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Introduction

Aim

We aim to find the number of ways to tile a $2 \times N$ floor, where N is a nonnegative integer, with 1×1 and 2×1 tiles. Specifically, we hope to obtain a closed-form solution in terms of N.

Generating Functions

Generating functions are a powerful tool used in several areas of math and are particularly useful in combinatorics and number theory (Morris, n.d.). This investigation will explore the applications of generating functions to combinatorics.

A generating function is a polynomial whose coefficients correspond to a sequence $\{a_i\}_{i=0}^{n}$ (Morris, n.d.), where a_i denotes the i^{th} term of the sequence. The general form of a generating function is:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n = \sum_{i=0}^{n} a_i x^i$$

where n is a nonnegative integer. Generating functions can also be infinite, in which case the polynomial would correspond to the infinite sequence $\{a_i\}_{i=0}^{\infty}$. In this case, we can write:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{i=0}^{\infty} a_i x^i$$

In combinatorics, mathematicians are able to encode information about a problem in a generating function. One of the biggest reasons for this is ease of algebraic manipulation. Polynomials can easily be multiplied, divided, subtracted, and added. Furthermore, polynomials often lend themselves to various useful factorizations and expansions. For instance, in combinatorics, $\binom{n}{i}$ denotes the number of ways to choose i distinct items from a set of n

items. Now consider the generating function for the sequence $\left\{\binom{n}{i}\right\}_{i=0}^{n}$:

$$f(x) = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n = \sum_{i=0}^{n}\binom{n}{i}x^i$$

The binomial theorem tells us that this can be easily factored as:

$$f(x) = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n = (x+1)^n$$

We can apply this generating function to derive various relationships. For example, plugging in $x = 1$ yields the identity:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

An important type of generating function we will investigate are generating functions corresponding to infinite geometric sequences. Recall that an infinite geometric sequence is a sequence of the form $\{a_0 r^i\}_{i=0}^{\infty}$ where a_0 is the first term and r is the common ratio between any two consecutive terms. The generating function representing such a sequence is of the form:

$$f(x) = a_0 + a_0 rx + a_0(rx)^2 + \cdots$$

where $|rx| < 1$ – this condition ensures that the series will converge, allowing us to write $f(x)$ as a rational function, a function that is a ratio between two polynomials. To see this, we multiply both sides of the above equation by rx to obtain:

$$f(x) = a_0 rx + a_0(rx)^2 + a_0(rx)^3 + \cdots$$

We can then write:

$$f(x) = a_0 + rx \cdot f(x)$$

Rearranging, we obtain:

$$f(x) = \frac{a_0}{1 - rx}$$

Thus, an infinite generating function such as $1 + x + x^2 + \cdots$ can be written as $\frac{1}{1-x}$ over the domain $(-1, 1)$ (since we must have $|rx| = |x| < 1$). Note how we just turned an infinite polynomial into an easily manageable rational function. Indeed, using geometric series we can often express generating functions as rational functions and vice-versa. For example, consider the rational function:

$$f(x) = \frac{3}{(1 + x)(2 - x)}$$

Using partial fractions decomposition, we can write this as:

$$f(x) = \frac{1}{1 + x} + \frac{1}{2 - x}$$

$$f(x) = \frac{1}{1 - (-x)} + \frac{1}{2}\left(\frac{1}{1 - \frac{x}{2}}\right)$$

$$f(x) = \left(1 - x + x^2 - x^3 + \cdots\right) + \frac{1}{2}\left(1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3 + \cdots\right)$$

$$f(x) = \left(1 + \frac{1}{2}\right) + \left(-1 + \frac{1}{2^2}\right)x + \left(1 + \frac{1}{2^3}\right)x^2 + \left(-1 + \frac{1}{2^4}\right)x^3 + \cdots$$

$$f(x) = \sum_{i=0}^{\infty}\left((-1)^i + \frac{1}{2^{i+1}}\right)x^i$$

Which is just an infinite generating function with the i^{th} term $a_i = (-1)^i + \frac{1}{2^{i+1}}$.

We can also perform calculus on generating functions. Consider the general form for a generating function:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

We saw earlier that by using partial fractions decomposition and geometric series, it is possible to find a closed-form expression for a_i. In fact, we can find another closed-form expression for a_i by repeatedly differentiating $f(x)$. Say we wanted to find the value of a_2. Taking the second derivative of $f(x)$ with respect to x, we obtain:

$$f''(x) = 0 + 0 + (2)(1)a_2 + (3)(2)a_3x + (4)(3)a_4x^2 + \cdots$$

The first two terms become 0 since their power of x in $f(x)$ was less than 2. Substitute $x = 0$ to obtain:

$$f''(0) = 0 + 0 + (2)(1)a_2 + 0 + 0 + \cdots$$

So,

$$a_2 = \frac{f''(0)}{2!}$$

We can generalize this process of differentiating, and then considering $x = 0$, for all a_i. In general, we find that:

$$a_i = \frac{f^{(i)}(0)}{i!}$$

where $f^{(i)}(x)$ denotes the i^{th} derivative of $f(x)$ with respect to x. In other words, given some infinitely differentiable function $f(x)$ (that is, $f^{(i)}(x)$ exists for all nonnegative integers i) that can be written as an infinite generating function, we can express the coefficients of said generating function in terms of the derivatives of $f(x)$. In fact, this is the key idea behind Maclaurin Series and, more generally, Taylor Series.

Recurrence Relations

A recurrence relation is an equation that defines a sequence in terms of its previous terms (Doerr and Levasseur, n.d.). Such a sequence is often referred to as a recurrence sequence. As we will see in this investigation, recurrence relations are very common in combinatorics. One of the most famous examples of a recurrence sequence is the Fibonacci sequence, defined as:

$$F_n = F_{n-1} + F_{n-2}$$

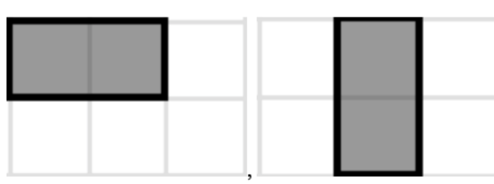
for $n \geq 2$, with $F_0 = 0$ and $F_1 = 1$. Notice how we must separately define F_0 and F_1. This is necessary for us to find F_2 and all subsequent terms afterwards. That aside, we can then calculate the first few terms of the sequence to be:

$$\{0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$$

While we can calculate the first few terms easily, computing F_{100} or F_{1000} using our recurrence relation would be cumbersome. If we had a closed-form expression for F_N – that is, an equation for F_N in terms of just N – calculating terms would become just one operation. It turns out we can find such a closed-form solution using generating functions, an elegant method that will be explored later in the investigation.

Main Body – Tiling a $2 \times N$ Floor

We are now familiar with the main tools we will use for this investigation: generating functions and recurrence relations. Using these tools, we are aiming to find a closed-form expression for the number of ways to tile a $2 \times N$ floor, where N is a nonnegative integer, using 2×1 and 1×1 tiles. This process is made difficult by several factors; for instance, the 2×1 tiles can be oriented both horizontally and vertically (the following diagrams were all created in Google Sheets):



Furthermore, horizontal 2×1 tiles in the top and bottom rows can touch as so:

This makes it difficult for us to separate the floor into different parts and consider these parts separately. We also don't know how many 2×1 and 1×1 tiles we can use – we just know they must cover the floor completely with no overlap between tiles. Needless to say, solving this problem purely with combinatorics methods would be tedious, if not impossible. Instead, let us tackle this problem with our newfound tools of recurrence relations and generating functions.

Determining a Recurrence Relation

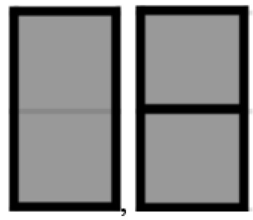
To start, let a_N denote the number of ways to tile a $2 \times N$ floor with 2×1 and 1×1 tiles. To find a closed-form expression for a_N, we will consider the generating function:

$$G(x) = \sum_{N=0}^{\infty} a_N x^N$$

We will first find a way to write $\{a_N\}_{N=0}^{\infty}$ as a recurrence sequence. As we saw in the introduction, this requires defining the first terms of the sequence ourselves. Thus, let us start by counting to find a_0, a_1, and a_2.

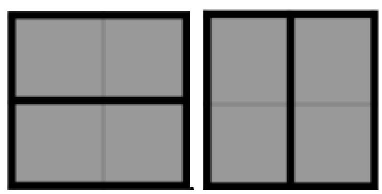
$\boldsymbol{a_0 = 1}$, since there is 1 way to tile a 2×0, or nonexistent, floor: not tiling at all.

$\boldsymbol{a_1 = 2}$. This is because there are only two ways to tile a 2×1 floor: with one 2×1 tile or two 1×1 tiles.

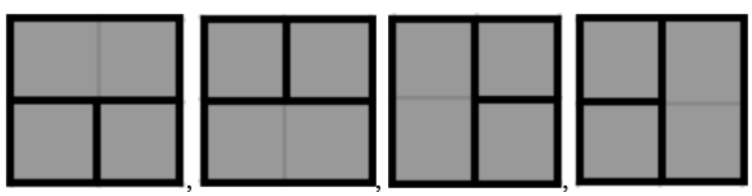


Lastly, $\boldsymbol{a_2 = 7}$. To see this, we will employ *casework*, a tool in combinatorics where a counting problem is split into smaller cases which are then considered separately. Here, we have 3 total cases, corresponding to the number of 2×1 tiles used when tiling a 2×2 floor.

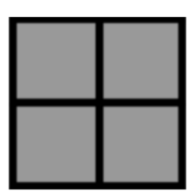
Case 1: Two 2×1 tiles. The tiles can be orientated either horizontally or vertically for a total 2 tilings:



Case 2: One 2×1 tile and two 1×1 tiles. The 2×1 tile can be placed at the top, bottom, left, and right edges for a total of 4 possible tilings:

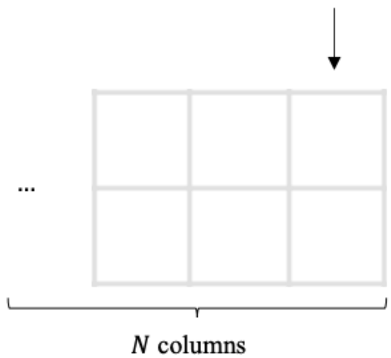


Case 3: No 2×1 tiles and four 1×1 tiles. Clearly, there is only 1 possible tiling:



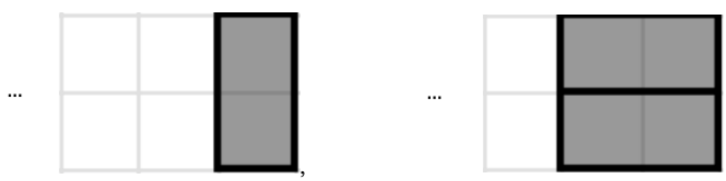
So $a_2 = 2 + 4 + 1 = 7$.

We now consider a_N where $N \geq 3$. To find a recurrence relation, the key insight is that we can examine what tiles occupy the rightmost column of the $2 \times N$ floor. The following diagram shows the three rightmost columns of such a floor – the rightmost column is marked with an arrow.



We again proceed with casework. Our cases here will explore all the possible positions of 1×1 tiles in the rightmost column.

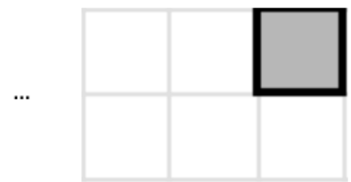
Case 1: no 1×1 tile occupies the rightmost column. Then the column must be occupied by either one vertical 2×1 tile or two horizontal 2×1 tiles:



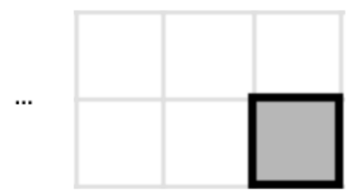
Consider the case with the vertical tile. The number of ways to tile the floor in this case is just the number of ways to tile the remaining $2 \times (N-1)$ area, or a_{N-1}. As to the case with the two horizontal tiles, the number of ways to tile the floor just becomes a_{N-2} as there is a $2 \times (N-2)$ area left to be covered. So, the total number of tilings for Case 1 is $a_{N-1} + a_{N-2}$.

Case 2: there is one 1×1 tile in the top row of the rightmost column. This case is harder; let us define another variable. Namely, let b_N denote the number of ways to tile a

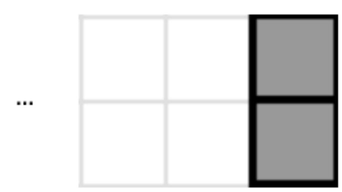
$2 \times N$ floor with a corner square already covered. Then the number of tilings for Case 2 is b_N.



Case 3: there is one 1×1 tile in the bottom row of the rightmost column. The remaining area is the same as in Case 2, so the number of tilings for this case is also b_N.



However, notice that there is a situation that we considered twice for Case 2 and Case 3. More specifically, we double-counted the case where the rightmost column is occupied by 1×1 tiles in both the top and bottom rows.



Luckily, we already have a variable representing the number of ways to tile the remaining $2 \times (N-1)$ area: a_{N-1}. So, we double-counted a total of a_{N-1} times. By the principle of inclusion-exclusion, the total number of tilings for Cases 2 and 3 is thus $b_N + b_N - a_{N-1}$.

Therefore, we have the relationship:

$$a_N = \underbrace{a_{N-1} + a_{N-2}}_{\text{Case 1}} + \underbrace{b_N + b_N - a_{N-1}}_{\text{Cases 2 and 3}}$$

Which simplifies to:

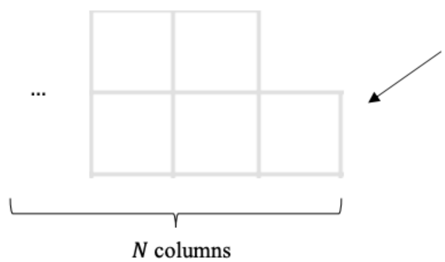
$$a_N = a_{N-2} + 2b_N$$

Or:

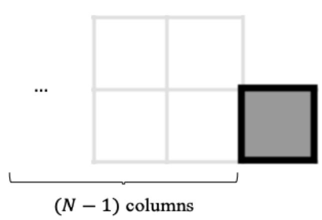
$$\boldsymbol{b_N = \frac{a_N - a_{N-2}}{2}} \tag{1.1}$$

with the original condition that $N \geq 3$.

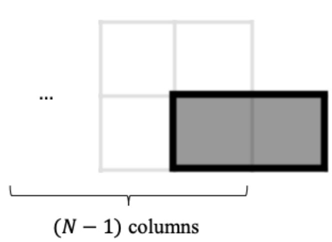
Now, recall that b_N denotes the number of ways to tile the following area:



To count the number of tilings of this area, we consider how the bottom-right corner square, marked by an arrow, is covered. If it is covered by a 1×1 tile, then the remaining $2 \times (N-1)$ area has a_{N-1} tilings.



If, instead, it is covered by a 2×1 tile, then the remaining area has b_{N-1} tilings.



So, we have the relationship:

$$\boldsymbol{b_N = b_{N-1} + a_{N-1}} \tag{1.2}$$

We now substitute $b_N = \frac{a_N - a_{N-2}}{2}$, the result from Equation (1.1). Also note that substituting $(N-1)$ for N into this equation yields $b_{N-1} = \frac{a_{N-1} - a_{N-3}}{2}$. Then, substituting for both b_N and b_{N-1} into Equation (1.2), we have:

$$\frac{a_N - a_{N-2}}{2} = \frac{a_{N-1} - a_{N-3}}{2} + a_{N-1}$$

Which simplifies to our final recurrence relation for a_N:

$$\boldsymbol{a_N = 3a_{N-1} + a_{N-2} - a_{N-3}}$$

for $N \geq 3$. Also, recall that $a_0 = 1$, $a_1 = 2$, and $a_2 = 7$.

Applying the Generating Function

We can finally attempt to find a closed-form expression for a_N by considering the generating function:

$$G(x) = \sum_{N=0}^{\infty} a_N x^N \tag{2.1}$$

We can multiply both sides by x^3 to obtain:

$$x^3 G(x) = \sum_{N=0}^{\infty} a_N x^{N+3} = \sum_{N=3}^{\infty} a_{N-3} x^N \tag{2.2}$$

Notice how the polynomial now starts at the x^3 term while the first coefficient is still a_0. The significance of this will soon be elaborated on; for now, notice how easy it was to perform this algebraic manipulation on the generating function. Now, since $a_0 = 1$ we can take out the first term of Equation (2.1) to obtain:

$$G(x) = 1 \cdot x^0 + \sum_{N=1}^{\infty} a_N x^N$$
$$G(x) - 1 = \sum_{N=1}^{\infty} a_N x^N$$

Multiplying both sides by x^2, we obtain:

$$x^2(G(x) - 1) = \sum_{N=1}^{\infty} a_N x^{N+2} = \sum_{N=3}^{\infty} a_{N-2} x^N \tag{2.3}$$

Like before, these algebraic manipulations may seem somewhat arbitrary, but we will examine shortly how they allow us to rewrite our infinite sum in terms of just x and $G(x)$. Additionally, since $a_0 = 1$ and $a_1 = 2$, we can take out the first two terms of Equation (2.1) and write:

$$G(x) = 1 \cdot x^0 + 2 \cdot x^1 + \sum_{N=2}^{\infty} a_N x^N$$
$$G(x) - 2x - 1 = \sum_{N=2}^{\infty} a_N x^N$$

$$3x(G(x) - 2x - 1) = \sum_{N=3}^{\infty} 3a_{N-1} x^N \tag{2.4}$$

Again, the justification for this will be provided shortly. Finally, since $a_0 = 1$, $a_1 = 2$, and $a_2 = 7$, we can take out the first three terms of Equation (2.1) to obtain:

$$G(x) = 1 \cdot x^0 + 2 \cdot x^1 + 7 \cdot x^2 + \sum_{N=3}^{\infty} a_N x^N$$

$$G(x) - 7x^2 - 2x - 1 = \sum_{N=3}^{\infty} a_N x^N \tag{2.5}$$

Now recall how we found that $a_N = 3a_{N-1} + a_{N-2} - a_{N-3}$ for $N \geq 3$. We can thus write Equation (2.5) as:

$$G(x) - 7x^2 - 2x - 1 = \sum_{N=3}^{\infty} (3a_{N-1} + a_{N-2} - a_{N-3})x^N$$
$$G(x) - 7x^2 - 2x - 1 = \sum_{N=3}^{\infty} 3a_{N-1}x^N + \sum_{N=3}^{\infty} a_{N-2}x^N - \sum_{N=3}^{\infty} a_{N-3}x^N$$

The rationale behind our previous algebraic manipulations is now clear – using Equations (2.2), (2.3) and (2.4), we can eliminate the infinite series and rewrite the right-hand side in terms of x and $G(x)$:

$$G(x) - 7x^2 - 2x - 1 = 3x(G(x) - 2x - 1) + x^2(G(x) - 1) - x^3 G(x)$$

Rearranging for $G(x)$, we obtain:

$$\boxed{G(x) = \frac{1 - x}{x^3 - x^2 - 3x + 1}}$$

Let us briefly verify our findings. Remember that $G(x)$ was defined as the generating function $\sum_{N=0}^{\infty} a_N x^N$. In fact, according to Wolfram Alpha (2024), our rational function is equivalent to the series expansion:

$$1 + 2x + 7x^2 + 22x^3 + 71x^4 + O(x^5)$$

Notice how the first three coefficients correspond to our calculated values of $a_0 = 1$, $a_1 = 2$, and $a_2 = 7$; our results so far are promising. We can now attempt the final part of this investigation, finding a closed form expression for a_N.

Finding a Closed-Form Expression

As outlined in the introduction, one way to find the closed-form expression for a_N is to expand the rational function $\frac{1-x}{x^3-x^2-3x+1}$ into the generating function in series form: $\sum_{N=0}^{\infty} a_N x^N$. This requires that we use partial fraction decomposition to separate $\frac{1-x}{x^3-x^2-3x+1}$ into an expression of the form:

$$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$$

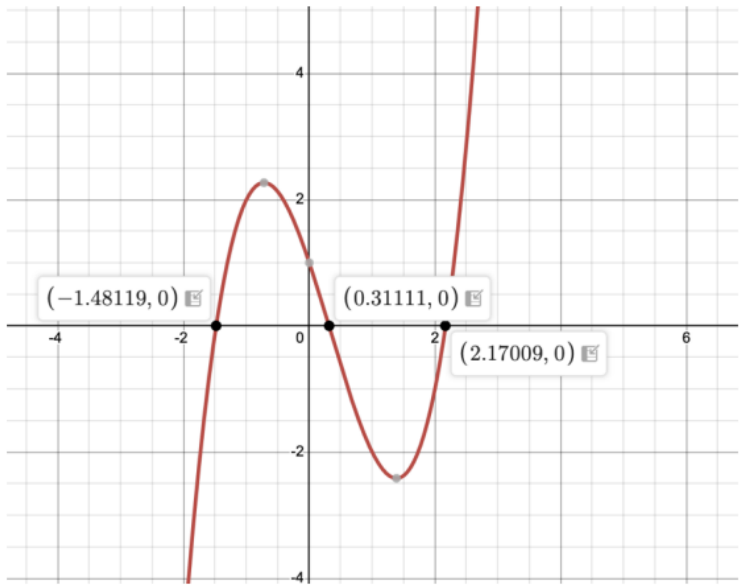
where A, B, C are constants and a, b, c are the roots of x^3-x^2-3x+1. However, this expression is a cubic, or a polynomial of degree 3, that cannot be easily factored. Indeed, the exact roots of the cubic are (Wolfram Alpha, 2024):

$$x=\frac{1}{3}-\frac{5(1+i\sqrt{3})}{3\sqrt[3]{1+3i\sqrt{111}}}-\frac{1}{6}(1-i\sqrt{3})\sqrt[3]{1+3i\sqrt{111}}$$

$$x=\frac{1}{3}-\frac{5(1-i\sqrt{3})}{3\sqrt[3]{1+3i\sqrt{111}}}-\frac{1}{6}(1+i\sqrt{3})\sqrt[3]{1+3i\sqrt{111}}$$

$$x=\frac{1}{3}\left(1+\frac{10}{\sqrt[3]{1+3i\sqrt{111}}}+\sqrt[3]{1+3i\sqrt{111}}\right)$$

Unfortunately, dealing with these roots is beyond the scope of this investigation. Let us instead graph $y=x^3-x^2-3x+1$ (Desmos, 2024) to approximate these roots:



Thus, we have:

$$x^3 - x^2 - 3x + 1 \approx (x - 2.17009)(x - 0.31111)(x + 1.48119)$$

And:

$$G(x) = \frac{1 - x}{x^3 - x^2 - 3x + 1} \approx \frac{1 - x}{(x - 2.17009)(x - 0.31111)(x + 1.48119)}$$

While we can attempt to perform partial fraction decomposition manually, it is much more efficient to use a computer. After all, online tools were already used to approximate the roots of our cubic. Using Wolfram Alpha (2024), we find that:

$$\frac{1 - x}{(x - 2.17009)(x - 0.31111)(x + 1.48119)} \approx \frac{0.206757}{0.31111 - x} + \frac{0.379141}{1.4812 + x} + \frac{0.172385}{2.1701 - x}$$

We can rearrange the right-hand side as so:

$$\frac{0.206757}{0.31111}\left(\frac{1}{1 - \frac{x}{0.31111}}\right) + \frac{0.379141}{1.4812}\left(\frac{1}{1 - \left(-\frac{x}{1.4812}\right)}\right) + \frac{0.172385}{2.1701}\left(\frac{1}{1 - \frac{x}{2.1701}}\right)$$

As mentioned earlier, we can turn this into a geometric series; indeed, the above expression can be written as:

$$\frac{0.206757}{0.31111}\sum_{N=0}^{\infty}\left(\frac{x}{0.31111}\right)^N+\frac{0.379141}{1.4812}\sum_{N=0}^{\infty}\left(-\frac{x}{1.4812}\right)^N+\frac{0.172385}{2.1701}\sum_{N=0}^{\infty}\left(\frac{x}{2.1701}\right)^N$$

Rewriting this as one sum and setting it approximately equal to our original generating function, we have:

$$\sum_{N=0}^{\infty} a_N x^N \approx \sum_{N=0}^{\infty}\left(\frac{0.206757}{0.31111^{N+1}}-\frac{0.379141}{(-1.4812)^{N+1}}+\frac{0.172385}{2.1701^{N+1}}\right)x^N$$

With this, we have an approximate closed-form expression for a_N:

$$\boxed{a_N \approx \frac{0.206757}{0.31111^{N+1}}-\frac{0.379141}{(-1.4812)^{N+1}}+\frac{0.172385}{2.1701^{N+1}}}$$

for $N \geq 0$. We calculated earlier that $a_2 = 7$. Computing a_2 with the above expression, we have:

$$a_2 \approx \frac{0.206757}{0.31111^3}-\frac{0.379141}{(-1.4812)^3}+\frac{0.172385}{2.1701^3}$$

$$a_2 \approx 6.9998$$

Indeed, this closed form expression provides very accurate approximations, at least for small N. The following table contains calculations for a_N using the closed-form solution as well the values from the actual recurrence relation: $a_N = 3a_{N-1} + a_{N-2} - a_{N-3}$.

	Recurrence Relation	Closed-Form (4 d.p.)
a_0	1	1.0000
a_1	2	1.9999
a_2	7	6.9998
a_3	22	21.9991
a_4	71	70.9966
a_5	228	227.9876
a_6	733	732.9549

The choice of 4 d.p. is arbitrary and serves to show precision. It is also important to note that a_N must be an integer since the number of ways to tile a floor is, trivially, an integer. This means that for any approximate value of a_N we compute with the closed-form expression, we should be rounding to the nearest integer. While this confirms the closed-form's validity for the values we've calculated above, at large N the error arising from approximation could lead us to round to incorrect integer values. This effectively defeats the main purpose of a closed-form expression, which is most useful for calculating a_N when N is large – after all, it is easy to directly calculate values for small N using the recurrence relation. While this inaccuracy could be reduced by finding more precise approximations, it is best that we find an exact closed-form expression for a_N.

Luckily, we know that we can find another closed-form expression for a_N using calculus. Namely, for an infinitely differentiable function $f(x)$ that can be written as a generating function $\sum_{i=0}^{\infty} a_i x^i$, we have:

$$a_i = \frac{f^{(i)}(0)}{i!}$$

Where $f^{(i)}(x)$ denotes the i^{th} derivative of $f(x)$. Thus, taking $G(x) = \frac{1-x}{x^3-x^2-3x+1} =$

$\sum_{N=0}^{\infty} a_N x^N$, we have the closed-form expression:

$$\boxed{a_N = \frac{\frac{d^N}{dx^N}\left(\frac{1-x}{x^3-x^2-3x+1}\right)\Big|_{x=0}}{N!}}$$

for $N \geq 0$.

Conclusion

The aim of this exploration was to find a closed-form expression for the number of ways to tile a $2 \times N$ floor, where N is a nonnegative integer, with 1×1 and 2×1 tiles. We sought to solve this combinatorics problem with tools from algebra and calculus, linking these different areas of math in the process. We used counting principles such as casework to derive a recurrence relation for the number of tilings. Next, we encoded this recurrence sequence in a generating function, an algebraic tool from continuous math. We then found a closed-form solution in two ways: firstly, with the help of computer tools, we used partial fractions decomposition and manipulated the generating function such that it corresponded to a geometric sequence. From this we derived an approximate closed-form expression. Secondly, we used ideas from calculus, namely Taylor Series, to attain a closed-form expression in terms of our generating function's derivatives.

An extension to this exploration could include amending or improving upon these limitations. For instance, a further investigation could attempt the partial decomposition of the function $\frac{1-x}{x^3-x^2-3x+1}$ without approximating the roots of $x^3 - x^2 - 3x + 1$ or even without using a computer. Additionally, one could find a closed-form expression for the number of ways to tile a floor of width larger than 2, for instance a $3 \times N$ or $4 \times N$ floor. It might even be plausible to find a general formula, in terms of M and N, for the number of tilings of a $M \times N$ floor (where M and N are nonnegative integers). Adjusting the size of the tiles used and/or possible tile sizes is yet another area for further analysis.

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