MATH1853 Tutorial 2

Who am I (Again and the last time :P)

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  - ➢Before coming, please send me an email about:
    When will you come and ***what question you have***

* Questions of Tutorial 2 are set by ***LIU Chang*** (lcon7@eee.hku.hk)

Content

** It is necessary for you to check the details of Gaussian Elimination after the tutorial*

- ***Question 1.2 (In detail)***
  - Augmented Matrix Form
  - Gaussian Elimination
  - **R**educed **R**ow **E**chelon **F**orm (RREF)
  - Number of solutions
- ***Question 3.4.5 (Check details yourself)***
  - How to get the general or particular solution
- ***Question 6 (Get a feeling)***
  - Linear combination
  - Linear dependence / independence

Question 1

1. Solve the System of Linear Equation in **Augmented Matrix Form**:

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 8 \\ 3x_1 + 2x_2 + 4x_3 &= 24 \\ x_1 + 3x_2 &= -1 \end{aligned}$$

Key Points:

- Augmented Matrix Form.
- Gaussian Elimination.
- Reduced Row Echelon Form (RREF).

Question 1

1. Solve the System of Linear Equation in **Augmented Matrix Form:**

$$
\begin{aligned}
2x_1 + x_2 + x_3 &= 8 \\
3x_1 + 2x_2 + 4x_3 &= 24 \\
x_1 + 3x_2 &= -1
\end{aligned}
$$

Solution :

Step 1: Build the *Augmented Matrix*

- Coefficients on the left
- Results on the right

$$
\left[\begin{array}{ccc|c}
2 & 1 & 1 & 8 \\
3 & 2 & 4 & 24 \\
1 & 3 & 0 & -1
\end{array}\right]
$$

Question 1

FAQ:

1) What is this doing in each step?
2) Do we have to show notations during our working like R3 + 7R2?
3) From Step 1 to step 2, we can change the order of the rows in any way we like?
4) …

Gaussian Elimination

- Exchange of two equations (rows in the augmented matrix.)
- Multiplication of an equation with a constant $\lambda \in \mathbb{R} \setminus \{0\}$
- Addition of two equations (rows)

1. Solve the System of Linear Equation in **Augmented Matrix Form:**

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 8 \\ 3x_1 + 2x_2 + 4x_3 &= 24 \\ x_1 + 3x_2 &= -1 \end{aligned}$$

Solution :

Step 1: Build the *Augmented Matrix*

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 8 \\ 3 & 2 & 4 & 24 \\ 1 & 3 & 0 & -1 \end{array}\right]$$

Step 2: Solve with *Gaussian Elimination*

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 8 \\ 3 & 2 & 4 & 24 \\ 1 & 3 & 0 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 2 & 1 & 1 & 8 \\ 3 & 2 & 4 & 24 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & -5 & 1 & 10 \\ 0 & -7 & 4 & 27 \end{array}\right] \xrightarrow{R_2/-5}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 1 & -\frac{1}{5} & -2 \\ 0 & -7 & 4 & 27 \end{array}\right] \xrightarrow{R_3+7R_2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 1 & -\frac{1}{5} & -2 \\ 0 & 0 & \frac{13}{5} & 13 \end{array}\right] \xrightarrow{R_3/\frac{13}{5}}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 1 & -\frac{1}{5} & -2 \\ 0 & 0 & 1 & 5 \end{array}\right] \xrightarrow{R_2+\frac{1}{5}R_3} \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{array}\right] \xrightarrow{R_1-3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{array}\right] \rightarrow \begin{cases} 1 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 2 \\ 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 = -1 \\ 0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 = 5 \end{cases}$$

$$\boxed{\begin{aligned} x_1 &= 2 \\ x_2 &= -1 \\ x_3 &= 5 \end{aligned}}$$

Question 1

FAQ:
1) What is this doing in each step?
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3) From Step 1 to step 2, we can change the order of the rows in any way we like?
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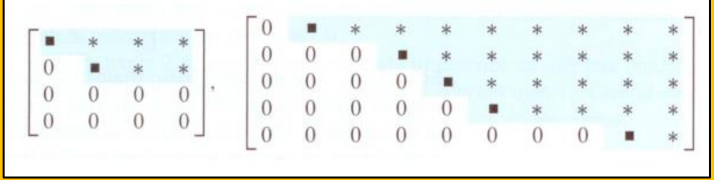
Reduced Row Echelon Form (RREF)

- It is in row-echelon form



- Every pivot is 1
- The pivot is the only nonzero entry in its column

➢ All rows that contain only zeros are at the bottom of the matrix.
➢ All rows that contain at least one nonzero element are on top of rows that contain only zeros.
➢ For nonzero rows, the first nonzero number from the left (pivot) is always strictly to the right of the pivot of the row above it.

1. Solve the System of Linear Equation in **Augmented Matrix Form:**

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 8 \\ 3x_1 + 2x_2 + 4x_3 &= 24 \\ x_1 + 3x_2 &= -1 \end{aligned}$$

Solution :

Step 1: Build the *Augmented Matrix*

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 8 \\ 3 & 2 & 4 & 24 \\ 1 & 3 & 0 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 8 \\ 3 & 2 & 4 & 24 \\ 1 & 3 & 0 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 2 & 1 & 1 & 8 \\ 3 & 2 & 4 & 24 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & -5 & 1 & 10 \\ 0 & -7 & 4 & 27 \end{array}\right] \xrightarrow{R_2/-5}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 1 & -\frac{1}{5} & -2 \\ 0 & -7 & 4 & 27 \end{array}\right] \xrightarrow{R_3+7R_2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 1 & -\frac{1}{5} & -2 \\ 0 & 0 & \frac{13}{5} & 13 \end{array}\right] \xrightarrow{R_3/\frac{13}{5}}$$

$$\boxed{\begin{aligned} x_1 &= 2 \\ x_2 &= -1 \\ x_3 &= 5 \end{aligned}}$$

Step 2: Solve with *Gaussian Elimination*

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 1 & -\frac{1}{5} & -2 \\ 0 & 0 & 1 & 5 \end{array}\right] \xrightarrow{R_2+\frac{1}{5}R_3} \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{array}\right] \xrightarrow{R_1-3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{array}\right] \to \begin{cases} 1 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 2 \\ 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 = -1 \\ 0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 = 5 \end{cases}$$

Question 1

FAQ:
1) What is this doing in each step?
2) Do we have to show notations during our working like R3 + 7R2?
3) From Step 1 to step 2, we can change the order of the rows in any way we like?
4) …

1. Solve the System of Linear Equation in **Augmented Matrix Form:**

$$2x_1 + x_2 + x_3 = 8$$
$$3x_1 + 2x_2 + 4x_3 = 24$$
$$x_1 + 3x_2 = -1$$

FAQ

I know what is Gaussian Elimination, I know what is RREF. ***BUT I don't know how to get RREF by using Gaussian Elimination….***

Solution :

Step 1: Build the *Augmented Matrix*

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 8 \\ 3 & 2 & 4 & 24 \\ 1 & 3 & 0 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 8 \\ 3 & 2 & 4 & 24 \\ 1 & 3 & 0 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 2 & 1 & 1 & 8 \\ 3 & 2 & 4 & 24 \end{array}\right] \boxed{\xrightarrow[R_3-3R_1]{R_2-2R_1}} \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & -5 & 1 & 10 \\ 0 & -7 & 4 & 27 \end{array}\right] \boxed{\xrightarrow{R_2/-5}}$$

$$x_1 = 2$$
$$x_2 = -1$$
$$x_3 = 5$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 1 & -\frac{1}{5} & -2 \\ 0 & -7 & 4 & 27 \end{array}\right] \boxed{\xrightarrow{R_3+7R_2}} \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 1 & -\frac{1}{5} & -2 \\ 0 & 0 & \frac{13}{5} & 13 \end{array}\right] \boxed{\xrightarrow{R_3/\frac{13}{5}}}$$

Why???

Step 2: Solve with *Gaussian Elimination*

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 1 & -\frac{1}{5} & -2 \\ 0 & 0 & 1 & 5 \end{array}\right] \boxed{\xrightarrow{R_2+\frac{1}{5}R_3}} \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{array}\right] \boxed{\xrightarrow{R_1-3R_2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{array}\right] \rightarrow \begin{cases} 1 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 2 \\ 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 = -1 \\ 0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 = 5 \end{cases}$$

Reference

- How to use Gaussian Elimination to get RREF matrix?

If you are ***patient*** enough, Gaussian elimination is quite easy, and ***can be done in a fixed process***.

https://www.dummies.com/education/math/calculus/how-to-use-gaussian-elimination-to-solve-systems-of-equations/

FAQ:

1) What method should we use in the exam?
2) Can I use other methods?
3) Why should we use Gaussian Elimination, it seems quite complex…

According to myself, I suggest you to master at least the following two methods:

1) Gaussian Elimination
2) Inverse matrix

You can choose the method you like if the questions do not have specific requirements.

You will learn the advantage of Gaussian Elimination when you are doing your HW…

Question 2

2. *(18 Dec. final)*

a. Solve the following system of linear equations:

$$\left[\begin{array}{rrrr|r} 3 & -8 & 1 & 5 & 0 \\ 1 & -3 & -2 & -1 & 6 \\ -2 & 1 & -4 & 1 & -12 \\ -1 & 4 & -1 & -3 & 2 \end{array}\right]$$

b. Under what constraint(s) will the following system of lienar equations have solution(s)? Next, please find the solution(s):

$$\left[\begin{array}{rrrr|r} 1 & -1 & -1 & 5 & 5 \\ 2 & -2 & -3 & 8 & 12 \\ 1 & -1 & 0 & 7 & a+5 \\ -2 & 2 & 5 & -4 & -16 \end{array}\right]$$

Key Points:

- Number of solutions.
- General solutions.
- Non-trivial solutions.

Question 2

2. *(18 Dec. final)*

a. Solve the following system of linear equations:

$$\left[\begin{array}{cccc|c} 3 & -8 & 1 & 5 & 0 \\ 1 & -3 & -2 & -1 & 6 \\ -2 & 1 & -4 & 1 & -12 \\ -1 & 4 & -1 & -3 & 2 \end{array}\right]$$

FAQ:
Can we always make the left side of augmented matrix an identity matrix?

Number of solutions:

- If there is row 0=b (b≠0), no solution
- Else if there is free variable, infinite solutions
- Else, unique solution

Solution :

$$\left[\begin{array}{cccc|c} 3 & -8 & 1 & 5 & 0 \\ 1 & -3 & -2 & -1 & 6 \\ -2 & 1 & -4 & 1 & -12 \\ -1 & 4 & -1 & -3 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & -3 & -2 & -1 & 6 \\ 3 & -8 & 1 & 5 & 0 \\ -2 & 1 & -4 & 1 & -12 \\ -1 & 4 & -1 & -3 & 2 \end{array}\right] \xrightarrow[R_4+R_1]{R_2-3R_1;R_3+2R_1}$$

$$\left[\begin{array}{cccc|c} 1 & -3 & -2 & -1 & 6 \\ 0 & 1 & 7 & 8 & -18 \\ 0 & -5 & -8 & -1 & 0 \\ 0 & 1 & -3 & -4 & 8 \end{array}\right] \xrightarrow[R_4-R_2]{R_3+5R_2} \left[\begin{array}{cccc|c} 1 & -3 & -2 & -1 & 6 \\ 0 & 1 & 7 & 8 & -18 \\ 0 & 0 & 27 & 39 & -90 \\ 0 & 0 & -10 & -12 & 26 \end{array}\right] \xrightarrow{R_3/27}$$

$$\left[\begin{array}{cccc|c} 1 & -3 & -2 & -1 & 6 \\ 0 & 1 & 7 & 8 & -18 \\ 0 & 0 & 1 & \frac{13}{9} & -\frac{10}{3} \\ 0 & 0 & -10 & -12 & 26 \end{array}\right] \xrightarrow{R_4+10R_3} \left[\begin{array}{cccc|c} 1 & -3 & -2 & -1 & 6 \\ 0 & 1 & 7 & 8 & -18 \\ 0 & 0 & 1 & \frac{13}{9} & -\frac{10}{3} \\ 0 & 0 & 0 & \frac{22}{9} & -\frac{22}{3} \end{array}\right] \xrightarrow{R_4/\frac{22}{9}}$$

$$\left[\begin{array}{cccc|c} 1 & -3 & -2 & -1 & 6 \\ 0 & 1 & 7 & 8 & -18 \\ 0 & 0 & 1 & \frac{13}{9} & -\frac{10}{3} \\ 0 & 0 & 0 & 1 & -3 \end{array}\right] \xrightarrow[R_3-\frac{13}{9}R_4]{R1+R_4;R_2-8R_4} \left[\begin{array}{cccc|c} 1 & -3 & -2 & 0 & 3 \\ 0 & 1 & 7 & 0 & 6 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right] \xrightarrow[R_2-7R_3]{R1+2R_3}$$

$$\left[\begin{array}{cccc|c} 1 & -3 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right] \xrightarrow{R1+3R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right]$$

$$\longrightarrow \begin{cases} 1\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 + 0\cdot x_4 = 2 \\ 0\cdot x_1 + 1\cdot x_2 + 0\cdot x_3 + 0\cdot x_4 = -1 \\ 0\cdot x_1 + 0\cdot x_2 + 1\cdot x_3 + 0\cdot x_4 = 1 \\ 0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 + 1\cdot x_4 = -3 \end{cases}$$

$$\boxed{\begin{aligned} x_1 &= 2 \\ x_2 &= -1 \\ x_3 &= 1 \\ x_4 &= -3 \end{aligned}}$$

Question 2

FAQ:

1) Why is x2 and x4 free variable?
2) Just set the free variables to zero?
3) Why x2 and x4 are replaced by alpha and beta?
4) How to find the vectors?
5) If the bottom raw is all zero, why should we write it? …

b. Under what constraint(s) will the following system of linear equations have solution(s)? Next, please find the solution(s):

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 5 & 5 \\ 2 & -2 & -3 & 8 & 12 \\ 1 & -1 & 0 & 7 & a+5 \\ -2 & 2 & 5 & -4 & -16 \end{array}\right]$$

Number of solutions:

- If there is row 0=b (b≠ 0), no solution
- Else if there is free variables, infinite solutions
- Else, unique solution

➢ ***Free variables***:
The variables whose ***columns*** in the RREF ***contain leading 1's are called leading variables***. A variable whose column in the RREF ***does not contain a leading 1 is called a <u>free variable</u>***.

➢ ***Non-trivial solutions***:
If all elements in the solutions are zero, the solutions are considered trivial. ***<u>Nonzero solutions</u>*** or examples are considered nontrivial.

Solution :

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 5 & 5 \\ 2 & -2 & -3 & 8 & 12 \\ 1 & -1 & 0 & 7 & a+5 \\ -2 & 2 & 5 & -4 & -16 \end{array}\right] \xrightarrow[R_4+2R_1]{R_2-2R_1;R_3-R_1} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 5 & 5 \\ 0 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & 2 & a \\ 0 & 0 & 3 & 6 & -6 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 5 & 5 \\ 0 & 0 & 1 & 2 & a \\ 0 & 0 & -1 & -2 & 2 \\ 0 & 0 & 3 & 6 & -6 \end{array}\right] \xrightarrow[R_4-3R_2]{R_3+R_2} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 5 & 5 \\ 0 & 0 & 1 & 2 & a \\ 0 & 0 & 0 & 0 & 2+a \\ 0 & 0 & 0 & 0 & -6-3a \end{array}\right] \xrightarrow{R_1+R_2}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 7 & 5+a \\ 0 & 0 & 1 & 2 & a \\ 0 & 0 & 0 & 0 & 2+a \\ 0 & 0 & 0 & 0 & -6-3a \end{array}\right] \longrightarrow \begin{cases} 1\cdot x_1 + (-1)\cdot x_2 + 0\cdot x_3 + 7\cdot x_4 = 3 \\ 0\cdot x_1 + 0\cdot x_2 + 1\cdot x_3 + 2\cdot x_4 = -2 \end{cases}$$

- If $a=-2$: $\left[\begin{array}{cccc|c} 1 & -1 & 0 & 7 & 3 \\ 0 & 0 & 1 & 2 & -2 \end{array}\right]$
- <u>*Free variables*</u>: $x_2,\ x_4$
- Find the variable x_3 from the ***equation 2*** of the system
  $$x_3 = -2 - 2x_4$$
- Find the variable x_1 from the ***equation 1*** of the system
  $$x_1 = 3 + x_2 - 7x_4$$

$$x = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 7 \\ 0 \\ 2 \\ -1 \end{bmatrix}$$

Answer:

$$\begin{aligned} x_1 &= 3 + 1x_2 - 7x_4 \\ x_2 &= 0 + 1x_2 + 0x_4 \\ x_3 &= -2 + 0x_2 - 2x_4 \\ x_4 &= 0 + 0x_2 + 1x_4 \end{aligned}$$

$$x = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -7 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Question 2

FAQ:
1) Why is x2 and x4 free variable?
2) Just set the free variables to zero?
3) Why x2 and x4 are replaced by alpha and beta?
4) How to find the vectors?
5) If the bottom raw is all zero, why should we write it? …

b. Under what constraint(s) will the following system of linear equations have solution(s)? Next, please find the solution(s):

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Solution :

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 5 & 5 \\ 2 & -2 & -3 & 8 & 12 \\ 1 & -1 & 0 & 7 & a+5 \\ -2 & 2 & 5 & -4 & -16 \end{array}\right] \xrightarrow[R_4+2R_1]{R_2-2R_1;R_3-R_1} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 5 & 5 \\ 0 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & 2 & a \\ 0 & 0 & 3 & 6 & -6 \end{array}\right] \rightarrow$$

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$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 7 & 5+a \\ 0 & 0 & 1 & 2 & a \\ 0 & 0 & 0 & 0 & 2+a \\ 0 & 0 & 0 & 0 & -6-3a \end{array}\right] \longrightarrow \begin{cases} 1\cdot x_1 + (-1)\cdot x_2 + 0\cdot x_3 + 7\cdot x_4 = 3 \\ 0\cdot x_1 + 0\cdot x_2 + 1\cdot x_3 + 2\cdot x_4 = -2 \end{cases}$$

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- *<u>Free variables</u>*: $x_2,\ x_4$
- Find the variable x_3 from the ***equation 2*** of the system
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- Find the variable x_1 from the ***equation 1*** of the system
$$x_1 = 3 + x_2 - 7x_4$$

$$x = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 7 \\ 0 \\ 2 \\ -1 \end{bmatrix}$$

$\cdot(-1)$ $\cdot(-1)$

Answer:

$$\begin{aligned} x_1 &= 3 + 1x_2 - 7x_4 \\ x_2 &= 0 + 1x_2 + 0x_4 \\ x_3 &= -2 + 0x_2 - 2x_4 \\ x_4 &= 0 + 0x_2 + 1x_4 \end{aligned}$$

$$x = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -7 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Keep the inner relationship in the vectors!!

Question 2

FAQ:
1) Why is x2 and x4 free variable?
2) Just set the free variables to zero?
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Example

$x_2 = 5 \cdot x_1$

1) $x_1 = 0, x_2 = 0$

2) $x_1 = 1, x_2 = 5$

Number of solutions:

- If there is row 0=b (b≠ 0), no solution
- Else if there is free variables, infinite solutions
- Else, unique solution

➢ ***Free variables***:
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Solution :

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$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 7 & 5+a \\ 0 & 0 & 1 & 2 & a \\ 0 & 0 & 0 & 0 & 2+a \\ 0 & 0 & 0 & 0 & -6-3a \end{array}\right] \longrightarrow \begin{cases} 1 \cdot x_1 + (-1) \cdot x_2 + 0 \cdot x_3 + 7 \cdot x_4 = \;\; 3 \\ 0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 2 \cdot x_4 = -2 \end{cases}$$

- If $a = -2$: $\left[\begin{array}{cccc|c} 1 & -1 & 0 & 7 & 3 \\ 0 & 0 & 1 & 2 & -2 \end{array}\right]$
- <u>*Free variables*</u>: $x_2, \; x_4$
- Find the variable x_3 from the ***equation 2*** of the system
  $$x_3 = -2 - 2x_4$$
- Find the variable x_1 from the ***equation 1*** of the system
  $$x_1 = 3 + x_2 - 7x_4$$

Answer:

$$\begin{aligned} x_1 &= 3 + 1x_2 - 7x_4 \\ x_2 &= 0 + 1x_2 + 0x_4 \\ x_3 &= -2 + 0x_2 - 2x_4 \\ x_4 &= 0 \; + 0x_2 + 1x_4 \end{aligned}$$

$$x = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -7 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$x = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 7 \\ 0 \\ 2 \\ -1 \end{bmatrix}$$

$\cdot (-1)$ $\qquad$ $\cdot (-1)$

Keep the inner relationship in the vectors!!

Question 2

FAQ:
1) Why is x2 and x4 free variable?
2) Just set the free variables to zero?
3) Why x2 and x4 are replaced by alpha and beta?
4) How to find the vectors?
5) If the bottom raw is all zero, why should we write it? …

b. Under what constraint(s) will the following system of linear equations have solution(s)? Next, please find the solution(s):

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 5 & 5 \\ 2 & -2 & -3 & 8 & 12 \\ 1 & -1 & 0 & 7 & a+5 \\ -2 & 2 & 5 & -4 & -16 \end{array}\right]$$

FAQ

If a system of linear equations has general solutions, it has infinite solutions?

Solution :

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 5 & 5 \\ 2 & -2 & -3 & 8 & 12 \\ 1 & -1 & 0 & 7 & a+5 \\ -2 & 2 & 5 & -4 & -16 \end{array}\right] \xrightarrow[R_4+2R_1]{R_2-2R_1;R_3-R_1} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 5 & 5 \\ 0 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & 2 & a \\ 0 & 0 & 3 & 6 & -6 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 5 & 5 \\ 0 & 0 & 1 & 2 & a \\ 0 & 0 & -1 & -2 & 2 \\ 0 & 0 & 3 & 6 & -6 \end{array}\right] \xrightarrow[R_4-3R_2]{R_3+R_2} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 5 & 5 \\ 0 & 0 & 1 & 2 & a \\ 0 & 0 & 0 & 0 & 2+a \\ 0 & 0 & 0 & 0 & -6-3a \end{array}\right] \xrightarrow{R_1+R_2}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 7 & 5+a \\ 0 & 0 & 1 & 2 & a \\ 0 & 0 & 0 & 0 & 2+a \\ 0 & 0 & 0 & 0 & -6-3a \end{array}\right] \longrightarrow \begin{cases} 1\cdot x_1 + (-1)\cdot x_2 + 0\cdot x_3 + 7\cdot x_4 = 3 \\ 0\cdot x_1 + 0\cdot x_2 + 1\cdot x_3 + 2\cdot x_4 = -2 \end{cases}$$

- If $a = -2$: $\left[\begin{array}{cccc|c} 1 & -1 & 0 & 7 & 3 \\ 0 & 0 & 1 & 2 & -2 \end{array}\right]$
- *Free variables*: $x_2,\ x_4$
- Find the variable x_3 from the ***equation 2*** of the system
  $$x_3 = -2 - 2x_4$$
- Find the variable x_1 from the ***equation 1*** of the system
  $$x_1 = 3 + x_2 - 7x_4$$

$$x = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 7 \\ 0 \\ 2 \\ -1 \end{bmatrix}$$

Answer:

$$\begin{aligned} x_1 &= 3 + 1x_2 - 7x_4 \\ x_2 &= 0 + 1x_2 + 0x_4 \\ x_3 &= -2 + 0x_2 - 2x_4 \\ x_4 &= 0 + 0x_2 + 1x_4 \end{aligned}$$

$$x = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -7 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Question 3

3. Find the set $\mathcal{S}$ of all solutions in x of the following linear systems $\mathbf{A}\mathbf{x} = \mathbf{b}$, where $\mathbf{A}$ and $\mathbf{b}$ are defined as follows:

a. $$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 2 & 5 & -7 & -5 \\ 2 & -1 & 1 & 3 \\ 5 & 2 & -4 & 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 4 \\ 6 \end{bmatrix}$$

b. $$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -3 & 0 \\ 2 & -1 & 0 & 1 & -1 \\ -1 & 2 & 0 & -2 & -1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 6 \\ 5 \\ -1 \end{bmatrix}$$

a) Solution :

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 2 & 5 & -7 & -5 & -2 \\ 2 & -1 & 1 & 3 & 4 \\ 5 & 2 & -4 & 2 & 6 \end{array}\right] \xrightarrow[R_4-5R_1]{R_2-2R_1;R_3-2R_1} \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & -3 & 3 & 5 & 2 \\ 0 & -3 & 1 & 7 & 1 \end{array}\right] \xrightarrow[R_4+R_2]{R_3+R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & 0 & -2 & 2 & -2 \\ 0 & 0 & -4 & 4 & -3 \end{array}\right] \xrightarrow{R_4-2R_3} \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & 0 & -2 & 2 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$0 \neq 1$
No solution

b) Solution :

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 1 & 1 & 0 & -3 & 0 & 6 \\ 2 & -1 & 0 & 1 & -1 & 5 \\ -1 & 2 & 0 & -2 & -1 & -1 \end{array}\right] \xrightarrow[R_4+R_1]{R_2-R_1;R_3-2R_1} \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & 2 & 0 & -3 & -1 & 3 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & 1 & 0 & -2 & 0 & 2 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & 2 & 0 & -3 & -1 & 3 \\ 0 & 1 & 0 & -2 & 0 & 2 \end{array}\right] \xrightarrow[R_4-R_2]{R_1+R_2;R_3-2R_2} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & -5 & 5 & 5 \\ 0 & 0 & 0 & -3 & 3 & 3 \end{array}\right] \xrightarrow[R_4/-3]{R_3/-5}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{array}\right] \xrightarrow[R_4-R_3]{R_1-R_3;R_2-R_3} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\longrightarrow \begin{cases} 1 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 + (-1) \cdot x_5 = 3 \\ 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 + (-2) \cdot x_5 = 0 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 + (-1) \cdot x_5 = -1 \end{cases}$$

- Find the variable x_4 from the ***equation 3*** of the system
$$x_4 = -1 + x_5$$
- Find the variable x_2 from the ***equation 2*** of the system
$$x_2 = 2x_5$$
- Find the variable x_1 from the ***equation 1*** of the system
$$x_1 = 3 + x_5$$

Answer:

$$\begin{aligned} x_1 &= 3 + x_5 \\ x_2 &= 2x_5 \\ x_3 &= x_3 \\ x_4 &= -1 + x_5 \\ x_5 &= x_5 \end{aligned}$$

$$x = \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Question 4

4. Using Gaussian elimination, find all solutions of the equation system $\mathbf{A}\mathbf{x} = \mathbf{b}$ with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Solution :

$$\left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{-R_3+R_1} \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right] \xrightarrow[R_2-R_3]{R_1-R_3} \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

$$\longrightarrow \begin{cases} 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 + 1 \cdot x_6 = 1 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 + 0 \cdot x_5 + 1 \cdot x_6 = -2 \\ 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 1 \cdot x_5 + (-1) \cdot x_6 = 1 \end{cases}$$

- Find the variable x_5 from the ***equation 3*** of the system

$$x_5 = 1 + x_6$$

- Find the variable x_4 from the ***equation 2*** of the system

$$x_4 = -2 - x_6$$

- Find the variable x_5 from the ***equation 1*** of the system

$$x_2 = 1 - x_6$$

Answer:

$$\begin{aligned} x_1 &= x_1 \\ x_2 &= 1 - x_6 \\ x_3 &= x_3 \\ x_4 &= -2 - x_6 \\ x_5 &= 1 + x_6 \\ x_6 &= x_6 \end{aligned}$$

General solution:

$$x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \lambda_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_3 \cdot \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

Question 5

5. Find all solutions in $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ of the equation system $\mathbf{A}\mathbf{x} = 12\mathbf{x}$,

where

$$\mathbf{A} = \begin{bmatrix} 6 & 4 & 3 \\ 6 & 0 & 9 \\ 0 & 8 & 0 \end{bmatrix}$$

and $\sum_{i=1}^{3} x_i = 1$.

Solution :

$$\mathrm{Ax} = \begin{bmatrix} 6 & 4 & 3 \\ 6 & 0 & 9 \\ 0 & 8 & 0 \end{bmatrix} x = 12x = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix} x \longrightarrow \left(\begin{bmatrix} 6 & 4 & 3 \\ 6 & 0 & 9 \\ 0 & 8 & 0 \end{bmatrix} - \begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix} \right) x = \begin{bmatrix} -6 & 4 & 3 \\ 6 & -12 & 9 \\ 0 & 8 & -12 \end{bmatrix} x = 0$$

$$\left[\begin{array}{ccc|c} -6 & 4 & 3 & 0 \\ 6 & -12 & 9 & 0 \\ 0 & 8 & -12 & 0 \end{array}\right] \xrightarrow{R_1 \cdot (-\frac{1}{6})} \left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & -\frac{1}{2} & 0 \\ 6 & -12 & 9 & 0 \\ 0 & 8 & -12 & 0 \end{array}\right] \xrightarrow{R_2 - 6 \cdot R_1}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & -\frac{1}{2} & 0 \\ 0 & -8 & 12 & 0 \\ 0 & 8 & -12 & 0 \end{array}\right] \xrightarrow{R_2 \cdot (-\frac{1}{8})} \left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 8 & -12 & 0 \end{array}\right] \xrightarrow{R_3 - 8 \cdot R_2}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 - (-\frac{2}{3}) \cdot R_2} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{3}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{cases} 1 \cdot x_1 + 0 \cdot x_2 + \left(-\frac{3}{2}\right) \cdot x_3 = 0 \\ 0 \cdot x_1 + 1 \cdot x_2 + \left(-\frac{3}{2}\right) \cdot x_3 = 0 \end{cases}$$

- Find the variable x_2 from the ***equation 2*** of the system

$$x_2 = \frac{3}{2} \cdot x_3$$

- Find the variable x_3 from the ***equation 3*** of the system

$$x_1 = \frac{3}{2} x_3$$

Answer:

$$x_1 = \frac{3}{2} \cdot x_3$$
$$x_2 = \frac{3}{2} \cdot x_3$$
$$x_3 = x_3$$

$$\boxed{x = \lambda \cdot \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix} (\lambda \neq 0) \text{ or } x = 0}$$

Because $\sum x_i = 1,\ x = \begin{bmatrix} \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \end{bmatrix}^T$.

Question 6

6. *(Linear Independence)* Consider $\mathbb{R}^4$ vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \\ 1 \end{bmatrix},$$

check whether they are linearly dependent.

Solution :

$$\lambda_1 \mathbf{x_1} + \lambda_2 \mathbf{x_2} + \lambda_3 \mathbf{x_3} = \mathbf{0}$$

solve for λ_1, λ_2 and λ_3

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & 1 & -2 & 0 \\ -3 & 0 & 1 & 0 \\ 4 & 2 & 1 & 0 \end{array}\right] \xrightarrow{Gaussian Elimination} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Only when $\lambda_1 = \lambda_2 = \lambda_3 = 0$ does the equation hold.
So they are ***linearly independent***.

Linear Combination

If $v = \lambda_1 x_1 + \cdots + \lambda_k x_k$, then we say v is a linear combination of the vectors $x_1, \dots, x_k$

- Example:

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$v = 1 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Linear Independence

If there is a non-trivial linear combination, such that $0 = \sum_{i=1}^{k} \lambda_i x_i$, with at least one $\lambda_i \neq 0$, then the vectors are ***linearly dependent***. Otherwise, they are ***linear independent***.

- Example:

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, x_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$0 = 1 \cdot x_1 + 1 \cdot x_2 + (-1) \cdot x_4$. ($x_1, x_2, x_4$: linearly dependent)

$0 = 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3$. ($x_1, x_2, x_3$: linearly independent)

Question 6

6. *(Linear Independence)* Consider $\mathbb{R}^4$ vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \\ 1 \end{bmatrix},$$

check whether they are linearly dependent.

Solution :

$$\lambda_1\mathbf{x_1} + \lambda_2\mathbf{x_2} + \lambda_3\mathbf{x_3} = \mathbf{0}$$

solve for λ_1, λ_2 and λ_3

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & 1 & -2 & 0 \\ -3 & 0 & 1 & 0 \\ 4 & 2 & 1 & 0 \end{array}\right] \xrightarrow{GaussianElimination} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$v = 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Only when $\lambda_1 = \lambda_2 = \lambda_3 = 0$ does the equation hold.
So they are ***linearly independent***.

Linear Combination

If $v = \lambda_1 x_1 + \cdots + \lambda_k x_k$, then we say v is a linear combination of the vectors $x_1, \ldots, x_k$

- Example:

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$v = 1 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Linear Independence

If there is a ***<u>non-trivial linear combination</u>***, such that $0 = \sum_{i=1}^{k} \lambda_i x_i$, with at least one $\lambda_i \neq 0$, then the vectors are ***linearly dependent***. Otherwise, they are ***linear independent***.

- Example:

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, x_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$0 = 1 \cdot x_1 + 1 \cdot x_2 + (-1) \cdot x_4$. ($x_1, x_2, x_4$: linearly dependent)

$0 = 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3$. ($x_1, x_2, x_3$: linearly independent)

Summary

- Augmented Matrix Form
  - ❑ Coefficient on the left
  - ❑ Results on the right
- Gaussian Elimination
- Reduced Row Echelon Form (RREF)
- Discussion of the Solutions
  - ❑ Number of solutions: None/Infinite/Unique
  - ❑ General solution: free variables, representatives
  - ❑ Nontrivial solution
- Linear combination/ dependence/ independence
  - ◘ Closely related to the solutions of a specific linear system