Concentration Inequalities for Multinoulli Random Variables

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1 Problem Formulation

We analyse the concentration properties of the random variable $Z_n \geq 0$ defined as:

$$Z_n := \max_{v \in [0,D]^S} \left\{ (\widehat{p}_n - p)^\intercal v \right\} \tag{1}$$

where $\widehat{p}_n \in \Delta^S$ is a random vector, $p \in \Delta^S$ is deterministic and $\Delta^S = \{x \in \mathbb{R}^S : \sum_{i=1}^S x_i = 1 \wedge x_i \geq 0\}$ is the $(S-1)$-dimensional simplex. It is easy to show that the maximum in Eq. 1 is equivalent to computing the (scaled) ℓ_1-norm of the vector $\widehat{p}_n - p$:

$$Z_n = \max_{u \in [-\frac{D}{2}, \frac{D}{2}]} \left\{ (\widehat{p}_n - p)^\intercal \left(u + \frac{D}{2} e \right) \right\} = \frac{D}{2} \|\widehat{p}_n - p\|_1 \tag{2}$$

where we have used the fact that $\frac{D}{2}(\widehat{p}_n - p)^\intercal e = 0$. As a consequence, Z_n is a bounded random variable in $[0, D]$. While the following discussion apply to Dirichlet distributions, we focus on $\widehat{p}_n \sim \frac{1}{n}$*Multinomial*$(n, p)$. The results previously available in the literature are summarized in the following.

The literature has analysed the concentration of the ℓ_1-discrepancy of the true distribution and the empirical one in this setting.

Proposition 1. *[Weissman et al., 2003] Let $p \in \Delta^S$ and $\widehat{p} \sim \frac{1}{n}$Multinomial$(n, p)$. Then, for any $S \geq 2$ and $\delta \in [0, 1]$:*

$$\mathbb{P}\left(\|\widehat{p} - p\|_1 \geq \sqrt{\frac{2S \ln(2/\delta)}{n}} \right) \leq \mathbb{P}\left(\|\widehat{p} - p\|_1 \geq \sqrt{\frac{2 \ln\left((2^S - 2)/\delta\right)}{n}} \right) \leq \delta \tag{3}$$

This concentration inequality is at the core of the proof of UCRL, see [Jaksch et al., 2010, App. C.1]. Another inequality is provided in [Devroye, 1983, Lem. 3].

Proposition 2. *[Devroye, 1983] Let $p \in \Delta^S$ and $\widehat{p} \sim \frac{1}{n}$Multinomial$(n, p)$. Then, for any $0 \leq \delta \leq 3 \exp(-4S/5)$:*

$$\mathbb{P}\left(\|\widehat{p}_n - p\|_1 \geq 5\sqrt{\frac{\ln(3/\delta)}{n}} \right) \leq \delta \tag{4}$$

While Prop. 1 shows an explicit dependence on the dimension of the random variable, such dependence is hidden in Prop. 2 by the constraint on δ. Note that for any $0 \leq \delta \leq 3\exp\left(-4S/5\right)$, $\sqrt{\frac{\ln(3/\delta)}{n}} > \sqrt{\frac{4S}{5n}}$. This shows that the ℓ_1-deviation always scales proportionally to the dimension of the random variable, i.e., as $\sqrt{S}$.

A better inequality. The natural question is whether is possible to derive a concentration inequality independent from the dimension of p by exploiting the correlation between $\widehat{p}$ and the maximizer vector v^*. This question has been recently addressed in [Agrawal and Jia, 2017, Lem. C.2]:

Lemma 3. *[Agrawal and Jia, 2017] Let $p \in \Delta^S$ and $\widehat{p} \sim \frac{1}{n}$Multinomial$(n,p)$. Then, for any $\delta \in [0,1]$:*

$$\mathbb{P}\left(\|\widehat{p}_n - p\|_1 \geq \sqrt{\frac{2\ln(1/\delta)}{n}} \right) \leq \delta$$

Their results resemble the one in Prop. 2 but removes the constraint on δ. As a consequence, the implicit or explicit dependence on the dimension S is removed. In the following, we will show (empirically and theoretically) that Lem. 3 may not be correct.

2 Theoretical Analysis (the asymptotic case)

In this section, we provide a counter-argument to the Lem. 3 in the asymptotic regime (i.e., $n \to +\infty$). The overall idea is to show that the expected value of Z_n asymptotically grows as $O(\sqrt{S})$ and Z_n itself is well concentrated around its expectation. As a result, we can deduce that all quantiles of Z_n grow as $O(\sqrt{S})$ as well.

We consider the true vector p to be uniform, i.e., $p = (\frac{1}{S}, \dots, \frac{1}{S})$ and $\widehat{p} \sim \frac{1}{n}$*Multinomial*$(n,p)$.[1] The following lemma provides a characterization of the variable $Z_S := \lim_{n\to+\infty} \sqrt{n} Z_n$.

Lemma 4. *Consider $S \in \mathbb{N}$, $\mathcal{S} = \{1, \dots, S\}$ and $p = (\frac{1}{S}, \dots, \frac{1}{S})$ be the uniform distribution on $\mathcal{S}$. Let e_S be the vector of ones of dimension S. Define $Y \sim \mathcal{N}(0, I_S - \frac{1}{S-1}N)$ where $N = e_S e_S^\mathsf{T} - I_S$ is the matrix with 0 in all the diagonal entry and 1 elsewhere, and $Y^+ = (\max(Y_i, 0))_{i\in\mathcal{S}}$. Then:*

$$Z_S = \lim_{n\to+\infty} \sqrt{n} Z_n \sim \|Y^+\|_1 D \sqrt{\frac{S-1}{S^2}}.$$

Furthermore,

$$\mathbb{E}[Z_S] = \sqrt{\frac{S-1}{S^2}} \cdot \mathbb{E}\left[\sum_{i=1}^{S} Y_i^+\right] = \sqrt{S-1} \cdot \mathbb{E}[Y_1^+] = \sqrt{\frac{S-1}{2\pi}}.$$

While the previous lemma may already suggest that Z_S should grow as $O(\sqrt{S})$ as its expectation, it is still possible that a large part of the distribution is concentrated around a value independent from S, with limited probability assigned to, e.g., values growing as $O(S)$, which could justify the $O(\sqrt{S})$ growth of the expectation. Thus, in order to conclude the analysis, we need to show that Z_S is concentrated "enough" around its expectation.

[1] The analysis holds also in the case $\widehat{p} \sim$ *Direchlet*(np).

Since the random variables Y_i are correlated, it is complicated to directly analyze the deviation of Z_S from its mean. Thus we first apply an orthogonal transformation on Y to obtain independent r.v. (recall that jointly normally distributed variables are independent if uncorrelated).

Lemma 5. *Consider the same settings of Lem. 4 and recall that $Y \sim \mathcal{N}(0, I_S - \frac{1}{S-1}N)$. There exists an orthogonal transformation $U \in O_S(\mathbb{R})$, s.t.*

$$W = \sqrt{\frac{S-1}{S}} UY \sim \mathcal{N}\left(0, \begin{bmatrix} I_{S-1} & 0 \\ 0 & 0 \end{bmatrix}\right).$$

By exploiting the transformation U we can write that $Z_S \sim g(W) := \frac{1}{\sqrt{S}} e_S^\mathsf{T} \left(U^\mathsf{T} W\right)^+$. Since W_i are i.i.d. standard Gaussian random variables and g is 1-Lipschitz, we can finally characterize the mean and the deviations of Z_S and derive the following anticoncentration inequality for Z_S.

Theorem 6. *Let $p \in \Delta^S = (\frac{1}{S}, \dots, \frac{1}{S})$ and $\widehat{p}_n \sim \frac{1}{n}$Multinomial$(n, p)$. Define $Z_n = \max_{v \in [0,D]} \left\{(\widehat{p}_n - p)^\mathsf{T} v\right\}$ and $Z_S = \lim_{n \to +\infty} \sqrt{n} Z_n$. Then, for any $\delta \in (0, 1)$:*

$$\mathbb{P}\left[Z_S \geq \sqrt{\frac{2(S-1)}{\pi}} - \sqrt{2\log(2/\delta)}\right] \geq 1 - \delta.$$

This result shows that every quantile of Z_S is dependent on the dimension of the random variable, i.e., $\sqrt{S}$. Similarly to Lem. 2, it is possible to lower bound the quantile by a dimension-free quantity at the price of having an exponential dependence on S in δ.

A Proof for the asymptotic scenario

In this section we report the proofs of lemmas and theorem stated in Sec. 2.

A.1 Proof of Lem. 4

Let $Y_{n,i} = \frac{1}{\sqrt{n}\sqrt{\frac{S-1}{S^2}}} \sum_{j=1}^{n} (X_i^j - \frac{1}{S})$ and $Y_n = (Y_{n,i})_{i \in \mathcal{S}}$. Then:

$$\sqrt{n} Z_n = \sqrt{n} \max_{v \in [0,D]^S} (\widehat{p} - p)^\mathsf{T} v = \sqrt{n} \max_{v \in [0,D]^S} \sum_{i=1}^{S} \frac{v_i}{n} \sum_{j=1}^{n} (X_i^j - \frac{1}{S})$$
$$= \max_{v \in [0,D]^S} \sum_{i=1}^{S} Y_{n,i} v_i \sqrt{\frac{S-1}{S^2}} = D \sqrt{\frac{S-1}{S^2}} \cdot e^\mathsf{T} Y_n^+,$$

where we used the fact that the v maximizing Z_n takes the largest value D for all positive components $Y_{n,i}$ and is equal to 0 otherwise. We recall that the covariance of the normalized multinoulli variable $Y_{n,i}$ with probabilities $p_i = 1/S$ is $I_S - \frac{1}{S-1}N$. As a result, a direct application of the central limit theorem gives $Y_n \overset{\mathcal{D}}{\rightarrow} Y \sim \mathcal{N}(0, I_S - \frac{1}{S-1}N)$. Then we can apply the functional CLT and obtain $Z_S = \lim_{n\rightarrow\infty} \sqrt{n} Z_n = \lim_{n\rightarrow\infty} e_S^\mathsf{T} Y_n^+ \sqrt{\frac{S-1}{S^2}} \overset{\mathcal{D}}{\sim} \sqrt{\frac{S-1}{S^2}} \cdot e_S^\mathsf{T} Y^+$, where Y^+ is a random vector obtained by truncating from below at 0 the multi-variate Gaussian vector Y. Since the marginal distribution of each random variable Y_i is $\mathcal{N}(0,1)$, i.e., are identically distributed (see definition in Lem. 4), Y_i^+ has a distribution composed by a Dirac distribution in 0 and a half normal distribution, and its expected value is $\mathbb{E}[Y_i^+] = 1/\sqrt{2\pi}$, while leads to the final statement on the expectation.

A.2 Proof of Lem. 5

Denote $\lambda(A)$ the set of eigenvalues of square matrix A. Let $B \in \mathbb{R}^{S\times S}$ such that $B = \begin{bmatrix} 0_{S,S-1} & e_S \end{bmatrix}$, where $0_{S,S-1} \in \mathbb{R}^{S\times(S-1)}$ is a matrix full of zeros. Then, we can write the eigenvalues of the covariance matrix of Y as

$$\lambda(I_S - \frac{1}{S-1}N) = \lambda(\frac{S}{S-1}I_S - \frac{1}{S-1}e_S e_S^\mathsf{T}) = \lambda\left(\frac{S}{S-1}I_S - \frac{1}{S-1}BB^\mathsf{T}\right)$$
$$= \frac{S}{S-1}\lambda\left(I_S - \frac{1}{S-1}B^\mathsf{T}B\right) = \frac{S}{S-1}\lambda\left(I_S - \begin{bmatrix} 0_{S-1} & 0 \\ 0 & 1 \end{bmatrix}\right),$$

where we use the fact that $\lambda(I - A^\mathsf{T}A) = \lambda(I - AA^\mathsf{T})$. As a result, the covariance of Y has one eigenvalue at 0 and eigenvalues equal to $\frac{S}{S-1}$ with multiplicity $S-1$. As a result, we can diagonalize it with an orthogonal matrix $U \in O_S(\mathbb{R})$ (obtained using the normalized eigenvectors) and obtain

$$U(I_S - \frac{1}{S-1}N)U^T = \begin{bmatrix} \frac{S}{S-1}I_{S-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Define $W = \sqrt{\frac{S-1}{S}} UY$, then:

$$\begin{aligned} Cov(W,W) &= \frac{S-1}{S} Cov(UY, UY) = \frac{S-1}{S} U Cov(Y,Y) U^T \\ &= \frac{S-1}{S} U (I_S - \frac{1}{S-1} N) U^T = \begin{bmatrix} I_{S-1} & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Thus $W \sim \mathcal{N}\left(0, \begin{bmatrix} I_{S-1} & 0 \\ 0 & 0 \end{bmatrix}\right)$.

A.3 Proof of Thm. 6

By exploiting Lem. 4 and Lem. 5 we can write:

$$Z_S \sim e_S^\intercal Y^+ \cdot \sqrt{\frac{S-1}{S^2}} = e_S^\intercal \left(\sqrt{\frac{S}{S-1}} U^\intercal W \right)^+ \cdot \sqrt{\frac{S-1}{S^2}} = e_S^\intercal \left(U^\intercal W \right)^+ \cdot \frac{1}{\sqrt{S}}$$

Let $g(\cdot) = e_S^\intercal \left(U^T \cdot \right)^+ \frac{1}{\sqrt{S}}$. Then g is 1-Lipschitz:

$$|g(x) - g(y)| \leq Lip(e_s^\intercal \cdot) Lip(U^\intercal \cdot) Lip((\cdot)^+) \frac{1}{\sqrt{S}} \|x - y\|_2 = \sqrt{S} \cdot 1 \cdot 1 \cdot \frac{1}{\sqrt{S}} \|x - y\|_2$$

where $Lip(f)$ denotes the Lipschitz constant of a function f and we exploit the fact that U is an orthonormal matrix.

We can now study the concentration of the variable Z_S. Given that W is a vector of i.i.d. standard Gaussian variables[2] and g is 1-Lipschitz, we can use [Wainwright, 2017, Thm. 2.4] to prove that for all $t > 0$:

$$\mathbb{P}(Z_S \geq \mathbb{E}[Z_S] - t) \geq 1 - \mathbb{P}(|Z_S - \mathbb{E}[Z_S]| \geq t) \geq 1 - 2e^{-\frac{t^2}{2}}.$$

Substituting the value of $\mathbb{E}[Z_S]$ and inverting the bound gives the desired statement.

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[2] Note that we can drop the last component of W since it is deterministically zero.