Confidence intervals, hypothesis testing, Monte Carlo, and generalized linear models

Roger Levy

9.S916: Statistical data analysis for scientific inference in cognitive science

16 April 2024

Credible intervals & confidence intervals

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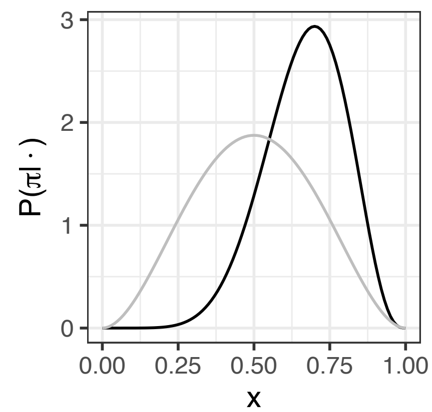

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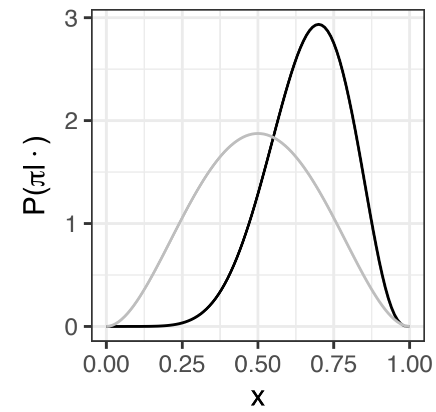

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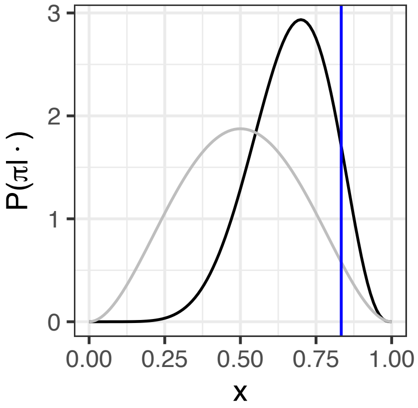

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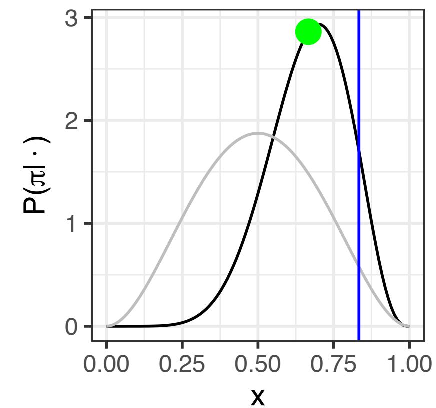

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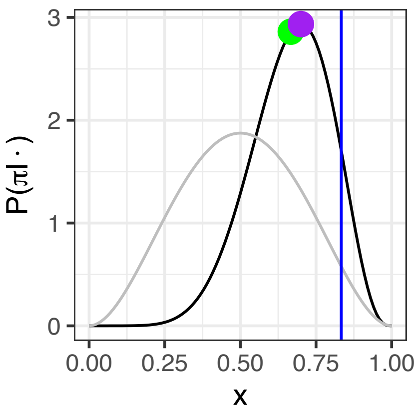

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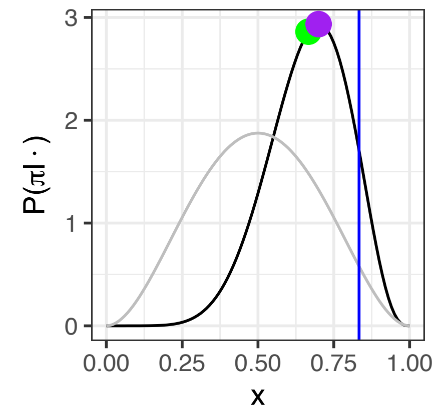

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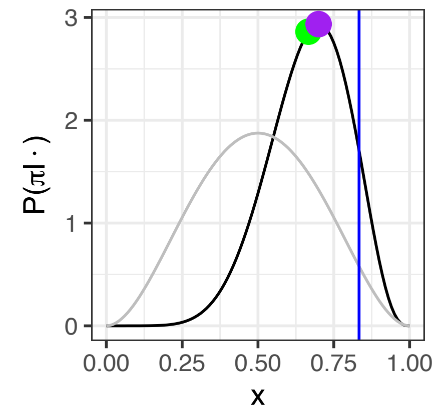

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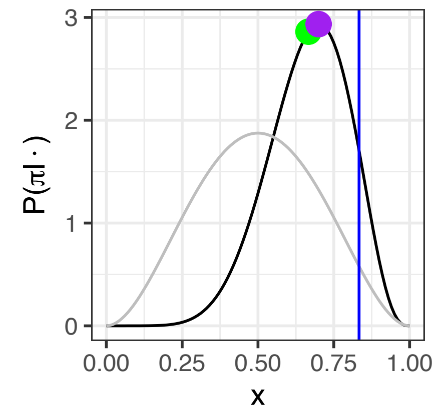

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- Credible intervals (Bayesian) and confidence intervals (frequentist) provide a bit more information about this uncertainty

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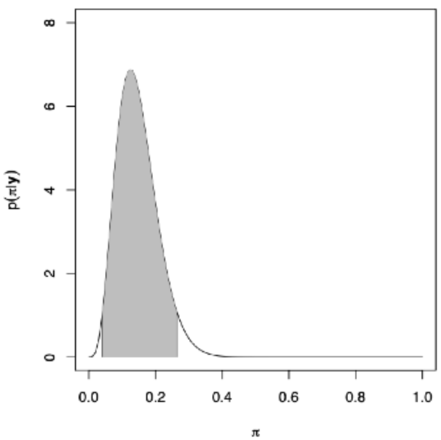

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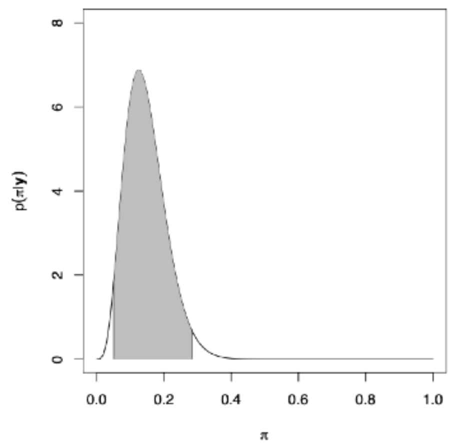

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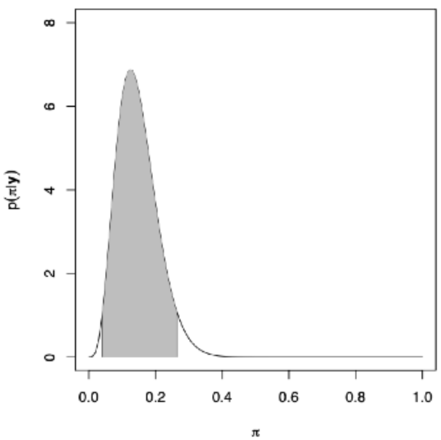

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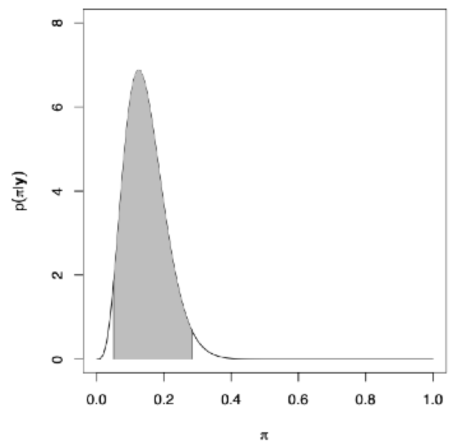

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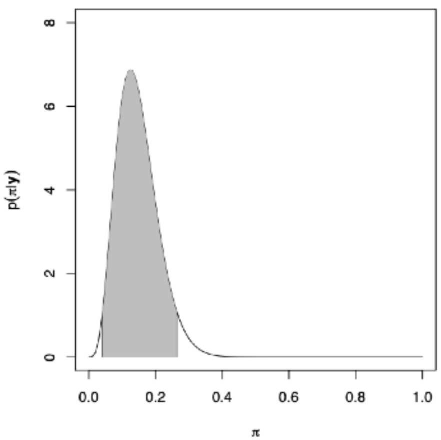

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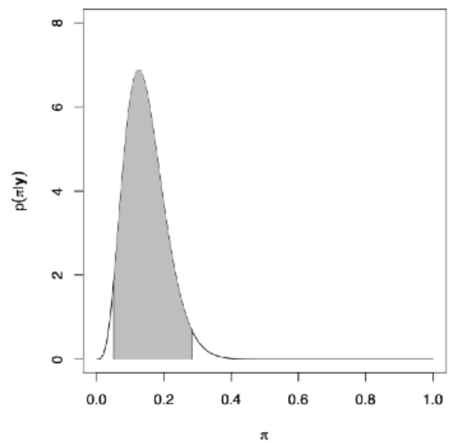

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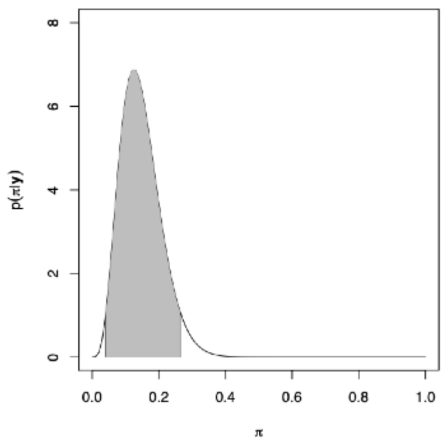

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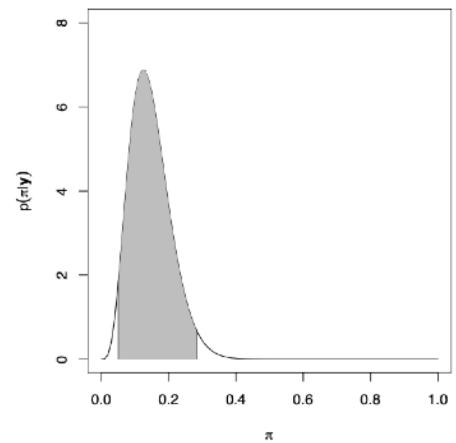

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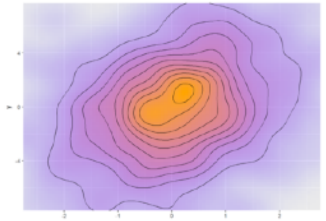

https://www.martinmodrak.cz/2018/02/19/taming-divergences-in-stan-models/

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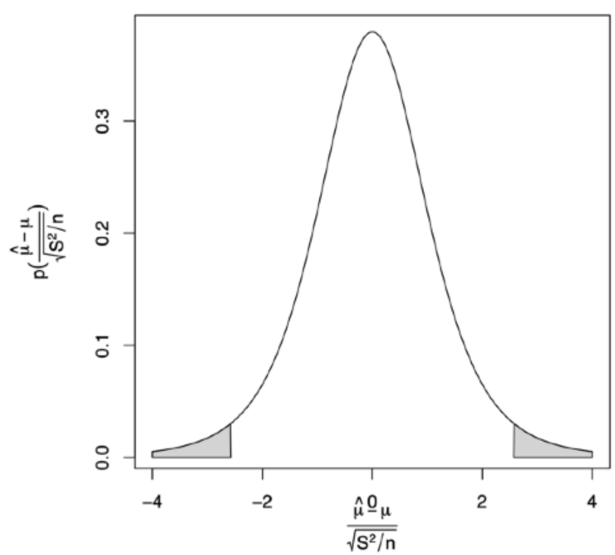

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Bayes Factor: $\dfrac{P(\boldsymbol{y}|H)}{P(\boldsymbol{y}|H')}$

Interpreting Bayes Factors

$$K = \frac{P(\boldsymbol{y}|H)}{P(\boldsymbol{y}|H')}$$

$\log_{10} K$	K	Strength of evidence
0 to 1/2	1 to 3.2	Not worth more than a bare mention
1/2 to 1	3.2 to 10	Substantial
1 to 2	10 to 100	Strong
> 2	> 100	Decisive

(Kass & Raftery, 1995; table from https://en.wikipedia.org/wiki/Bayes_factor)

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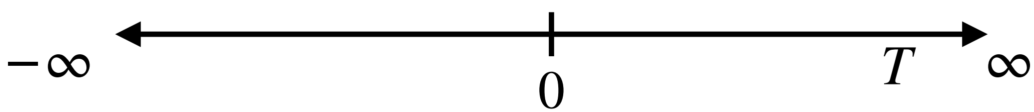

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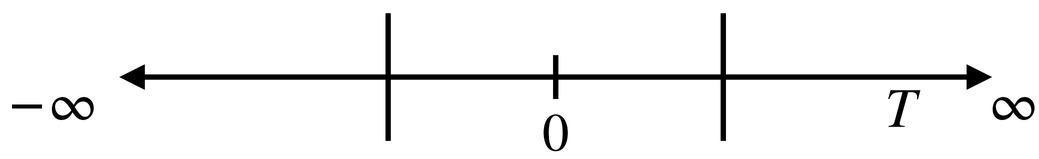

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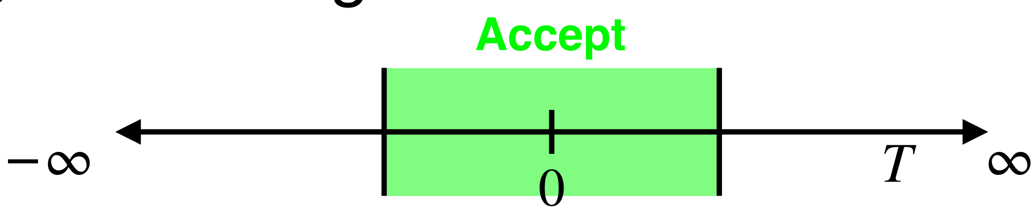

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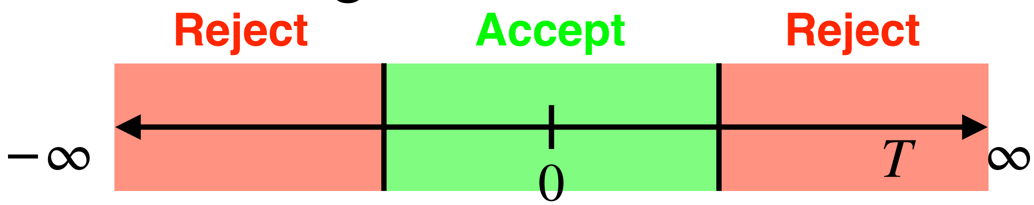

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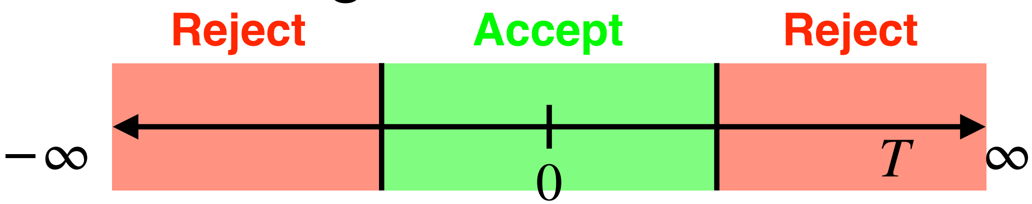

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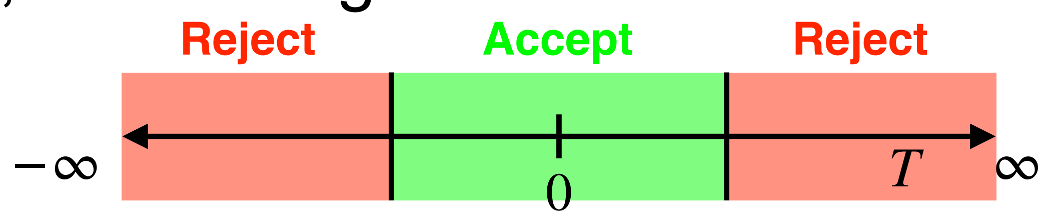

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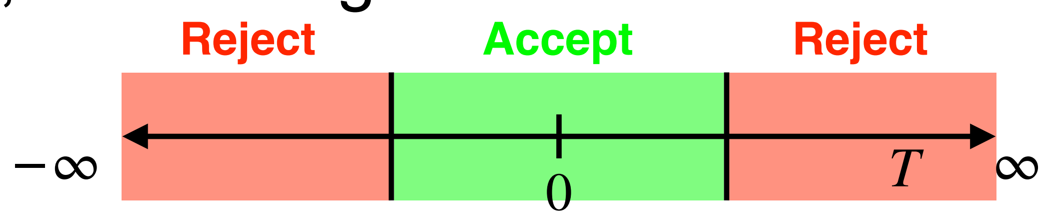

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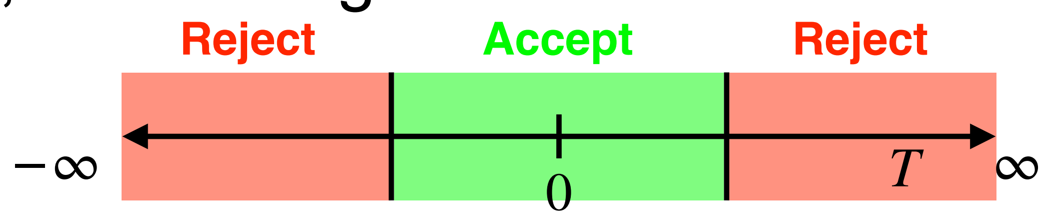

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Power

The **Gaussian**, or **normal**, distribution

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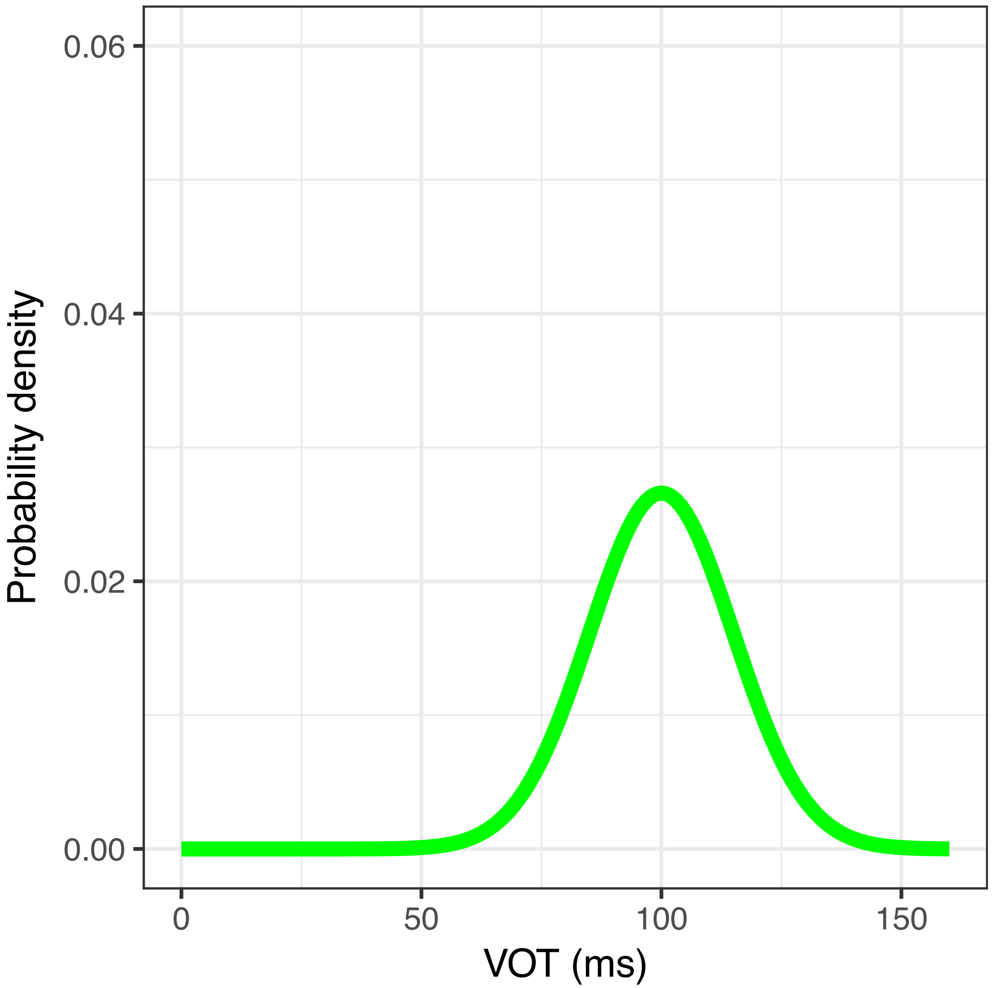

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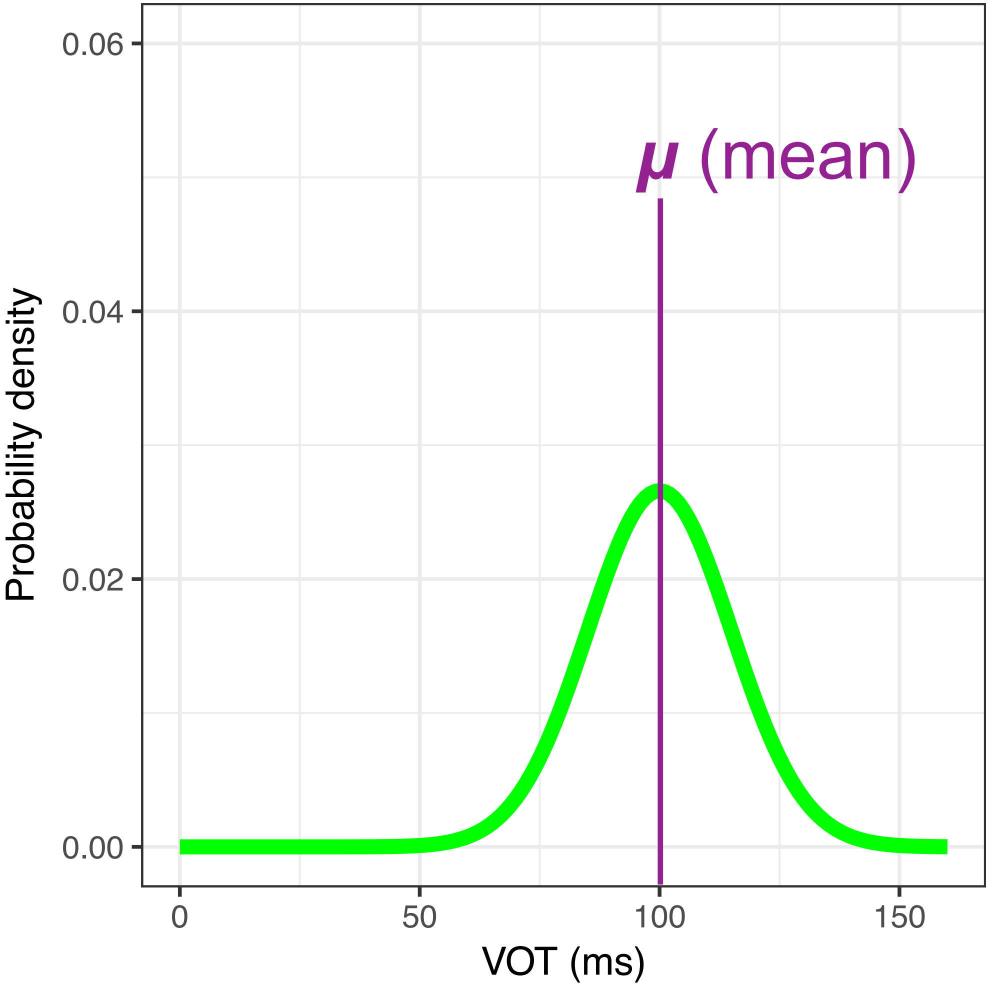

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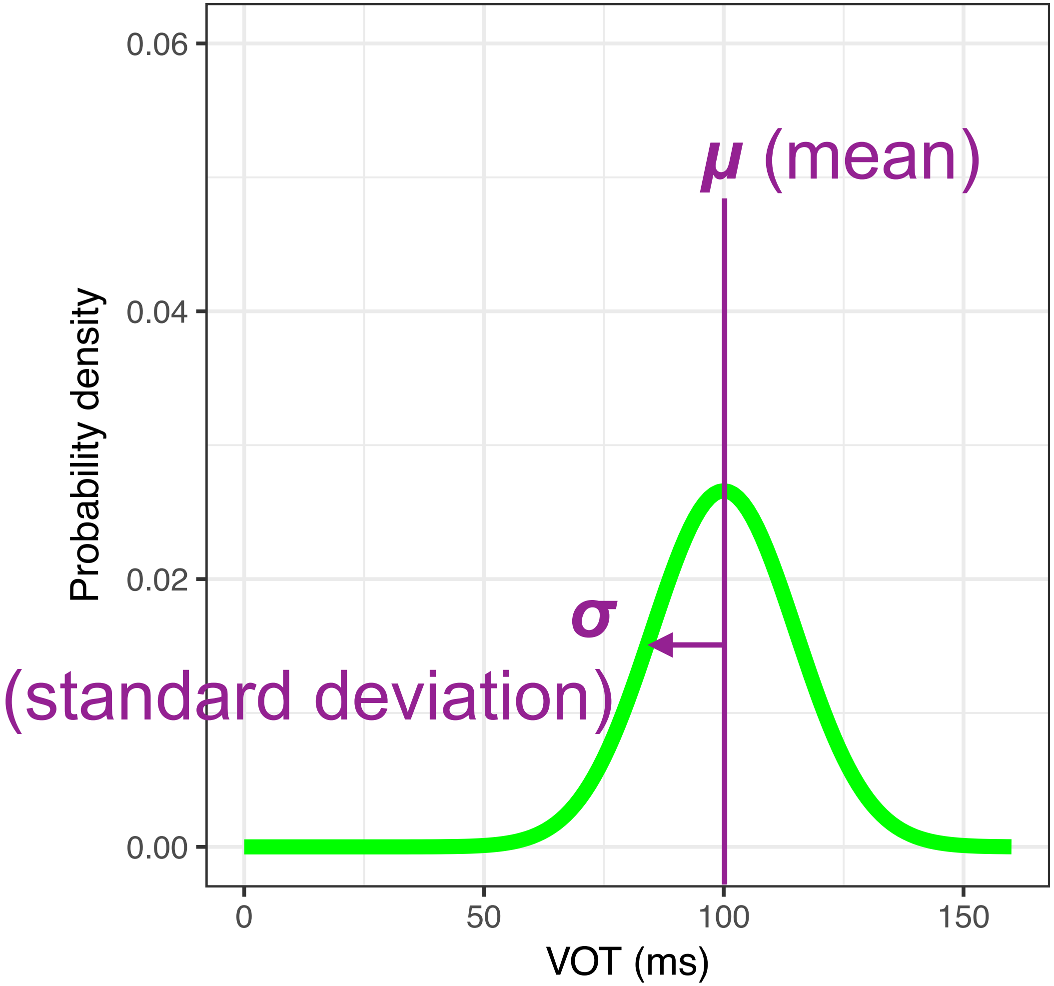

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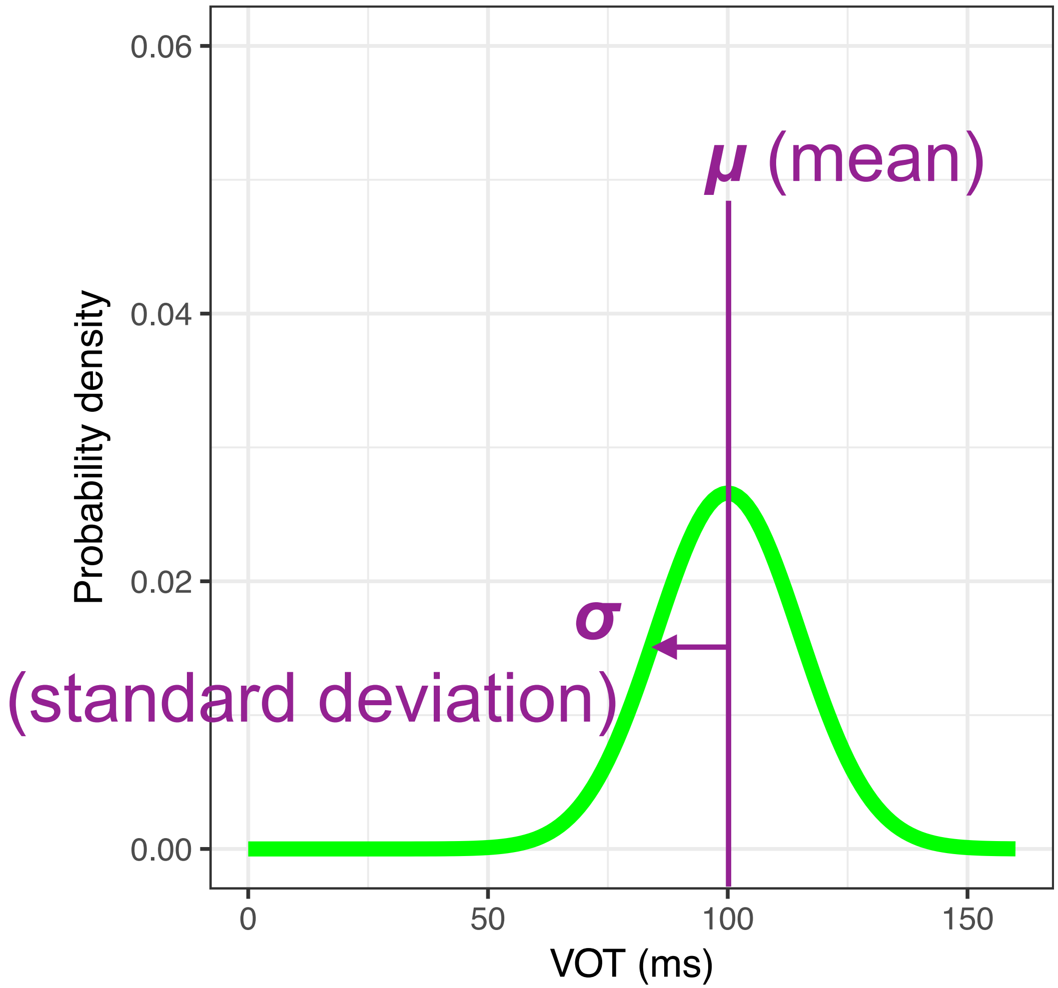

$$p(x \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]$$

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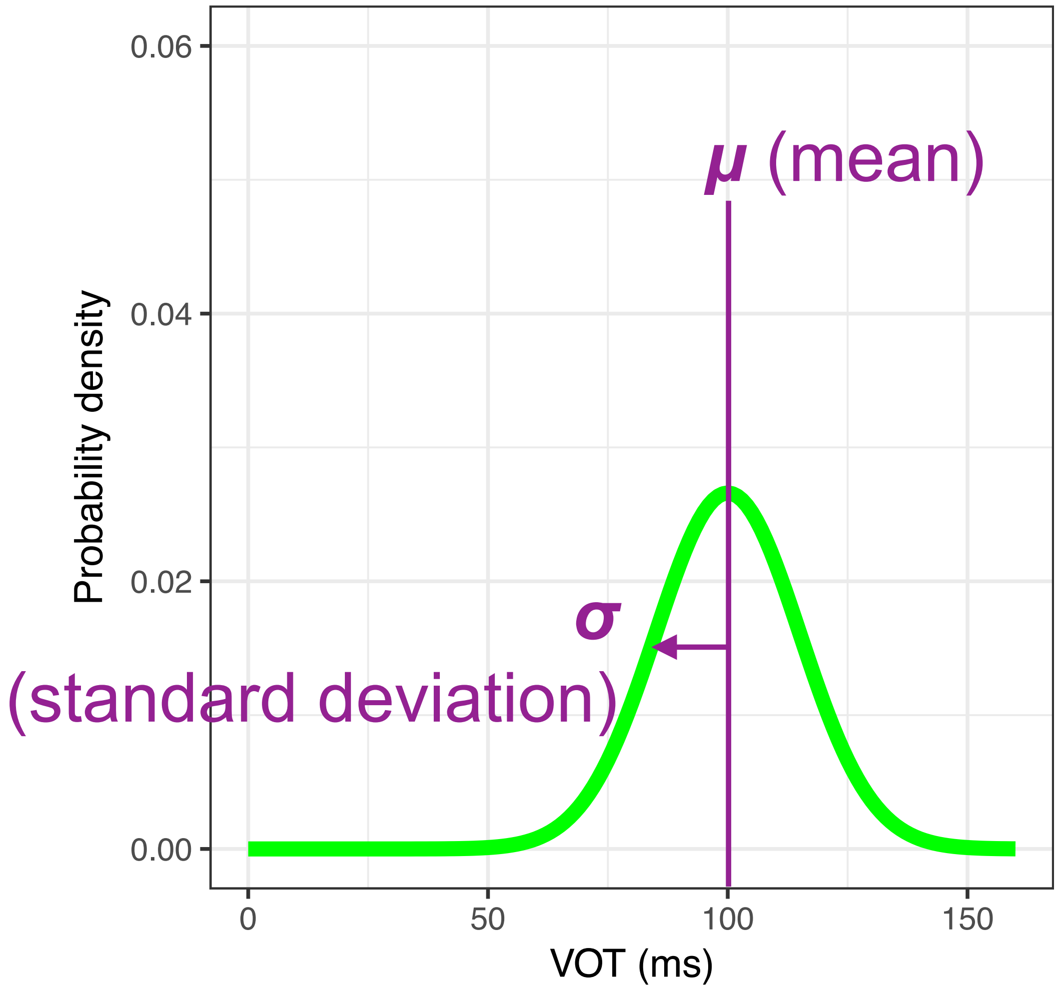

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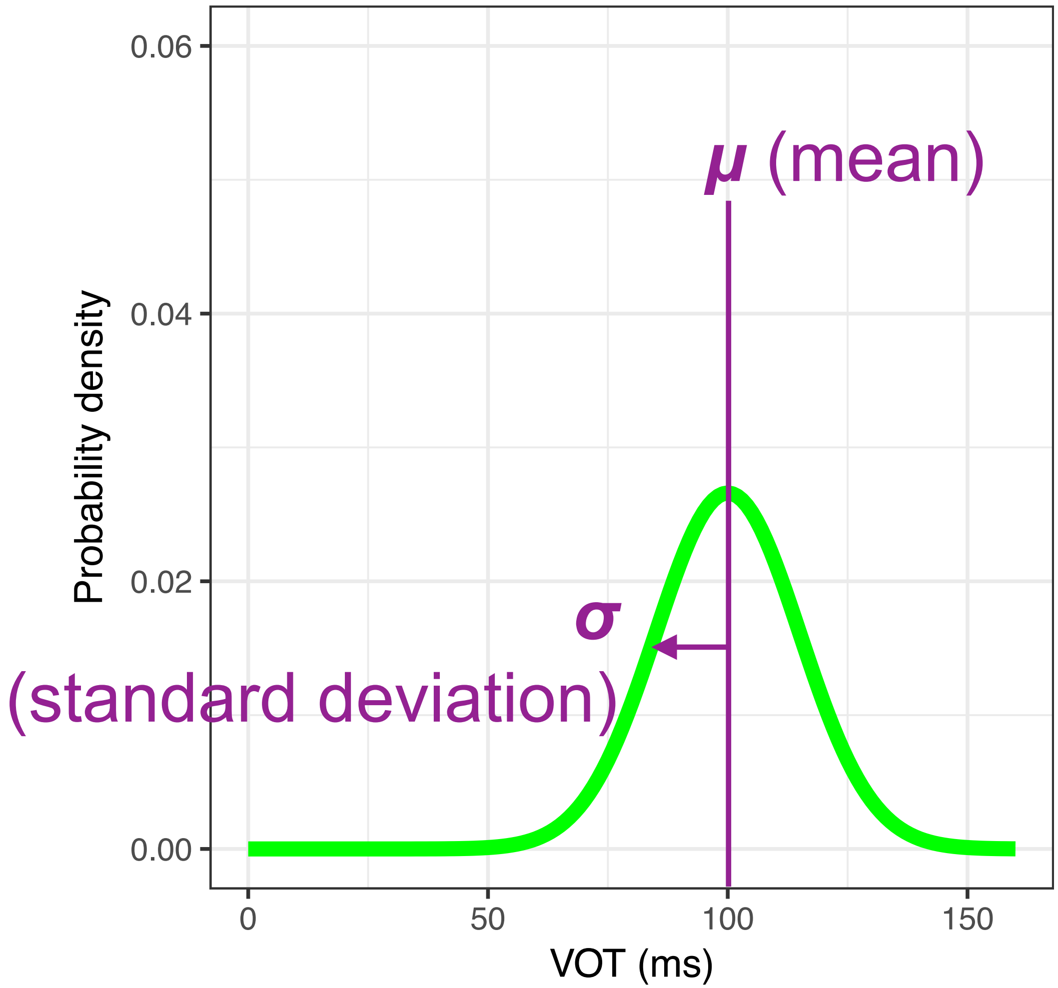

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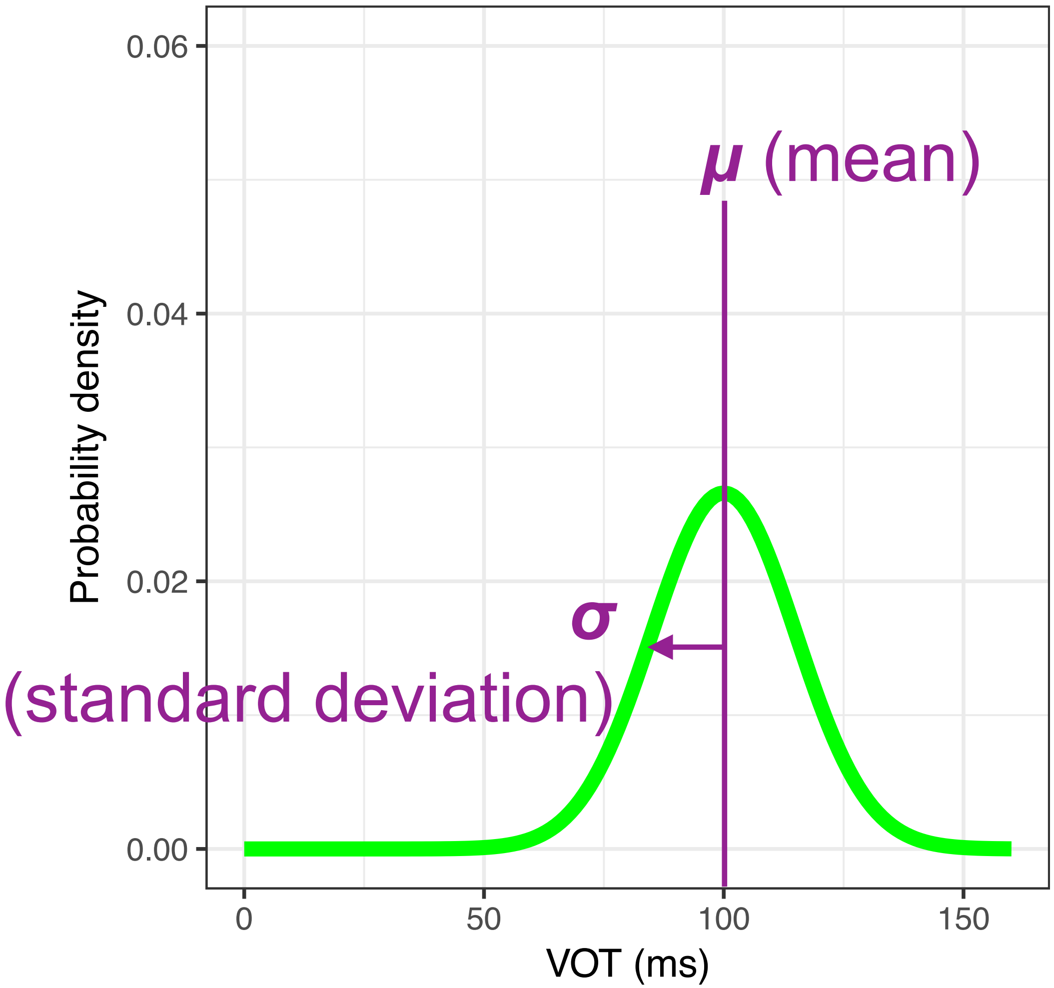

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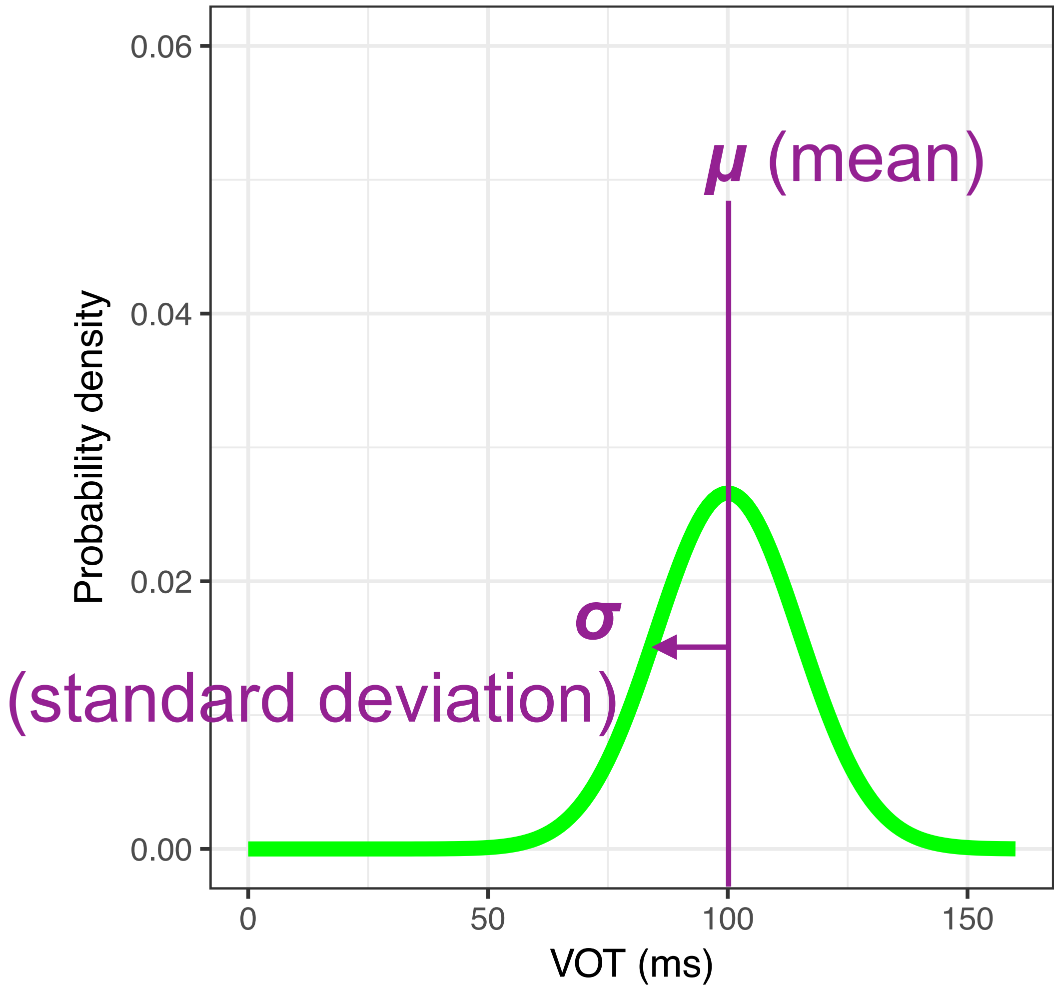

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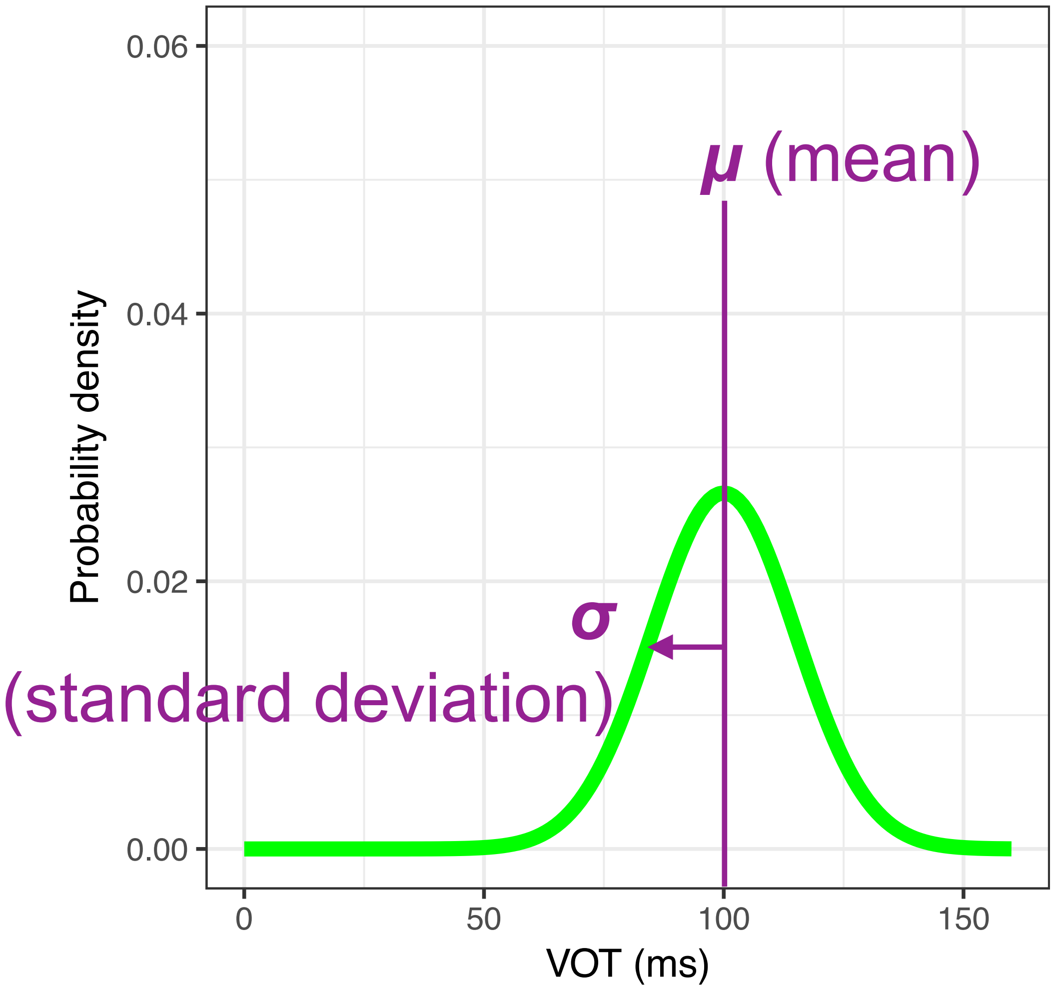

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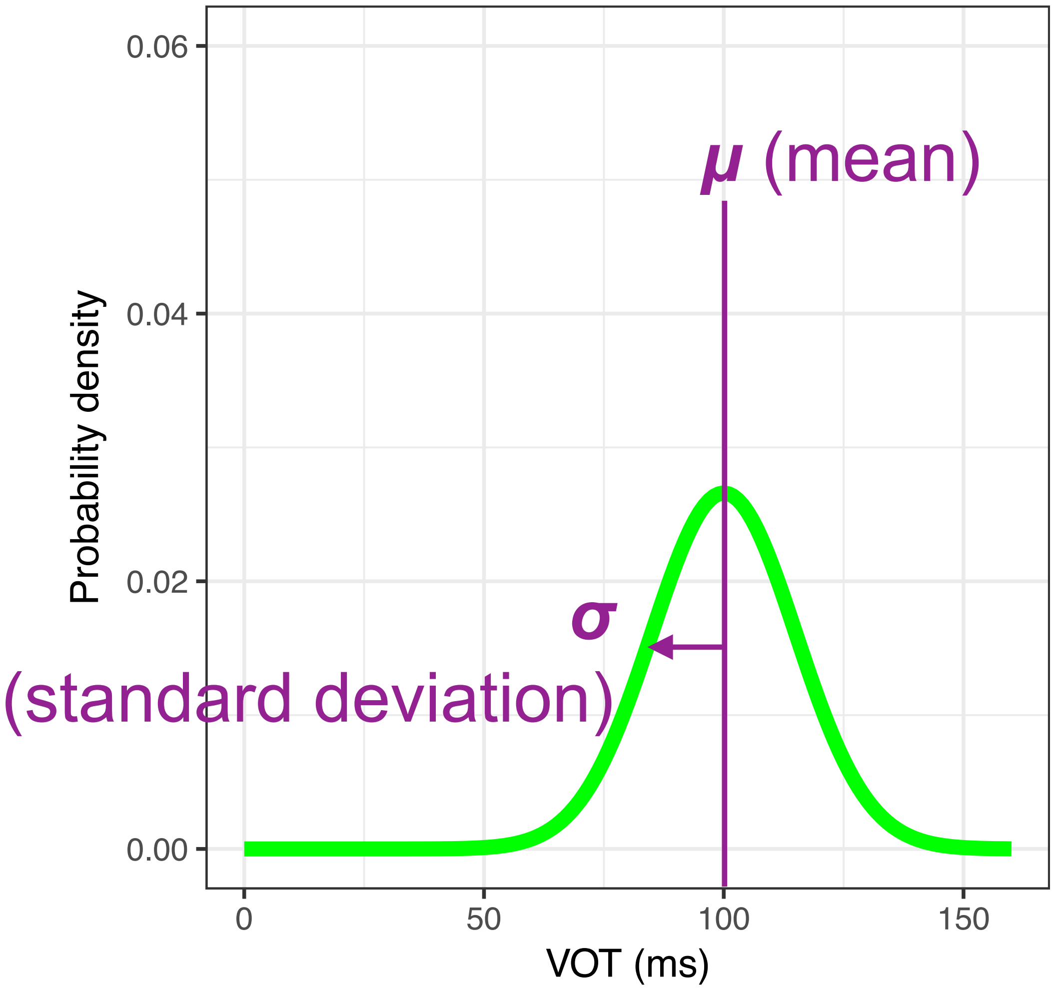

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The *t*-test: three variants

- **One sample (Student's) test:** Does the underlying population mean of a sample differ from zero?
- **Two-sample test (unpaired):** do the underlying population means of two samples differ from one another?
- **Two-sample test (paired):** You have a sample of individuals from the population and take measurements from each member of the sample in two different conditions. Do the underlying population means in the two conditions differ from one another?

William Sealy Gosset, a.k.a. Student

One-sample t-test

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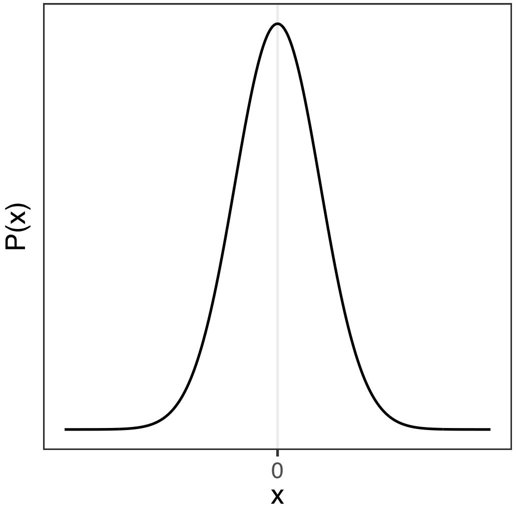

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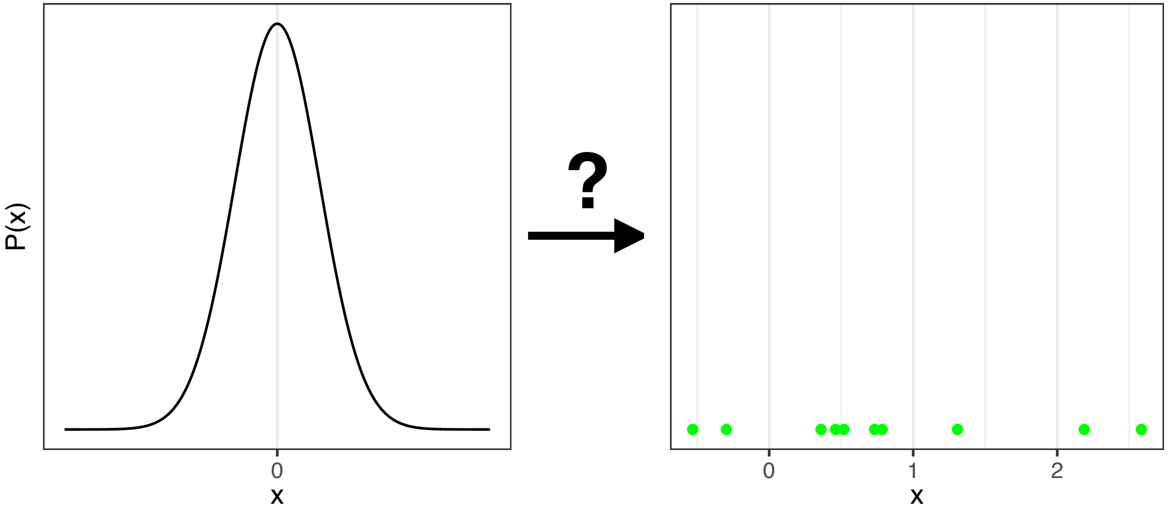

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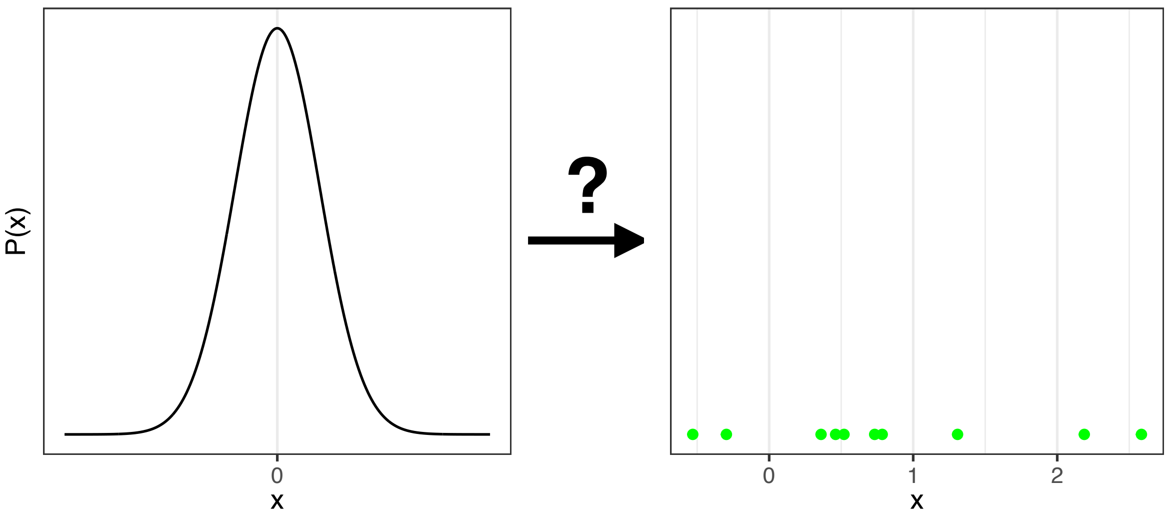

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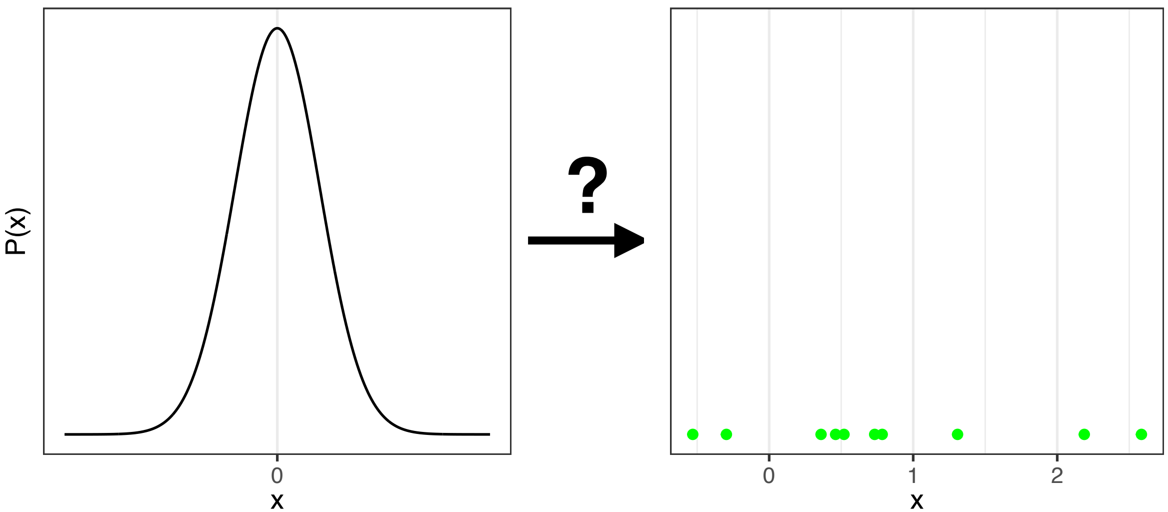

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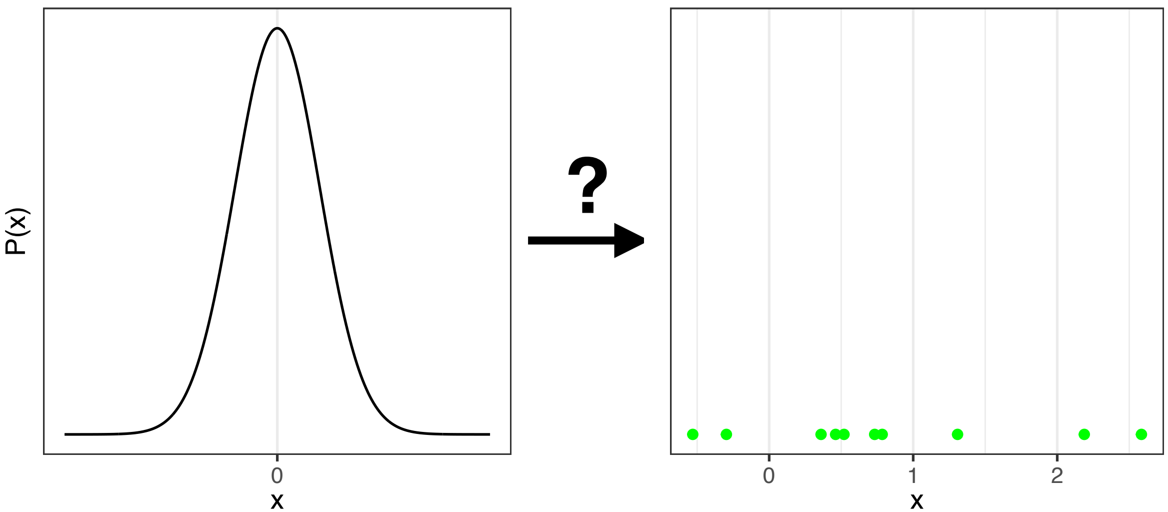

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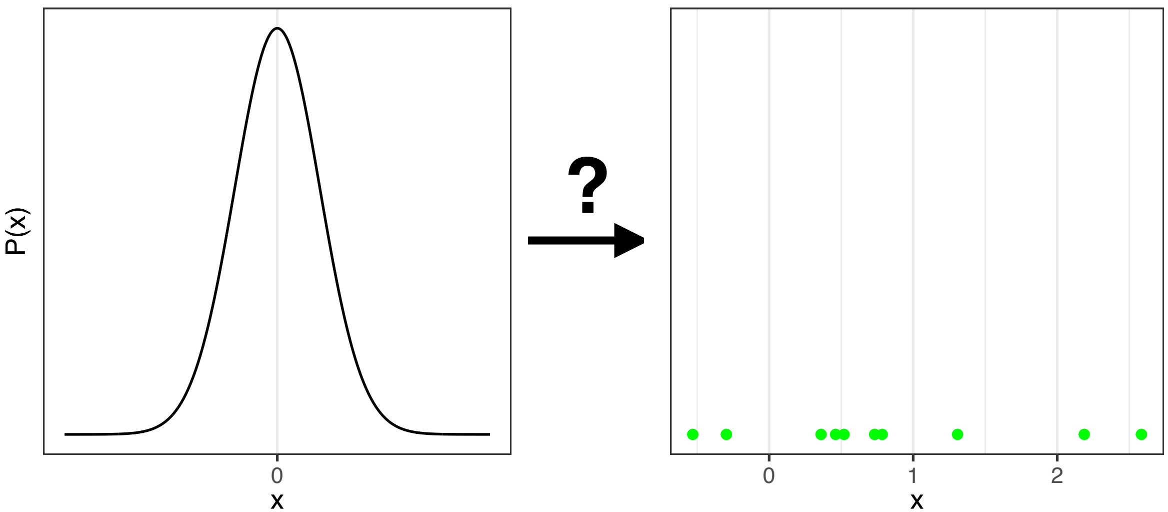

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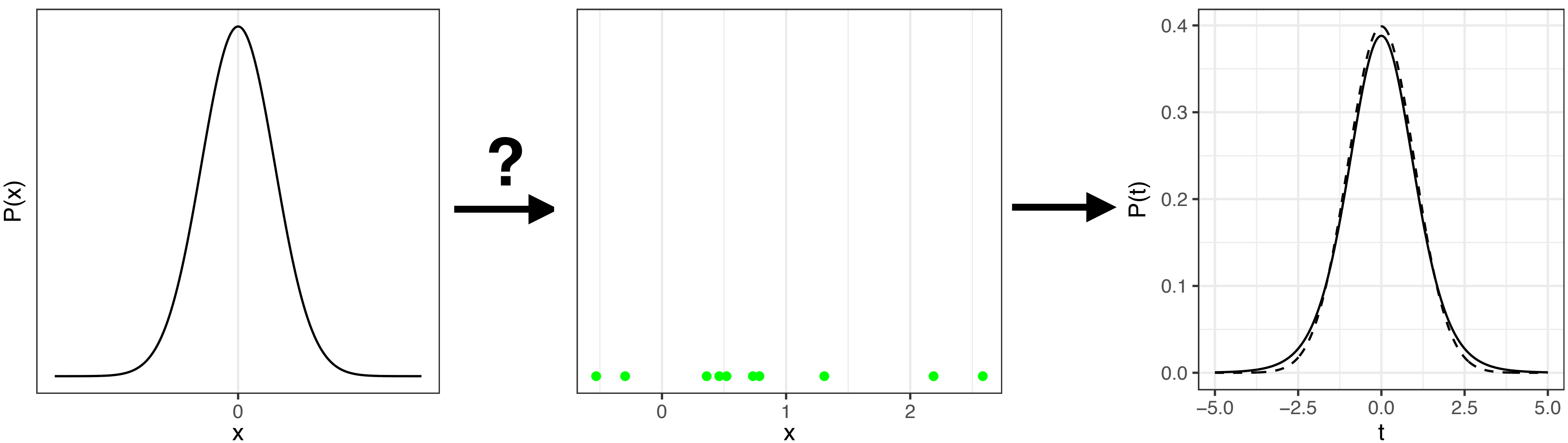

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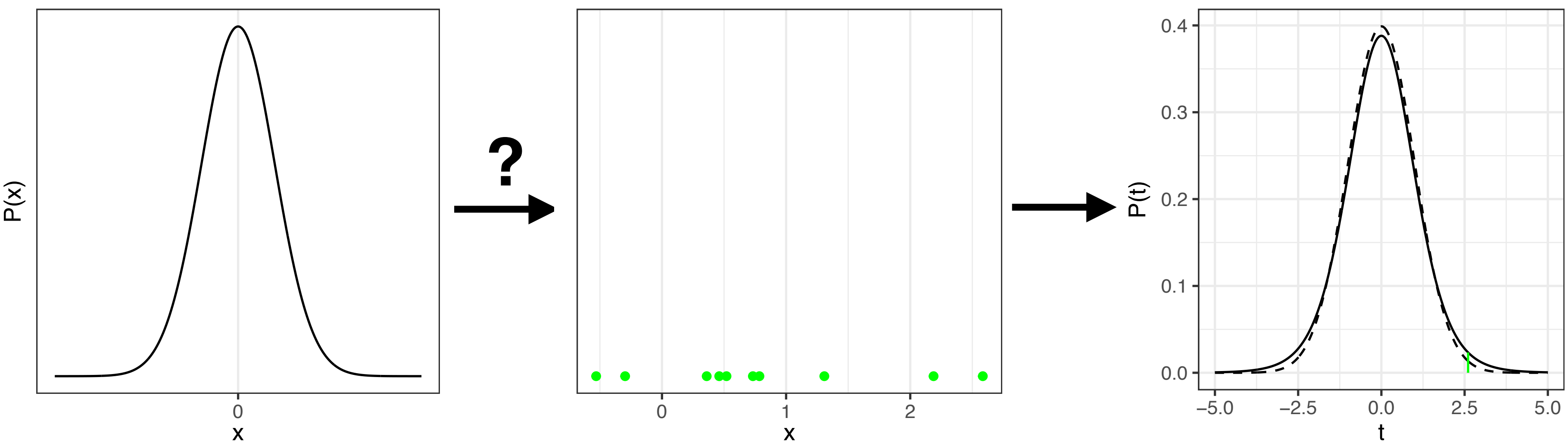

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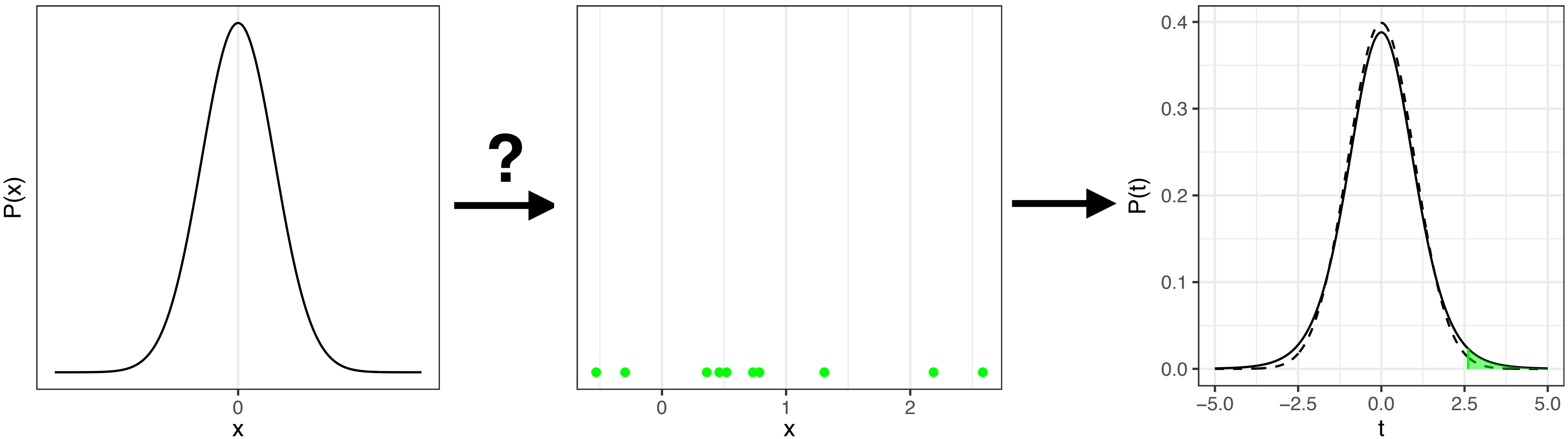

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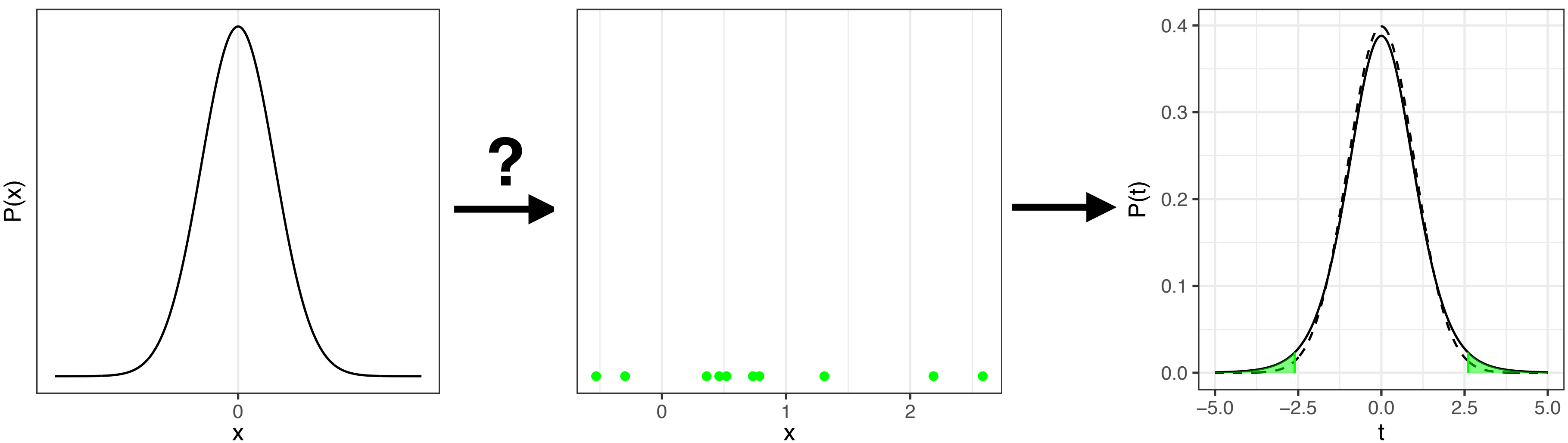

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t-distributed with a complex number of degrees of freedom whose formula can easily be looked up

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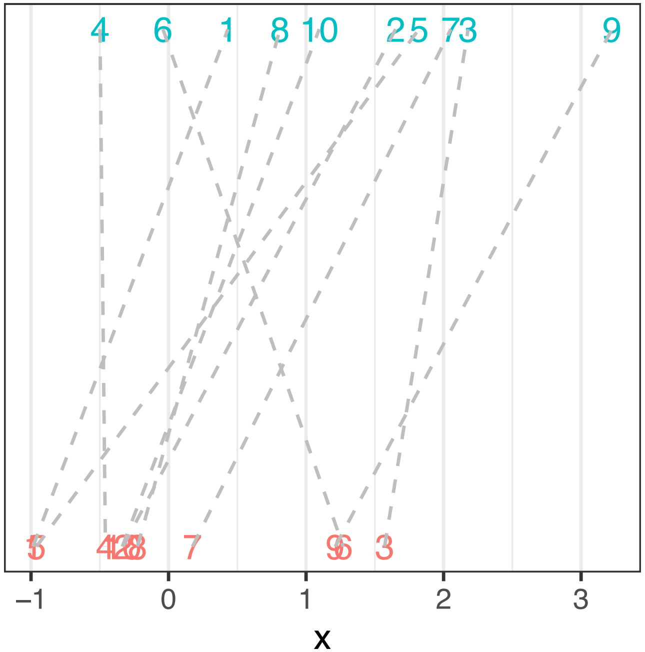

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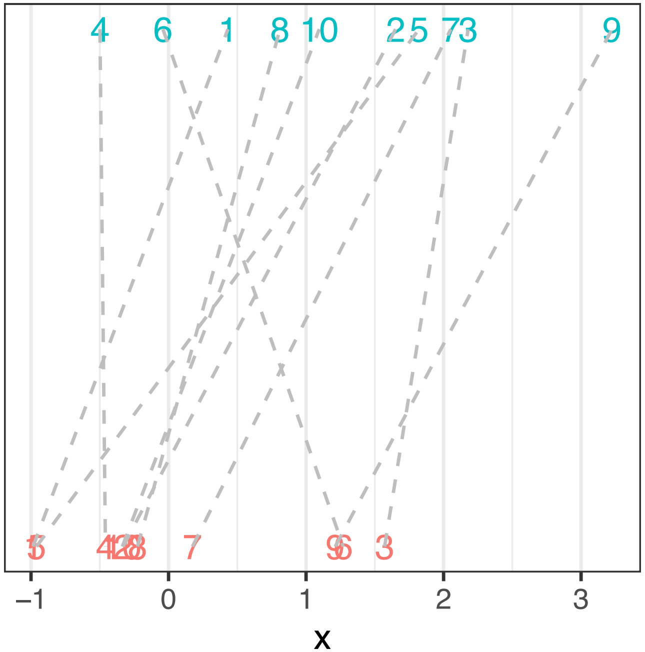

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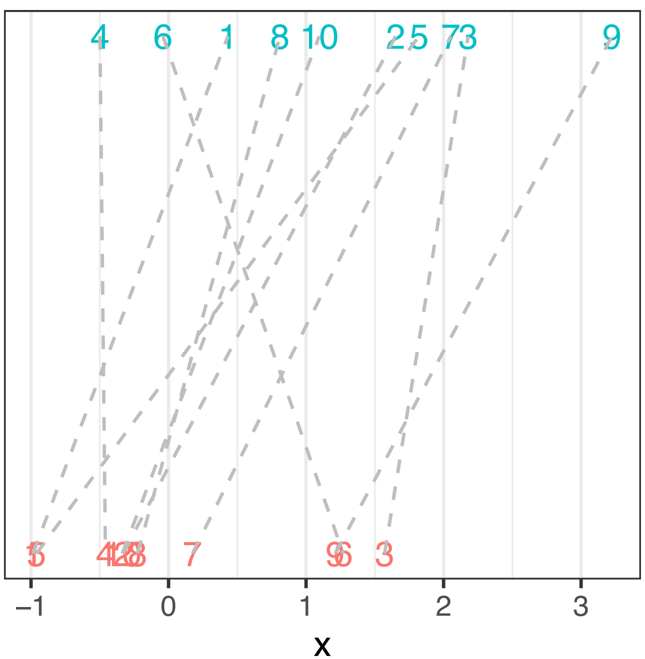
  - (Sufficient condition: paired measurements are **bivariate normal** – a distr. we haven't yet covered)



Paired two-sample t-test

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  - In a sample of **units** from a population; for each unit we have two **measurements** $\langle x_1, x_2 \rangle$ on the same scale
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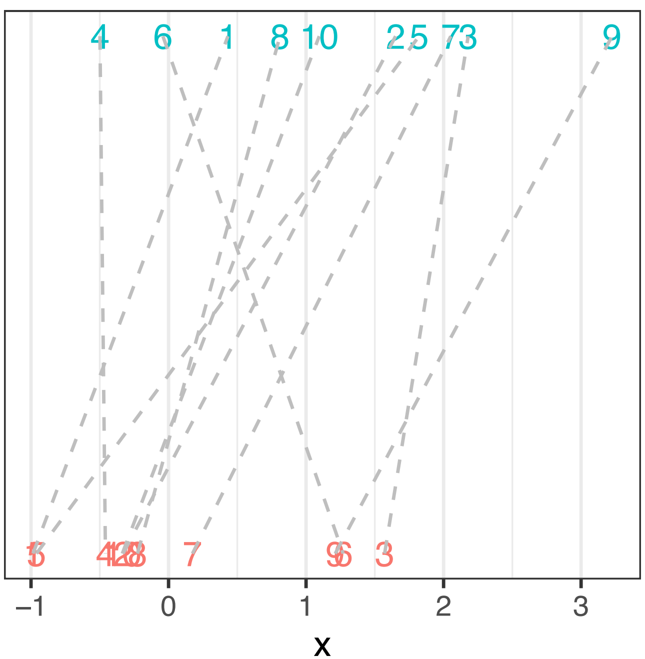

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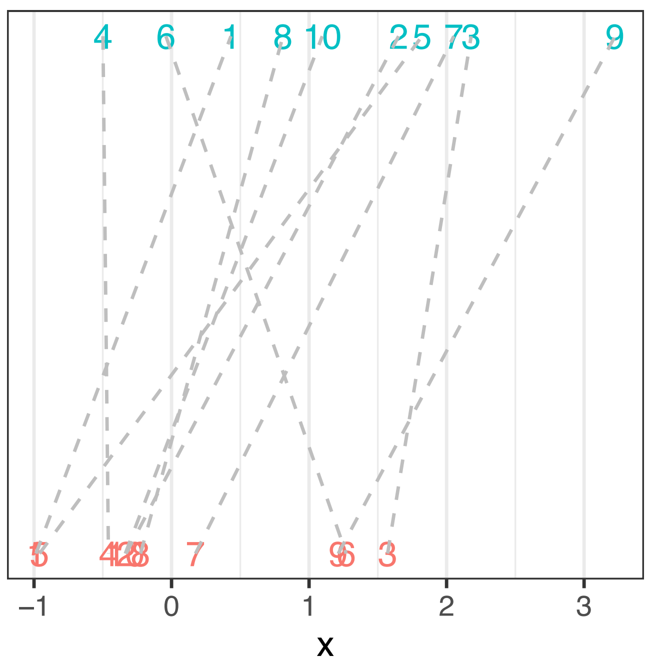

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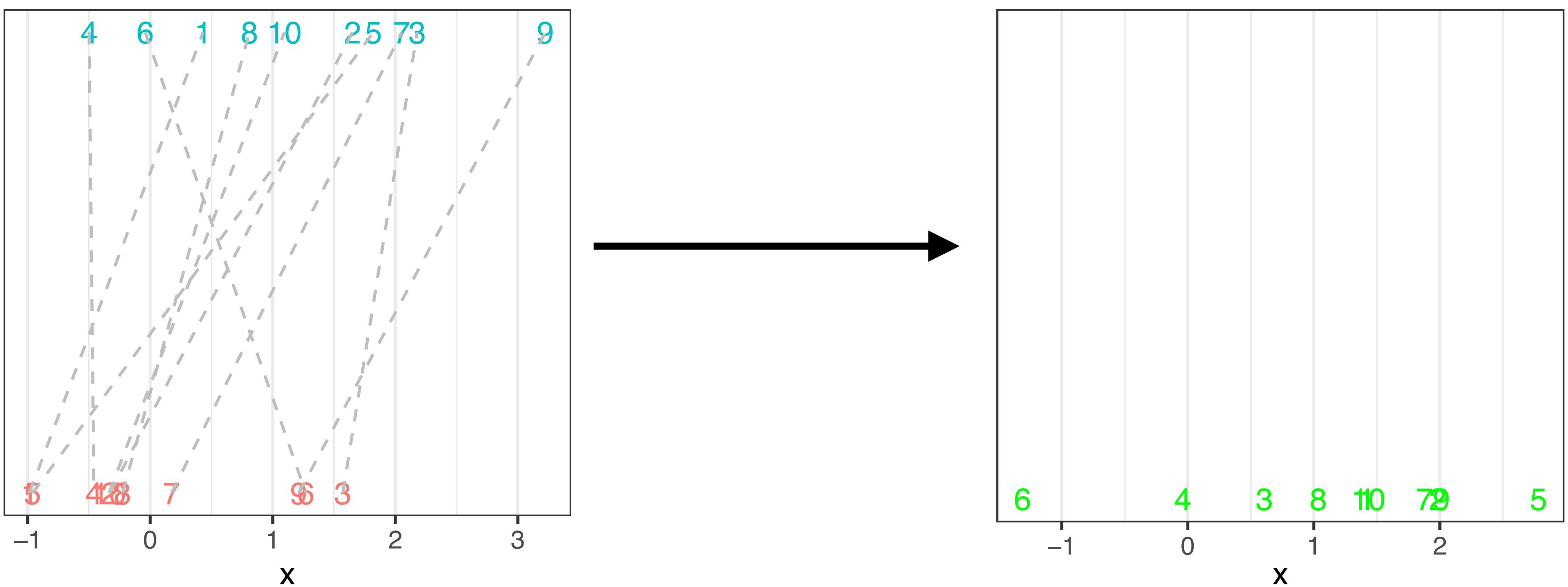

The likelihood ratio test

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- The **likelihood ratio**:

$$\Lambda^* = \frac{\max \mathrm{Lik}_{H_0}(\boldsymbol{y})}{\max \mathrm{Lik}_{H_A}(\boldsymbol{y})}$$

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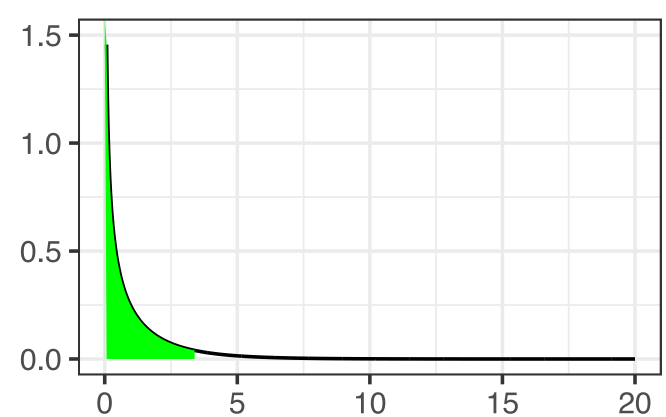

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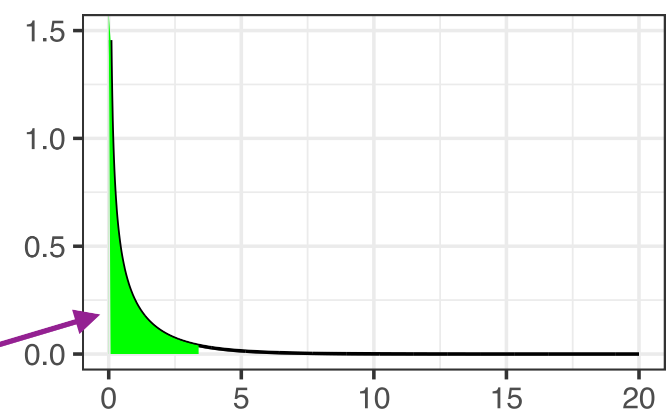

93.5% of probability mass under $G^2 \rightarrow p = 0.065$

Simulation and approximate computation

Simulation and approximate computation

- All the statistical analysis that I've shown you so far has involved **exact** computation using **analytic expressions**
- This is facilitated by:
  - **Strong assumptions** regarding the data generating process (e.g., iid normal data for t-tests); and/or
  - **Conjugate priors** for Bayesian inference (e.g., Beta prior for Bernoulli/binomial data)
- But often, exact computation is not possible
- Solution: use more computationally intensive methods that don't rely on these strong assumptions. Examples:
  - Bootstrapped confidence intervals
  - Nonparametric statistical tests
  - Monte Carlo methods **(today)**

Monte Carlo methods, or "probabilistic simulation"

- Generally speaking:
  1. Define a domain of possible inputs
  2. Generate n iid random inputs from a probability distribution on the domain
  3. Perform a deterministic computation on each randomly generated input
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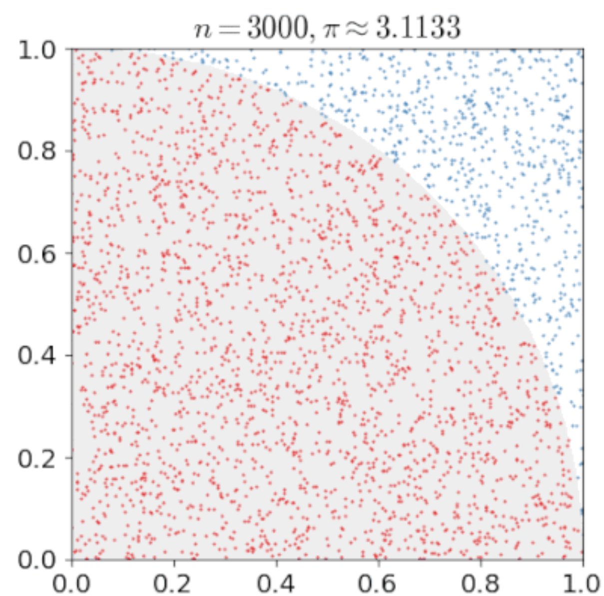
- As n grows larger, the simulated result approaches the true value



https://en.wikipedia.org/wiki/Monte_Carlo_method

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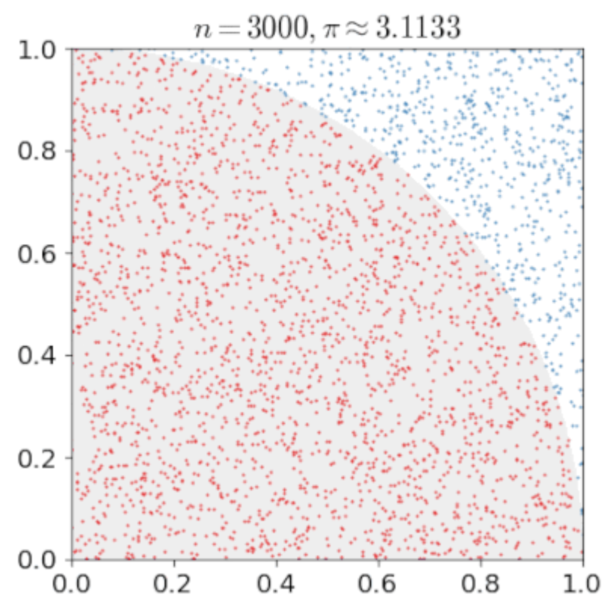

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Simple example of Monte Carlo

- Suppose I want to do a two-sample *t*-test but my data aren't normally distributed

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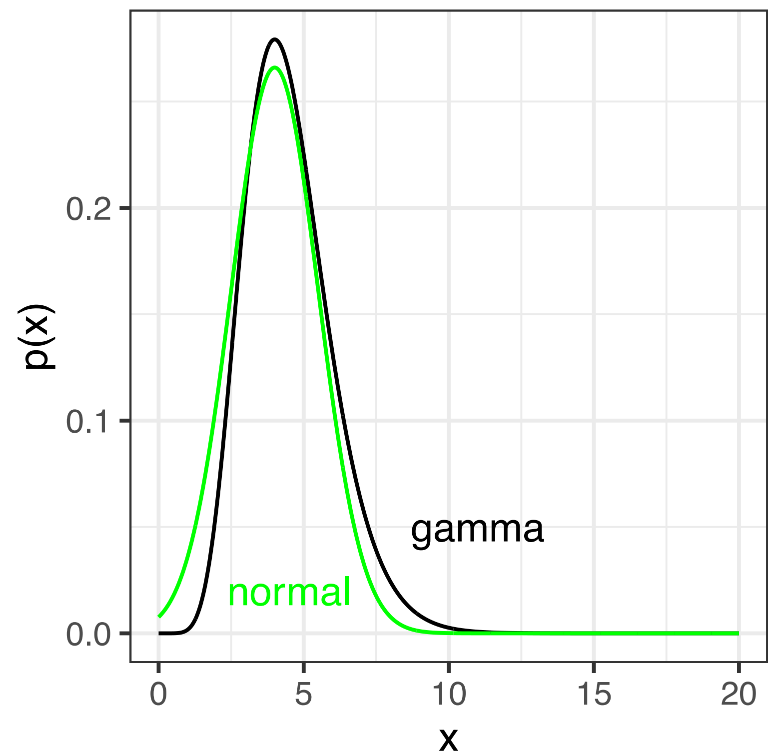
Student's *t*-test

From Wikipedia, the free encyclopedia

The ***t*-test** is any statistical hypothesis test in which the test statistic follows a Student's *t*-distribution under the null hypothesis.

A *t*-test is the most commonly applied when the test statistic would follow a normal distribution if the value of a scaling term in the test statistic were known. When the scaling term is unknown and is replaced by an estimate based on the data, the test statistics (under certain conditions) follow a Student's *t* distribution. The *t*-test can be used, for example, to determine if the means of two sets of data are significantly different from each other.



- **How bad will this be for my *t*-test????**

Monte Carlo, in action

```r
library(ggplot2)
library(tidyverse)

# Manually compute Student t-statistic
f <- function(seed,N=100,shape=9,scale=0.5) {
  set.seed(seed)
  y1 <- rgamma(N,shape=shape,scale=scale)
  y2 <- rgamma(N,shape=shape,scale=scale)
  s_p <- sqrt( (var(y1) + var(y2)) / 2 )
  t_statistic <- ( mean(y1) - mean(y2) ) / ( s_p*sqrt(2/N ) )
  return(t_statistic)
}

N <- 100
Ts <- sapply(1:10000,f)

t_reference <- tibble(x=seq(-4,4,by=0.01),t=dt(x,df=2*(N-1)))

ggplot(data=tibble(t=Ts),aes(x=t)) +
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Compare against Student's *t* distribution

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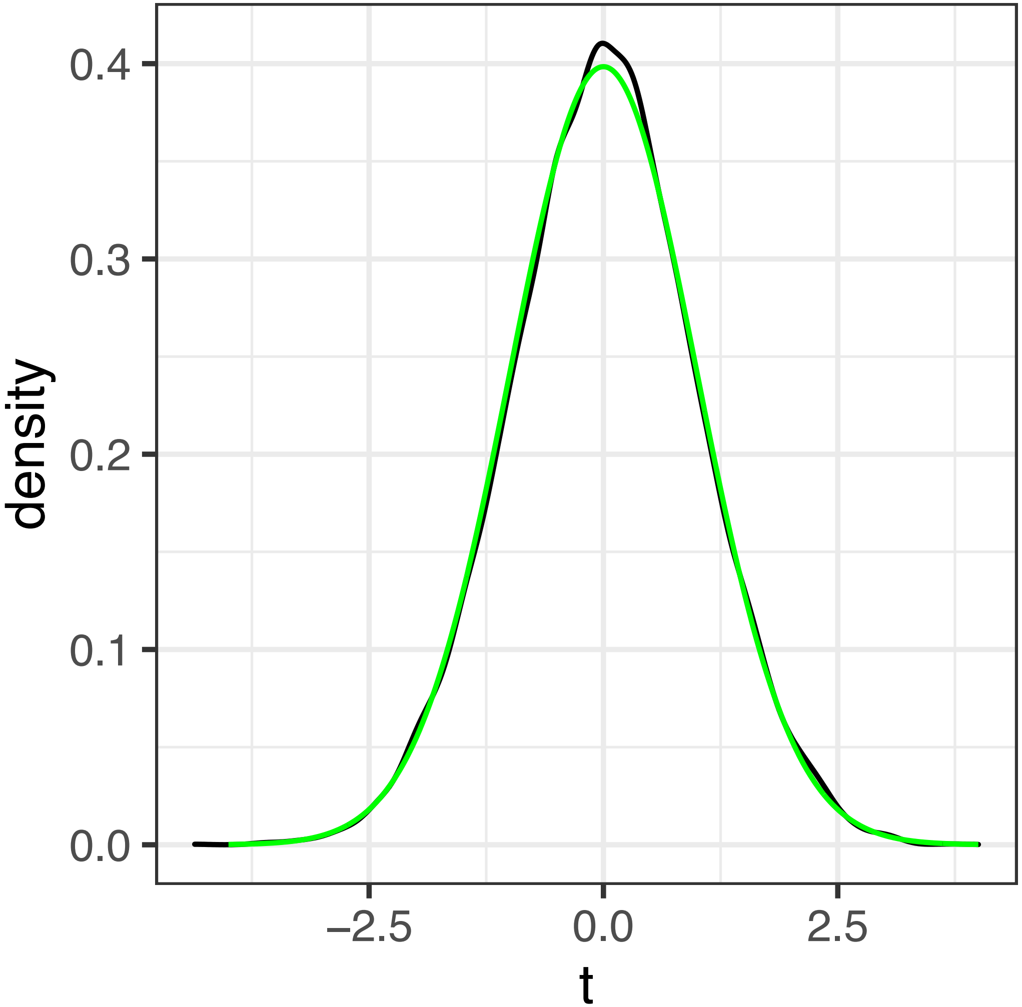

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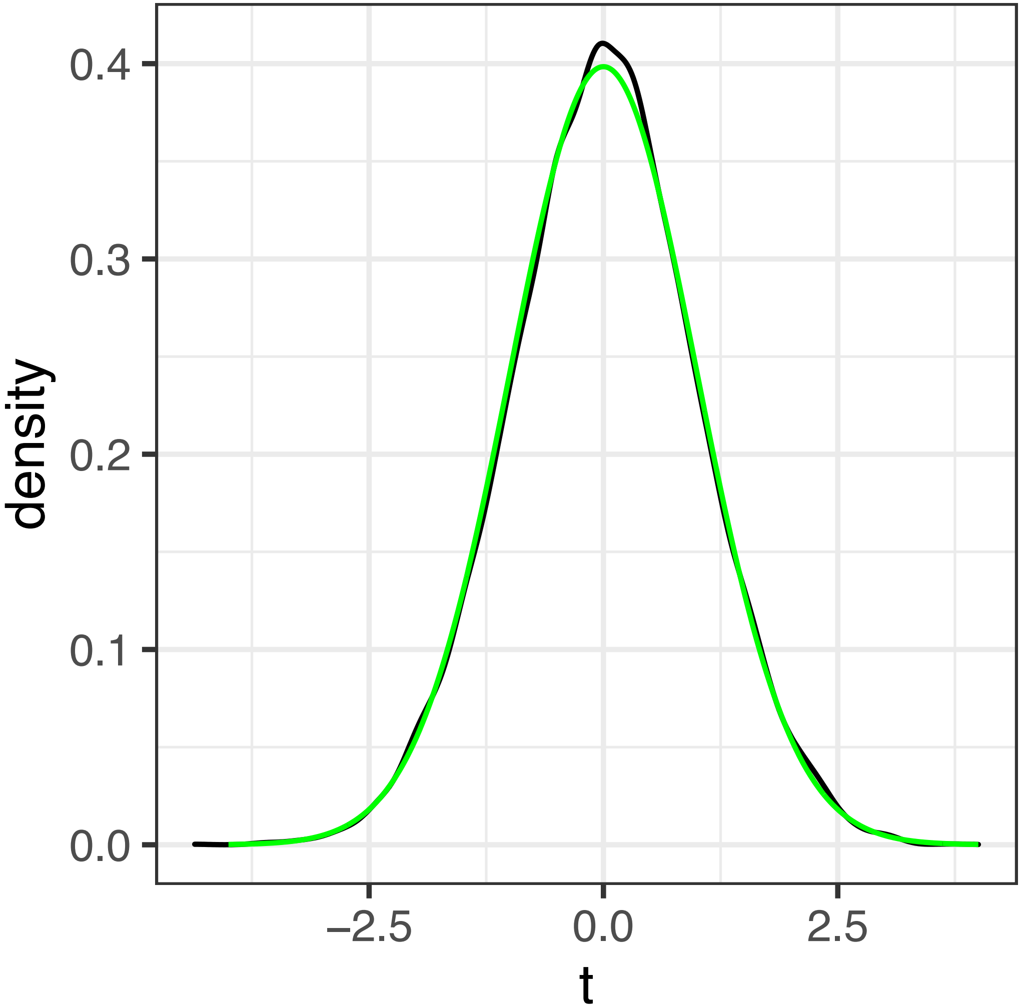

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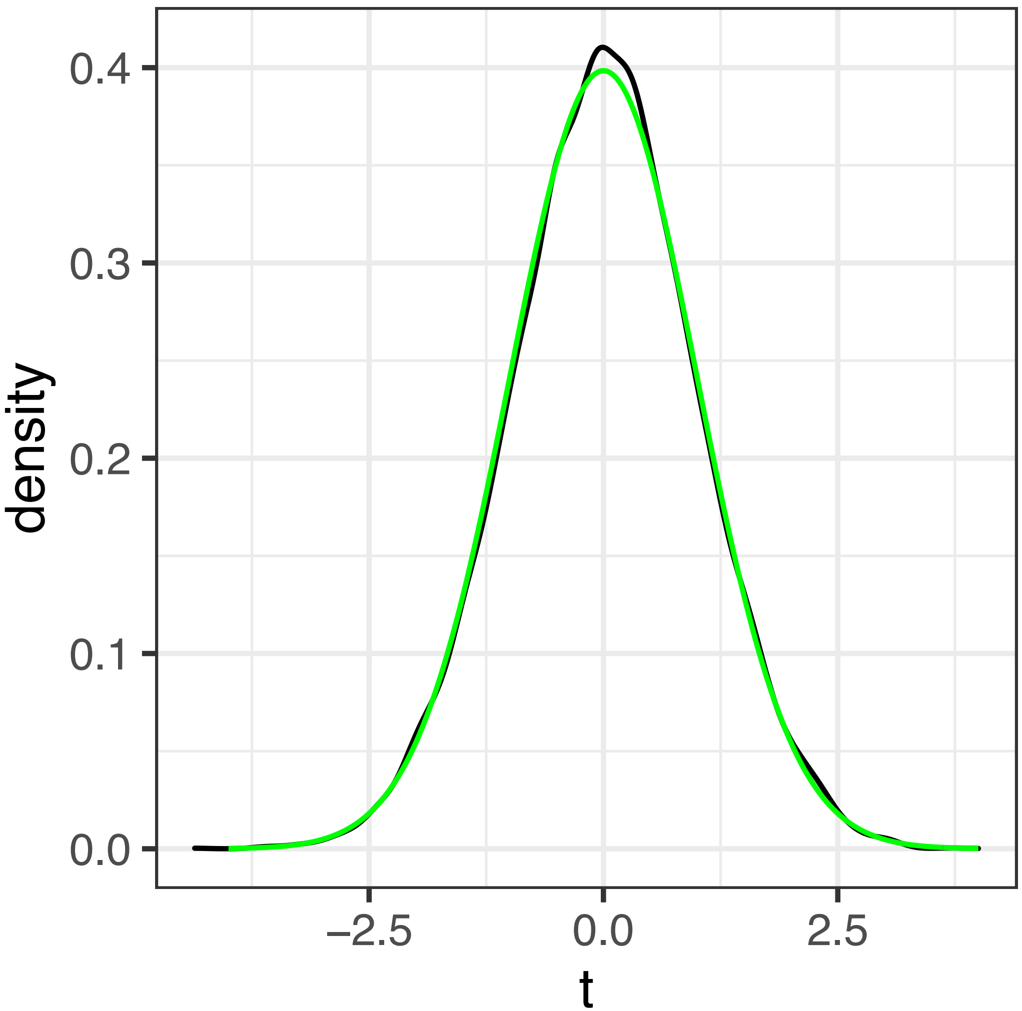

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- This exemplifies what is meant when people say that the *t* test is **robust to deviations from normality**