Directed Acyclic Graphical Models, and Causal Models
9.S918: Statistical Inference in Brain and Cognitive Sciences
Spring 2024

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Today's content

- Conditional Independence
- Bayes Nets (a.k.a. directed acyclic graphical models, DAGs)

(Conditional) Independence

Events A and B are said to be Conditionally Independent given information C if

$$P(A, B|C) = P(A|C)P(B|C)$$

Conditional independence of A and B given C is often expressed as

$$A \perp B|C$$

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- The intuitiveness is because a causal interpretation of the framework is natural—and, indeed, this is formalized in the causal treatment of Bayes nets

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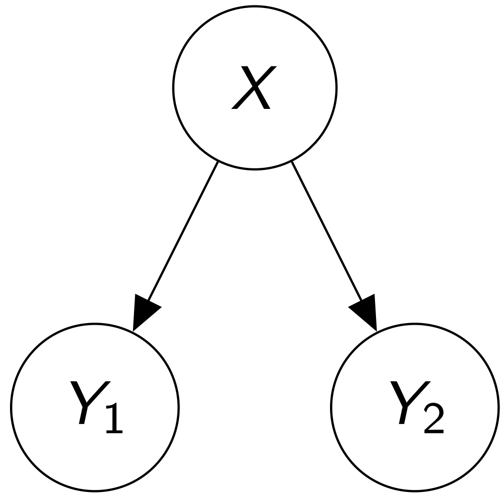
- Receiving a coin from the factory and flipping it twice is **sampling** (or **taking a sample**) from the joint distribution $P(X, Y_1, Y_2)$

This generative process is a Bayes Net

The directed acyclic graphical model (DAG), or Bayes net:



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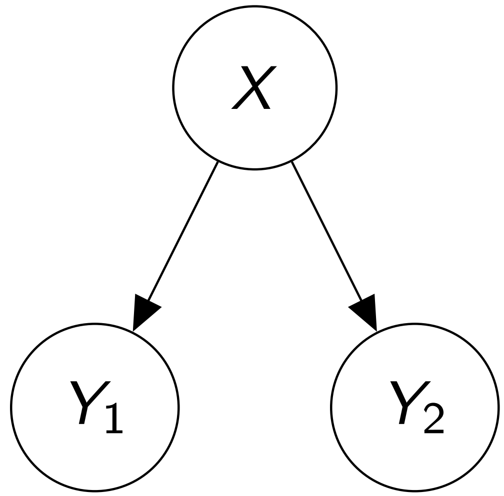

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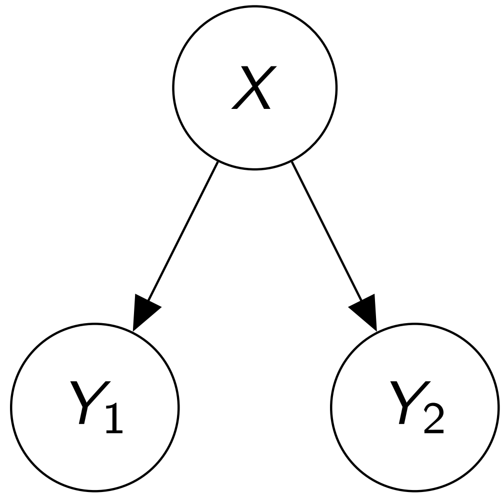

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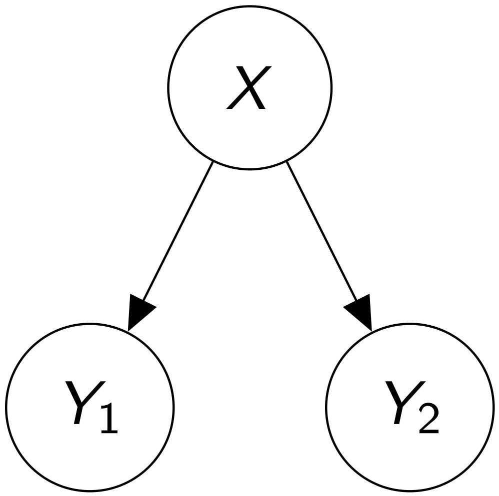

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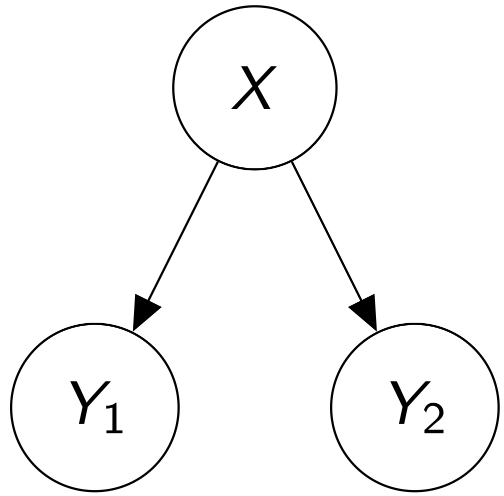

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- The answer to this question is **No!**
  - $P(Y_2 = H) = \frac{1}{2}$ (you can see this by symmetry)
  - But $P(Y_2 = H | Y_1 = H) = \overbrace{\frac{1}{3} \times \frac{1}{2}}^{\text{Coin was fair}} + \overbrace{\frac{2}{3} \times 1}^{\text{Coin was 2-H}} = \frac{5}{6}$

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- A node on the path has non-converging arrows if two edges on the path connect to it, but at least one does not point to it.
- A third disjoint node set C d-separates A and B if for every path between A and B, either:
  1. there is some node N on the path whose arrows do not converge and which *is* in C; or
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Major types of d-separation

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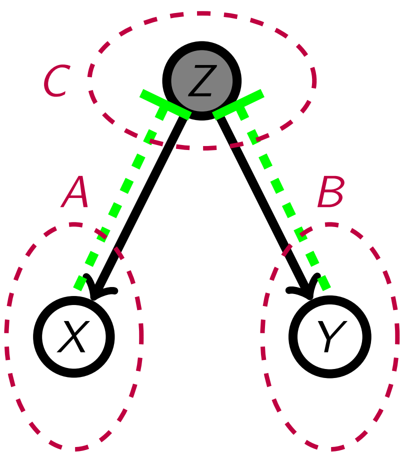
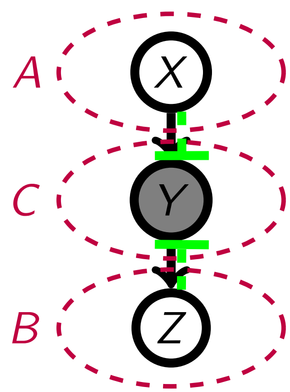
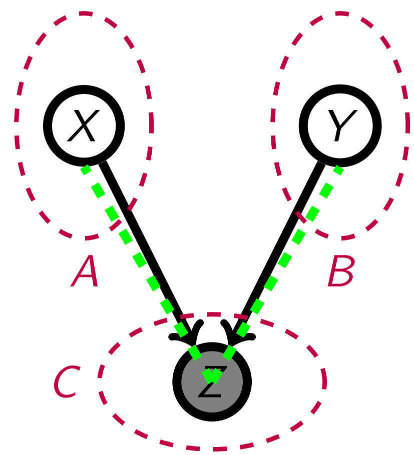


(Shaded node=in C)

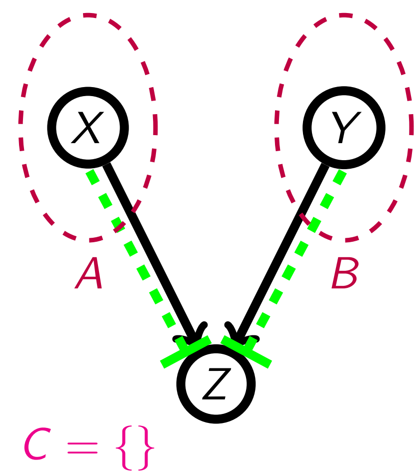
Intervening d-separation (from knowing Y)



Explaining away: knowing Z prevents d-separation



D-separation in the absence of knowledge of Z



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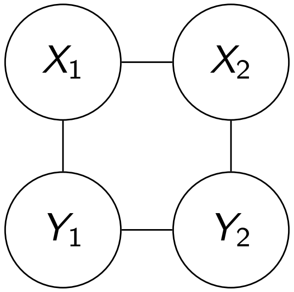
  - **Example:** let X_1, X_2, Y_1, Y_2 each be $0/1$ random variable, and let the joint distribution reflect the constraint that $Y_1 = (X_1 == X_2)$ and $Y_2 = \text{xor}(X_1, X_2)$. This gives us $Y_1 \perp Y_2 | \{X_1, X_2\}$, but you won't be able to write a Bayes net involving these four variables such that $\{X_1, X_2\}$ d-separates Y_1 and Y_2.

Conditional independencies not expressible in a Bayes net

- **Example:** let X_1, X_2, Y_1, Y_2 each be binary 0/1 random variables, in the following arrangement on an **undirected** graph:



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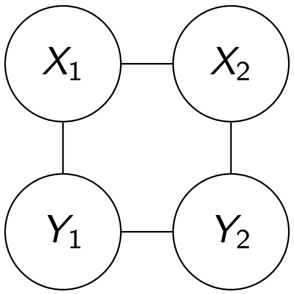

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- Suppose the joint distribution is determined entirely by adjacent nodes "liking" to have the same value. Formally, for example:

$$P(X_1, X_2, Y_1, Y_2) \propto \prod_{i=1}^{4} \left(\frac{1}{2}\right)^{f_i(X_1, X_2, Y_1, Y_2)}$$

(Most probable outcomes, each with prob. 0.195: either all 0s, or all 1s)

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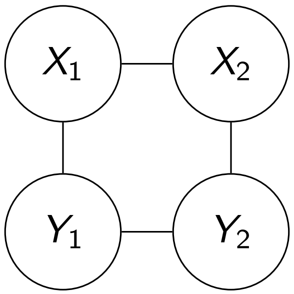

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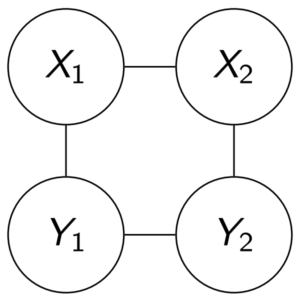

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- But this set of conditional independencies cannot be expressed in a Bayes Net.

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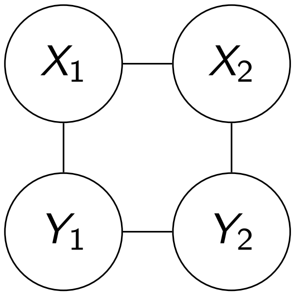

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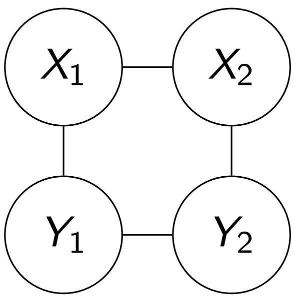

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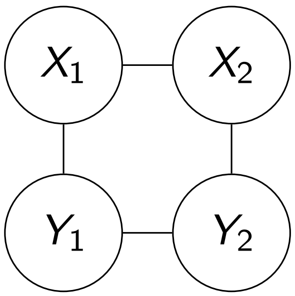

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- We won't look at these further, but you can read about them in books and papers about graphical models (e.g., (Bishop, 2006, Section 8.3)