Continuity of Functions

[Neighborhood] $U_\varepsilon(a)=\{x\in X : d(a,x)<\varepsilon\}$ **[Acc. Point a of S]** 3.1: $(\forall\varepsilon>0)(\exists s\in S)s\in U_\varepsilon(a)\setminus\{a\}$

[Open/Closed] 3.4: *A subset S of a metric space X is…*

…closed iff S contains all it's Acc. Points: $(\forall x\in X\setminus S)(\exists\varepsilon>0)U_\varepsilon(x)\cap S=\varnothing$

…open iff $X\setminus S$ *is closed*: $(\forall s\in S)(\exists\varepsilon>0)U_\varepsilon(s)\subseteq S$

[Topology] 3.5: *Let O denote the set of open subsets of X then* (X,O) *is a topological space iff it satisfies*

(O1) $\varnothing\subseteq O\wedge X\subseteq O$ (O2) $F\subseteq O\Rightarrow\bigcup F\in O$ (O3) $F\subseteq O\wedge F$ finite $\Rightarrow\bigcap F\in O$

[Connected] 3.8: *A nonempty metric space X is…*

…disconnected iff: $(\exists S\subseteq X)S\neq\varnothing\wedge S\neq X\wedge S$ open $\wedge$ closed

…connected otherwise: $(\forall S\subseteq X)$S open $\wedge$ closed $\Rightarrow S=\varnothing\vee S=X$

[Preservance] 3.15: $(\forall f:X\mapsto Y)C\subseteq X$ connected $\wedge$ f continuous $\Rightarrow f(C)\subseteq Y$ connected

[Topological Characterization of Real Intervals] 3.9: $(\forall S\subseteq\mathbb{R})S$ connected $\Leftrightarrow S$ is an interval

[Inverse Image] 3.13: *For* $f:X\mapsto Y$*:* $(\forall S\subseteq Y)\ f^{-1}(S)=\{x\in X : f(x)\in S\}$

[Continuity] 3.12: *A function* $f:X\mapsto Y$ *between metric spaces is continuous iff any of the three defs:*

…metric: $(\forall\varepsilon>0)(\exists\delta>0)\ f(U_\delta(x))\subseteq U_\varepsilon(f(x))$

…topological 3.13: $(\forall S\subseteq Y)S$ open in $Y\Rightarrow f^{-1}(S)$ open in X

…convergence 3.13: $(\forall(x_n)_n:\mathbb{N}\mapsto X)\lim_n x_n=x\Rightarrow\lim_n f(x_n)=f(x)$

[Comp] 3.20: $f:X\mapsto Y, g:Y\mapsto Z$ continuous $\Rightarrow g\circ f:X\mapsto Z$ continuous

[Metric] *the metric function with a fixed point* $(\forall c\in X)\ f:X\mapsto\mathbb{R}:x\mapsto d(x,c)$ *is continuous*

[Add, Mult] 3.21: $f,g:X\mapsto\mathbb{C}$ continuous $\Rightarrow f+g,\ fg,(f/g$ iff $0\notin g(X))$ continuous

[Intermediate Value] 3.17: *For* $f:X\mapsto\mathbb{R}$ *continuous with* $X\subseteq\mathbb{R}$ *the following holds*

$(\forall[a,b]\subseteq X)\ f(a)\le y\le f(b)\Rightarrow(\exists x\subseteq[a,b])\,y=f(x)$

[Surjectivity] 3.18: *For* $f:I\mapsto J$ *continuous with nonempty real intervals* I,J *the following holds*

J open $\wedge\ f(I)$ unbounded $\Rightarrow f$ surjective

[Special Functions: Exp, Sin, Cos, Log]

$\exp:\mathbb{C}\mapsto\mathbb{C}:x\mapsto e^x=\lim_n\left(1+\frac{x}{n}\right)^n=\sum_{n=0}^{\infty}\frac{1}{n!}x^n\approx 2.7182...^x$

$\cos:[0,2\pi]\mapsto[-1,1]:x\mapsto\cos(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}=\frac{1}{2}\left(e^{xi}+e^{-xi}\right)$

$\sin:[0,2\pi]\mapsto[-1,1]:x\mapsto\sin(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}=\frac{1}{2i}\left(e^{xi}-e^{-xi}\right)$

$\ln:\mathbb{R}^+\mapsto\mathbb{R}:x\mapsto\ln(x)$ is the inverse of $\exp:\mathbb{R}\mapsto\mathbb{R}^+:x\mapsto e^x$

$e^{ix}=\cos(x)+i\sin(x)$

$e^{x+y}=e^xe^y;\ e^{x*\ln(a)}=a^x$

$\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)$

$\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)$

$\ln(xy)=\ln(x)+\ln(y);\ \frac{\ln y}{\ln a}=\log_a y$

All X are metric spaces, S subsets, $(c\in\mathbb{C})$ $(\varepsilon\in\mathbb{R})$ $(N,n,m\in\mathbb{N})$