Is Univalence Inevitable?

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Probably.

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  - Algebraic Geometry
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  - BTW Axiom of Choice is true.

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- Formalize concepts, not statements. (univalent = faithful)
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- Based on types-as-spaces interpretation.
- New principles, e.g., the Univalence Axiom.

Insight 1: h-levels

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4: $\mathtt{is\text{-}2\text{-}Gpd}\,A :\equiv \prod_{x,y:A} \mathtt{isGpd}(x =_A y)$ 2-CATEGORY THEORY

$\vdots$

Insight 2: Univalence Axiom

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$$\mathtt{isEquiv}\, f :\equiv \prod_{b:B} \mathtt{isContr} \left(\sum_{a:A} (f\ a =_B b) \right)$$

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The Univalence Axiom

For any A and B, the map $w_{A,B}$ is an equivalence.

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- Replacement for the Axiom of Choice.
- **Not** about synthetic homotopy theory.

Univalent categories (cf. Ahrens-K-Shulman, 2015)

A category C consists of:

- a type of objects $C_0 : \mathtt{U}$;
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Theorem

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- For set-theoretic categories: $\Leftrightarrow$ Axiom of Choice.
- For univalent categories: true.

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Thanks for listening!