The Works

Lemma: Limit Comparison Test
Let $(a_n)_n, (b_n)_n$ *be positive real sequences such that*

$$0 < \lim_{n\to\infty} \frac{a_n}{b_n} = L \quad \textit{then} \quad \sum_n^\infty a_n \text{ converges} \Leftrightarrow \sum_n^\infty b_n \text{ converges}$$

Proof: By the characterization of limits (2.17) we have

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} \Leftrightarrow (\exists L)(\forall \varepsilon > 0)(\exists N)(\forall n > N)\, d(\frac{a_n}{b_n}, L) < \varepsilon$$

We set $\varepsilon = L/2$ and let d be the natural metric then we get

$$d(\frac{a_n}{b_n}, L) < \varepsilon \Rightarrow d(\frac{a_n}{b_n}, L) < \frac{L}{2} \Rightarrow \left|\frac{a_n}{b_n} - L\right| < \frac{L}{2}$$

We solve the absolute value, add L and take the inequalities times b_n so

$$\left|\frac{a_n}{b_n} - L\right| < \frac{L}{2} \Rightarrow -\frac{L}{2} < \frac{a_n}{b_n} - L < \frac{L}{2} \Rightarrow \frac{1}{2}L < \frac{a_n}{b_n} < \frac{3}{2}L \Rightarrow \left(\frac{1}{2}L\right)b_n < a_n < \left(\frac{3}{2}L\right)b_n$$

We can now prove the actual equivalence we have implied.
First the left side $\sum_n^\infty a_n$ converges $\Rightarrow \sum_n^\infty b_n$ converges

$$\sum_n^\infty a_n \text{ converges} \overbrace{\Rightarrow}^{\text{comparison (2.32)}} \sum_n^\infty \left(\frac{1}{2}L\right)b_n \text{ converges} \overbrace{\Rightarrow}^{\text{constant (2.40)}} \sum_n^\infty b_n \text{ converges}$$

The right side is very similar $\sum_n^\infty b_n$ converges $\Rightarrow \sum_n^\infty a_n$ converges

$$\sum_n^\infty b_n \text{ converges} \overbrace{\Rightarrow}^{\text{constant (2.40)}} \sum_n^\infty \left(\frac{3}{2}L\right)b_n \text{ converges} \overbrace{\Rightarrow}^{\text{comparison (2.32)}} \sum_n^\infty a_n \text{ converges}$$

Remark: *If* $L = 0$ *then* $\sum_n^\infty b_n$ converges $\Rightarrow \sum_n^\infty a_n$ converges *still holds.*

[Examples of Convergence with the LCT]

$$a_n = \frac{1}{n^2 - 9n + 31} \wedge b_n = \frac{1}{n^2} \Rightarrow \sum_n^\infty a_n \text{ converges} \qquad a_n = \frac{1}{2^n - 1} \wedge b_n = \frac{1}{2^n} \Rightarrow \sum_n^\infty a_n \text{ converges}$$

$$a_n = \frac{n-1}{n^3 - 1} \wedge b_n = \frac{1}{n^2} \Rightarrow \sum_n^\infty a_n \text{ converges} \qquad a_n = \frac{1 + 2^n}{1 + 3^n} \wedge b_n = \left(\frac{2}{3}\right)^n \Rightarrow \sum_n^\infty a_n \text{ converges}$$

Hint: Find a suitable b_n by keeping only the **highest power** in numerator and denominator!

[Other Examples of Convergence] These all converge:

$$\sum_{n=0}^\infty \frac{1}{n!} x^n = e^x \qquad \sum_{n=1}^\infty (-1)^{n+1} \frac{1}{n} \approx .6931... \qquad (\forall p > 1) \sum_{n=1}^\infty \frac{1}{n^p}$$

$$\sum_{n=1}^\infty \frac{n!}{n^n} \textit{ by ratio} \qquad \sum_{n=1}^\infty \frac{c^n}{n!} \textit{ by ratio} \qquad \sum_{n=1}^\infty \frac{n^2}{2^n} \textit{ by root}$$