Basic Set Theory

$\{\ldots\}$ **[Sets]** *Let X be a set.* ($\mathbb{N} \subseteq \mathbb{N}_0 \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$)

$\mathbb{N} \Leftrightarrow \{1, 1+1, 1+1+1, \ldots\}$ Fails: ADD(N)(I), MULT(I)(0)(S)

$\mathbb{N}_0 \Leftrightarrow \mathbb{N} \cup \{0\}$ Fails: ADD(I), MULT(I)

$\mathbb{Z} \Leftrightarrow \{x \in \mathbb{R} : x = 0 \vee x \in \mathbb{N} \vee -x \in \mathbb{N}\}$ Fails: MULT(I)

$\mathbb{Q} \Leftrightarrow \{x \in \mathbb{R} : (\exists z \in \mathbb{Z})(\exists n \in \mathbb{N})\, x = \frac{z}{n}\}$ Fails: LUB

$\bigcup U = \{x \in X : (\exists V \subseteq U)\, x \in V\}$ $\bigcap U = \{x \in X : (\forall V \subseteq U)\, x \in V\}$

(countable) X finite $\vee$ $|X| = |\mathbb{N}|$

$f(x)$ **[Functions]** *Let f be a function that maps X into Y* $f : X \to Y \Leftrightarrow (\forall x \in X)(\exists!\, y \in Y)\, f(x) = y$

(inj) $(\forall x, x' \in X)\, x \neq x' \Rightarrow f(x) \neq f(x')$ (inj') $(\exists g : Y \to X)\, g \circ f = id_X$

(sur) $(\forall y \in Y)(\exists x \in X)\, y = f(x)$ (sur') $(\exists g : Y \to X)\, f \circ g = id_Y$

(bij) $(\forall y \in Y)(\exists \mid x \in X)\, y = f(x)$ (bij') $(\exists f^{-1} : Y \to X)(\forall x \in X)\, x = f^{-1}(f(x)) \wedge (\forall y \in Y)\, y = f(f^{-1}(y))$

$(X, <)$ **[Order]** *Let (S,<) be a total order on the set S*

(t) $(\forall x, y \in S)(x < y) \wedge (y < z) \Rightarrow x < z$ (i) $(\forall x, y \in S)\, x < y \Rightarrow x \neq y$

(c) $(\forall x, y \in S)\, x \neq y \Leftrightarrow (x < y) \vee (y < x)$ (c') $(\forall x, y \in S)(x = y)\operatorname{xor}(x < y)\operatorname{xor}(y < x)$

[Bounds] *Let b element S be a bound of X subset S*

(ub) $\mathrm{B}(X) \Leftrightarrow \{b \in S : (\forall x \in X)\, x \leq b\}$ (max) $(\exists b \in \mathrm{B}(X))\, b \in X \Rightarrow b = \max X$

(sup) $[(\forall s \in S)\, s < b \Rightarrow (\exists x \in X)\, s < x] \Leftrightarrow b = \sup X$

(lub) $(\forall X \subseteq S)(X \neq \varnothing) \wedge (\mathrm{B}(X) \neq \varnothing) \Rightarrow (\exists s \in S)\, s = \sup X$

$\mathbb{K}$ **[Fields]** *Let K be a field then K satisfies ADD,MULT,DIST*

ADD (C) $(\forall x, y \in K)\, x + y = y + x$ (A) $(\forall x, y, z \in K)(x + y) + z = x + (y + z)$

(N) $(\exists 0 \in K)(\forall x \in K)\, 0 + x = x = x + 0$ (I) $(\forall x \in K)(\exists y \in k)\, x + y = y + x = 0$

MULT (C) $(\forall x, y \in K)\, xy = yx$ (A) $(\forall x, y, z \in K)(xy)z = x(yz)$

(N) $(\exists 1 \in K)(\forall x \in K)\, 1x = x = x1$ (I) $(\exists 0)(\forall x \in K)\, x \neq 0 \Rightarrow (\exists y \in K)\, xy = yx = 1$

(0) $00 = 0$ (S) $0 \neq 1$

(DIST) $(\forall x, y, z \in K)\, x(y + z) = xy + xz \wedge (x + y)z = xz + yz$

Ordered Fields also need: (OADD) $(\forall x, y, z \in K)\, x < y \Rightarrow x + z < y + z$

(OMULT) $(\forall x, y, z \in K)(x < y) \wedge (0 < z) \Rightarrow xz < yz$

$\mathbb{C}$ **[Complex]** $\mathbb{C} = \mathbb{R} \times \mathbb{R}$ *is a field satisfying ADD and MULT in the following way*

(ADD) $(u, v) + (x, y) = (u + x, v + y)$ Inverse: $(-x, -y)$

(MULT) $(u, v)(x, y) = (ux - vy, uy + vx)$ Inverse: $\left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right)$

Conjugate: $(\forall z \in \mathbb{C})\, z = z_R + z_I i \Rightarrow \bar{z} = z_R - z_I i$ Absolute: $(\forall z \in \mathbb{C})\, |z| \Leftrightarrow \sqrt{z\bar{z}} = \sqrt{z_R^2 + z_I^2}$

(X, d) **[Metric Space]** *Let X be a set and* $d : X^2 \to \mathbb{R}_0^+$ *be a metric then* (X, d) *is a metric space*

(pos) $(\forall x, y \in X)\, 0 \leq d(x, y)$ $(\forall x, y \in X)\, d(x, y) = 0 \Leftrightarrow x = y$

(sym) $(\forall x, y \in X)\, d(x, y) = d(y, x)$

(tri) $(\forall x, y, z \in X)\, d(x, z) \leq d(x, y) + d(y, z)$