Bivector Bases and Cross Products

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A bivector basis is an ordered pairing of pairs of basis vectors. Each pair of vector basis elements forms an oriented area basis element. These basis elements are used in applications such as fluid dynamics, where they provide the signs of flux passing through test surfaces, and for defining rotations in higher dimensions (also finite element solvers)

Closely related to the question of constructing cyclic bivector bases on a space is the question of constructing a vector cross product on a space.

1 Cyclic Bivector Bases

A cyclically oriented bivector basis is one in which the area basis elements are symmetric with respect to cyclic permutation of the basis elements. For example, consider a vector basis e_a, e_b, e_c, which we will initially label as e_1, e_2, e_3.

The right-hand rule for three-dimensional spaces defines positive pairings to be (e_1, e_2), (e_2, e_3), and (e_3, e_1), such that for our initial basis numbering the absolute pairs (e_a, e_b), (e_b, e_c), and (e_c, e_a) are all positive. If we relabel each vector basis from e_i to e_{i+1} (so that e_a becomes e_2, etc.,), the vector pairings (e_1, e_2), (e_2, e_3), and (e_3, e_1) again make (e_a, e_b), (e_b, e_c), and (e_c, e_a) all positive.

In contrast, the lexigraphical construction for a bivector basis from an ordered vector basis defines a bivector as positive if the first vector has a lower index number than the second basis vector, such that (e_1, e_2), (e_1, e_3), and (e_2, e_3) are defined as positive, which makes (e_c, e_a) negative. This basis is not cyclical: if we relabel e_i to e_{i+1}, (e_c, e_a) becomes the positive (e_1, e_2), and (e_b, e_c) becomes the negative (e_3, e_1).

In the development of numerical computational tools, it is common to progress from two-dimensional to three-dimensional and then n-dimensional implementations. In moving from two dimensions to three dimensions, especially in physically-motivated problems, it is common to extend the single-element bivector basis (e1, e2) to three dimension using the right hand rule described above. The next step, moving to general n-dimensional formulations then poses a problem: How to extend the right-hand-rule basis to four or more dimensions. After some trial and error, it becomes clear to the student that there is no satisfactory way to make this extension, and they convert to a lexigraphic ordering (which requires careful examination of the project to ensure that all assumptions of a right-hand rule are replaced with the lexigraphic rule).

It is not immediately apparent, however, *why* we cannot extend the right hand rule to four dimensional spaces. Our first theorem in the paper addresses this point:

Theorem 1.1 *Cyclically oriented bases can only be constructed on odd-dimensional spaces.*

Proof: An oriented basis on n-dimensional space corresponds to a directed complete graph with n nodes. Cyclic orientation with respect to an ordering of the nodes additionally requires that this graph be rotationally symmetric. A necessary condition for rotational symmetry is that there are no sources or sinks in the graph, i.e., that the graph contains (equivalently, that its edges can be induced by) an Eulerian circuit through the n nodes. As per Euler's theorem, this condition is only possible in a graph with an even number of edges on each node; for a complete graph, this means that there must be an odd number of nodes, and thus that it must correspond to an odd-dimensional space.

Some geometric intuition in support of the above proof: The complete graph on an ordered n-node space consists of the polygon defined by the nodes and all of its inscribed stars, as illustrated in Fig. 1. On odd-dimensional spaces, this graph can be made into a rotationally-symmetric directed graph by assigning a traversal direction to the polygon and each of the stars. On even-numbered spaces, one of the stars is degenerate (cross-shaped) and cannot be followed without retracing another segment, so no symmetric graph can be constructed.

Theorem 1.2 *There are $2^{(n-1)/2}(n-2)!$ distinct cyclic bivector bases that can be constructed on an n-dimensional space.*

Proof: The polygon and its inscribed stars are together $\frac{n-1}{2}$ loops, which can individually be traversed in any direction, for a total of $2^{(n-1)/2}$ loop

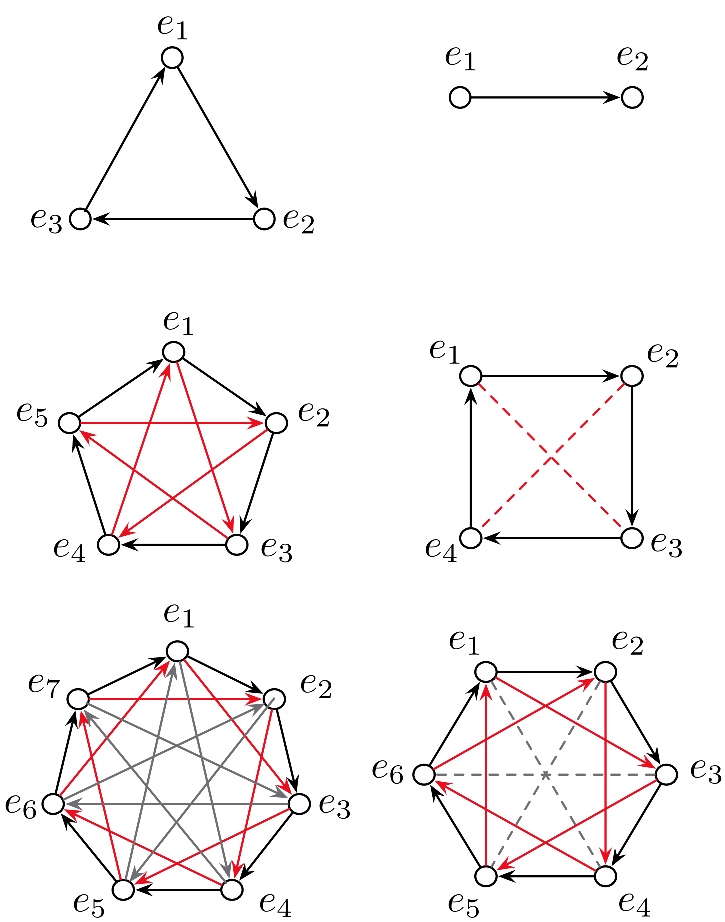

Figure 1: A complete graph with n nodes consists of the convex n-gon and its inscribed stars. When n is odd, this graph can be turned into a directed graph that is symmetric with respect to cyclic permutations of the basis numbering, but when n is even, the degenerate star, or "cross" at the center of the graph prevents the construction of a cyclically symmetric directed graph. Because construction of bivector bases is equivalent to construction of complete directed graphs, cyclically symmetric bivector bases can thus only be constructed on odd-dimensional spaces.

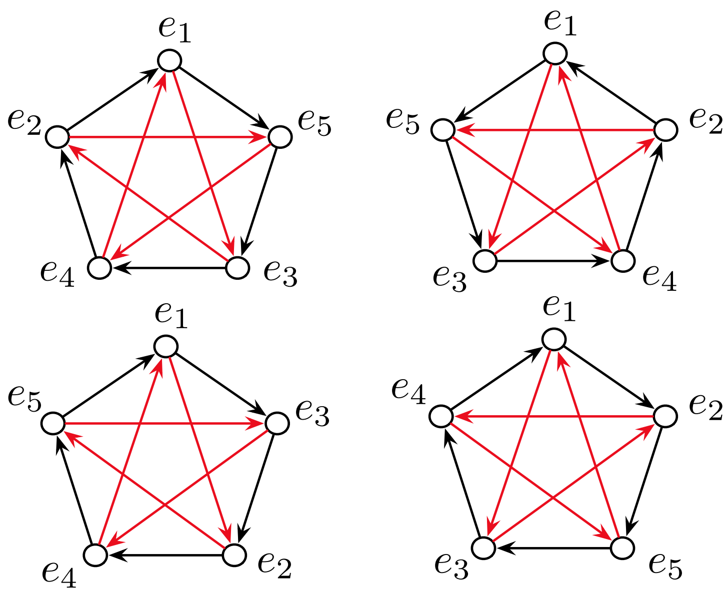

Figure 2: Permutations that change the choice of e_2 produce the same directed graph as is produced by some combination of permutations of the $n-2$ other nodes and flipping the signs with which the polygon and stars are traversed. For example, on a five-dimensional space, (top) swapping e_2 and e_5 while maintaining the directions of both cycles is equivalent to swapping e_3 and e_4 while flipping the directions of both the pentagon and the pentagram, and (bottom) swapping e_2 and e_5 and maintaining the directions of both cycles is equivalent to swapping e_3, e_4, and e_5 while flipping the direction of the pentagram.

sign combinations. Taking the first two points as fixed, there are $(n-2)!$ *order-permutations of the other points. (Taking only the first point as fixed would double-count with some of the sign reversals.)*

2 Cross product

The standard requirements for a cross product are that it is

Bilinear, orthogonal, and have magnitude equal to the product of the magnitudes when the input vectors are orthogonal, with consequences of anticommutativity and the other identities

In the context of our bivector basis discussion above, we can equivalently describe a cross product as an operation that assigns a basis vector to each basis bivector, $(e_i, e_j) \to e_k, k \neq i, j$. For geometric consistency under cyclic relabeling of the basis vectors, both the construction of the basis and the assignment must be rotationally symmetric on the graph. All such constructions correspond to assigning a cyclic order to the inscribed stars on the graph and a traversal direction for each, then for each bivector edge in one star, finding e_k by continuing one edge from e_j in the next star.

2.1 Spaces that could admit a cross product

Theorem 2.1 *Cross products can only be constructed on spaces with "doubly odd" dimensions.*

Proof: Assigning a rotationally invariant cyclic ordering to a set requires that the set have an odd number of elements, because with an even number of elements it is undefined which of two diametrically opposed points is "before" the other. The there are $\frac{n-1}{2}$ polygon and stars on an n-dimensional space, and therefore they can only be assigned a symmetric cyclic ordering when $\frac{n-1}{2}$ is odd and n is thus "doubly odd".

2.2 Collisions when attempting to construct a cross product

There are $(\frac{n-1}{2} - 1)!$ possible polygon-star orderings that we can use to construct the cross product. Together with the direction- and basis-ordering permutations identified in Theorem 1.2, there are $2^{(n-1)/2)}(\frac{n-1}{2} - 1)!(n-2)!$ possible ways in which we could possibly construct a cross product. Not all of these direction-star-basis orderings produce valid cross products. Many have "collisions" in which two bivectors containing a shared basis vector are assigned the same output vector (up to a sign),

$$(e_i, e_j) \to e_k \qquad \text{and} \qquad (e_i, e_\ell) \to \pm e_k \tag{1}$$

A "cross product" incorporating such an assignment would result in a product

$$e_i \times (e_j + e_\ell) = 2e_k \qquad \text{or} \qquad e_i \times (e_j + e_\ell) = 0, \tag{2}$$

neither of which is compatible with the condition that the magnitude of the products must be equal to the product of the magnitudes when the inputs are orthogonal, as

$$1 \cdot \sqrt{2} \neq 2 \qquad \text{and} \qquad 1 \cdot \sqrt{2} \neq 0. \tag{3}$$

Such collisions occur when there exists a single directed star for which traversing a single edge produces the same transformation as is produced by traversing one edge on each of a consecutive pair of directed stars.

The conditions for this lack of overlapping sums are easily checked for the lowest-dimensional doubly-odd-dimensioned spaces:

Three dimensions. In three dimensions, the polygon-star set is just the polygon and it is its own "next item" in the polygon-star ordering.. This polygon can be followed in either direction, such that the candidate cross product sequences and their collision tests are

Polygon-star sequence	Collision tests	Validity
1	$1+1=2$	Pass
2	$2+2=4\equiv 1$	Pass

(4)

and so three-dimensional spaces admit the construction of two cross products, distinguished by the order in which the basis elements are traversed: $e_1 \times e_2 = e_3$ and $e_1 \times e_2 = -e_3$.

Five dimensions. In five dimensions, the polygon-star set is the pentagon and the pentagram, and cross product candidates assign to each leg of the pentagon the point reached by following the next leg of the pentagram, and vise versa. This construction means that all cross-product candidates on a five-dimensional space are self-colliding, with pairs of cross products $e_i \times e_{j1}$ and $e_i \times e_{j2}$ both mapping to the same e_k, as illustrated in Fig. 3 for the positively oriented pentagon cases. The full set of collision tests for the five-dimensional case is

Polygon-star sequence	Collision tests	Validity
12	$\begin{bmatrix} 1+2=3 \\ 2+1=3 \end{bmatrix}$	Fail
13	$\begin{bmatrix} 1+3=4 \\ 3+1=4 \end{bmatrix}$	Fail
42	$\begin{bmatrix} 4+2=6\equiv 1 \\ 2+4=6\equiv 1 \end{bmatrix}$	Fail
43	$\begin{bmatrix} 4+3=7\equiv 2 \\ 3+4=7\equiv 2 \end{bmatrix}$	Fail

(5)

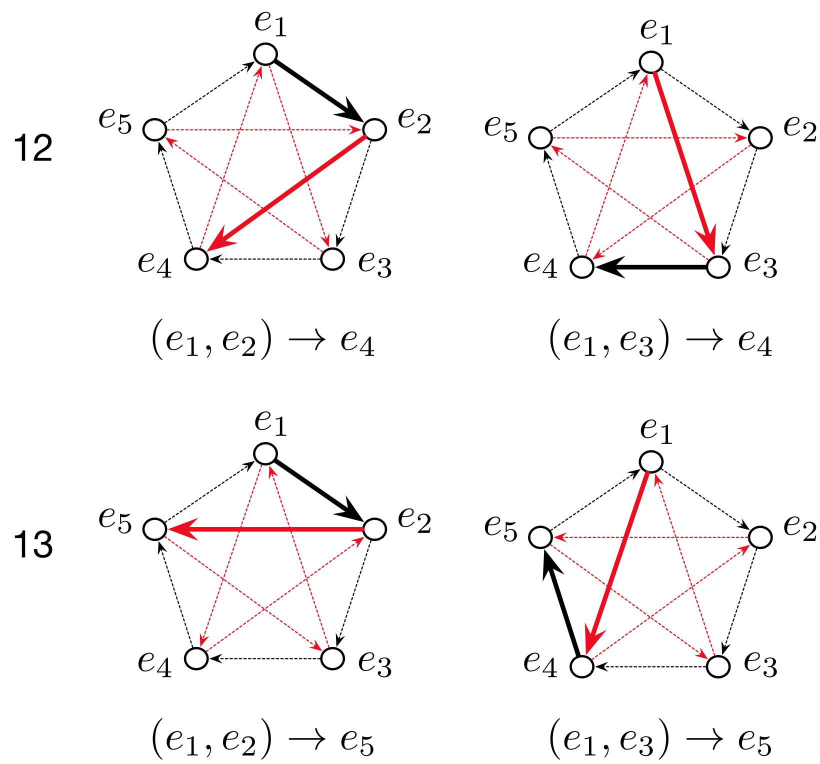

Figure 3: All cross-product candidates on a five-dimensional space are self-colliding, because they involve moving once along the polygon and once along the pentagram, in the same direction for both pentagon legs and both pentagram legs

Seven dimensions. In seven dimensions (the next-smallest doubly-odd space), there are two stars: a "wide" star in which each edge increases the index count by two or by five (negative two in a seven-element cycle), and a "narrow" star which increases the index count by three or four. The candidate cross products and their collision tests with the polygon counting forward and a polygon-wide narrow order are

Polygon-star sequence	Collision tests	Validity
123	$1+2=3$	Fail
124	$\begin{bmatrix} 1+2=3 \\ 2+4=6 \\ 4+1=5 \end{bmatrix}$	Pass
153	$5+3=8\equiv 1$	Fail
154	$4+1=5$	Fail

(6)

with only a single candidate producing a valid cross product,

$$e_i \times e_{i+1} = e_{i+3} \tag{7a}$$
$$e_i \times e_{i+2} = e_{i+6} \tag{7b}$$
$$e_i \times e_{i+4} = e_{i+6} \tag{7c}$$

Changing the direction of the polygon traversal,

Polygon-star sequence	Collision tests	Validity
623	$1+2=6+3=9\equiv 2$	Fail
624	$2+4=6$	Fail
653	$\begin{bmatrix} 6+5=11\equiv 4 \\ 5+3=8\equiv 1 \\ 3+9=9\equiv 2 \end{bmatrix}$	Pass
654	$6+5=11\equiv 4$	Fail

(8)

again leaves only a single candidate that produces a valid cross product,

$$e_i \times e_{i+6} = e_{i+4} \tag{9a}$$
$$e_i \times e_{i+5} = e_{i+1} \tag{9b}$$
$$e_i \times e_{i+3} = e_{i+2}, \tag{9c}$$

which is the sign reversal of all traversals in the original passing candidate.

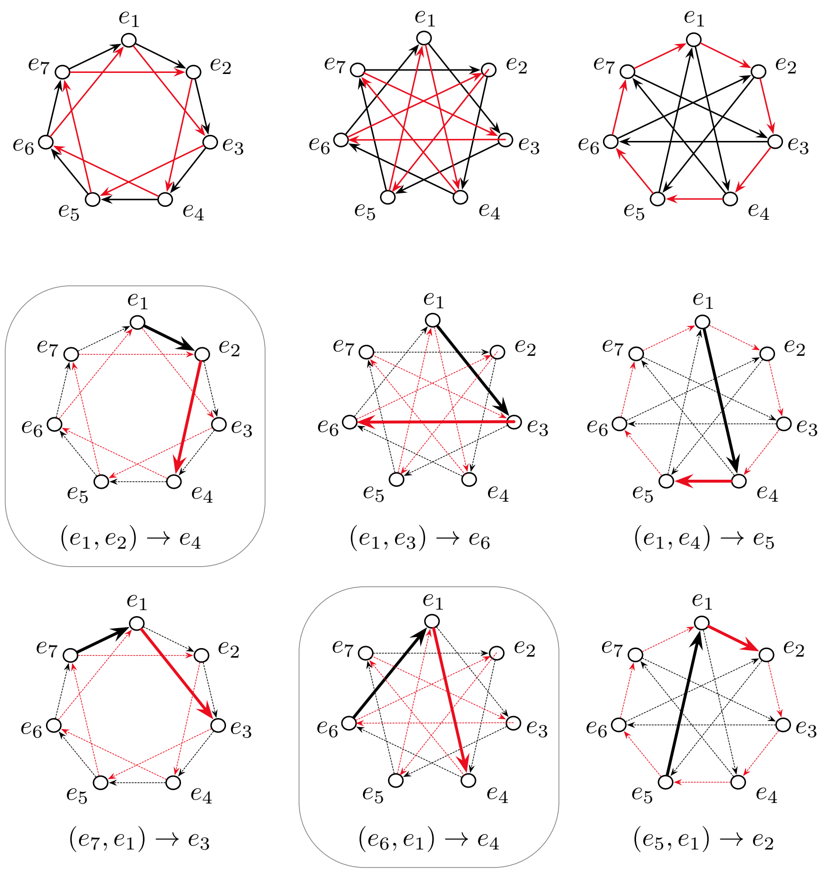

Figure 4: Some cross-product candidates on a seven-dimensional space are self-colliding, with the cross products whose first and second arguments are e_i both producing the same output basis element. Here, with a polygon-star ordering 123, both $e_1 \times e_2$ and $e_6 \times e_1$ map to e_4, meaning that this ordering does not define a valid cross product.

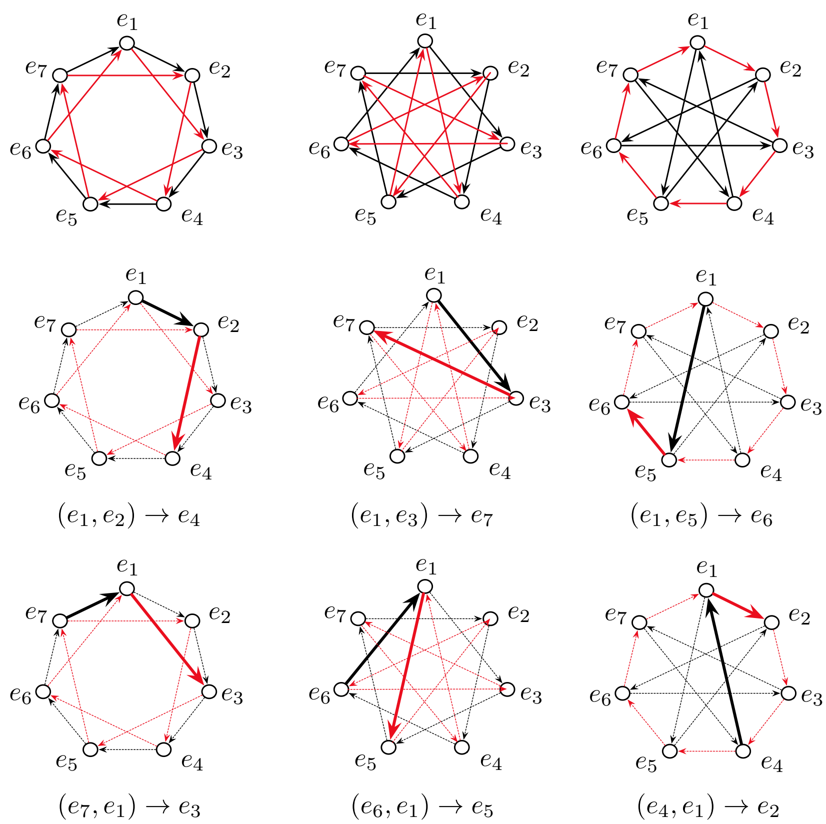

Figure 5: Changing the polygon-star ordering from 123 to 124 eliminates the collision, making each cross

Changing the order of the polygon-star sequences as

$$
\begin{array}{c|c|c}
\text{Polygon-star sequence} & \text{Collision tests} & \text{Validity} \\
\hline\hline
132 & 1+2=3 & \text{Fail} \\
142 & \begin{bmatrix} 1+2=3 \\ 2+4=6 \\ 4+1=5 \end{bmatrix} & \text{Pass} \\
135 & 5+3=8\equiv 1 & \text{Fail} \\
145 & 4+1=5 & \text{Fail}
\end{array}
\tag{10}
$$

and

$$
\begin{array}{c|c|c}
\text{Polygon-star sequence} & \text{Collision tests} & \text{Validity} \\
\hline\hline
632 & 1+2=6+3=9\equiv 2 & \text{Fail} \\
642 & 2+4=6 & \text{Fail} \\
635 & \begin{bmatrix} 6+5=11\equiv 4 \\ 5+3=8\equiv 1 \\ 3+9=9\equiv 2 \end{bmatrix} & \text{Pass} \\
645 & 6+5=11\equiv 4 & \text{Fail}
\end{array}
\tag{11}
$$

again produces a single valid cross product for each polygon traversal direction,

$$e_i \times e_{i+1} = e_{i+3} \tag{12a}$$

$$e_i \times e_{i+4} = e_{i+6} \tag{12b}$$

$$e_i \times e_{i+2} = e_{i+5}, \tag{12c}$$

and

$$e_i \times e_{i+6} = e_{i+4} \tag{13a}$$

$$e_i \times e_{i+3} = e_{i+1} \tag{13b}$$

$$e_i \times e_{i+5} = e_{i+2}, \tag{13c}$$

with the interesting feature that changing the order of the polygon-star sequence, while producing distinct cross products, does not change the set of star directions needed to produce a valid cross product – all the valid cases are either re-orderings of 124 or $635 \equiv -(124)$.

Taking these validity tests together, we see that the $2^{(7-1)/2} = 2^(3)$ direction-permutations for bivector pairings on seven-dimensional space are reduced to just 2 sets of signs that correspond to valid cross products (which is consistent with the two signs we found for the cross product in three dimensions), and that the $(\frac{7-1}{2} - 1)! = 2$ permutations of the polygon-star order produce distinct valid cross products. Combining this with the

$(7-2)! = 120$ edge-fixed permutations of the basis ordering, we can conclude that there are

$$2 \cdot 2 \cdot 120 = 480 \tag{14}$$

valid cross products amongst the

$$8 \cdot 2 \cdot 120 = 1920 \tag{15}$$

distinct bivector pairings in seven-dimensional space, which is consistent with previous results in the literature.

2.3 Higher-dimensional spaces

Beyond confirming the previous results that geometrically consistent cross products can be defined in seven dimensions, we can extend our reasoning above to also rule out the possibility of such a cross product existing in dimensions greater than seven.

Based on the analysis, we can restrict our attention to direction-order sequences starting with 1 and with stars ordered by their increasing increment when followed in the same direction as the polygon. The general form of such sequences is

$$1 \quad \frac{2}{n-2} \quad \frac{3}{n-3} \quad \cdots \quad \frac{(n-1)/2-1}{(n-1)/2+2} \quad \frac{(n-1)/2}{(n-1)/2+1} \tag{16}$$

with either the positive (top) or negative (bottom) parity element selected from each pair.

Three conditions on these sequences together ensure that there are no paths through the sequence that produce valid cross products:

1. Paths through this sequence that produce valid cross products must alternate parity, because equences that contain consecutive elements with the same parity fail the validity test, because one of those numbers will sum with 1 to produce the other.

2. Paths through this sequence that produce valid cross products cannot start with $n-2$, because with alternating parity, the next element is 3, and $(n-2)+3 \equiv 1$, making these sequences invalid.

3. Paths through this sequence that produce valid cross products cannot start with 2, because with alternating parity, the last two elements are $(n-1)/2-1$ and $(n-1)/2+1$ or $(n-1)/2+2$ and $(n-1)/2$, both of which are separated by 2, again making the sequences invalid, and leaving no remaining candidates.

Seven-dimensional spaces escape this fate because 2 fills the dual role of starting the sequence and acting as the $(n-1)/2-1$ element in the final pair, such that 2 is not available to add to $(n-1)/2-1$ to produce $(n-1)/2+1$.