Bivector Bases and Cross Products

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A bivector basis is an ordered pairing of pairs of basis vectors. Each pair of vector basis elements forms an oriented area basis element. These basis elements are used in applications such as fluid dynamics, where they provide the signs of flux passing through test surfaces, and for defining rotations in higher dimensions (also finite element solvers)

Closely related to the question of constructing cyclic bivector bases on a space is the question of constructing a vector cross product on a space.

1 Vector and Bivector Bases

Vector basis: Select a set of n orthogonal unit vectors to define oriented directions. Any vector can thus be described as a sum of the

Bivector basis: A set of oriented planes defined by ordered pairs of elements from the vector basis. Canonically, the positive side of the bivector basis element (e_a, e_b) is the side from which e_b is seen as being at a counter-clockwise right angle from e_b.[1] For example, Figure 1 illustrates two bivector bases in each of two-dimensional and three-dimensional space, where different choices of which basis vector in each pair is first in the bivector flip the bivector's orientation.

The three-dimensional bivector bases illustrated in Figure 1 are the two most commonly encountered, the *right-hand rule basis* and the *lexigraphic basis*. The right-hand rule basis is defined by using cyclic ordering to determine which vector comes first in each pair (one before two, two before three, and three before one), and has the property that when each pair of vectors

[1] This is the planar righthand rule for orientation (which is related to, but distinct from, the righthand rule for selecting a bivector basis discussed below). Using a lefthanded rule in which e_b is clockwise from e_a changes the sign of the basis but also the signs of bivector-valued functions relative to the basis, so that calculations made with them remain unchanged.

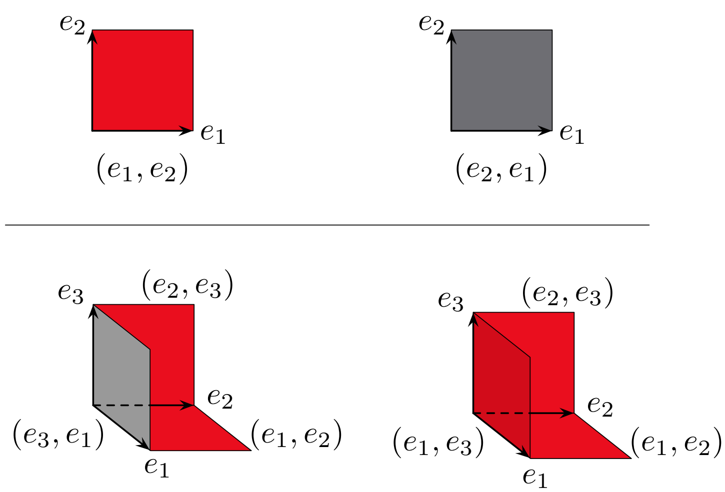

Figure 1: Bivector bases are composed of oriented sets of two basis vectors, and define oriented areas basis areas in the planes containing those vectors. The sign of this orientation depends on the order in which the basis vectors are paired, so that if e_1 is pointing right and e_2 is pointing out, we are viewing the positive side of (e_1, e_2) and the negative side of (e_1, e_2). Similarly, in three dimensions constructing the basis with the cyclic ordering (e_3, e_1) or the lexigraphic ordering (e_1, e_3) flips the sign of the corresponding bivector basis element.

is viewed from the planar-right-hand-rule perspective described above, the third basis vector points towards the viewer. The lexigraphic basis does not use cyclic ordering, instead taking one as being before three; as discussed below, this ordering is not as "geometrically pure" as the right-hand-rule, but it is more generalizable when extending analysis beyond three dimensions

On n-dimensional space, there are $n(n-1)/2$ pairs of basis vectors, each of which could be used in either order to construct the bivector basis, resulting in $2^{n(n-1)/2}$ possible sets of bivector bases corresponding to any given vector basis. A convenient means of visualizing the construction of such bivector bases, especially in more than three dimensions, is as directed complete graphs on n-dimensional space, with each node representing a basis vector, each edge representing a pairing of basis vectors, and each direction specifying which of the vectors is first in the pairing. For example, the first two rows in Figure 2 correspond to the bases illustrated in Figure 1, and the third row illustrates two possible constructions of bivector bases in four

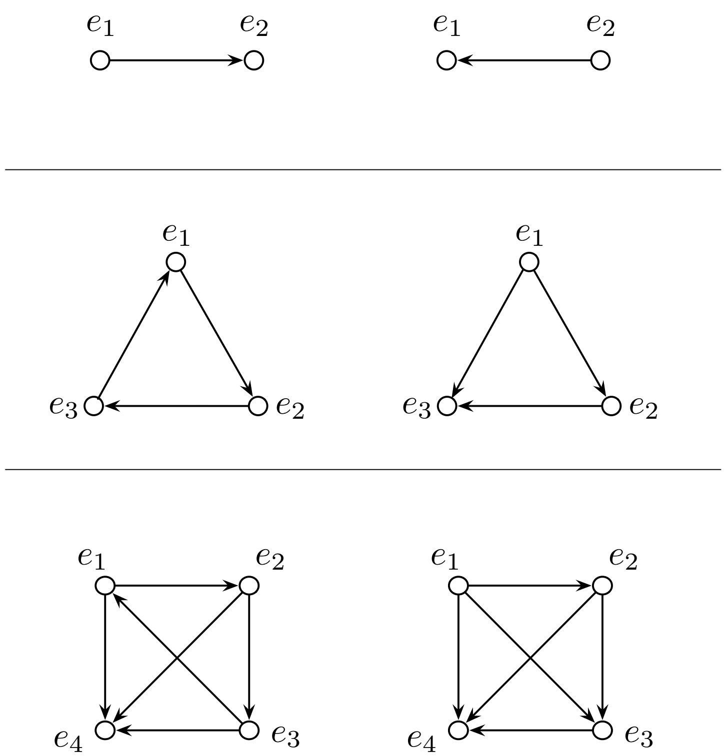

Figure 2: Bivector bases on n-dimensional space can be thought of

dimensions, one formed by extending the three-dimensional right-hand-rule basis by adding a fourth basis vector placed after all of the other bases, and the second the lexographic basis.

1.1 Ordered Bases

1.2 Cyclic Bivector Bases

A cyclically oriented bivector basis is one in which the area basis elements are symmetric with respect to cyclic permutation of the basis elements. For example, consider a vector basis e_a, e_b, e_c, which we will initially label as e_1, e_2, e_3.

The right-hand rule for three-dimensional spaces defines positive pairings to be (e_1, e_2), (e_2, e_3), and (e_3, e_1), such that for our initial basis numbering the absolute pairs (e_a, e_b), (e_b, e_c), and (e_c, e_a) are all positive. If we relabel each vector basis from e_i to e_{i+1} (so that e_a becomes e_2, etc.,), the vector pairings (e_1, e_2), (e_2, e_3), and (e_3, e_1) again make (e_a, e_b), (e_b, e_c), and (e_c, e_a) all positive.

In contrast, the lexigraphical construction for a bivector basis from an ordered vector basis defines a bivector as positive if the first vector has a lower index number than the second basis vector, such that (e_1, e_2), (e_1, e_3), and (e_2, e_3) are defined as positive, which makes (e_c, e_a) negative. This basis is not cyclical: if we relabel e_i to e_{i+1}, (e_c, e_a) becomes the positive (e_1, e_2), and (e_b, e_c) becomes the negative (e_3, e_1).

In the development of numerical computational tools, it is common to progress from two-dimensional to three-dimensional and then n-dimensional implementations. In moving from two dimensions to three dimensions, especially in physically-motivated problems, it is common to extend the single-element bivector basis (e1, e2) to three dimension using the right hand rule described above. The next step, moving to general n-dimensional formulations then poses a problem: How to extend the right-hand-rule basis to four or more dimensions. After some trial and error, it becomes clear to the student that there is no satisfactory way to make this extension, and they convert to a lexigraphic ordering (which requires careful examination of the project to ensure that all assumptions of a right-hand rule are replaced with the lexigraphic rule).

It is not immediately apparent, however, *why* we cannot extend the right hand rule to four dimensional spaces. Our first theorem in the paper addresses this point:

Theorem 1.1 *Cyclically oriented bases can only be constructed on odd-dimensional spaces.*

Proof: An oriented basis on n-dimensional space corresponds to a directed complete graph with n nodes. Cyclic orientation with respect to an ordering of the nodes additionally requires that this graph be rotationally symmetric. A necessary condition for rotational symmetry is that there are no sources or sinks in the graph, i.e., that the directed graph can be constructed by following an Eulerian circuit through the n nodes. As per Euler's theorem, this condition is only possible in a graph with an even number of edges on each node; for a complete graph, this means that there must be an odd number of nodes, and thus that it must correspond to an odd-dimensional space.

Some geometric intuition in support of the above proof: The complete graph on an ordered n-node space consists of the polygon defined by the nodes and all of its inscribed stars, as illustrated in Figure 3. On odd-dimensional spaces, this graph can be made into a rotationally-symmetric directed graph by assigning a traversal direction to the polygon and each of the stars. On even-numbered spaces, one of the stars is degenerate (cross-shaped) and cannot be followed without retracing another segment, so no symmetric graph can be constructed.

Theorem 1.2 *There are* $2^{(n-1)/2}(n-2)!$ *distinct cyclic bivector bases that can be constructed on an* n*-dimensional space.*

Proof: The polygon and its inscribed stars are together $\frac{n-1}{2}$ *loops, which can individually be traversed in any direction, for a total of* $2^{(n-1)/2}$ *loop sign combinations.*

There are $(n-2)!$ *distinct permutations of how we can assign Taking the first two points as fixed, there are* $(n-2)!$ *permutations of how we assign the basis vectors to poin. Taking only the first point as fixed and including permutations of the second pwould double-count with some of the sign reversals, as illustrated for two examples in five dimensions in Figure 4.*

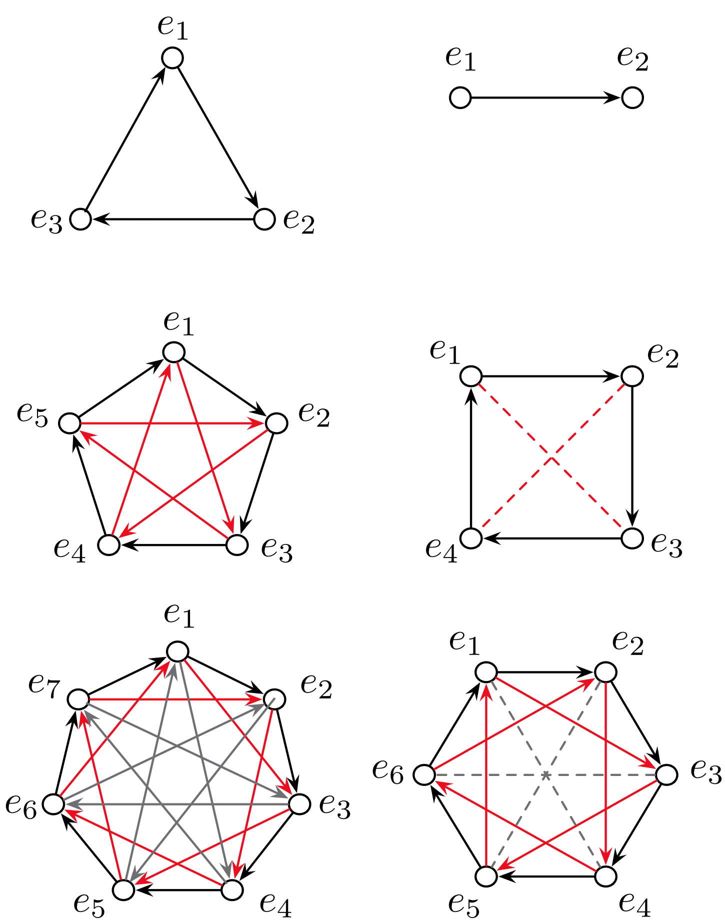

Figure 3: A complete graph with n nodes consists of the convex n-gon and its inscribed stars. When n is odd, this graph can be turned into a directed graph that is symmetric with respect to cyclic permutations of the basis numbering, but when n is even, the degenerate star, or "cross" at the center of the graph prevents the construction of a cyclically symmetric directed graph. Because construction of bivector bases is equivalent to construction of complete directed graphs, cyclically symmetric bivector bases can thus only be constructed on odd-dimensional spaces.

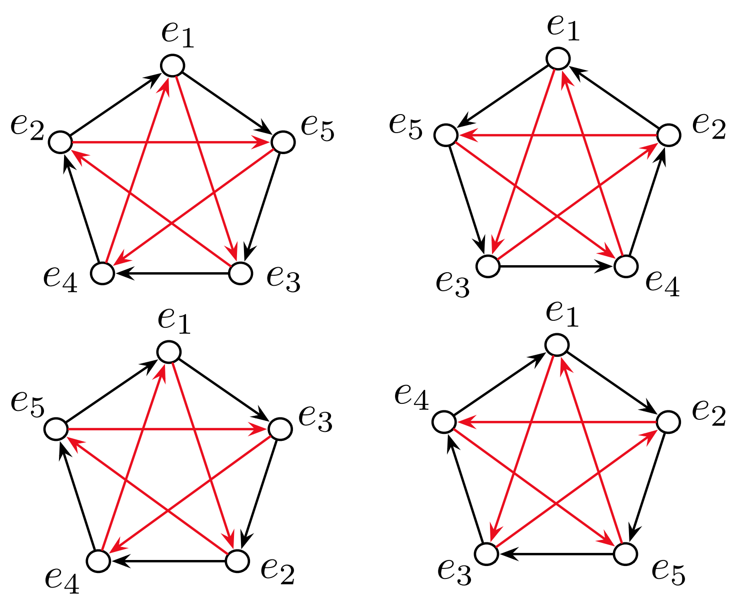

Figure 4: Permutations that change the choice of e_2 produce the same directed graph as is produced by some combination of permutations of the $n-2$ other nodes and flipping the signs with which the polygon and stars are traversed. For example, on a five-dimensional space, (top) swapping e_2 and e_5 while maintaining the directions of both cycles is equivalent to swapping e_3 and e_4 while flipping the directions of both the pentagon and the pentagram, and (bottom) swapping e_2 and e_5 and maintaining the directions of both cycles is equivalent to swapping e_3, e_4, and e_5 while flipping the direction of the pentagram.

2 Cross product

The standard definition for a cross product is that it is a bilinear map taking pairs of vectors to a third vector,

$$v_a \times v_b = v_c, \tag{1}$$

with this product satisfying two additional properties,

- **Orthogonality**: The cross product of two vectors must be orthogonal to those vectors,
$$v_a \cdot (v_a \times v_b) = \mathbf{0} \tag{2}$$
- **Area magnitude**: The magnitude of the cross product between two vectors is equal to the area of the parallelogram they define,
$$|v_a \times v_b| = |v_a|\,|v_b| \sin\theta, \tag{3}$$
where θ is the angle between the vectors.

This definition has a number of well-known consequences, in particular that the cross product operation must be anticommutative (or "alternating"), such that reversing the order in which vectors are crossed flips the sign of the output,
$$v_i \times v_j = -(v_j \times v_i). \tag{4}$$

In three dimensions, each pair of vectors has a unique normal direction (up to the choice of sign), so there are two cross-product functions that can be constructed on three-dimensional space – the standard "right-hand rule" cross product and its counterpart, which flips the sign of the output. In spaces with more than three dimensions, however, any pair of vectors has multiple normal vectors, and the definition of the cross product potentially admits multiple functions that differ by more than the sign of the output. This situation naturally leads to the questions

- What spaces beyond three-dimensions admit cross products?
- How many cross products can be defined on such spaces?

Previous work in this area has proved that three and seven are the only dimensions that admit a cross product, and that on seven-dimensional spaces there are 480 distinct cross products that can be constructed. The standard proofs for these facts rely on properties related to the rotations of quaternions and octonions (each one dimension higher than the spaces that admit cross products) or on the projective geometry of the Fano plane. In the remainder of this paper, we will demonstrate that these conclusions about the existence and number of cross products in n-dimensional spaces can be reached simply from the properties of graphs with n nodes.

1. Candidate cross products can only be defined on spaces of odd-dimension.
2. Candidate cross products can only be defined on spaces of "doubly odd" dimension (with $(n-1)/2$ odd), or on five-dimensional space.
3. None of the five-dimensional candidates produce valid cross products.
4. Several of the seven-dimensional candidates do produce valid cross products.
5. None of the candidates from spaces of greater than seven dimensions can possibly produce a valid cross product.

2.1 Cross product definition tables

The bilinear aspect of a cross product means that it can be defined by specifying a multiplication table in which each pair of basis vectors e_i and e_j maps to a third basis vector e_k,

$$e_i \times e_j = e_k, \tag{5}$$

such that each component of the output vector can be found by summing the corresponding component products of the inputs and subtracting the reversed-order component products,

$$v_c^k = \sum_{e_i \times e_j = e_k} v_a^i v_b^j - v_a^j v_b^i. \tag{6}$$

Cross products are geometrically distinct from each other (meaning that the planes defined by each bivector are assigned a different set of positive/negative orientations) if their multiplication tables are different from each other, and equivalent to each other if they result in the same assignment of orientations.

2.2 Cross products as directed complete graphs

In the context of our bivector basis discussion in §1.2, a cross product multiplication table can be treated as assigning a basis vector to each basis bivector,

$$(e_i, e_j) \to e_k, \tag{7}$$

with $k \neq i, j$.

For geometric consistency under cyclic relabeling of the basis vectors, both the construction of the basis and the assignment must be rotationally symmetric on the graph. All such constructions correspond to assigning a cyclic order to the inscribed stars on the graph and a traversal direction for each, then for each bivector edge in one star, finding e_k by continuing one edge from e_j in the next star.

Theorem 2.1 *Cross products can only be constructed on spaces with "doubly odd" dimensions.*

Proof: Assigning a rotationally invariant cyclic ordering to a set requires that the set have an odd number of elements, because with an even number of elements it is undefined which of two diametrically opposed points is "before" the other. The there are $\frac{n-1}{2}$ polygon and stars on an n-dimensional space, and therefore they can only be assigned a symmetric cyclic ordering when $\frac{n-1}{2}$ is odd and n is thus "doubly odd".

2.3 Collisions when attempting to construct a cross product

There are $(\frac{n-1}{2} - 1)!$ possible polygon-star orderings that we can use to construct the cross product. Together with the direction- and basis-ordering permutations identified in Theorem 1.2, there are $2^{(n-1)/2}(\frac{n-1}{2}-1)!(n-2)!$ possible ways in which we could possibly construct a cross product. Not all of these direction-star-basis orderings produce valid cross products. Many have "collisions" in which two bivectors containing a shared basis vector are assigned the same output vector (up to a sign),

$$(e_i, e_j) \to e_k \qquad \text{and} \qquad (e_i, e_\ell) \to \pm e_k \tag{8}$$

A "cross product" incorporating such an assignment would result in a product

$$e_i \times (e_j + e_\ell) = 2e_k \qquad \text{or} \qquad e_i \times (e_j + e_\ell) = 0, \tag{9}$$

neither of which is compatible with the condition that the magnitude of the products must be equal to the product of the magnitudes when the inputs are orthogonal, as

$$1 \cdot \sqrt{2} \neq 2 \qquad \text{and} \qquad 1 \cdot \sqrt{2} \neq 0. \tag{10}$$

Such collisions occur when there exists a single directed star for which traversing a single edge produces the same transformation as is produced by traversing one edge on each of a consecutive pair of directed stars.

The conditions for this lack of overlapping sums are easily checked for the lowest-dimensional doubly-odd-dimensioned spaces:

Table 1: Collision tests for cross-product candidates in three dimensions.

Polygon-star sequence	Collision tests	Validity
1	$1+1=2$	Pass
2	$2+2=4\equiv 1$	Pass

Three dimensions. In three dimensions, the polygon-star set is just the polygon, which is thus its own "next item" in the polygon-star ordering.. This polygon can be followed by incrementing the index count by one or two units (equivalently, indexing by one unit in either direction), with both the bivector-pairing and output-assignment traversals constrained to be in the same direction.

The candidate cross product sequences and their collision tests are given in Table 1. Both candidate sequences pass, meaning that three-dimensional spaces admit the construction of two cross products distinguished by the order in which the basis elements are traversed, as illustrated in Figure 5. These cross products are the traditional righthand-rule cross product

$$e_i \times e_{i+1} = e_{i+2} \tag{11}$$

and the lefhand-rule cross product, which can be expressed as

$$e_i \times e_{i+2} = e_{i+1} \qquad \text{or} \qquad e_i \times e_{i-1} = e_{i-2} \qquad \text{or} \qquad e_i \times e_{i-1} = e_{i+1}. \tag{12}$$

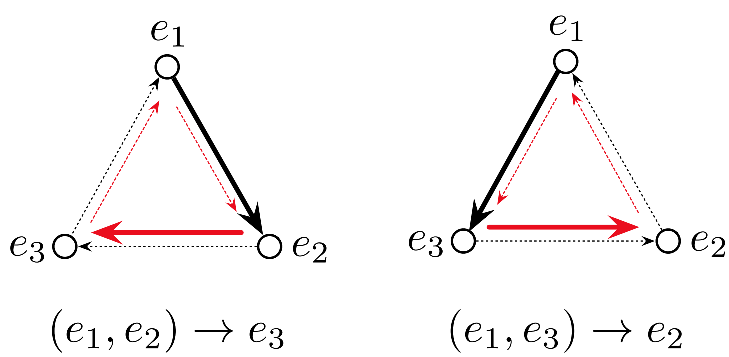

Figure 5: The two cross products that can be defined in three dimensions take the positive bivector pairings as either clockwise or counterclockwise, and the output as the next basis vector in the same direction around the triangle.

Table 2: Collision tests for cross-product candidates in five dimensions.

Polygon-star sequence	Collision tests	Validity
12	$\begin{bmatrix} 1+2=3 \\ 2+1=3 \end{bmatrix}$	Fail
13	$\begin{bmatrix} 1+3=4 \\ 3+1=4 \end{bmatrix}$	Fail
42	$\begin{bmatrix} 4+2=6\equiv 1 \\ 2+4=6\equiv 1 \end{bmatrix}$	Fail
43	$\begin{bmatrix} 4+3=7\equiv 2 \\ 3+4=7\equiv 2 \end{bmatrix}$	Fail

Five dimensions. In five dimensions, the polygon-star set is the pentagon and the pentagram, and cross product candidates assign to each leg of the pentagon the point reached by following the next leg of the pentagram, and vise versa. This construction means that all cross-product candidates on a five-dimensional space are self-colliding, with pairs of cross products $e_i \times e_{j1}$ and $e_i \times e_{j2}$ both mapping to the same e_k, as detailed in Table 2, and as illustrated in Figure 6 for the positively oriented pentagon cases.

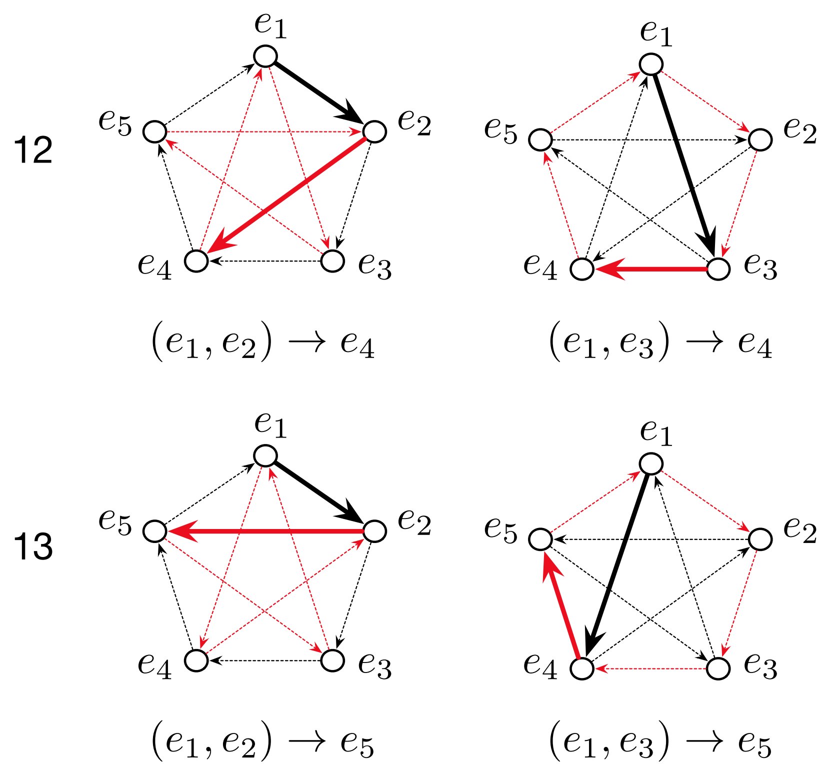

Figure 6: All cross-product candidates on a five-dimensional space are self-colliding, because they involve moving once along the polygon and once along the pentagram, in the same direction for both pentagon legs and both pentagram legs

Seven dimensions. In seven dimensions, there are two inscribed stars: a "wide" star in which each edge increases the index count by two or by five (negative two in a seven-element cycle), and a "narrow" star which increases the index count by three or four.

The candidate cross products and their collision tests with the a positive polygon direction and a polygon-wide-narrow ordering are detailed in the first block of Table 3. Three out of these four candidates have collisions (with the collisions for the 123 sequence illustrated in Figure 7), and only a single candidate from this set (illustrated in Figure 8) produces a valid cross product,

$$e_i \times e_{i+1} = e_{i+3} \tag{13a}$$
$$e_i \times e_{i+2} = e_{i+6} \tag{13b}$$
$$e_i \times e_{i+4} = e_{i+5} \tag{13c}$$

Flipping the direction of the polygon traversal again leaves only a single candidate that produces a valid cross product,

$$e_i \times e_{i+6} = e_{i+4} \tag{14a}$$
$$e_i \times e_{i+5} = e_{i+1} \tag{14b}$$
$$e_i \times e_{i+3} = e_{i+2}, \tag{14c}$$

which is the sign reversal of all traversals in the original passing candidate.

Changing the order of the polygon-star sequences again produces a single valid cross product for each polygon traversal direction,

$$e_i \times e_{i+1} = e_{i+3} \tag{15a}$$
$$e_i \times e_{i+4} = e_{i+6} \tag{15b}$$
$$e_i \times e_{i+2} = e_{i+5}, \tag{15c}$$

and

$$e_i \times e_{i+6} = e_{i+4} \tag{16a}$$
$$e_i \times e_{i+3} = e_{i+1} \tag{16b}$$
$$e_i \times e_{i+5} = e_{i+2}, \tag{16c}$$

with the interesting feature that changing the order of the polygon-star sequence, while producing distinct cross products, does not change the set of star directions needed to produce a valid cross product – all the valid cases are either re-orderings of 124 or 635, with the second set equivalent to flipping all the traversal directions on the first set.

Table 3: Collision tests for cross-product candidates in seven dimensions.

Polygon-star sequence		Collision tests	Validity
Polygon + Wide star ± Narrow star ±	123	$1+2=3$	Fail
	124	$\begin{bmatrix}1+2=3\\2+4=6\\4+1=5\end{bmatrix}$	Pass
	153	$5+3=8\equiv 1$	Fail
	154	$4+1=5$	Fail
Polygon − Wide star ± Narrow star ±	623	$1+2=6+3=9\equiv 2$	Fail
	624	$2+4=6$	Fail
	653	$\begin{bmatrix}6+5=11\equiv 4\\5+3=8\equiv 1\\3+9=9\equiv 2\end{bmatrix}$	Pass
	654	$6+5=11\equiv 4$	Fail
Polygon + Narrow star ± Wide star ±	132	$1+2=3$	Fail
	142	$\begin{bmatrix}1+2=3\\2+4=6\\4+1=5\end{bmatrix}$	Pass
	135	$5+3=8\equiv 1$	Fail
	145	$4+1=5$	Fail
Polygon − Narrow star ± Wide star ±	632	$1+2=6+3=9\equiv 2$	Fail
	642	$2+4=6$	Fail
	635	$\begin{bmatrix}6+5=11\equiv 4\\5+3=8\equiv 1\\3+9=9\equiv 2\end{bmatrix}$	Pass
	645	$6+5=11\equiv 4$	Fail

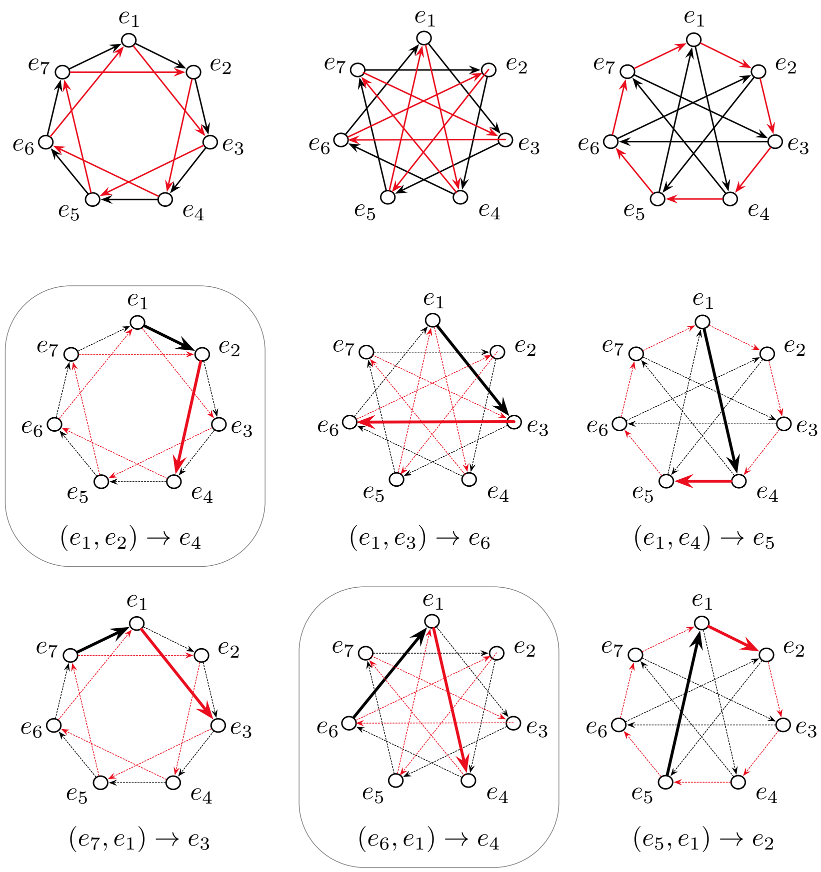

Figure 7: Some cross-product candidates on a seven-dimensional space are self-colliding, with the cross products whose first and second arguments are e_i both producing the same output basis element. Here, with a polygon-star ordering 123, both $e_1 \times e_2$ and $e_6 \times e_1$ map to e_4, meaning that this ordering does not define a valid cross product.

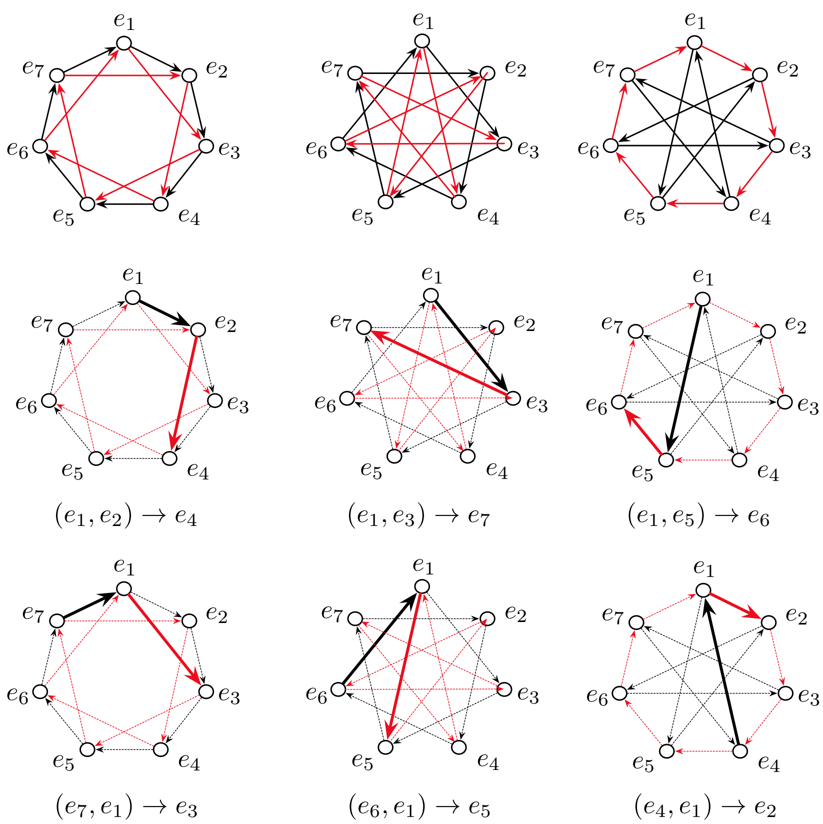

Figure 8: Changing the polygon-star ordering from 123 to 124 eliminates the collision, making each cross

Taking these validity tests together, we see that the $2^{(7-1)/2} = 2^{(3)}$ direction-permutations for bivector pairings on seven-dimensional space are reduced to just 2 sets of signs that correspond to valid cross products (which is consistent with the two signs we found for the cross product in three dimensions), and that the $(\frac{7-1}{2} - 1)! = 2$ permutations of the polygon-star order produce distinct valid cross products. Combining this with the $(7-2)! = 120$ edge-fixed permutations of the basis ordering, we can conclude that there are

$$2 \cdot 2 \cdot 120 = 480 \tag{17}$$

valid cross products amongst the

$$8 \cdot 2 \cdot 120 = 1920 \tag{18}$$

distinct bivector pairings in seven-dimensional space, which is consistent with previous results in the literature.

2.4 Higher-dimensional spaces

Beyond confirming the previous results that geometrically consistent cross products can be defined in seven dimensions but not five, we can extend our reasoning above to demonstrate the impossibility of such a cross product existing in dimensions greater than seven.

To start, we observe that the polygon-star sequences for an n-dimensional space (with n odd) can be constructed by taking either the top or bottom element from each entry in

$$\frac{1}{n-1} \quad \frac{2}{n-2} \quad \frac{3}{n-3} \quad \cdots \quad \frac{(n-1)/2-1}{(n-1)/2+2} \quad \frac{(n-1)/2}{(n-1)/2+1}, \tag{19}$$

where the pairs of entries correspond to the positive and negative directions around each polygon or star.

Because of the direction-reversal symmetry in the collision condition, we can restrict our attention to direction-order sequences starting with 1,

$$1 \quad \frac{2}{n-2} \quad \frac{3}{n-3} \quad \cdots \quad \frac{(n-1)/2-1}{(n-1)/2+2} \quad \frac{(n-1)/2}{(n-1)/2+1} \tag{20}$$

Three conditions on these sequences together prevent there from being any paths through this sequence that produce valid cross products:

1. Paths through this sequence that produce valid cross products must alternate between positive/negative direction elements, because sequences that contain consecutive elements with the same direction

fail the validity test, because one of those numbers will sum with 1 to produce the other.

2. Paths through this sequence that produce valid cross products cannot start with $n-2$, because with alternating direction, the next element is 3, and $(n-2)+3 \equiv 1$, making these sequences invalid.

3. Paths through this sequence that produce valid cross products cannot start with 2, because with alternating parity, the last two elements are $(n-1)/2-1$ and $(n-1)/2+1$ or $(n-1)/2+2$ and $(n-1)/2$, both of which are separated by 2, again making the sequences invalid, and leaving no remaining candidates.

Seven-dimensional spaces escape this fate because 2 fills the dual role of starting the sequence and acting as the $(n-1)/2-1$ element in the final pair, such that 2 is not available to add to $(n-1)/2-1$ to produce $(n-1)/2+1$.