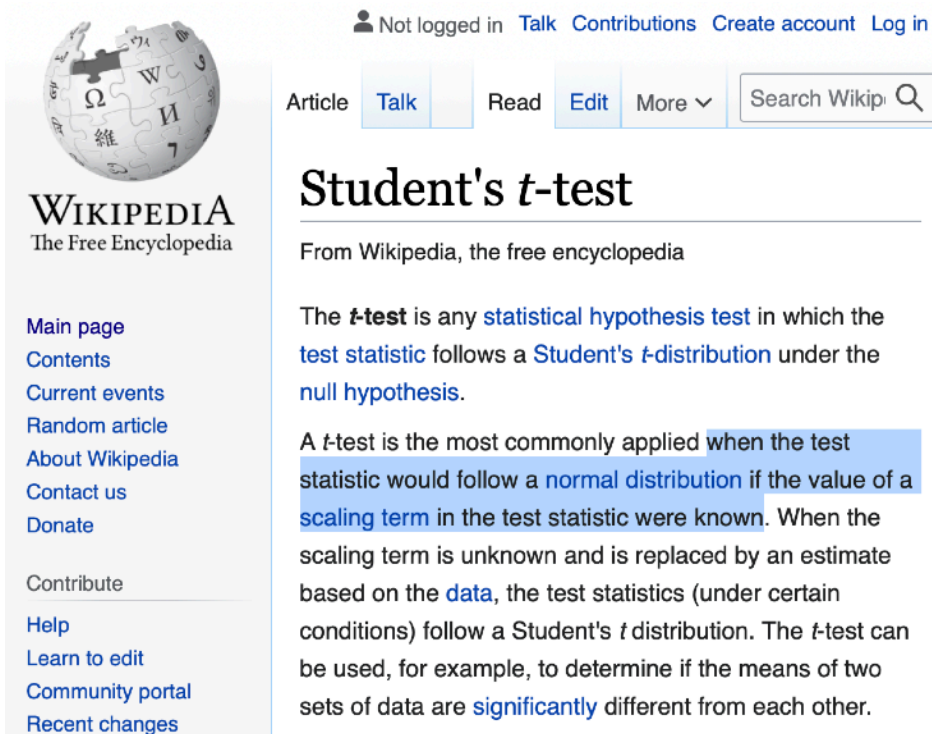
Monte Carlo methods, or "probabilistic simulation"

- Generally speaking:
 1. Define a domain of possible inputs
 2. Generate n iid random inputs from a probability distribution on the domain
 3. Perform a deterministic computation on each randomly generated input
 4. Aggregate the results of the deterministic computation
- As n grows larger, the simulated result approaches the true value

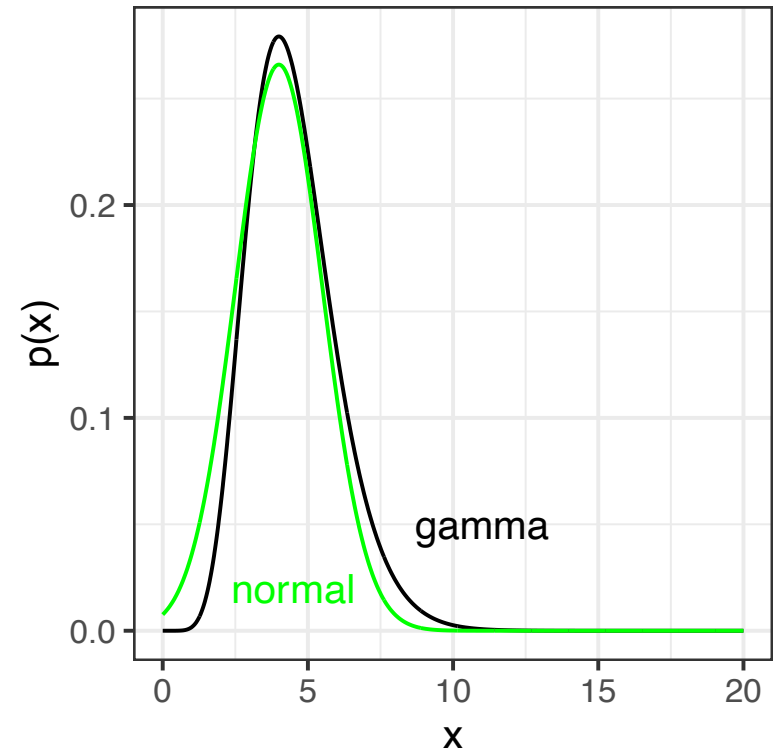


Simple example of Monte Carlo

- Suppose I want to do a two-sample t -test but my data aren't normally distributed



The screenshot shows the Wikipedia page for "Student's t -test". The page header includes the Wikipedia logo and navigation links like "Not logged in", "Talk", "Contributions", "Create account", and "Log in". The article title is "Student's t -test". Below the title, it says "From Wikipedia, the free encyclopedia". The main text explains that the t -test is a statistical hypothesis test where the test statistic follows a Student's t -distribution under the null hypothesis. A highlighted section states: "A t -test is the most commonly applied when the test statistic would follow a normal distribution if the value of a scaling term in the test statistic were known. When the scaling term is unknown and is replaced by an estimate based on the data, the test statistics (under certain conditions) follow a Student's t distribution. The t -test can be used, for example, to determine if the means of two sets of data are significantly different from each other."



- How bad will this be for my t -test????

Monte Carlo, in action

```
1 library(ggplot2)
2 library(tidyverse)
3
4 # Manually compute Student t-statistic
5 f <- function(seed,N=100,shape=9,scale=0.5) {
6   set.seed(seed)
7   y1 <- rgamma(N,shape=shape,scale=scale)
8   y2 <- rgamma(N,shape=shape,scale=scale)
9   s_p <- sqrt( (var(y1) + var(y2)) / 2 )
10  t_statistic <- ( mean(y1) - mean(y2) ) / ( s_p*sqrt(2/N) )
11  return(t_statistic)
12 }
13
14 N <- 100
15 Ts <- sapply(1:10000,f)
16
17 t_reference <- tibble(x=seq(-4,4,by=0.01),t=dt(x,df=2*(N-1)))
18
19 ggplot(data=tibble(t=Ts),aes(x=t)) +
20   geom_density() +
21   geom_line(data=t_reference,aes(x=x,y=t),color="green",linetype="dashed")
```

Monte Carlo, in action

```
1 library(ggplot2)
2 library(tidyverse)
3
4 # Manually compute Student t-statistic
5 f <- function(seed,N=100,shape=9,scale=0.5) {
6   set.seed(seed) ← Reproducibility!
7   y1 <- rgamma(N,shape=shape,scale=scale)
8   y2 <- rgamma(N,shape=shape,scale=scale)
9   s_p <- sqrt( (var(y1) + var(y2)) / 2 )
10  t_statistic <- ( mean(y1) - mean(y2) ) / ( s_p*sqrt(2/N) )
11  return(t_statistic)
12 }
13
14 N <- 100
15 Ts <- sapply(1:10000,f)
16
17 t_reference <- tibble(x=seq(-4,4,by=0.01),t=dt(x,df=2*(N-1)))
18
19 ggplot(data=tibble(t=Ts),aes(x=t)) +
20   geom_density() +
21   geom_line(data=t_reference,aes(x=x,y=t),color="green",linetype="dashed")
```

Monte Carlo, in action

```
1 library(ggplot2)
2 library(tidyverse)
3
4 # Manually compute Student t-statistic
5 f <- function(seed,N=100,shape=9,scale=0.5) {
6   set.seed(seed)
7   y1 <- rgamma(N,shape=shape,scale=scale)
8   y2 <- rgamma(N,shape=shape,scale=scale)
9   s_p <- sqrt( (var(y1) + var(y2)) / 2 )
10  t_statistic <- ( mean(y1) - mean(y2) ) / ( s_p*sqrt(2/N) )
11  return(t_statistic)
12 }
13
14 N <- 100
15 Ts <- sapply(1:10000,f)
16
17 t_reference <- tibble(x=seq(-4,4,by=0.01),t=dt(x,df=2*(N-1)))
18
19 ggplot(data=tibble(t=Ts),aes(x=t)) +
20   geom_density() +
21   geom_line(data=t_reference,aes(x=x,y=t),color="green",linetype="dashed")
```

Reproducibility!

Monte Carlo simulation

Monte Carlo, in action

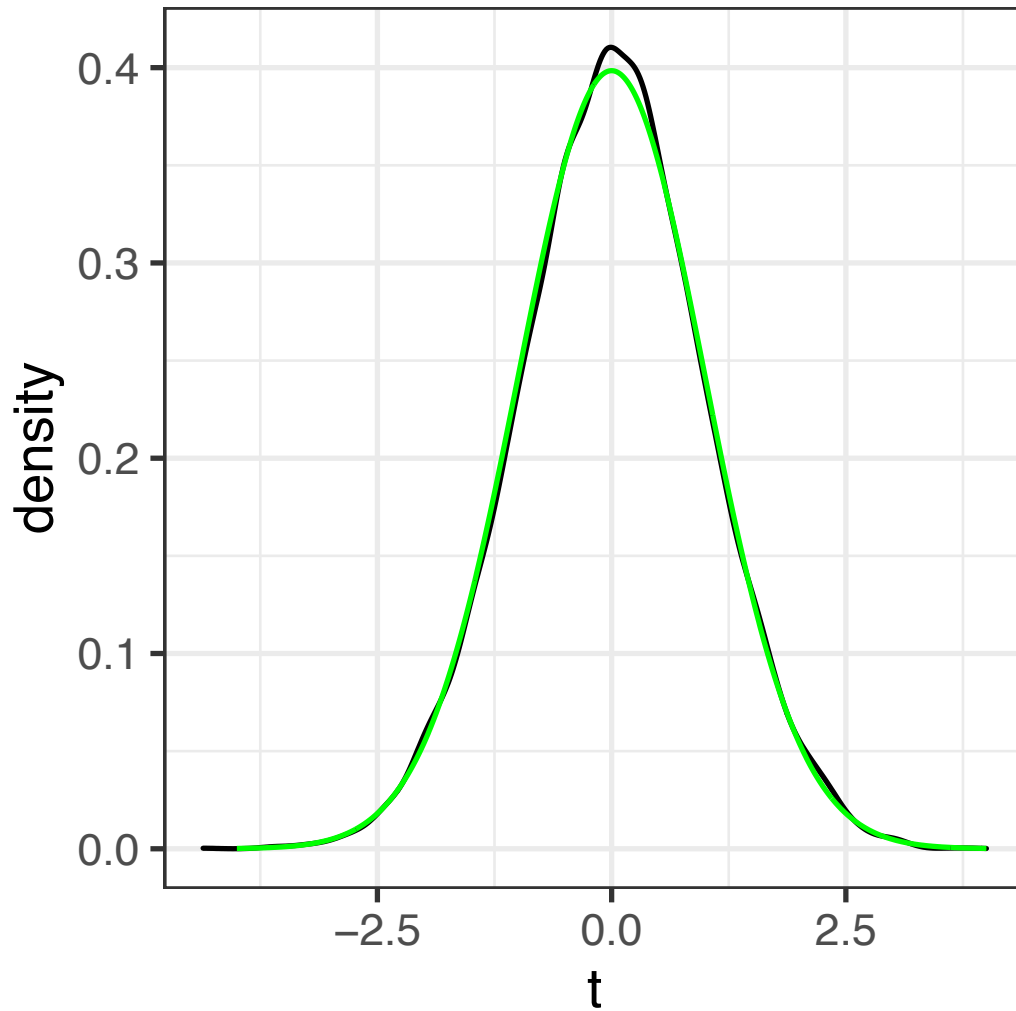
```
1 library(ggplot2)
2 library(tidyverse)
3
4 # Manually compute Student t-statistic
5 f <- function(seed,N=100,shape=9,scale=0.5) {
6   set.seed(seed)
7   y1 <- rgamma(N,shape=shape,scale=scale)
8   y2 <- rgamma(N,shape=shape,scale=scale)
9   s_p <- sqrt( (var(y1) + var(y2)) / 2 )
10  t_statistic <- ( mean(y1) - mean(y2) ) / ( s_p*sqrt(2/N) )
11  return(t_statistic)
12 }
13
14 N <- 100
15 Ts <- sapply(1:10000,f)
16
17 t_reference <- tibble(x=seq(-4,4,by=0.01),t=dt(x,df=2*(N-1)))
18
19 ggplot(data=tibble(t=Ts),aes(x=t)) +
20   geom_density() +
21   geom_line(data=t_reference,aes(x=x,y=t),color="green",linetype="dashed")
```

Reproducibility!

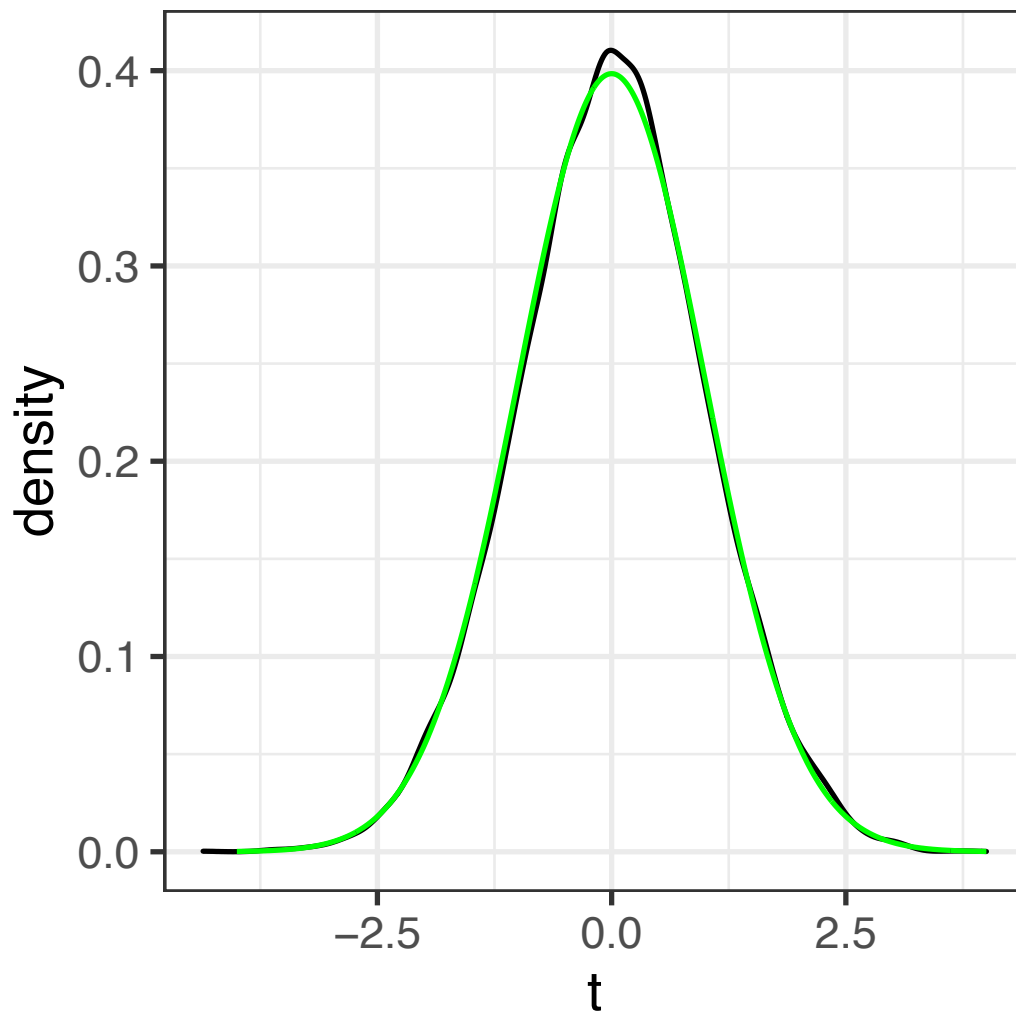
Monte Carlo simulation

Compare against Student's t distribution

Monte Carlo, in action

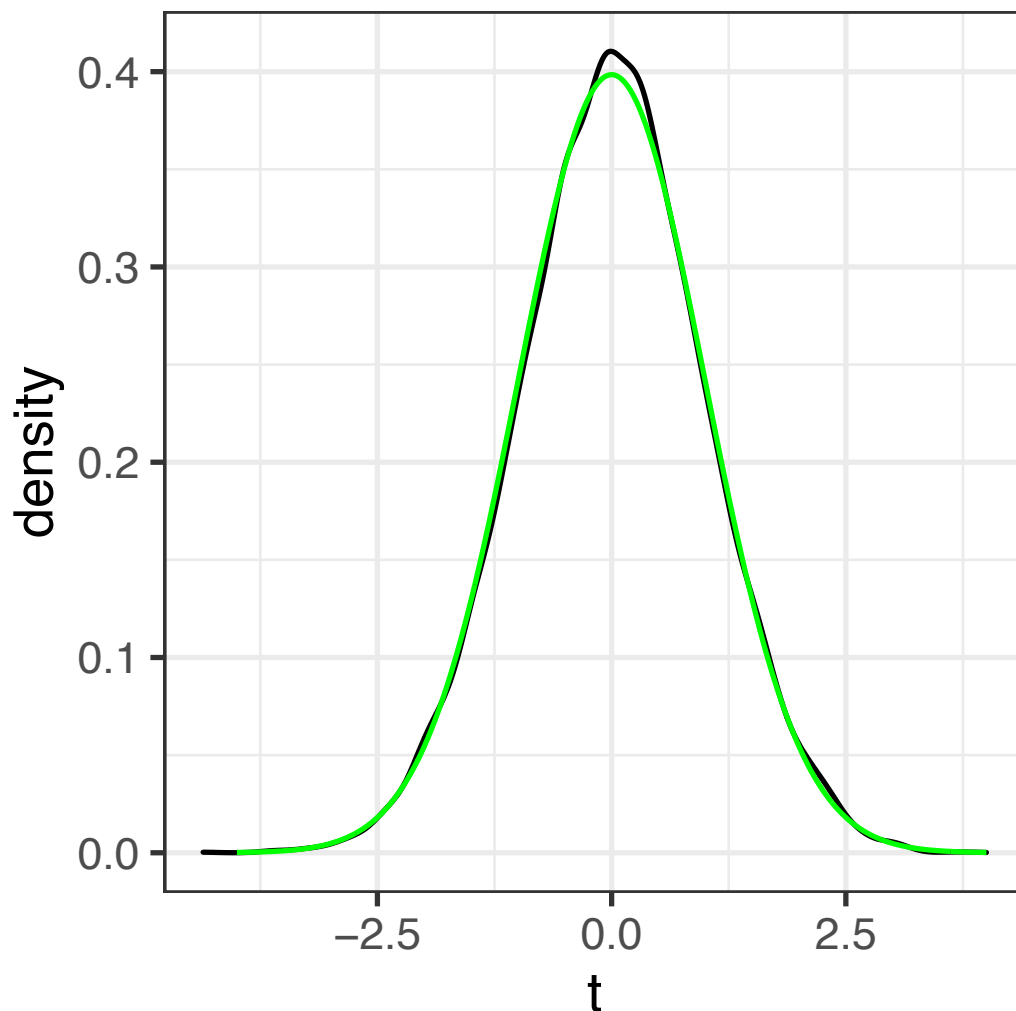


Monte Carlo, in action



- The t distribution is still a pretty good approximation of the distribution of the t statistic, even when the underlying distribution is gamma!

Monte Carlo, in action



- The t distribution is still a pretty good approximation of the distribution of the t statistic, even when the underlying distribution is gamma!
- This exemplifies what is meant when people say that the t test is **robust to deviations from normality**