times—was a lost cause. All of the suggestive proofs, all of the promising clues starting with the quadrature of the lune, turned out to be illusory. Compass and straightedge alone are inadequate for turning circles into squares.

And what did history have to say about lunes? Our great theorem above showed Hippocrates squaring a particular lune, and he managed to do two other kinds as well. Thus, as of 440 B.C., three types of lunes were known to be quadrable. At this point, progress stopped for over two millennia until, in 1771, the great Leonhard Euler (1707–1783)—who will be the object of our attention in Chapters 9 and 10—found two more kinds of lunes that were squarable. There the matter rested until the twentieth century when N. G. Tschebatorew and A. W. Dorodnow proved that these five are the *only* squarable lunes! All other lunes, such as the one that generated Alexander's harsh criticism cited earlier, share with the circle the impossibility of being squared.

So the final chapter in the story of Hippocrates and his lunes has been written, and it has been a rather perverse story at that. At first, intuition suggested that curved figures could not be squared with compass and straightedge. Hippocrates' lunes turned intuition upside down, and the search was on for quadratures galore. But, in the end, the negative results of Lindemann, Tschebatorew, and Dorodnow showed that intuition had not been so flawed after all. The quadrature of curvilinear figures, far from being the norm, must forever remain the exception.

2 Chapter

Euclid's Proof of the Pythagorean Theorem

(ca. 300 B.C.)

The *Elements* of Euclid

A century and a half passed between Hippocrates and Euclid. During this span, Greek civilization grew and matured, enriched by the writings of Plato and Aristotle, of Aristophanes and Thucydides, even as it underwent the turmoil of the Peloponnesian Wars and the glory of the Greek empire under Alexander the Great. By 300 B.C., Greek culture had spread across the Mediterranean world and beyond. In the West, Greece reigned supreme.

The period from 440 B.C. to 300 B.C. saw a number of individuals contribute significantly to the development of mathematics. Among these were Plato (427–347 B.C.) and Eudoxus (ca. 408–355 B.C.), although only the latter was truly a mathematician.

Plato, the great philosopher of Athens, deserves mention here not so much for the mathematics he created as for the enthusiasm and status he imparted to the subject. As a youth, Plato had studied in Athens under Socrates and is of course our primary source of information about his esteemed teacher. For a number of years Plato roamed the world, meeting the great thinkers and formulating his own philosophical positions. In 387 B.C., he returned to his native Athens and founded the Academy. Devoted to learning and contemplation, the Academy attracted talented scholars from near and far, and under Plato's guidance it became the intellectual center of the classical world.

Of the many subjects studied at the Academy, none was more highly regarded than mathematics. The subject certainly appealed to Plato's sense of beauty and order and represented an abstract, ideal world unsullied by the humdrum demands of day-to-day existence. Moreover, Plato considered mathematics to be the perfect training for the mind, its logical rigor demanding the ultimate in concentration, cleverness, and care. Legend has it that across the arched entryway to his prestigious Academy were the words "Let no man ignorant of geometry enter here." Explicit sexism notwithstanding, this motto reflected the view that only those who had first demonstrated a mathematical maturity were capable of facing the intellectual challenge of the Academy. We might say that Plato regarded geometry as the ideal entrance requirement, the Scholastic Aptitude Test of his day.

Although very little original mathematics is now attributed to Plato, the Academy produced many capable mathematicians and one indisputably great one, Eudoxus of Cnidos. Eudoxus came to Athens about the time the Academy was being created and attended the lectures of Plato himself. Eudoxus' poverty forced him to live in Piraeus, on the outskirts of Athens, and make the daily round-trip journey to and from the Academy, thus distinguishing him as one of the first commuters (although we are unsure whether he had to pay out-of-city-state tuition). Later in his career, he traveled to Egypt and returned to his native Cnidos, all the while assimilating the discoveries of science and constantly extending its frontiers. Particularly interested in astronomy, Eudoxus devised complex explanations of lunar and planetary motion whose influence was felt until the Copernican revolution in the sixteenth century. Never willing to accept divine or mystical explanations for natural phenomena, he instead tried to subject them to observation and rational analysis. Thus, Sir Thomas Heath said of Eudoxus, "He was a *man of science* if ever there was one."

In mathematics, Eudoxus is remembered for two major contributions. One was his theory of proportion, and the other his method of exhaustion. The former provided a logical victory over the impasse created by the Pythagoreans' discovery of incommensurable magnitudes. This impasse was especially apparent in geometric theorems about similar triangles, theorems that had initially been proved under the assumption that *any* two magnitudes were commensurable. When this assumption was destroyed, so too were the existing proofs of some of geometry's foremost theorems. What resulted is sometimes called the

"logical scandal" of Greek geometry. That is, while people still believed that the theorems were correct as stated, they no longer were in possession of sound proofs with which to support this belief. It was Eudoxus who developed a valid theory of proportions and thereby supplied the long-sought proofs. His theory, which must have brought a collective sigh of relief from the Greek mathematical world, is now most readily found in Book V of Euclid's *Elements*.

Eudoxus' other great contribution, the method of exhaustion, found immediate application in the determination of areas and volumes of the more sophisticated geometric figures. The general strategy was to approach an irregular figure by means of a succession of known elementary ones, each providing a better approximation than its predecessor. We can think, for instance, of a circle as being a totally curvilinear, and thus quite intractable, plane figure. But, if we inscribe within it a square, and then double the number of sides of the square to get an octagon, and then again double the number of sides to get a 16-gon, and so on, we will find these relatively simple polygons ever more closely approximating the circle itself. In Eudoxean terms, the polygons are "exhausting" the circle from within.

This process is, in fact, precisely how Archimedes determined the area of a circle, as we shall see in the great theorem of Chapter 4. It is to Eudoxus that he owed this fundamental logical tool. In addition, Archimedes credited Eudoxus with using the method of exhaustion to prove that the volume of "any cone is one third part of the cylinder which has the same base with the cone and equal height," a theorem that is by no means trivial. The reader familiar with higher mathematics will recognize in the method of exhaustion the geometric forerunner of the modern notion of "limit," which in turn lies at the heart of the calculus. Eudoxus' contribution was a significant one, and he is usually regarded as being the finest mathematician of antiquity next to the unsurpassed Archimedes himself.

It was during the latter third of the fourth century B.C. that Alexander the Great emerged from Macedonia and set out to conquer the world. His conquests carried him to Egypt where, in 332 B.C., he established the city of Alexandria at the mouth of the Nile River. This city grew rapidly, reportedly reaching a population of half a million in the next three decades. Of particular importance was the formation of the great Alexandrian Library that soon supplanted the Academy as the world's foremost center of scholarship. At one point, the facility had over 600,000 papyrus rolls, a collection far more complete and astounding than anything the world had ever seen. Indeed, Alexandria would remain the intellectual focus of the Mediterranean world through the Greek and Roman periods until its final destruction in A.D. 641 at the hands of the Arabs.

Among the scholars attracted to Alexandria around 300 B.C. was a man named Euclid, who came to set up a school of mathematics. We know very little about his life either before or after his arrival on the African coast, but it appears that he received his training at the Academy from the followers of Plato. Be that as it may, Euclid's influence was so profound that virtually all subsequent Greek mathematicians had some connection or other with the Alexandrian School.

What Euclid did that established him as one of the greatest names in mathematics history was to write the *Elements*. This work had a profound impact on Western thought as it was studied, analyzed, and edited for century upon century, down to modern times. It has been said that of all books from Western civilization, only the Bible has received more intense scrutiny than Euclid's *Elements*.

The highly acclaimed *Elements* was simply a huge collection—divided into 13 books—of 465 propositions from plane and solid geometry and from number theory. Today, it is generally agreed that relatively few of these theorems were of Euclid's own invention. Rather, from the known body of Greek mathematics, he created a superbly organized treatise that was so successful and so revered that it thoroughly obliterated all preceding works of its kind. Euclid's text soon became the standard. Consequently, a mathematician's reference to I.47 can only mean the 47th proposition of the first book of the *Elements*; there is no more need to *say* that we are talking about the *Elements* than there is to specify that I Kings 7:23 is referring to the Bible.

Actually the parallel is quite accurate, for no book has come closer to being the "bible of mathematics" than Euclid's spectacular creation. Down through the centuries, over 2000 editions of the *Elements* have appeared, a figure that must make the authors of today's mathematics textbooks drool with envy. As noted, it was highly successful even in its own day. After the fall of Rome, the Arab scholars carried it off to Baghdad, and when it reappeared in Europe during the Renaissance, its impact was profound. The work was studied by the great Italian scholars of the sixteenth century and by a young Cambridge student named Isaac Newton a century later. We have a passage from Carl Sandburg's biography of Abraham Lincoln that recounts how, when a young lawyer trying to sharpen his reasoning skills, the largely unschooled Lincoln

> . . . bought the *Elements* of Euclid, a book twenty-three centuries old. . . [It] went into his carpetbag as he went out on the circuit. At night . . . he read Euclid by the light of a candle after others had dropped off to sleep.

It has often been noted that Lincoln's prose was influenced and enriched by his study of Shakespeare and the Bible. It is likewise obvious that many of his political arguments echo the logical development of a Euclidean proposition.

And Bertrand Russell (1872–1970) had his own fond memories of the *Elements*. In his autobiography, Russell penned this remarkable recollection:

> At the age of eleven, I began Euclid, with my brother as tutor. This was one of the great events of my life, as dazzling as first love.

As we consider the *Elements* in this chapter and the next, we should be aware that we proceed along paths that so many others have trod. Only a very few classics—the *Iliad* and *Odyssey* come to mind—share such a heritage. The propositions we shall examine were studied by Archimedes and Cicero, by Newton and Leibniz, by Napoleon and Lincoln. It is a bit daunting to place oneself in this long, long line of students.

Euclid's great genius was not so much in creating a new mathematics as in presenting the old mathematics in a thoroughly clear, organized, and logical fashion. This is no small accomplishment. It is important to recognize the *Elements* as more than just mathematical theorems and their proofs; after all, mathematicians as far back as Thales had been furnishing proofs of propositions. Euclid gave us a splendid *axiomatic* development of his subject, and this is a critical distinction. He began the *Elements* with a few basics: 23 definitions, 5 postulates, and 5 common notions or general axioms. These were the foundations, the "givens," of his system. He could use them at any time he chose. From these basics, he proved his first proposition. With this behind him, he could then blend his definitions, postulates, common notions, and this first proposition into a proof of his second. And on it went.

Consequently, Euclid did not just furnish proofs; he furnished them within this axiomatic framework. The advantages of such a development are significant. For one thing, it avoids circularity in reasoning. Each proposition has a clear, unambiguous string of predecessors leading back to the original axioms. Those familiar with computers could even draw a flow chart showing precisely which results went into the proof of a given theorem. This approach is far superior to "plunging in" to prove a proposition, for in such a case it is never clear which previous results can and cannot be used. The great danger from starting in the middle, as it were, is that to prove theorem A, one might need to use result B, which, it may turn out, cannot be proved without using theorem A itself. This results in a circular argument, the logical equivalent of a snake swallowing its own tail; in mathematics it surely leads to no good.

But the axiomatic approach has another benefit. Since we can clearly pick out the predecessors of any proposition, we have an immediate sense of what happens if we should alter or eliminate one of our basic postulates. If, for instance, we have proved theorem A without ever using either postulate C or any result previously proved by means of postulate C, then we are assured that our theorem A remains valid even if postulate C is discarded. While this might seem a bit esoteric, just such an issue arose with respect to Euclid's controversial fifth postulate and led to one of the longest and most profound debates in the history of mathematics. This matter is examined in the Epilogue of the current chapter.

Thus, the axiomatic development of the *Elements* was of major importance. Even though Euclid did not quite pull this off flawlessly, the high level of logical sophistication and his obvious success at weaving the pieces of his mathematics into a continuous fabric from the basic assumptions to the most sophisticated conclusions served as a model for all subsequent mathematical work. To this day, in the arcane fields of topology or abstract algebra or functional analysis, mathematicians will first present the axioms and then proceed, step-by-step, to build up their wonderful theories. It is the echo of Euclid, 23 centuries after he lived.

Book I: Preliminaries

In this chapter, we shall focus only on the first book of the *Elements*; subsequent books will be the topic of Chapter 3. Book I began abruptly with a list of definitions from plane geometry. (All Euclidean quotations are taken from Sir Thomas Heath's encyclopedic edition *The Thirteen Books of Euclid's Elements.*) Among the first few definitions were:

□ **Definition 1** A *point* is that which has no part.

□ **Definition 2** A *line* is breadthless length.

□ **Definition 4** A *straight line* is a line which lies evenly with the points on itself.

Today's students of Euclid find these statements unacceptable and a bit quaint. Obviously, in any logical system, not every term can be defined, since definitions themselves are composed of terms, which in turn must be defined. If a mathematician tries to give a definition for *everything*, he or she is condemned to a huge circular jumble. What, for instance, did Euclid mean by "breadthless"? What is the technical meaning of lying "evenly with the points on itself"?

From a modern viewpoint, a logical system begins with a few undefined terms to which all subsequent definitions relate. One surely tries to keep the number of these undefined terms to a minimum, but their presence is unavoidable. For modern geometers, then, the notions of "point" and "straight line" remain undefined. Statements such as

Euclid's may serve to convey some image in our minds, and this is not without merit; but as precise, logical definitions, these first few items are unsatisfactory.

Fortunately, his later definitions were more successful. A few of these figure prominently in our discussion of Book I and deserve comment.

□ **Definition 10** When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the equal angles is *right* and the straight line standing on the other is called a *perpendicular* to that on which it stands.

It may come as a surprise to modern readers that Euclid did not define a right angle in terms of 90°; in fact, nowhere in the *Elements* is "degree" ever mentioned as a unit of angular measure. The only angular measure that plays any significant role in the book is the right angle, and as we can see, Euclid defined this as one of two equal adjacent angles along a straight line.

□ **Definition 15** A *circle* is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another.

Clearly, the "one point" within the circle is the circle's center, and the equal "straight lines" he referred to are the radii.

In definitions 19 through 22, Euclid defined *triangles* (plane figures contained by three straight lines), *quadrilaterals* (those contained by four), and such specific subclasses as *equilateral* triangles (triangles with three sides equal) and *isosceles* triangles (those with "two of its sides alone equal"). His final definition proved to be critical:

□ **Definition 23** *Parallel* straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Notice that Euclid avoided defining parallels in terms of their being everywhere equidistant. His definition was far simpler and less fraught with logical pitfalls: parallels were simply lines in the same plane that never intersect.

With the definitions behind him, Euclid gave a list of five postulates for his geometry. Recall, these were to be the givens, the self-evident truths of his system. He certainly had to select them judiciously and to avoid overlap or internal inconsistency.

POSTULATE 1 [It is possible] to draw a straight line from any point to any point.

POSTULATE 2 [It is possible] to produce a finite straight line continuously in a straight line.

A moment's thought shows that the first two postulates permitted precisely the sorts of constructions one can make with an unmarked straightedge. For instance, if the geometer wanted to connect two points with a straight line—a task physically accomplished with a straightedge—then Postulate 1 provided the logical justification for doing so.

POSTULATE 3 [It is possible] to describe a circle with any center and distance (i.e., radius).

Here was the corresponding logical basis for pulling out a compass and drawing a circle, provided one first had a given point to be the center and a given distance to serve as radius. Thus, the first three postulates, together, justified all pertinent uses of the Euclidean tools.

Or did they? Those who think back to their own geometry training will recall an additional use of the compass, namely, as a means of transferring a fixed length from one part of the plane to another. That is, given a line segment whose length was to be copied elsewhere, one puts the point of the compass at one end of the segment and the pencil tip at the other; then, holding the device rigidly, we lift the compass and carry it to the desired spot. It is a simple and highly useful procedure. However, in playing by Euclid's rules, it was not permitted, for nowhere did he give a postulate allowing this kind of transfer of length. As a result, mathematicians often refer to the Euclidean compass as "collapsible." That is, although it is perfectly capable of drawing circles (as Postulate 3 guaranteed), upon lifting it from the plane, it falls shut, unable to remain open once it is removed.

What is one to make of this situation? Why did Euclid not insert an additional postulate to support this very important transfer of lengths? The answer is simple: he did not need to *assume* such a technique as a postulate, for he *proved* it as the third proposition of Book I. That is, Euclid introduced a clever technique for transferring lengths even if his compass "collapsed" upon lifting it from the page, and then he proved why his technique worked. It is to Euclid's great credit that he avoided assuming what he could in fact derive, and thereby kept his postulates to a bare minimum.

POSTULATE 4 All right angles are equal to one another.

This postulate did not relate to a construction. Rather, it provided a uniform standard of comparison throughout Euclid's geometry. Right angles had been introduced in Definition 10, and now Euclid was assuming that any two such angles, regardless of where they were situated in the plane, were equal. With this behind him, Euclid arrived at by far the most controversial statement in Greek mathematics:

POSTULATE 5 If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

As shown in Figure 2.1, this postulate is saying that if $\alpha + \beta < 2$ right angles, then lines *AB* and *CD* meet toward the right. Postulate 5 is often called Euclid's parallel postulate. This is a bit of a misnomer, since actually the postulate gave conditions under which two lines meet and thus, according to Definition 23, is more accurately called the nonparallel postulate.

Clearly, this postulate was quite unlike the others. It was longer to state, required a diagram to understand, and seemed far from being a self-evident truth. The postulate appeared too complicated to be included in the same category as the innocuous "All right angles are

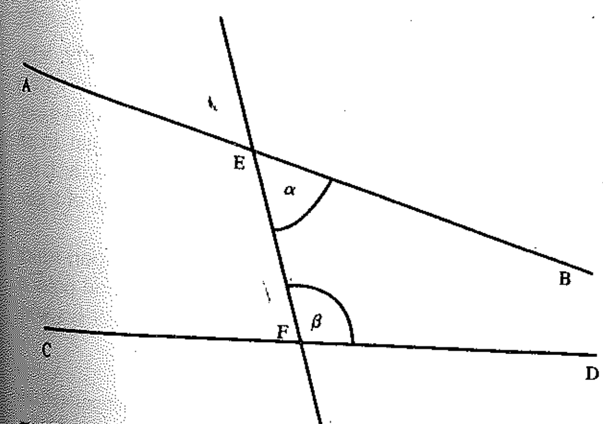

FIGURE 2.1

equal." In fact, many mathematicians felt in their bones that the fifth postulate was, in reality, a theorem. They sensed that, just as Euclid did not need to assume that lengths could be transferred with a compass, neither did he have to assume this postulate; he should simply have been able to prove it from the more elementary properties of geometry. There is evidence that Euclid himself was a bit uneasy about this matter, for in his development of Book I he avoided using the parallel postulate as long as he could. That is, whereas he felt perfectly content to use any of his other postulates as early and often as he needed, Euclid put off the use of his fifth postulate through his first 28 propositions. As shown in the Epilogue, however, it was one thing to be skeptical of the need for such a postulate but quite another to furnish the actual proof.

With this controversial statement behind him, Euclid completed his preliminaries with a list of five common notions. These too were meant to be self-evident truths but were of a more general nature, not specific to geometry. They were

- **Common Notion 1** Things which are equal to the same thing are also equal to one another.
- **Common Notion 2** If equals be added to equals, the wholes are equal.
- **Common Notion 3** If equals be subtracted from equals, the remainders are equal.
- **Common Notion 4** Things which coincide with one another are equal to one another.
- **Common Notion 5** The whole is greater than the part.

Of these, only the fourth raised some eyebrows. Apparently, what Euclid meant by it was that, if one figure could be moved rigidly from one portion of the plane and then be placed down upon a second figure so as to coincide perfectly, then the two figures were equal in all aspects—that is, they had equal angles, equal sides, and so forth. It has long been observed that Common Notion 4, having something of a geometric character, belonged among the postulates.

This, then, was the foundation of assumed statements upon which the entire edifice of the *Elements* was to be built. It is a good point at which to return to the young Bertrand Russell for another of his wonderful autobiographical confessions:

> I had been told that Euclid proved things, and was much disappointed that he started with axioms. At first, I refused to accept them unless my brother could offer me some reason for doing so, but he said, "If you don't accept them, we cannot go on," and, as I wished to go on, I reluctantly admitted them.

Book I: The Early Propositions

With the preliminaries behind him, Euclid was ready to prove the first of 48 propositions in Book I. Only those propositions of particular interest or importance are discussed here, the goal being to arrive at Propositions I.47 and I.48, which stand as the logical climax of the first book.

If someone were about to develop geometry from a few selected axioms, what would be his or her very first proposition? For Euclid, it was

PROPOSITION I.1 On a given finite straight line, to construct an equilateral triangle.

PROOF Euclid began with the given segment AB, as shown in Figure 2.2. Using A as center and AB as radius, he constructed a circle; then, with B as center and AB again as radius, he constructed a second circle. Both constructions, of course, made use of Postulate 3, and neither required the compass to remain open when lifted from the page. Letting C be the point where the circles intersect, Euclid invoked Postulate 1 to draw lines CA and CB and then claimed that $\triangle ABC$ was equilateral. For, by Definition 15, $\overline{AC} = \overline{AB}$ and $\overline{BC} = \overline{AB}$ since these are radii of their respective circles. Then, since Common Notion 1 states that things equal to the same thing are themselves equal, we conclude that $\overline{AC} = \overline{AB} = \overline{BC}$ and so the triangle is equilateral by definition.

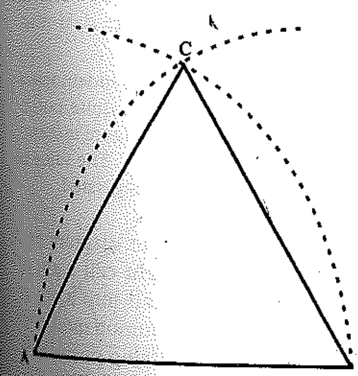

FIGURE 2.2