Brief review of elementary statistics: parameter estimation, confidence intervals, hypothesis testing

Roger Levy

9.S916: Statistical data analysis for scientific inference in cognitive science

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- In general, here we will use $\mathbf{y}$ to refer to observed-outcome **data** and θ to refer to the model parameters to be estimated

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- Good estimators have favorable **bias–variance** tradeoff

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(repeat slide from lecture 3)

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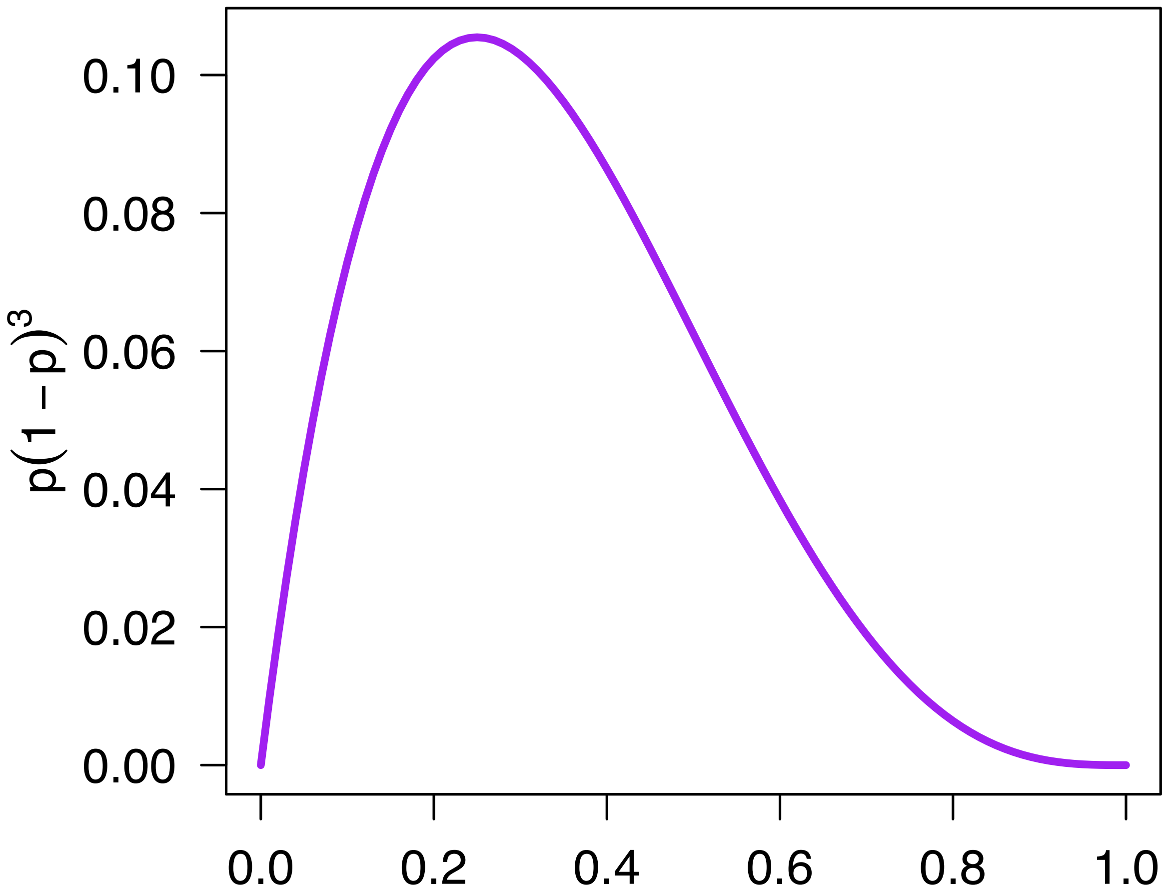

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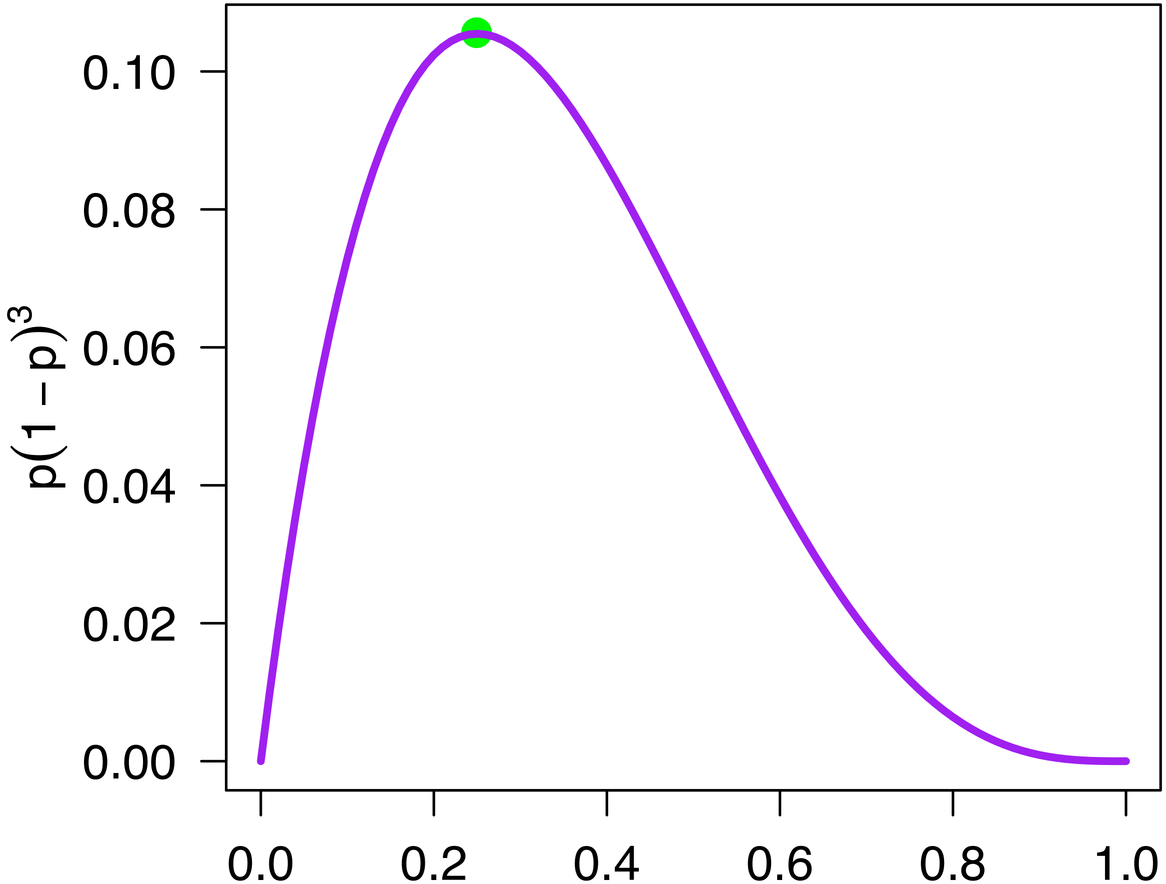

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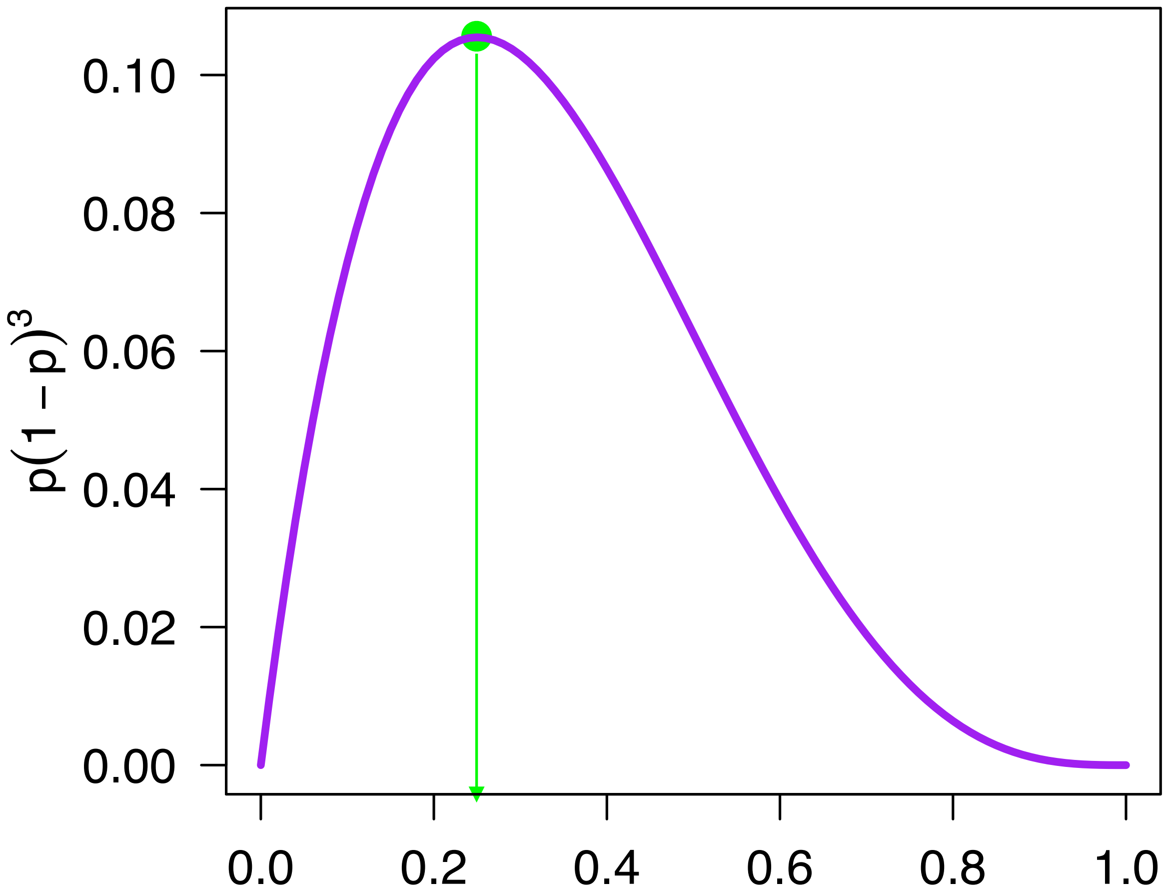

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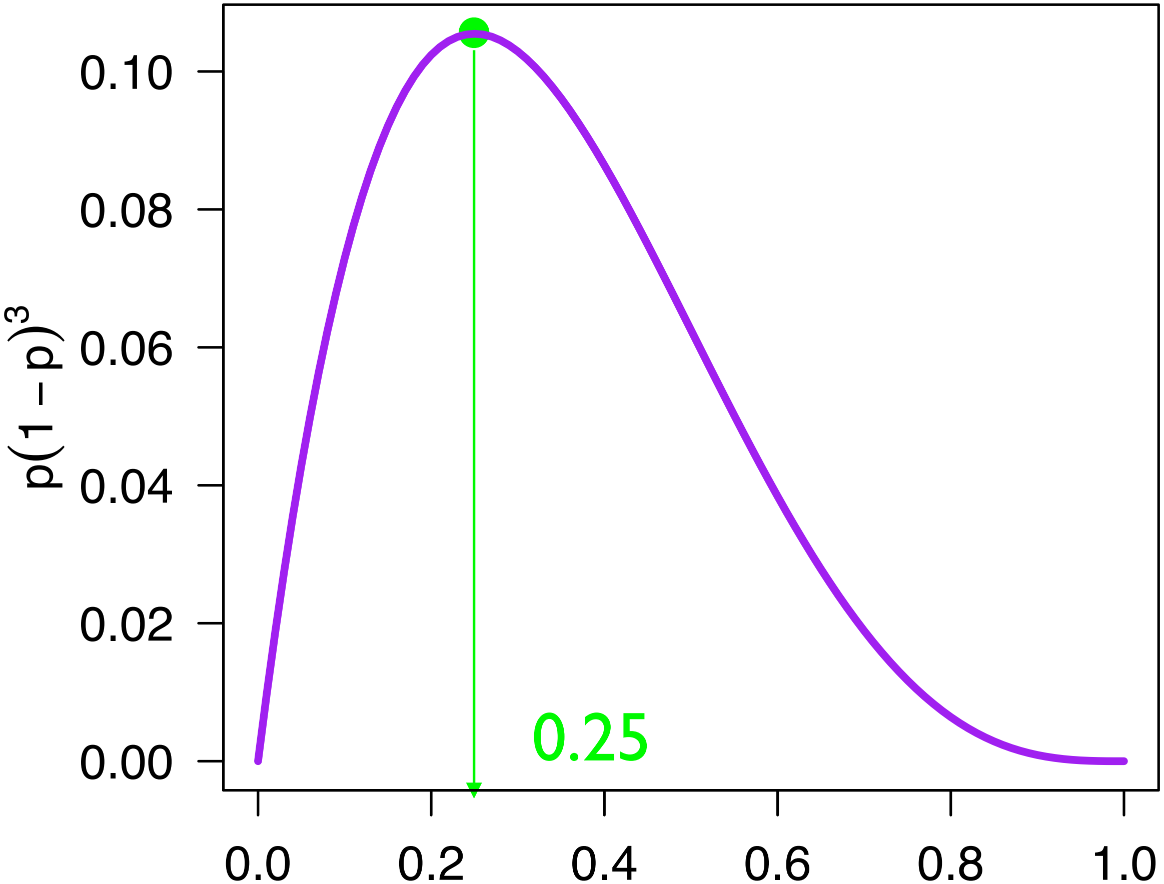

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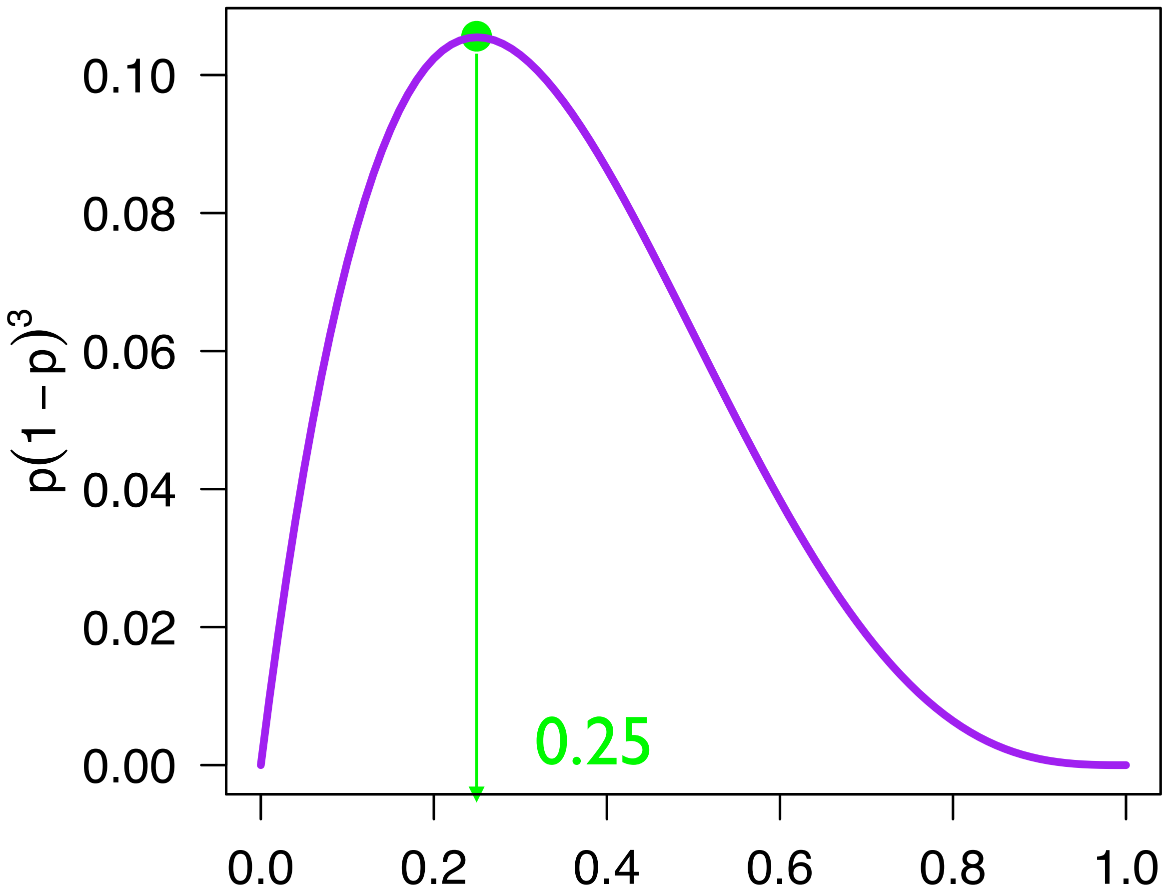

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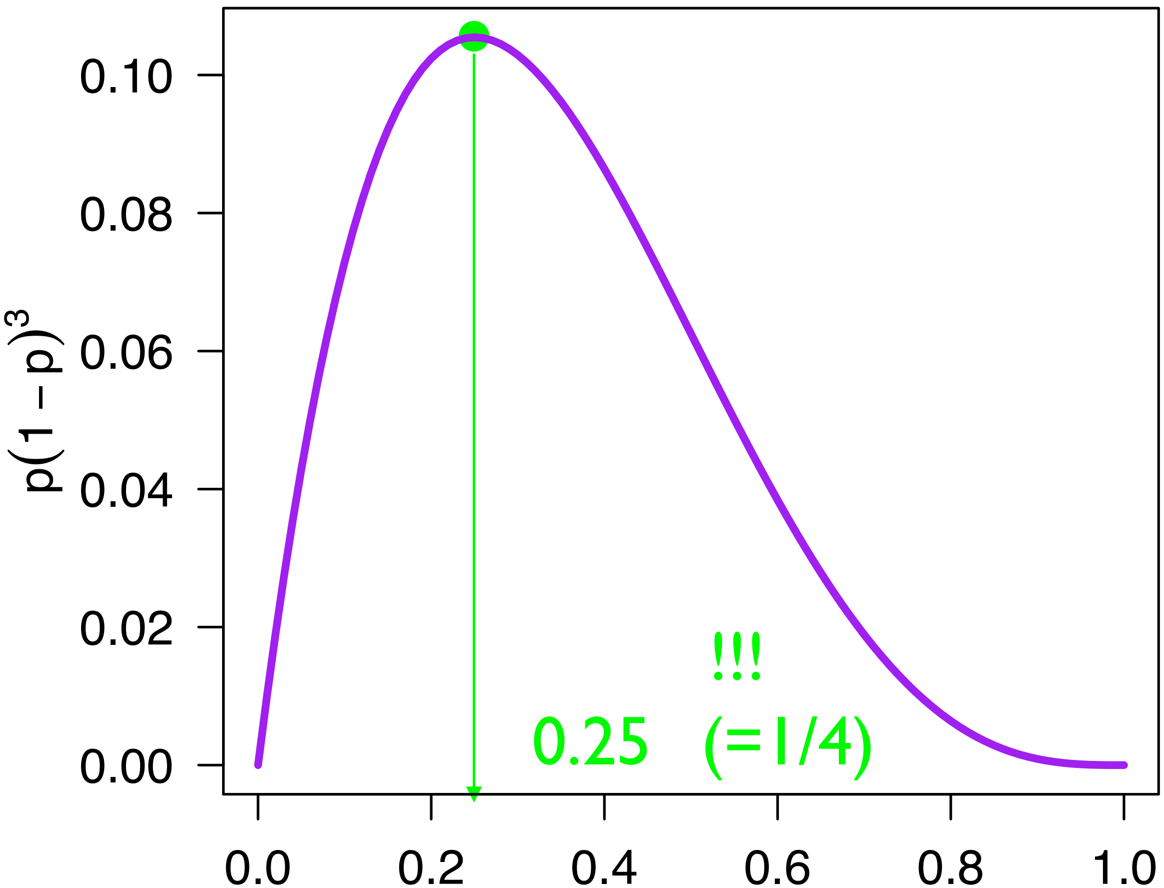

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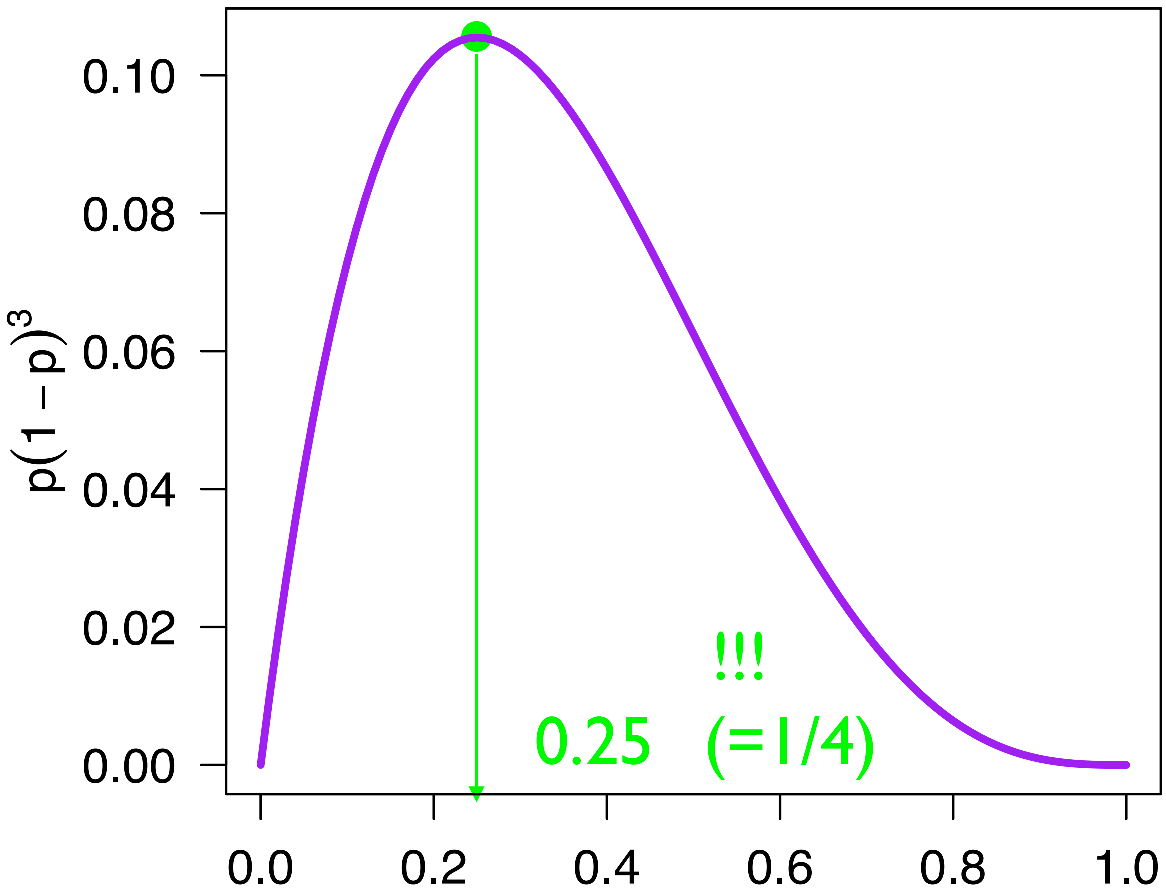

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The **MLE** also turns out to be the *relative frequency estimate* (RFE)

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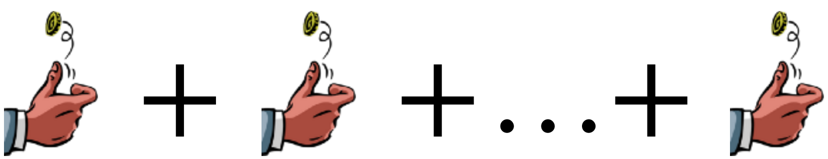

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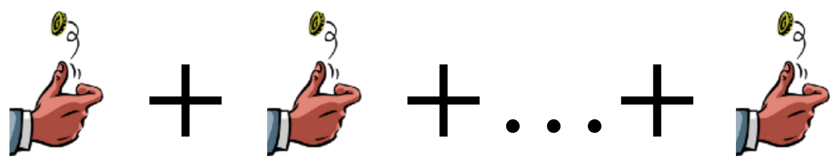

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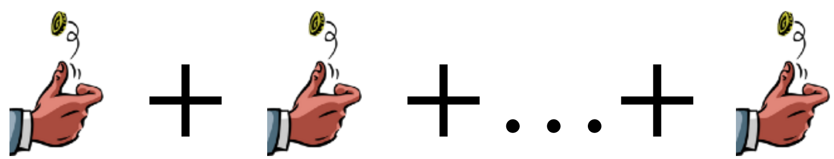

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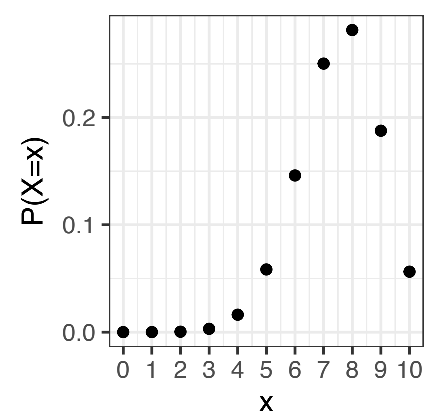

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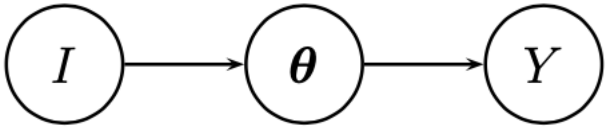
Bayesian parameter estimation



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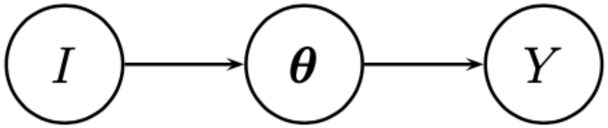

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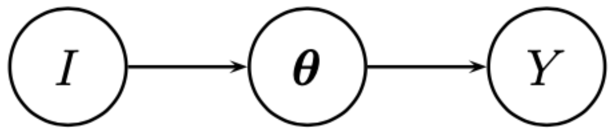

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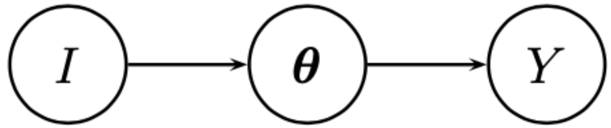

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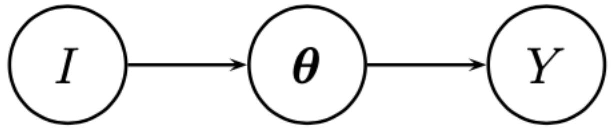

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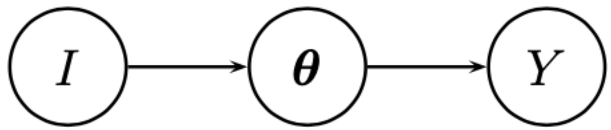

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- Then, if we assume a parametric form for $P(\mathbf{y} | \theta)$, we just need the prior $P(\theta | I)$

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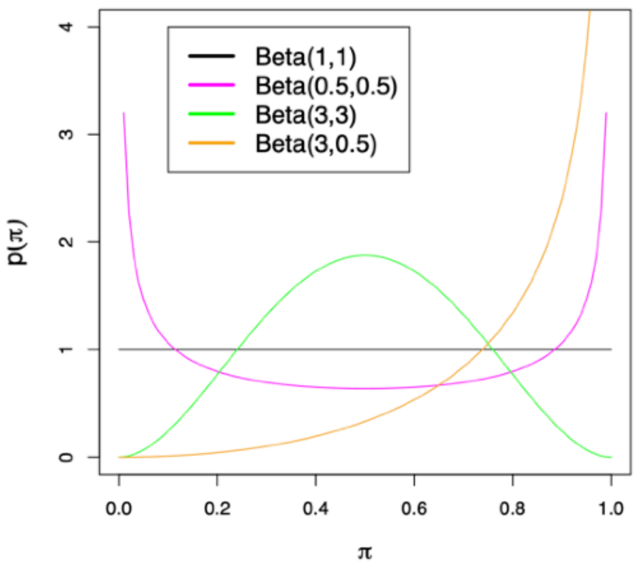

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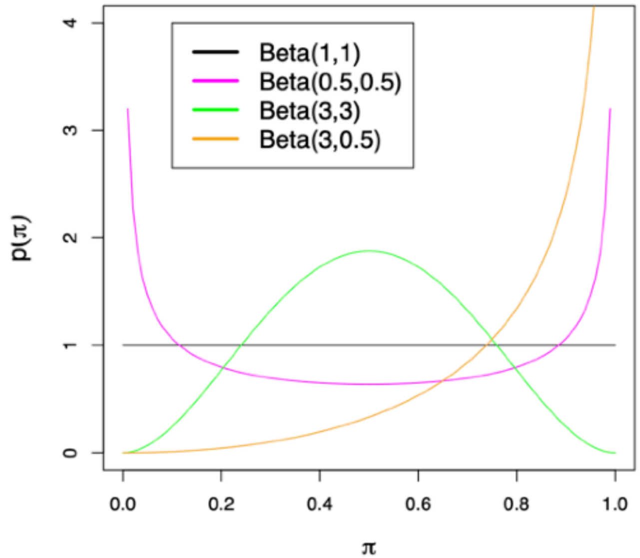

- Cool thing about the beta distribution: the posterior is also beta distributed! For $\mathbf{y} = m$ successes in n trials:

$$P(\pi|\boldsymbol{y},\alpha_1,\alpha_2) \propto \overbrace{\pi^m(1-\pi)^{n-m}}^{\text{Likelihood}}\overbrace{\pi^{\alpha_1-1}(1-\pi)^{\alpha_2-1}}^{\text{Prior}}$$

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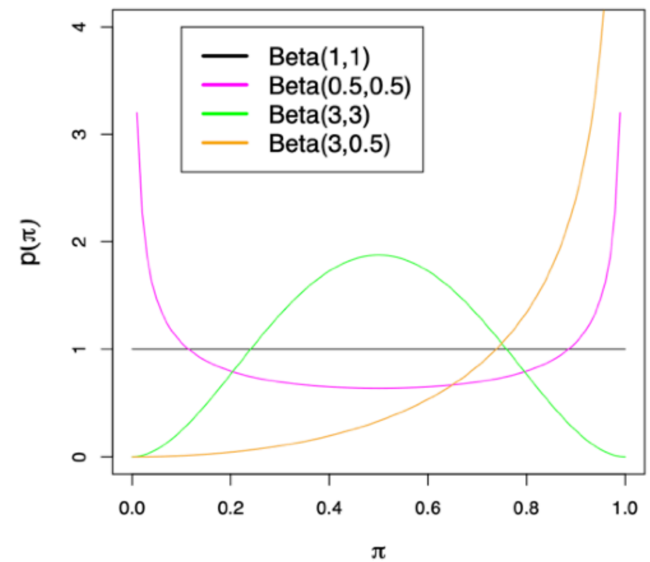

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- Cool thing about the beta distribution: the posterior is also beta distributed! For $\mathbf{y} = m$ successes in n trials:

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Example for coin flips: the beta distribution

- Express background knowledge I as two "pseudo-count" parameters α_1, α_2
- The beta distribution has form

$$P(\pi|\alpha_1,\alpha_2) = \boxed{\frac{1}{B(\alpha_1,\alpha_2)}}\boxed{\pi^{\alpha_1-1}(1-\pi)^{\alpha_2-1}}$$

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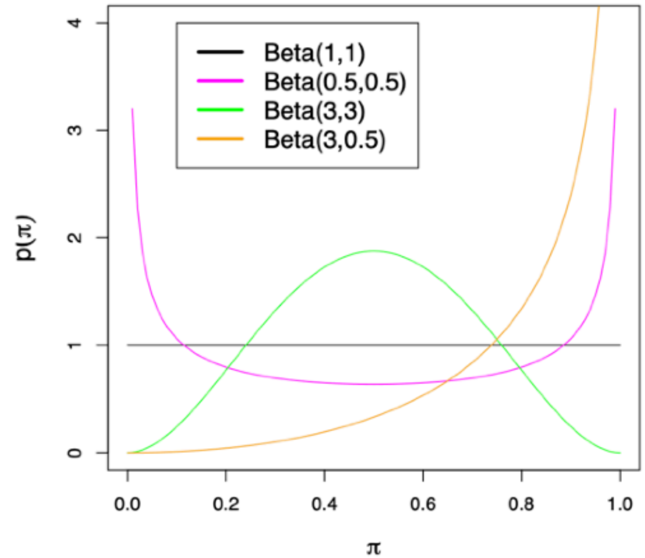

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- This property is called **conjugacy** and is convenient where available!

Example of Bayesian parameter estimation

https://commons.wikimedia.org/wiki/
File:Post_medieval_coin,_Sixpence_of_William_III_(FindID_605939).jpg

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- I inspect my coin and notice serious irregularities!

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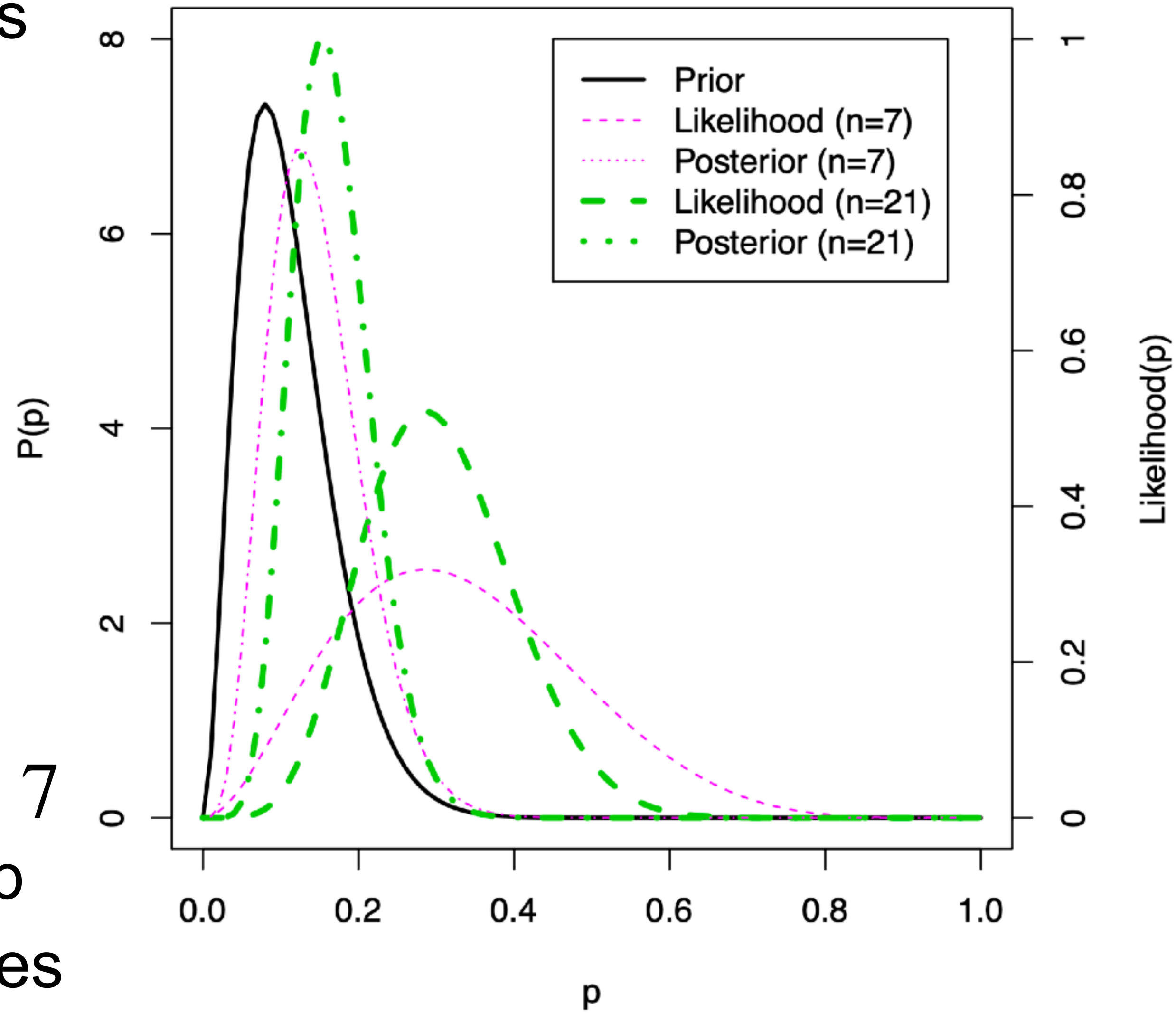

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Posterior prediction

$P(\text{heads}) = \pi$
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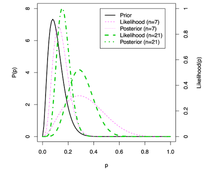

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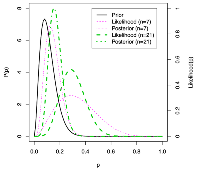

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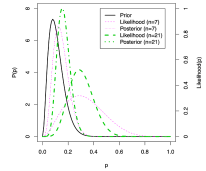

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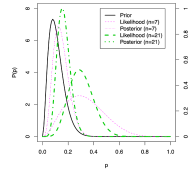

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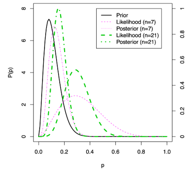

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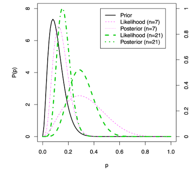

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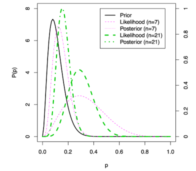

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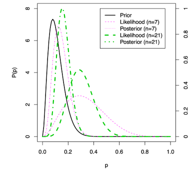

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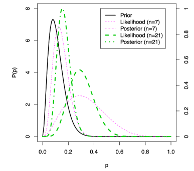

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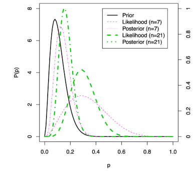

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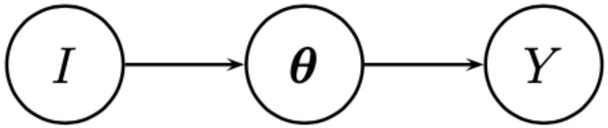

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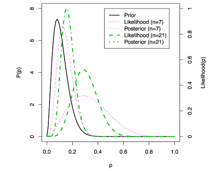

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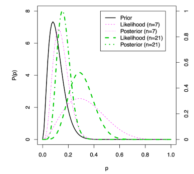
Beta distribution  **Our example**

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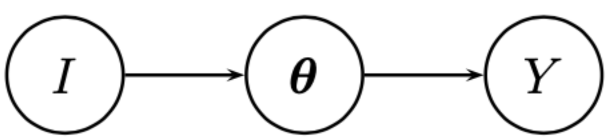

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→The **Beta-Binomial** model: $P(r | k, I, \boldsymbol{y}) = \binom{k}{r} \dfrac{B(\alpha_1 + m + r, \alpha_2 + n - m + k - r)}{B(\alpha_1 + m, \alpha_2 + n - m)}$

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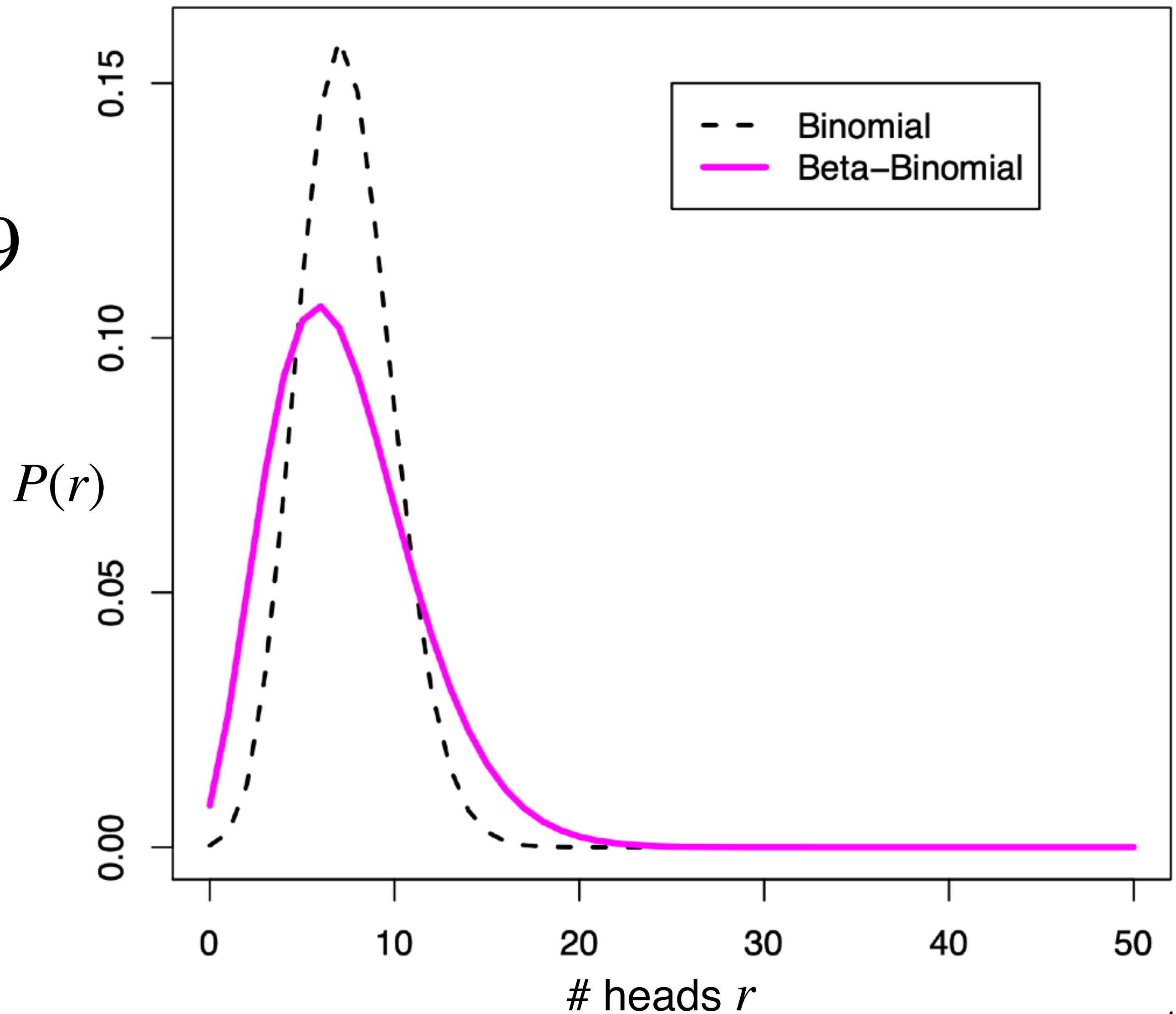

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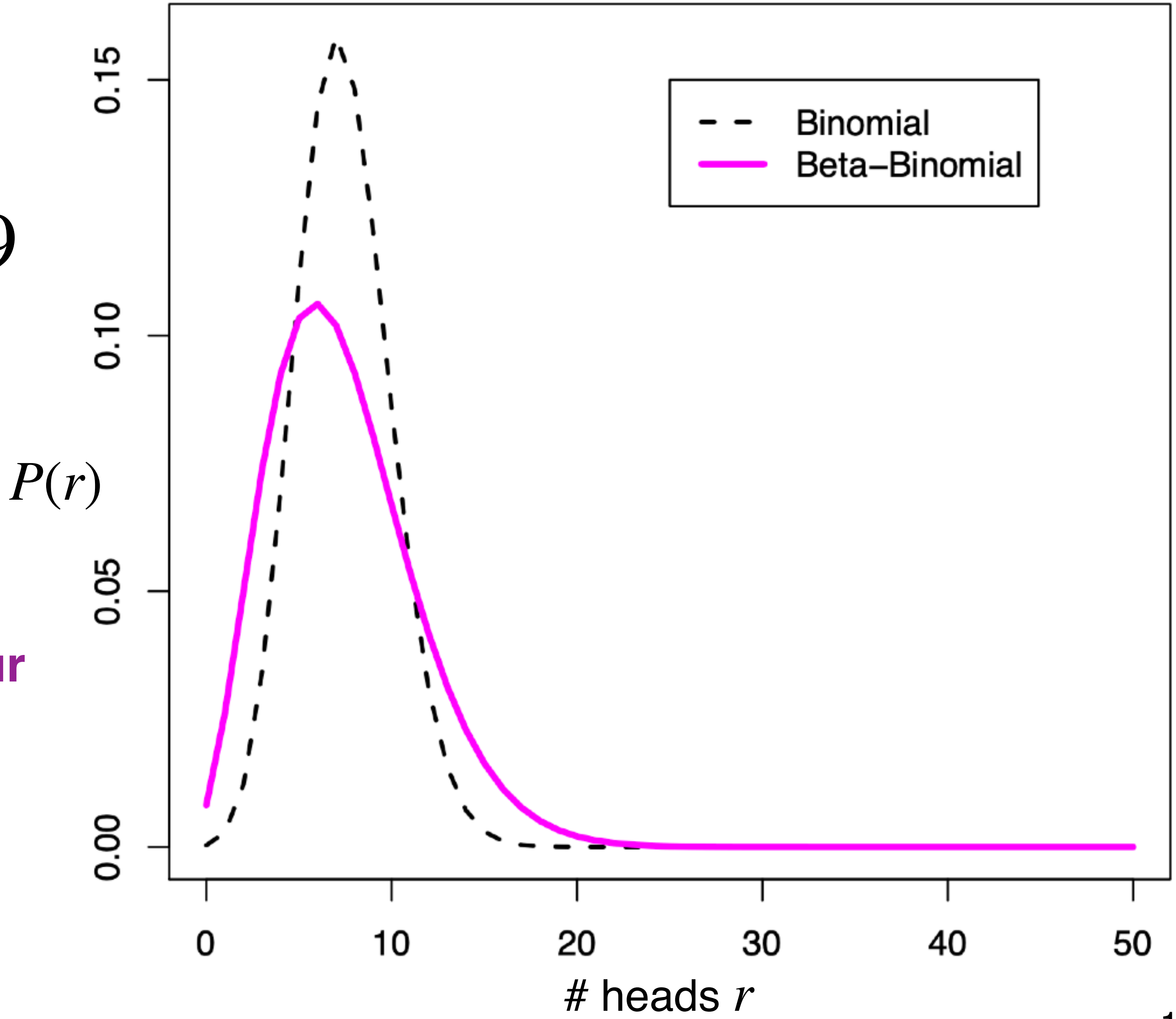

Bayesian inference takes into account our uncertainty about model parameters θ, leading to more hedged predictions

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- If your qualitative conclusions depend on choice of prior, it is a reason to be wary of the robustness of your analysis!
- As data become plentiful*, choice of prior *often but not always* recedes in importance

*What counts as "plentiful" depends on size of the model and structure of the data