Convergence of Sequences and Series

$\lim\limits_{n\to\infty}$ **Sequences**

[Characterized] 2.17: *A sequence* $(x_n)_n$ *in a metric space* (X,d) *converges iff*
$(\exists x)(\forall \varepsilon>0)(\exists N)(\forall n>N)\, d(x,x_n)<\varepsilon \Leftrightarrow x=\lim\limits_{n\to\infty} x_n$

[Monotone] 2.25: *A **bounded** and **eventually monotone** sequence of **reals** converges*
$(\exists x)\, x=\sup\{x_n\}_n \wedge (\exists N)(\forall n,m : N\le m\le n)\, x_m \le x_n \Rightarrow x=\lim_n x_n$

[Add, Mult] 2.40: *Let* $x=\lim_n x_n,\ y=\lim_n y_n$ *be limits of **convergent** complex sequences then*
$x+y=\lim_n (x_n+y_n)$ $\quad xy=\lim_n (x_n y_n)$ $\quad (\exists N)(\forall n>N)\, x_n\ne 0 \Rightarrow x^{-1}=\lim_n (x_n^{-1})$

[Cauchy] 2.41: *A **real** or **complex** sequence converges iff it is a **cauchy** sequence of the form*
$(x_n)_n \text{converges} \Leftrightarrow (\forall\varepsilon>0)(\exists N)(\forall n,k\in\mathbb{N}_0)\, n>N \Rightarrow |x_n - x_{n+k}|<\varepsilon$

[Upper Lim] 2.46: *Let* $(x_n)_n$ *be a **bounded real** sequence then*

$$x=\inf\{\sup\{x_n,x_{n+1},x_{n+2},\ldots\}\}_n=\overline{\lim}_n x_n \Leftrightarrow \begin{array}{l}(\forall\varepsilon>0)(\exists N)(\forall n>N)\, x_n<x+\varepsilon\\ (\forall\varepsilon>0,N)(\exists n>N)\, x-\varepsilon<x_n\end{array}$$

2.11: $x=\lim_n x_n \Leftrightarrow \overline{\lim}_n x_n = \underline{\lim}_n x_n = x$ $\qquad \overline{\lim}_n (-1)^n x_n = 0 \Rightarrow \lim (-1)^n x_n = 0$

[Examples] $0=\lim_n 1/n=\lim_n 1/\sqrt[n]{n!}=\lim_n n!/n^n=\lim_n c^n/n!$**;** $1=\lim_n \sqrt[n]{n}=\lim_n \sin c/c$**;**

$\sum\limits_n^\infty$ **Series**

[Partial Sum] 2.6: $(\forall x\ne 1)\sum_{m=0}^n cx^m = c\frac{1-x^{n+1}}{1-x}$ $\qquad$ **[Binom]** 2.11: $(a+b)^n=\sum_{k=0}^n \frac{n!}{k!(n-k)!}a^{n-k}b^k$

[Geometric] 2.22: $c=0\vee |x|<1 \Leftrightarrow \sum_{n=0}^\infty cx^n = \frac{c}{1-x}$ $\qquad$ **[A.Harmonic]** 2.30: $\sum_{n=1}^\infty (-1)^{n+1}\frac{1}{n}\approx .6931\ldots$

[Misc] 2.20: $\sum_n^\infty x_n \text{converges} \Rightarrow \lim_n x_n = 0$ $\qquad \lim_n x_n \ne 0 \Rightarrow \sum_n^\infty x_n \text{diverges}$

[Leibniz] 2.29: *An **alternating** infinite series converges if it satisfies these three conditions*
(i) $\sum_{n=0}^\infty x_n \text{alternates}$ $\qquad$ (ii) $\lim\limits_{n\to\infty} x_n = 0$ $\qquad$ (iii) $(\forall n)|x_n|>|x_{n+1}|$

[Comparison] 2.32: *Let* $\sum_n^\infty x_n$ *and* $\sum_n^\infty y_n$ *be **nonnegative real** series where* $(\exists N)(\forall n\ge N)\, x_n\le y_n$ *then*
$\sum_n^\infty y_n \text{converges} \Rightarrow \sum_n^\infty x_n \text{converges}$ $\qquad \sum_n^\infty x_n \text{diverges} \Rightarrow \sum_n^\infty y_n \text{diverges}$

[Add, Mult] 2.40: *Let* $x=\sum_n^\infty x_n,\ y=\sum_n^\infty y_n$ *be limits of **convergent** complex series then*
$x+y=\sum_n^\infty (x_n+y_n)$ $\qquad (\forall c\in\mathbb{C})\, cx=\sum_n^\infty cx_n$ $\qquad \sum_n^\infty |x_n|, \sum_n^\infty |y_n| \text{converge} \Rightarrow \sum_n^\infty |x_n y_n|$ converges

[Cauchy] 2.48: *A **real** or **complex** series converges iff it is a **cauchy** series of the form*
$\sum_n^\infty x_n \text{converges} \Leftrightarrow (\forall\varepsilon>0)(\exists N)(\forall n,k\in\mathbb{N}_0)\, n>N \Rightarrow |x_n+x_{n+1}+\ldots+x_{n+k}|<\varepsilon$

[Absolute] 2.50: $\sum_n^\infty |x_n|$ converges $\Rightarrow \sum_n^\infty x_n \text{converges}$

[Root-Test] 2.54: *For a complex series* $\overline{\lim}_n \sqrt[n]{|x_n|}<1 \Rightarrow \sum_n^\infty |x_n|$ converges

[Ratio-Test] 2.56: *For a series with* $x_n\ne 0$ $\left(\overline{\lim}_n \sqrt[n]{|x_n|} \le\right) \overline{\lim}_n \frac{|x_{n+1}|}{|x_n|}<1 \Rightarrow \sum_n^\infty |x_n|$ converges

[Power] 2.58: *For a **real** or **complex** power series* $p(x):=\sum_{n=0}^\infty a_n x^n$ $L:=\overline{\lim}_n \sqrt[n]{|a_n|}\ge 0$
$L=0\Rightarrow (\forall x)(\exists p(x))$ $\quad L>0 \Rightarrow (\forall x<L^{-1})(\exists p(x)) \wedge (\forall x>L^{-1})\neg(\exists p(x))$ $\quad \neg(\exists L)\Rightarrow (\forall x\ne 0)\neg(\exists p(x))$

[Convolution] 2.66: *For two **absolutely** convergent series* $\sum_{n=0}^\infty a_n, \sum_{n=0}^\infty b_n$ *this holds*

$$\left(\sum_{n=0}^\infty a_n\right)\left(\sum_{n=0}^\infty b_n\right)=\sum_{n=0}^\infty \sum_{p=0}^n a_p b_{n-p}$$

All $(c\in\mathbb{C})$ $(\varepsilon\in\mathbb{R})$ $(N,n,m\in\mathbb{N})$