Directed Acyclic Graphical Models, and Causal Models
9.S918: Statistical Inference in Brain and Cognitive Sciences
Spring 2024

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Today's content

- Conditional Independence
- Bayes Nets (a.k.a. directed acyclic graphical models, DAGs)

(Conditional) Independence

Events A and B are said to be Conditionally Independent given information C if

$$P(A, B|C) = P(A|C)P(B|C)$$

Conditional independence of A and B given C is often expressed as

$$A \perp B|C$$

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- And I hope that you'll agree that the framework is intuitive too!
- The intuitiveness is because a causal interpretation of the framework is natural—and, indeed, this is formalized in the causal treatment of Bayes nets

The coin factory

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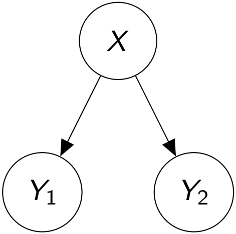
  - You receive the coin and flip it twice, with H(eads)/T(ails) outcomes Y_1 and Y_2
- Receiving a coin from the factory and flipping it twice is **sampling** (or **taking a sample**) from the joint distribution $P(X, Y_1, Y_2)$

This generative process is a Bayes Net

The directed acyclic graphical model (DAG), or Bayes net:



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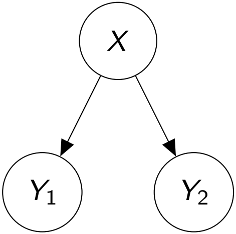

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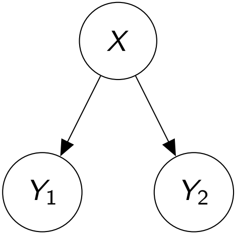

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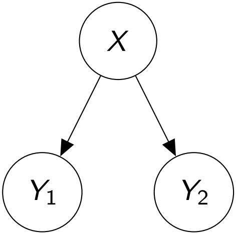

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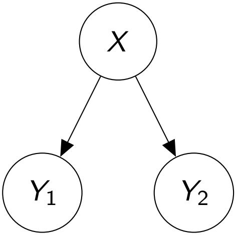

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- The answer to this question is **No!**
  - $P(Y_2 = H) = \frac{1}{2}$ (you can see this by symmetry)
  - But $P(Y_2 = H | Y_1 = H) = \overbrace{\frac{1}{3} \times \frac{1}{2}}^{\text{Coin was fair}} + \overbrace{\frac{2}{3} \times 1}^{\text{Coin was 2-H}} = \frac{5}{6}$

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- A node on the path has non-converging arrows if two edges on the path connect to it, but at least one does not point to it.
- A third disjoint node set C d-separates A and B if for every path between A and B, either:
    1. there is some node N on the path whose arrows do not converge and which *is* in C; or
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Major types of d-separation

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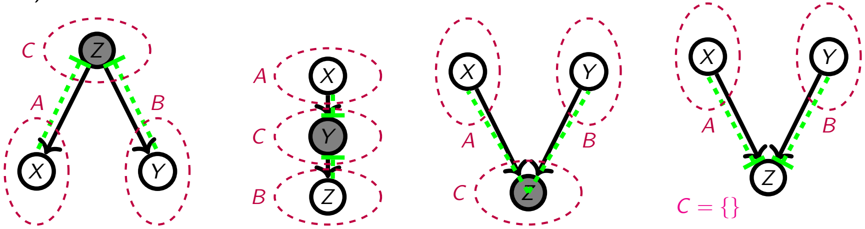
Common-cause d-separation (from knowing Z)

Mediating d-separation (from knowing Y)

Explaining away: knowing Z prevents d-separation

D-separation in the absence of knowledge of Z



(Shaded node=in C)

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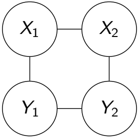
  - **Example:** let X_1, X_2, Y_1, Y_2 each be 0/1 random variable, and let the joint distribution reflect the constraint that $Y_1 = (X_1 == X_2)$ and $Y_2 = \text{xor}(X_1, X_2)$. This gives us $Y_1 \perp Y_2 | \{X_1, X_2\}$, but you won't be able to write a Bayes net involving these four variables such that $\{X_1, X_2\}$ d-separates Y_1 and Y_2.

Conditional independencies not expressible in a Bayes net

- **Example:** let X_1, X_2, Y_1, Y_2 each be binary 0/1 random variables, in the following arrangement on an **undirected** graph:



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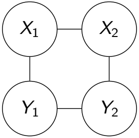

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- Suppose the joint distribution is determined entirely by adjacent nodes "liking" to have the same value. Formally, for example:

$$P(X_1, X_2, Y_1, Y_2) \propto \prod_{i=1}^{4} \left(\frac{1}{2}\right)^{f_i(X_1, X_2, Y_1, Y_2)}$$

(Most probable outcomes, each with prob. 0.195: either all 0s, or all 1s)

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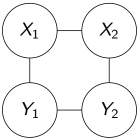

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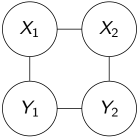

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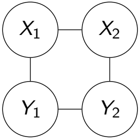
$$X_1 \perp Y_2 | \{X_2, Y_1\} \qquad\qquad X_2 \perp Y_1 | \{X_1, Y_2\}$$

- But this set of conditional independencies cannot be expressed in a Bayes Net.

Conditional independencies not expressible in a Bayes net



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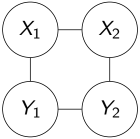

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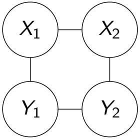

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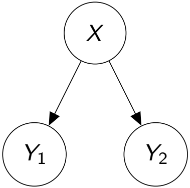
- We won't look at these further, but you can read about them in books and papers about graphical models (e.g., (Bishop, 2006, Section 8.3)

Back to our example



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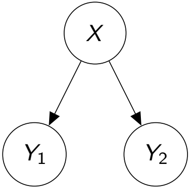
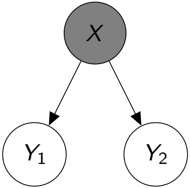


- *Without looking at the coin before flipping it*, the outcome Y_1 of the first flip gives me information about the type of coin, and affects my beliefs about the outcome of Y_2

Back to our example



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- But if I *look* at the coin before flipping it, Y_1 and Y_2 are rendered independent

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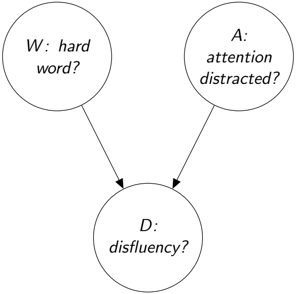
A reasonable graphical model:



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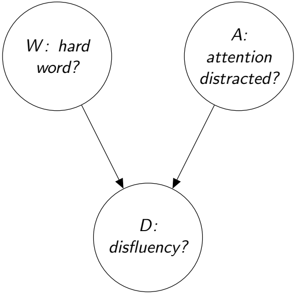

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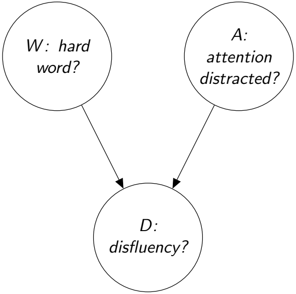

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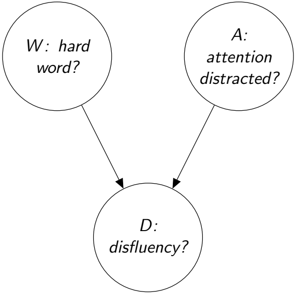
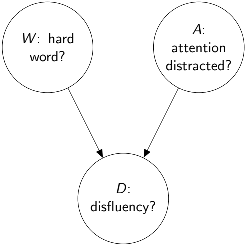


- Without knowledge of D, there's no reason to expect that W and A are correlated
- But hearing a disfluency *demands a cause*
- Knowing that there was a distraction *explains away* the disfluency, reducing the probability that the speaker was planning to utter a hard word

An example of the disfluency model



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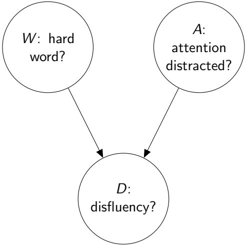

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- Hard words and distractions both induce disfluencies; having both makes a disfluency *really* likely

W	A	D=no disfluency	D=disfluency
easy	undistracted	0.99	0.01
easy	distracted	0.7	0.3
hard	undistracted	0.85	0.15
hard	distracted	0.4	0.6

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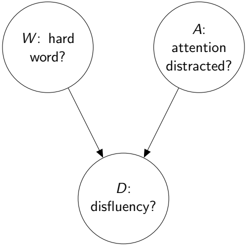

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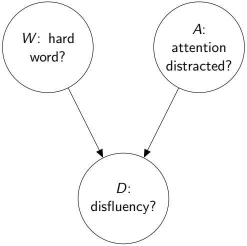

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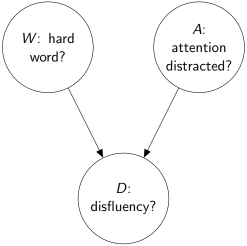

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- That is, what is $P(W = \text{hard}|D = \text{disfluent}, A = \text{distracted})$?

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Fortunately, there is automated machinery to "turn the Bayesian crank":

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P(W = \text{hard}) &= 0.25 \\
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P(W = \text{hard} | D = \text{disfluent}, A = \text{distracted}) &= 0.40
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- Knowing that the speaker was distracted (A) *decreased* the probability that the speaker was about to utter a hard word (W)—A **explained** D **away**.
- A caveat: the type of relationship among A, W, and D will depend on the values one finds in the probability table!

$$
\begin{aligned}
&P(W) \\
&P(A) \\
&P(D|W, A)
\end{aligned}
$$

Summary thus far

Key points:

- Bayes' Rule is a compelling framework for modeling inference under uncertainty
- DAGs/Bayes Nets are a broad class of models for specifying joint probability distributions with conditional independencies
- Classic Bayes Net references: Pearl (1988, 2000); Jordan (1998); Russell and Norvig (2003, Chapter 14); Bishop (2006, Chapter 8).

An example of the disfluency model

$$\boxed{P(W = \text{hard}|D = \text{disfluent}, A = \text{distracted})}$$

hard	W=hard
easy	W=easy
disfl	D=disfluent
distr	A=distracted
undistr	A=undistracted

$$P(\text{hard}|\text{disfl, distr}) = \frac{P(\text{disfl}|\text{hard, distr})P(\text{hard}|\text{distr})}{P(\text{disfl}|\text{distr})} \quad \text{(Bayes' Rule)}$$

$$= \frac{P(\text{disfl}|\text{hard, distr})P(\text{hard})}{P(\text{disfl}|\text{distr})} \quad \text{(Independence from the DAG)}$$

$$P(\text{disfl}|\text{distr}) = \sum_{w'} P(\text{disfl}|W = w')P(W = w') \quad \text{(Marginalization)}$$

$$= P(\text{disfl}|\text{hard})P(\text{hard}) + P(\text{disfl}|\text{easy})P(\text{easy})$$

$$= 0.6 \times 0.25 + 0.3 \times 0.75$$

$$= 0.375$$

$$P(\text{hard}|\text{disfl, distr}) = \frac{0.6 \times 0.25}{0.375}$$

$$= 0.4$$

An example of the disfluency model

$$\boxed{P(W = \text{hard} | D = \text{disfluent})}$$

$$
\begin{aligned}
P(\text{hard}|\text{disfl}) &= \frac{P(\text{disfl}|\text{hard})P(\text{hard})}{P(\text{disfl})} && \text{(Bayes' Rule)}\\
P(\text{disfl}|\text{hard}) &= \sum_{a'} P(\text{disfl}|A = a', \text{hard})P(A = a'|\text{hard})\\
&= P(\text{disfl}|A = \text{distr}, \text{hard})P(A = \text{distr}|\text{hard}) + P(\text{disfl}|\text{undistr}, \text{hard})P(\text{undistr}|\text{hard})\\
&= 0.6 \times 0.15 + 0.15 \times 0.85\\
&= 0.2175\\
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&= 0.3 \times 0.15 + 0.01 \times 0.85\\
&= 0.0535\\
P(\text{disfl}) &= 0.2175 \times 0.25 + 0.0535 \times 0.75\\
&= 0.0945\\
P(\text{hard}|\text{disfl}) &= \frac{0.2175 \times 0.25}{0.0945}\\
&= 0.575396825396825
\end{aligned}
$$

Interventions

- Suppose we have a collection of random variables V that follow some joint probability distribution. We define an "intervention" operator that can be conditioned on in probabilistic queries (Pearl, 2009):

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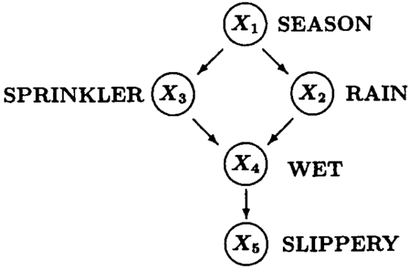
- In general, $P(V|X)$ and $P(V|\mathrm{Do}(X = x))$ will **NOT** be the same distribution. $P(V|\mathrm{Do}(X = x))$, also notated as $P_x(V)$, is sometimes called an interventional distribution.

Causal Bayes Nets and interventions as "graph surgery"

- If V can be organized into a causal Bayes Net G, then the relationship between the base joint distribution (no interventions) and the set of interventional distributions can be characterized succinctly.



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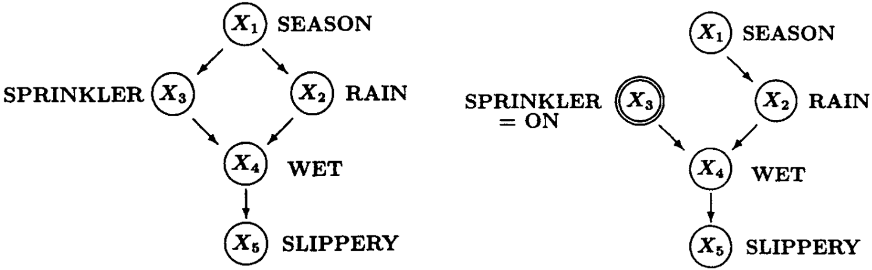

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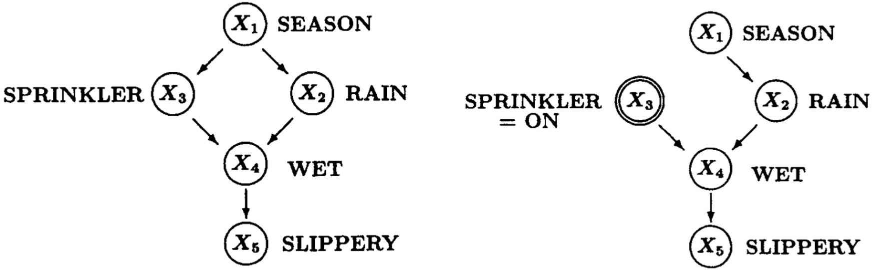

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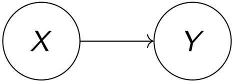

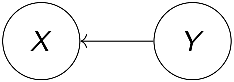

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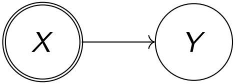

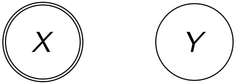

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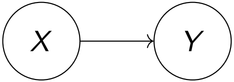

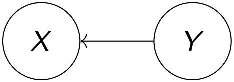

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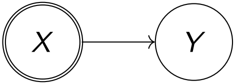

X = smoking
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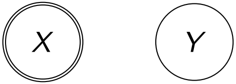

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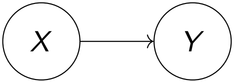

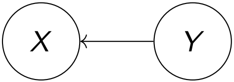

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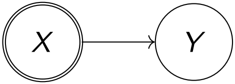

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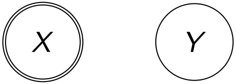

X = thermometer reading
Y = ambient temperature

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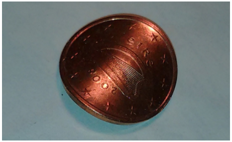
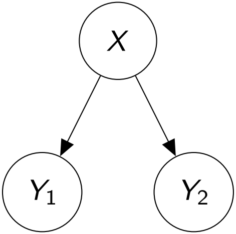
Back to our previous example, a bit modified





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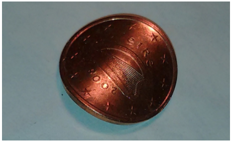

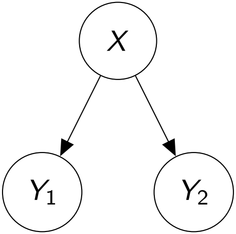

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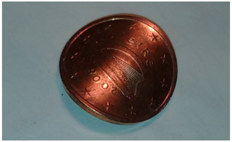

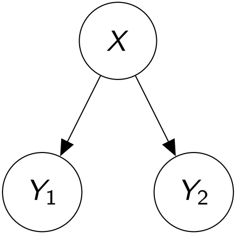

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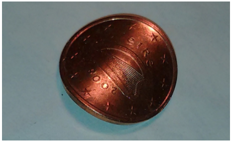

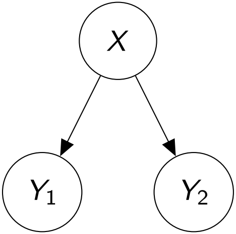

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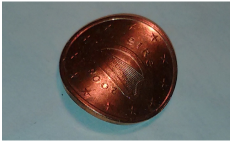

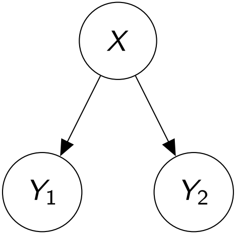

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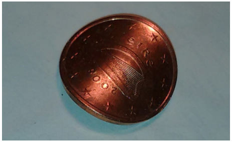
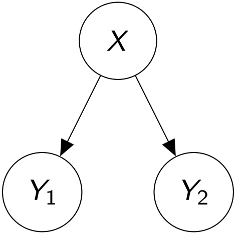
- Generative process:

Back to our previous example, a bit modified





- Imagine a factory that produces three types of coins in equal volumes:
  - Fair coins;
  - A slightly **bent** coin that lands heads with $^3/_5$ probability;
  - A slightly bent coin that lands tails with $^3/_5$ probability.
- Generative process:
  - The factory produces a coin of type X and sends it to you;

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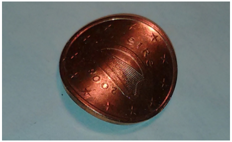

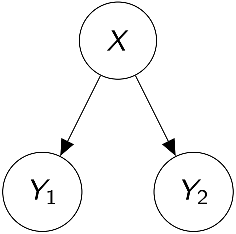

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- Generative process:
  - The factory produces a coin of type X and sends it to you;
  - You receive the coin and flip it twice, with H(eads)/T(ails) outcomes Y_1 and Y_2

Predictive value $\neq$ influence through intervention

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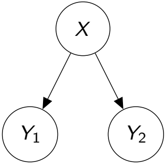

- The outcome of the first coin flip Y_1 has **predictive value** for the outcome of the second coin flip Y_2, and vice versa

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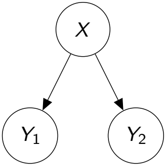

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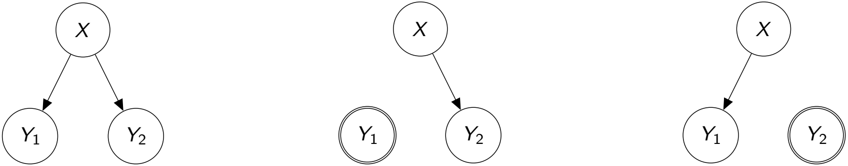

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- I could learn this **association** from observing pairs of flips of coins from the coin factory
- But I cannot intervene on either variable to influence the other, because neither is causally upstream of the other!

Implications

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- Typically, organizing information this way will help ensure that our statistical inferences actually answer our scientific questions of interest
- Traditional statistical tools are *associational*, so we need this top-down machinery (here, the mind of the scientist!) to ensure that they're being deployed appropriately
- We must also stay cognizant of possible "unseen" latent causes, and that we may be uncertain about the true causal relationship among our observable variables

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