9.S918: Statistical Inference for Brain and Cognitive Sciences, Pset 1
due 16 April 2024

9 April 2024

Note: I will be adding one more problem to this assignment shortly.

1 Incremental inference about possessor animacy

English has two CONSTRUCTIONS for grammatically expressing possession within a noun phrase, as exemplified in (1)–(2) below:

(1) the queen's crown (PRENOMINAL or 'S GENITIVE: possessor comes before the possessed noun)

(2) the crown of the queen (POSTNOMINAL or *of* GENITIVE: possessor comes after the possessed noun)

There is a correlation between the ANIMACY of the possessor and the preferred construction: animate possessors, as above, tend to be preferred prenominally relative to inanimate possessors, as in (3)– (4) below (Futrell & Levy, 2019; Rosenbach, 2005):

(3) the book's cover (Prenominal)

(4) the cover of the book (Postnominal)

Here is a pair of conditional probabilities that reflects this correlation:

$$P(\text{Possessor is }\textbf{prenominal}|\text{Possessor is }\textbf{animate}) = 0.9$$
$$P(\text{Possessor is }\textbf{prenominal}|\text{Possessor is }\textbf{inanimate}) = 0.25$$

Now consider the cognitive state of language comprehenders mid-sentence who have heard each of the three respective example sentence fragments, where the nouns that have been uttered are unfamiliar words:

1) the sneg of...
2) a...
3) a tufa's dax...

Task: Based on the knowledge encoded in the probabilities above, plot the probability in each of these three cases that the comprehender should assign to the possessor being animate, as a function of the prior probability P(Possessor is **animate**). That is, your plots will have the prior probability P(Possessor is **animate**) on the x-axis, and the posterior probability P(Possessor is **animate**|the provided sentence fragment) on the y-axis. Show your work in setting up the computations.

2 Variance of linear combinations of random variables, and of the MLE of a Bernoulli random variable

If random variables X and Y are conditionally independent given Z, then the variance of the sum $X+Y$ conditional on Z is $\mathrm{Var}[X+Y|Z] = \mathrm{Var}[X|Z] + \mathrm{Var}[Y|Z]$. (Note that it is common not to specify exactly what is conditioned on in discussions of this topic, so one will often see $\mathrm{Var}[X+Y| = \mathrm{Var}[X] + \mathrm{Var}[Y]$.)

1. **Task:** Suppose X and Y are not conditionally independent given Z. If they are positively correlated, will the variance of their sum be greater than or less than if they were independent? What if they are negatively correlated? Give an argument based on your intuitions. Then, look up the general formula for the variance of the sum of two random variables. Based on that formula, explain whether your intuitive argument is correct.

2. Suppose we apply a linear transformation to a random variable X: $X' = cX + b$. Then $\mathrm{Var}(X') = b^2 X$. **Task:** use the information provided thus far in this problem to show that the variance of the maximum likelihood estimate of the success parameter p of a Bernoulli random variable from n is $\mathrm{Var}(\hat{p}_{\mathrm{MLE}}) = \frac{p(1-p)}{n}$.

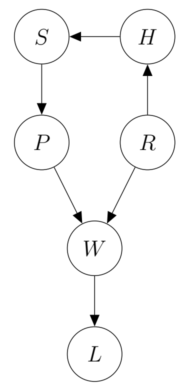
3 Conditional Independence

In our coverage of exchangeability and conditional exchangeability, we made use of the fact that for random variables X, Y, Z, if $X \perp Y | Z$ then $P(Y|Z,X) = P(Y|Z)$. **Task:** Prove that this is true based on the definition of conditional independence we gave in class, namely that if $X \perp Y | Z$ then $P(X,Y|Z) = P(X|Z)P(Y|Z)$.

4 Identifying conditional independencies in (causal) Bayes nets using d-separation

Below is a modification of a classic example of a (causal) Bayes net widely used in the AI literature (e.g., Pearl, 2009; Russell & Norvig, 2016), involving the sprinklers, rain, and whether one's lawn gets wet. In this version of the scenario, the sprinklers are automatically controlled by a humidity-sensitive sensor: twice a week, they will go off if the average humidity over the preceding three days has dropped below a certain threshold set by the homeowner. Rain not only increases the humidity but also gets the lawn and footpath wet. If the footpath gets wet, it becomes slippery. For good measure, the homeowner has a separate hygrometer that measures and logs the humidity near the sensor, to make sure the sensor is working properly.

Variable	**Meaning**
R	Whether it **R**ained recently
H	Recent **H**umidity
S	Whether the sprinklers' humidity **S**ensor detected a need to water the lawn
P	Whether the s**P**rinklers went off
W	Whether the lawn and footpath to the front door got **W**et
L	Whether the footpath is s**L**ippery



Task: We will be interested in what conditional independencies among variables in the network hold, given various types of information. For this exercise, when I ask, "Are A and B conditionally independent given C?", you may find that a useful way to think about that question is: suppose I already know C, and based on it I have some beliefs about A. If I additionally observe B, can it further change my beliefs about A? If yes, then A and B are **not** conditionally independent given C.

1. Are R and P conditionally independent, given no additional information? Answer both based on your intuition regarding the scenario and on the semantics of the Bayes net.

2. The homeowner is out of town but reads the hygrometer's log for the past three days (the hygrometer logs to the cloud). Are R and P conditionally independent now?

3. After reading the hygrometer's log, the homeowner gets a text message from the cat sitter, who reports slipping on the footpath on the way to the house. Are R and P conditionally independent now?

4. For each of the preceding three questions, suppose the sensor has broken so that it randomly sets off the sprinkler with 50% probability every three days (and the homeowner knows it). Does this change the answers to any of the above three questions? Why?

References

Futrell, R., & Levy, R. P. (2019). Do RNNs learn human-like abstract word order preferences? *Proceedings of the Society for Computation in Linguistics (SCiL) 2019*, *2*, 50–59.

Pearl, J. (2009). *Causality: Models, reasoning, and inference* (Second). Cambridge.
Rosenbach, A. (2005). Animacy versus weight as determinants of grammatical variation in English. *Language*, *81*(3), 613–644.
Russell, S. J., & Norvig, P. (2016). *Artificial intelligence: A modern approach*. Pearson.