MATH1853 Tutorial 4

Matrix Determinant & Cramer’s Rule

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During Tutorial …

- : Examples we will go through together in the tutorial
- : Exercises you should do yourself in the tutorial
- The remaining questions: you can do them after tutorial as a practice

If you cannot follow the tutorial, you can stop me at anytime.

Question 1 - Laplace Expansion

1. Compute the determinant of each of the following matrices:

(a) $A = \begin{bmatrix} 6 & 5 \\ 2 & 3 \end{bmatrix}$

(b) $A = \begin{bmatrix} 4 & -5 \\ -1 & -2 \end{bmatrix}$

(c) $A = \begin{bmatrix} 2 & -3 \\ 4 & 7 \end{bmatrix}$

(d) $A = \begin{bmatrix} 2 & 3 & -4 \\ 4 & 0 & 5 \\ 5 & 1 & 6 \end{bmatrix}$

(e) $A = \begin{bmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{bmatrix}$

(f) $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 4 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

(g) $A = \begin{bmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{bmatrix}$

(h) $A = \begin{bmatrix} 3 & 5 & -8 & 4 \\ 0 & -2 & 3 & -7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

(i) $A = \begin{bmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{bmatrix}$

Question 1 - Laplace Expansion

$\det\left(\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}\right)$?

Consider a matrix $A \in R^{n\times n}$, then for all i, $j = 1,2,\cdots,n$:

- Expansion along ***column*** j

$$det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} det(A_{i,j})$$

$$\begin{bmatrix} 1 & \cancel{5} & \cancel{0} \\ \cancel{2} & 4 & -1 \\ \cancel{0} & -2 & 0 \end{bmatrix} \quad \begin{bmatrix} \cancel{1} & 5 & 0 \\ 2 & \cancel{4} & \cancel{-1} \\ \cancel{0} & -2 & 0 \end{bmatrix} \quad \begin{bmatrix} \cancel{1} & 5 & 0 \\ \cancel{2} & 4 & -1 \\ 0 & \cancel{-2} & \cancel{0} \end{bmatrix}$$

$$\det(A) = \boxed{(-1)^{1+1}\cdot 1\cdot \det\left(\begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix}\right)} + \boxed{(-1)^{1+2}\cdot 2\cdot \det\left(\begin{bmatrix} 5 & 0 \\ -2 & 0 \end{bmatrix}\right)} + \boxed{(-1)^{1+3}\cdot 0\cdot \det\left(\begin{bmatrix} 5 & 0 \\ 4 & -1 \end{bmatrix}\right)}$$

$$= 1\cdot 1\cdot \big(4\cdot 0-(-1)\cdot(-2)\big) + (-1)\cdot 2\cdot \big(5\cdot 0-0\cdot(-2)\big) + 1\cdot 0\cdot (5\cdot(-1)-0\cdot 4)$$

$$= -2+0+0 = -2$$

$A_{i,j}$ is the submatrix obtained by deleting the i-th row and j-th column from A.

Question 1 - Laplace Expansion

$\det\left(\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}\right)$?

Consider a matrix $A \in R^{n \times n}$, then for all $\mathrm{i}, j = 1,2,\cdots,n$:

- Expansion along ***row*** i

$$det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} det(A_{i,j})$$

1	5	0
2	4	-1
0	-2	0

1	5	0
2	4	-1
0	-2	0

1	5	0
2	4	-1
0	-2	0

$$\det(A) = \boxed{(-1)^{1+1} \cdot 1 \cdot \det\left(\begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix}\right)} + \boxed{(-1)^{1+2} \cdot 5 \cdot \det\left(\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}\right)} + \boxed{(-1)^{1+3} \cdot 0 \cdot \det\left(\begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}\right)}$$

$$= 1 \cdot 1 \cdot \big(4 \cdot 0 - (-1) \cdot (-2)\big) + (-1) \cdot 5 \cdot \big(2 \cdot 0 - 0 \cdot (-1)\big) + 1 \cdot 0 \cdot (2 \cdot (-2) - 0 \cdot 4)$$

$$= -2 + 0 + 0 = -2$$

$A_{i,j}$ is the submatrix obtained by deleting the i-th row and j-th column from A.

Question 1 - Laplace Expansion

Consider a matrix $A \in R^{n \times n}$, then for all i, $j = 1,2,\cdots,n$:

- Expansion along column j

$$det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} det(A_{i,j})$$

- Expansion along row i

$$det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} det(A_{i,j})$$

- Notes:

The determinant of a n x n matrix can be computed by a cofactor expansion along ***any row or any column.***

Commonly along the row or column ***with the most zeros***.

$A_{i,j}$ is the submatrix obtained by deleting the i-th row and j-th column from A.

Question 1 - Laplace Expansion

(g) $A = \begin{bmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{bmatrix}$

Solution

$$\text{(g) } \det(A) = (-1)^{3+1} \cdot 2 \cdot \det\left(\begin{bmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 8 \end{bmatrix}\right)$$

$$= 1 \cdot 2 \cdot \left((-1)^{1+3} \cdot 5 \cdot \det\left(\begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}\right)\right) = \ 2 \cdot (5 \cdot 1) = 10$$

Question 1 – 5min Practice

(b) $A = \begin{bmatrix} 4 & -5 \\ -1 & -2 \end{bmatrix}$

(e) $A = \begin{bmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{bmatrix}$

(h) $A = \begin{bmatrix} 3 & 5 & -8 & 4 \\ 0 & -2 & 3 & -7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Question 1 - Laplace Expansion

(b) $A = \begin{bmatrix} 4 & -5 \\ -1 & -2 \end{bmatrix}$

(e) $A = \begin{bmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{bmatrix}$

(h) $A = \begin{bmatrix} 3 & 5 & -8 & 4 \\ 0 & -2 & 3 & -7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Solution

(b) $\det(A) = 4 \cdot (-2) - (-5) \cdot (-1) = -8 - 5 = -13$

(e) $$\begin{aligned}\det(A) &= (-1)^{1+1} \cdot 3 \cdot \det\left(\begin{bmatrix} 3 & 2 \\ 5 & -1 \end{bmatrix}\right) + (-1)^{1+2} \cdot 2 \cdot \det\left(\begin{bmatrix} 0 & 4 \\ 5 & -1 \end{bmatrix}\right) \\ &= 1 \cdot 3 \cdot (3 \cdot (-1) - 2 \cdot 5) + (-1) \cdot 1 \cdot (0 \cdot (-1) - 5 \cdot 4) \\ &= -39 + 40 = 1\end{aligned}$$

(h) $$\begin{aligned}\det(A) &= (-1)^{1+1} \cdot 3 \cdot \det\left(\begin{bmatrix} -2 & 3 & -7 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}\right) \\ &= 1 \cdot 3 \cdot \left((-1)^{3+3} \cdot 2 \cdot \det\left(\begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix}\right)\right) = 3 \cdot (-4) = -12\end{aligned}$$

Question 2 – Use Row Reduction

2. Compute the determinant of each of the following matrices using row reduction:

(a) $A = \begin{bmatrix} 3 & 8 & 6 \\ -2 & -3 & 1 \\ 5 & 10 & 5 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1/2 & -1 & -1/3 \\ 3/4 & 1/2 & -1 \\ 1 & -4 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 4 & 10 & -4 & -1 \end{bmatrix}$

(d) $A = \begin{bmatrix} 3 & -2 & -5 & 4 \\ 1 & -2 & -2 & 3 \\ -2 & 4 & 7 & -3 \\ 2 & -3 & -5 & 8 \end{bmatrix}$

Question 2 – Use Row Reduction

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{vmatrix}$$

1. Adding a multiple of a column/row to another one does not change $\det(A)$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \xrightarrow{R_2 - 4 \cdot R_1} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{vmatrix}$$

2. Multiplication of a column/row with $\lambda \in \mathbb{R}$ scales $\det(A)$ by λ. In particular, $\det(\lambda A) = \lambda^n \det(A)$

$$\begin{vmatrix} 2 \cdot 1 & 0 \\ 0 & 2 \cdot 1 \end{vmatrix} = 2^2 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 4$$

Question 2 – Use Row Reduction

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \to \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{vmatrix}$$

3. Swapping two rows/columns changes the sign of $\det(A)$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \to \begin{vmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{vmatrix} \det\left(\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}\right) = -\det\left(\begin{vmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{vmatrix}\right)$$

4. $\det(A) = \det(A^T)$

$$\det\left(\begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}\right) = 6 = \det\left(\begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}^T\right) = \det(\begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{bmatrix})$$

5. $\det(AB) = \det(A)\det(B)$

$$\det\left(\begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}\right) \cdot \det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

Question 2 – Use Row Reduction

(a) $A = \begin{bmatrix} 3 & 8 & 6 \\ -2 & -3 & 1 \\ 5 & 10 & 5 \end{bmatrix}$

Solution

$$\begin{vmatrix} 3 & 8 & 6 \\ -2 & -3 & 1 \\ 5 & 10 & 5 \end{vmatrix} \xrightarrow{R_2+\frac{2}{3}\cdot R1} \begin{vmatrix} 3 & 8 & 6 \\ 0 & \frac{7}{3} & 5 \\ 5 & 10 & 5 \end{vmatrix} \xrightarrow{R_3-\frac{5}{3}\cdot R_1} \begin{vmatrix} 3 & 8 & 6 \\ 0 & \frac{7}{3} & 5 \\ 0 & -\frac{10}{3} & -5 \end{vmatrix} \xrightarrow{R_3+\frac{10}{7}\cdot R_2} \begin{vmatrix} 3 & 8 & 6 \\ 0 & \frac{7}{3} & 5 \\ 0 & 0 & \frac{15}{7} \end{vmatrix}$$

$$\det(A) = 3 \cdot \frac{7}{3} \cdot \left(\frac{15}{7}\right) = 15$$

Question 2 – 5min Practice

(c) $A = \begin{bmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 4 & 10 & -4 & -1 \end{bmatrix}$

Question 2 – Use Row Reduction

(c) $A = \begin{bmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 4 & 10 & -4 & -1 \end{bmatrix}$

Solution

$$\begin{vmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 4 & 10 & -4 & -1 \end{vmatrix} \xrightarrow{R_2 - \frac{3}{2}\cdot R_1} \begin{vmatrix} 2 & 5 & -3 & -1 \\ 0 & -\frac{15}{2} & \frac{11}{2} & -\frac{3}{2} \\ -6 & 0 & -4 & 9 \\ 4 & 10 & -4 & -1 \end{vmatrix} \xrightarrow{R_3 + 3\cdot R_1}$$

$$\begin{vmatrix} 2 & 5 & -3 & -1 \\ 0 & -\frac{15}{2} & \frac{11}{2} & -\frac{3}{2} \\ 0 & 15 & -13 & 6 \\ 4 & 10 & -4 & -1 \end{vmatrix} \xrightarrow{R_4 - 2\cdot R_1} \begin{vmatrix} 2 & 5 & -3 & -1 \\ 0 & -\frac{15}{2} & \frac{11}{2} & -\frac{3}{2} \\ 0 & 15 & -13 & 6 \\ 0 & 0 & 2 & 1 \end{vmatrix} \xrightarrow{R_3 + 2\cdot R_2}$$

$$\begin{vmatrix} 2 & 5 & -3 & -1 \\ 0 & -\frac{15}{2} & \frac{11}{2} & -\frac{3}{2} \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 2 & 1 \end{vmatrix} \xrightarrow{R_4 + R_2} \begin{vmatrix} 2 & 5 & -3 & -1 \\ 0 & -\frac{15}{2} & \frac{11}{2} & -\frac{3}{2} \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & 4 \end{vmatrix}$$

$$\det(A) = 2 \cdot \left(-\frac{15}{2}\right) \cdot (-2) \cdot 4 = 120$$

Question 3 – Cramer's Rule

3. Solve the following linear systems by Cramer's Rule

a. $\begin{cases} 5x_1 + 7x_2 = 3 \\ 2x_1 + 4x_2 = 1 \end{cases}$

b. $\begin{cases} 3x_1 - 2x_2 = 7 \\ -5x_1 + 6x_2 = -5 \end{cases}$

c. $\begin{cases} 2x_1 = 5 + x_2 \\ 3 + 3x_1 + 2x_2 = 0 \end{cases}$

d. $\begin{cases} 2x_1 - 4x_2 = 7 \\ 3x_1 - 6x_2 = 5 \end{cases}$

e. $\begin{cases} 2x_1 + x_2 = 7 \\ -3x_1 + x_3 = -8 \\ x_2 + 2x_3 = -3 \end{cases}$

f. $\begin{cases} x_1 + 2x_3 = 5 \\ -2x_1 + x_2 - 3x_3 = 2 \\ 4x_1 + x_2 + 8x_3 = 8 \end{cases}$

Question 3

Let A be an invertible $n \times n$ matrix. For any b in R^n, the unique solution x of $Ax = b$ has entries given by

$$x_i = \frac{\det A_i(b)}{\det A}, \mathrm{i} = 1,2, \dots, \mathrm{n}$$

Where $A_i(b)$ is defined as a matrix ***replacing the ith column of A by the vector b***, $A_i(b) = [a_i, \cdots, a_{i-1}, b, a_{i+1}, \cdots, a_n]$

e. $\begin{cases} 2x_1 + x_2 = 7 \\ -3x_1 + x_3 = -8 \\ x_2 + 2x_3 = -3 \end{cases}$

Solution

$$\det A = \begin{vmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} = (-1)^{1+1} \cdot 2 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} + (-1)^{1+2} \cdot 1 \cdot \begin{vmatrix} -3 & 1 \\ 0 & 2 \end{vmatrix} = -2 + 6 = 4$$

$$\det A_1(b) = \begin{vmatrix} 7 & 1 & 0 \\ -8 & 0 & 1 \\ -3 & 1 & 2 \end{vmatrix} = (-1)^{1+1} \cdot 7 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} + (-1)^{1+2} \cdot 1 \cdot \begin{vmatrix} -8 & 1 \\ -3 & 2 \end{vmatrix} = -7 + 13 = 6 \qquad x_1 = \frac{6}{4} = \frac{3}{2}$$

$$\det A_2(b) = \begin{vmatrix} 2 & 7 & 0 \\ -3 & -8 & 1 \\ 0 & -3 & 2 \end{vmatrix} = (-1)^{1+1} \cdot 2 \cdot \begin{vmatrix} -8 & 1 \\ -3 & 2 \end{vmatrix} + (-1)^{1+2} \cdot 7 \cdot \begin{vmatrix} -3 & 1 \\ 0 & 2 \end{vmatrix} = -26 + 42 = 16 \qquad x_2 = \frac{16}{4} = 4$$

$$\det A_3(b) = \begin{vmatrix} 2 & 1 & 7 \\ -3 & 0 & -8 \\ 0 & 1 & -3 \end{vmatrix} = (-1)^{2+1} \cdot (-3) \cdot \begin{vmatrix} 1 & 7 \\ 1 & -3 \end{vmatrix} + (-1)^{2+3} \cdot (-8) \cdot \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = -30 + 16 = -14 \qquad x_3 = \frac{-14}{4} = -\frac{7}{2}$$

Question 3 – 5min Practice

f. $\begin{cases} x_1 + 2x_3 = 5 \\ -2x_1 + x_2 - 3x_3 = 2 \\ 4x_1 + x_2 + 8x_3 = 8 \end{cases}$

Question 3 – Cramer's Rule

f. $\begin{cases} x_1 + 2x_3 = 5 \\ -2x_1 + x_2 - 3x_3 = 2 \\ 4x_1 + x_2 + 8x_3 = 8 \end{cases}$

Solution

$$\mathrm{detA} = \begin{vmatrix} 1 & 0 & 2 \\ -2 & 1 & -3 \\ 4 & 1 & 8 \end{vmatrix} = -1$$

$$\mathrm{detA}_1(\mathrm{b}) = \begin{vmatrix} 5 & 0 & 2 \\ 2 & 1 & -3 \\ 8 & 1 & 8 \end{vmatrix} = 43 \qquad x_1 = \frac{43}{-1} = -43$$

$$\mathrm{detA}_2(\mathrm{b}) = \begin{vmatrix} 1 & 5 & 2 \\ -2 & 2 & -3 \\ 4 & 8 & 8 \end{vmatrix} = 12 \qquad x_2 = \frac{12}{-1} = -12$$

$$\mathrm{detA}_3(\mathrm{b}) = \begin{vmatrix} 1 & 0 & 5 \\ -2 & 1 & 2 \\ 4 & 1 & 8 \end{vmatrix} = -24 \qquad x_3 = \frac{-24}{-1} = 24$$

Question 3 – Rule of Sarrus

$$
\begin{array}{l} +aei \\ +dhc \\ +gbf \\ -gec \\ -ahf \\ -dbi \end{array}
\quad
\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}
\begin{array}{ccc} \\ \\ \\ a & b & c \\ d & e & f \end{array}
$$

e. $\begin{cases} 2x_1 + x_2 = 7 \\ -3x_1 + x_3 = -8 \\ x_2 + 2x_3 = -3 \end{cases}$

Solution

$$\text{detA} = \begin{vmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} = (-1)^{1+1} \cdot 2 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} + (-1)^{1+2} \cdot 1 \cdot \begin{vmatrix} -3 & 1 \\ 0 & 2 \end{vmatrix} = -2 + 6 = 4$$

$$\text{detA} = \begin{vmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 2 \cdot 0 \cdot 2 + (-3) \cdot 1 \cdot 0 + 0 \cdot 1 \cdot 1 - 0 \cdot 0 \cdot 0 - 1 \cdot 1 \cdot 2 - 2 \cdot 1 \cdot (-3) = 0 + 0 + 0 - 0 - 2 + 6 = 4$$

$$\begin{array}{ccc} 2 & 1 & 0 \\ -3 & 0 & 1 \end{array}$$

Question 3 – 5min Practice

$$+aei \quad +dhc \quad +gbf \quad -gec \quad -ahf \quad -dbi$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$
$$\begin{matrix} a & b & c \\ d & e & f \end{matrix}$$

e. $\begin{cases} 2x_1 + x_2 = 7 \\ -3x_1 + x_3 = -8 \\ x_2 + 2x_3 = -3 \end{cases}$

Solution

$$detA_1(b) = \begin{vmatrix} 7 & 1 & 0 \\ -8 & 0 & 1 \\ -3 & 1 & 2 \end{vmatrix} =$$

$$\det A_2(b) = \begin{vmatrix} 2 & 7 & 0 \\ -3 & -8 & 1 \\ 0 & -3 & 2 \end{vmatrix} =$$

$$\det A_3(b) = \begin{vmatrix} 2 & 1 & 7 \\ -3 & 0 & -8 \\ 0 & 1 & -3 \end{vmatrix} =$$

Question 3 – Rule of Sarrus

$$\begin{array}{l} +aei \\ +dhc \\ +gbf \\ -gec \\ -ahf \\ -dbi \end{array} \quad \begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & i \\ a & b & c \\ d & e & f \end{array}$$

e. $\begin{cases} 2x_1 + x_2 = 7 \\ -3x_1 + x_3 = -8 \\ x_2 + 2x_3 = -3 \end{cases}$

Solution

$$detA_1(b) = \begin{vmatrix} 7 & 1 & 0 \\ -8 & 0 & 1 \\ -3 & 1 & 2 \end{vmatrix} = 0 + 0 - 3 \ - 0 \ - 7 + 16 = 6$$

$$\begin{array}{ccc} 7 & 1 & 0 \\ -8 & 0 & 1 \end{array}$$

$$\det A_2(b) = \begin{vmatrix} 2 & 7 & 0 \\ -3 & -8 & 1 \\ 0 & -3 & 2 \end{vmatrix} = -32 + 0 + 0 \ - 0 + 6 + 42 = 16$$

$$\begin{array}{ccc} 2 & 7 & 0 \\ -3 & -8 & 1 \end{array}$$

$$\det A_3(b) = \begin{vmatrix} 2 & 1 & 7 \\ -3 & 0 & -8 \\ 0 & 1 & -3 \end{vmatrix} = 0 - 21 + 0 \ - 0 + 16 - 9 = -14$$

$$\begin{array}{ccc} 2 & 1 & 7 \\ -3 & 0 & -8 \end{array}$$

Summary

- ***Laplace Expansion***
  - Expansion can be conducted ***along any row and column***
  - Commonly along the row or column <u>***with the most zeros***</u>
- ***Use Row Reduction to Compute Matrix Determinate***
  - Linear transformation will not change the determinant
  - $\det(\lambda A) = \lambda^n \cdot \det(A)\ A \in R^{n\times n}$
  - $\det(A^T) = \det(A)$
  - $\det(AB) = \det(A) \cdot \det(B)$
- ***Use Cramer's Rule to solve Linear Systems***
  - $x_i = \frac{\det A_i(b)}{\det A}$
  - Rule of Sarrus